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Multiple solutions to a class of inclusion problems with operator involving $p(x)$-Laplacian

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Abstract: In this paper, we prove the existence of at least two nontrivial solutions for a nonlinear elliptic problem involving $p(x)$-Laplacian-like operator and nonsmooth potentials. Our approach is variational and it is based on the nonsmooth critical point theory for locally Lipschitz functions.

Keywords: $p(x)$-Laplacian-like; differential inclusion; nonlinear eigenvalue problem; multiple solutions.

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§1 Introduction

In this paper we are concerned with the following Dirichlet-type differential inclusion problems

$$\begin{cases} -\mathrm{div}\left((1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}})|\nabla u|^{p(x)-2}\nabla u\right) \in \lambda\partial F(x,u), \text{ a.e. in } \Omega, \\ u=0, \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $\lambda > 0$ is a real number, $p(x) \in C(\overline{\Omega})$, $1 < p^- \leq p(x) < +\infty$ and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz with respect to the second variable (in general it can be nonsmooth), and $\partial F(x,t)$ is the subdifferential with respect to the t-variable in the sense of Clarke [1].

Parabolic and elliptic problems with variable exponents have attracted in recent years a lot of interest of mathematicians around the world. For example, [2–14] and the references therein. The wide study of such kind of problems is motivated by various applications related

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to electrorheological fluids (an important class of non-Newtonian fluids) [2, 15, 16], image processing [17], elasticity [18], and also mathematical biology [19].

In a recent paper [20], by using the nonsmooth three critical points theorem and assuming suitable conditions for nonsmooth potential F, we proved the existence of three solutions of (P). In this paper our goal is to prove the existence of at least two solutions for the problem (P) as the parameter $\lambda > \lambda_0$ for some constant λ_0.

Next, we assume that $F(x,t)$ satisfies the following general conditions:

($\mathbf{f_1}$) $|w| \leq c_1 + c_2|t|^{\alpha(x)-1}$, for almost all $x \in \Omega$, all $t \in \mathbb{R}$ and $w \in \partial F(x,t)$;

($\mathbf{f_2}$) There exist $\gamma \in C(\overline{\Omega})$ with $p^+ < \gamma(x) < p^*(x)$ and $\mu \in L^\infty(\Omega)$, such that

$$\limsup_{t\to 0} \frac{\langle w,t\rangle}{|t|^{\gamma(x)}} < \mu(x),$$

uniformly for almost all $x \in \Omega$ and all $w \in \partial F(x,t)$;

($\mathbf{f_3}$) There exist $t_0 > r_0 > 0$ and $x_0 \in \Omega$ such that

$$F(x,t_0) > \delta_0 > 0, \text{a.e. } x \in B_{r_0}(x_0),$$

where $B_{r_0}(x_0) := \{x \in \Omega : |x - x_0| \leq r_0\} \subset \Omega$.

The paper is organized as follows. We first introduce some basic preliminary results and a well-known lemma in Section 2, including the variable exponent Lebesgue and Sobolev spaces. In Section 3, we give the main result and its proof. In Section 4, we give the summary of this paper.

§2 Preliminaries

In this part, we introduce some definitions and results which will be used in the next section.

Firstly, we introduce some theories of Lebesgue–Sobolev space with variable exponent. The detailed description can be found in [21–24].

Write

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- = \min_{x\in\overline{\Omega}} h(x), \quad h^+ = \max_{x\in\overline{\Omega}} p(x) \text{ for any } h \in C_+(\overline{\Omega}).$$

Obviously, $1 < h^- \leq h^+ < +\infty$.

Denote by $\mathcal{U}$ (Ω) the set of all measurable real functions defined on Ω. Two functions in $\mathcal{U}(\Omega)$ are considered to be one element of $\mathcal{U}(\Omega)$, when they are equal almost everywhere.

For $p \in C_+(\overline{\Omega})$, define

$$L^{p(x)}(\Omega) = \{u \in \mathcal{U}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty\},$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)}$ =inf$\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}$, and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

Denote $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Hereafter, let

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)}, & p(x) < N, \\ +\infty, & p(x) \geq N. \end{cases}$$

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. Denote by $L^{q(x)}(\Omega)$ the conjugate Lebesgue space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then the Hölder type inequality

$$\int_\Omega |uv|dx \leq (\tfrac{1}{p^-} + \tfrac{1}{q^-})|u|_{L^{p(x)}(\Omega)}|v|_{L^{q(x)}(\Omega)}, \quad u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega)$$

holds. Furthermore, define mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)}dx,$$

then the following relations hold

$$| \, u \, |_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+},$$
$$| \, u \, |_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}.$$

Proposition 2.1 [21] If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.

Consider the following function:

$$J(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx, u \in W_0^{1,p(x)}(\Omega).$$

We know that (see [1]), $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$. If we denote $A = J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, then

$$\langle A(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) (\nabla u, \nabla v)_{\mathbb{R}^N} dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

Proposition 2.2 [24] Set $X = W_0^{1,p(x)}(\Omega)$, A is as above, then

(1) $A : X \to X^*$ is a convex, bounded and strictly monotone operator;

(2) $A : X \to X^*$ is a mapping of type $(S)_+$, i.e., $u_n \xrightarrow{w} u$ in X and $\limsup\limits_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$, implies $u_n \to u$ in X;

(3) $A : X \to X^*$ is a homeomorphism.

Let X be a Banach space and X^* be its topological dual space and we denote $\langle \cdot, \cdot \rangle$ as the duality bracket for pair (X^*, X). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, we can find a neighbourhood U of x and a constant $k > 0$ (depending on U), such that $|\varphi(y) - \varphi(z)| \leq k\|y - z\|, \ \forall y, z \in U$.

For a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we define

$$\varphi^0(x; h) = \limsup_{x' \to x; \lambda \downarrow 0} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is obvious that the function $h \mapsto \varphi^0(x; h)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$, defined by

$$\partial\varphi(x) = \{x^* \in X^*; \langle x^*, h\rangle \leq \varphi^0(x; h), \ \forall h \in X\}.$$

The multifunction $\partial\varphi : X \to 2^{X^*}$ is called the generalized subdifferential of φ.

If φ is also convex, then $\partial\varphi(x)$ coincides with subdifferential in the sense of convex analysis, defined by

$$\partial_C\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi(x+h) - \varphi(x) \text{ in } h \in X\}.$$

If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$.

A point $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. It is easily seen that, if $x \in X$ is a local minimum of φ, then $0 \in \partial\varphi(x)$.

A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the nonsmooth C-condition at level $c \in \mathbb{R}$(the nonsmooth C-condition for short), if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$, such that $\varphi(x_n) \to c$ and $(1 + \|x_n\|)m(x_n) \to 0$, as $n \to +\infty$, there is a strongly convergent subsequence, where $m(x_n) = \{\|x^*\|_* : x^* \in \partial\varphi(x_n)\}$. If this condition is satisfied at every level $c \in \mathbb{R}$, then we say that φ satisfies the nonsmooth C-condition.

Finally, in order to prove our result in the next section, we introduce the following lemma:

Lemma 2.1 [25] Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz function and $x_0, x_1 \in X$. If there exists a bounded open neighbourhood U of x_0, such that $x_1 \in X\backslash U$, $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\limits_{\partial U}\varphi$ and φ satisfies the nonsmooth C-condition at level c, where $c = \inf\limits_{\gamma\in\mathcal{T}}\max\limits_{t\in[0,1]}\varphi(\gamma(t))$, $\mathcal{T}= \{\gamma \in C([0,1];X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, then c is a critical value of φ and $c \geq \inf\limits_{\partial U}\varphi$.

§3 The main results and proof of the theorem

In this part, we will prove that for (P) there also exist two weak solutions for the general case.

Our hypotheses on nonsmooth potential $F(x,t)$ are as follows.

H(F): $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $F(x, 0) = 0$ a.e. on Ω and satisfies the following facts:

(1) for all $t \in \mathbb{R}$, $x \mapsto F(x,t)$ is measurable;

(2) for almost all $x \in \Omega$, $t \mapsto F(x,t)$ is locally Lipschitz.

We consider the energy function $\varphi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ for the problem (P), defined by

$$\varphi(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x, u(x))dx, \forall u \in W_0^{1,p(x)}(\Omega).$$

Lemma 3.1. Assume H(F) and $(\mathbf{f_1})$. Then φ is locally Lipschitz in $W_0^{1,p(x)}(\Omega)$.

Proof: By $J \in C^1(W_0^{1,p(x)}(\Omega), R)$, we have

$$J(u_1) - J(u_2) = J'(\overline{u}) \cdot (u_1 - u_2),$$

where $\overline{u} = tu_1 + (1-t)u_2$, $t \in (0,1)$.

Let $B_r = \{x \in X : \|u - u_0\|_{W_0^{1,p(x)}} \leq r\}$.

Note that B_r is w-compact. Then we obtain that there exists a positive constant M, such that $\|J'(\overline{u})\|_{W^{-1,q(x)}(\Omega)} \leq M$, for sufficiently small r.

Therefore, for any $u_1, u_2 \in B_r$, we have

$$\begin{aligned}|J(u_1) - J(u_2)| =& |J'(\overline{u}) \cdot (u_1 - u_2)|\\ \leq& \|J'(\overline{u})\|_{W^{-1,q(x)}(\Omega)}\|u_1 - u_2\|_{W_0^{1,p(x)}}\\ \leq& M\|u_1 - u_2\|_{W_0^{1,p(x)}}.\end{aligned}$$

On the other hand, by $(\mathbf{f_1})$ and Lebourg's mean value theorem we have

$$|F(x,u_1)-F(x,u_2)|\le c_1|u_1-u_2|+c_2|\overline{u}|^{\alpha(x)-1}|u_1-u_2|.$$

Hence,

$$\begin{aligned}&\left|\int_\Omega F(x,u_1)dx-\int_\Omega F(x,u_2)dx\right|\\ \le& c_1\int_\Omega|u_1-u_2|dx+c_2\int_\Omega|\overline{u}|^{\alpha(x)-1}|u_1-u_2|dx\\ \le& c_2|u_1-u_2|_{\alpha(x)}+c_4||\overline{u}|^{\alpha(x)-1}|_{\alpha'(x)}|u_1-u_2|_{\alpha(x)},\end{aligned}$$

where $\frac{1}{\alpha'(x)}+\frac{1}{\alpha(x)}=1$.

It is immediate that

$$\int_\Omega(|\overline{u}|^{\alpha(x)-1})^{\alpha'(x)}=\int_\Omega|\overline{u}|^{\alpha(x)}dx\le\begin{cases}|\overline{u}|^{\alpha^+}_{\alpha(x)}\le c\|\overline{u}\|^{\alpha^+}, & |\overline{u}|_{\alpha(x)}>1,\\ |\overline{u}|^{\alpha^+}_{\alpha(x)}\le c\|\overline{u}\|^{\alpha^-}, & |\overline{u}|_{\alpha(x)}<1.\end{cases}$$

is bounded.

So,

$$\left|\int_\Omega F(x,u_1)dx-\int_\Omega F(x,u_2)dx\right|\le c_5|u_1-u_2|_{\alpha(x)}\le c\|u_1-u_2\|,$$

since $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$ is a compact embedding.

Therefore, φ is locally Lipschitz.

Theorem 3.1. If H(j), $(\mathbf{f_1})$, $(\mathbf{f_2})$, $(\mathbf{f_3})$ hold and $\alpha^+<p^-$, then there exists $\lambda_0>0$ such that for each $\lambda>\lambda_0$, problem (P) has at least two nontrivial solutions.

Proof: The proof is divided into five steps as follows.

Step 1. We will show that φ is coercive in the step.

Firstly, on account of (f_1), we have

$$|F(x,t)|\le c_1|t|+c_2|t|^{\alpha(x)}, \tag{1}$$

for almost all $x\in\Omega$ and $t\in\mathbb{R}$.

Since $1<\alpha(x)\le\alpha^+<p^-<p^*(x)$, $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$, then there exists $c_6>0$ such that

$$|u|_{\alpha(x)}\le c_6\|u\|,\quad u\in W_0^{1,p(x)}(\Omega).$$

Therefore, for any $|u|_{\alpha(x)}>1$ and $\|u\|>1$, we have

$$\int_\Omega|u|^{\alpha(x)}dx\le|u|^{\alpha^+}_{\alpha(x)}\le c_6^{\alpha^+}\|u\|^{\alpha^+}. \tag{2}$$

In view of (1), (2), the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned}\varphi(u)=&\int_\Omega\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}\right)dx-\lambda\int_\Omega F(x,u)dx\\ \ge&\frac{2}{p^+}\int_\Omega|\nabla u|^{p(x)}dx-\lambda c_1\int_\Omega|u|dx-\lambda c_2c_6^{\alpha^+}\|u\|^{\alpha^+}\\ \ge&\frac{2}{p^+}\|u\|^{p^-}-2\lambda c_1|1|_{\alpha'(x)}|u|_{\alpha(x)}-\lambda c_2c_6^{\alpha^+}\|u\|^{\alpha^+}\\ \ge&\frac{2}{p^+}\|u\|^{p^-}-2\lambda c_1c_6|1|_{\alpha'(x)}\|u\|-\lambda c_2c_6^{\alpha^+}\|u\|^{\alpha^+}\to\infty,\text{ as }\|u\|\to\infty.\end{aligned}$$

Step 2. We will show that the φ is weakly lower semi-continuous.

Let $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$, by Proposition 2.1, we obtain the following results:

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega);$$
$$u_n \to u \text{ in } L^{p(x)}(\Omega);$$
$$u_n \to u \text{ for a.e. } x \in \Omega;$$
$$F(x, u_n(x)) \to F(x, u(x)) \text{ for a.e. } x \in \Omega.$$

Applying the Fatou Lemma, we have

$$\limsup_{n\to\infty} \int_\Omega F(x, u_n(x))dx \le \int_\Omega F(x, u(x))dx.$$

Thus,

$$\begin{aligned}\liminf_{n\to\infty} \varphi(u_n) &= \int_\Omega \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + \sqrt{1 + |\nabla u_n|^{2p(x)}} \right) dx - \lambda \limsup_{n\to\infty} \int_\Omega F(x, u_n)dx \\ &\ge \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x, u)dx = \varphi(u).\end{aligned}$$

Hence, by the Weierstrass Theorem, we deduce that there exists a global minimizer $u_0 \in W_0^{1,p(x)}(\Omega)$ such that

$$\varphi(u_0) = \min_{u \in W_0^{1,p(x)}(\Omega)} \varphi(u).$$

Step 3. We will show that there exists $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, $\varphi(u_0) < 0$.

In view of condition $(\mathbf{f_3})$, there exists $t_0 \in \mathbb{R}$ such that $F(x, t_0) > \delta_0 > 0$, a.e. $x \in B_{r_0}(x_0)$. It is clear that

$$0 < M_1 := \max_{|t| \le |\xi_0|} \{c_1|t| + c_2|t|^{\alpha^+}, c_1|t| + c_2|t|^{\alpha^-}\} < +\infty.$$

Now we denote

$$t_0 = \left(\frac{M_1}{\delta_0 + M_1} \right)^{\frac{1}{N}}, \quad K(t) := \left(\frac{t_0}{r_0(1-t)} \right)^{p^+}$$

and

$$\lambda_0 = \max_{t \in [t_1, t_2]} \frac{3K(t)(1 - t^N)}{\delta_0 t^N - M_1(1 - t^N)},$$

where $t_0 < t_1 < t_2 < 1$ and δ_0 is given in the condition $(\mathbf{f_3})$. A direct calculation shows that the function $t \mapsto \delta_0 t^N - M_1(1 - t^N)$ is positive whenever $t > t_0$ and $\delta_0 t_0^N - M_1(1 - t_0^N) = 0$. Thus λ_0 is well defined and $\lambda_0 > 0$.

Next, we will show that for each $\lambda > \lambda_0$, the problem (P) has two nontrivial solutions. In order to do this, for $t \in [t_1, t_2]$, we define

$$\xi_t(x) = \begin{cases} 0, & \text{if } x \in \Omega \backslash B_{r_0}(x_0), \\ t_0, & \text{if } x \in B_{tr_0}(x_0), \\ \dfrac{t_0}{r_0(1-t)}(r_0 - |x - x_0|), & \text{if } x \in B_{r_0}(x_0) \backslash B_{tr_0}(x_0). \end{cases}$$

Hypotheses $(\mathbf{f_1})$ and $(\mathbf{f_3})$ imply that

$$\begin{aligned}\int_\Omega F(x, \xi_t(x))dx =& \int_{B_{tr_0}(x_0)} F(x, \xi_t(x))dx + \int_{B_{r_0}(x_0) \backslash B_{tr_0}(x_0)} F(x, \xi_t(x))dx \\ \ge& w_N r_0^N t^N \delta_0 - M_1(1 - t^N) w_N r_0^N \\ =& w_N r_0^N (\delta_0 t^N - M_1(1 - t^N)).\end{aligned}$$

Thus, for $t \in [t_1, t_2]$,

$$
\begin{aligned}
\varphi(\xi_t) =& \int_\Omega \frac{1}{p(x)} \left(|\nabla \xi_t|^{p(x)} + \sqrt{1 + |\nabla \xi_t|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x, \xi_t(x)) dx \\
\leq& \frac{3}{p^-} \int_{B_{r_0}(x_0) \setminus B_{tr_0}(x_0)} |\nabla \xi_t|^{p(x)} dx - \lambda w_N r_0^N (\delta_0 t^N - M_1(1 - t^N)) \\
\leq& 3[\frac{t_0}{r_0(1-t)}]^{p^+} w_N r_0^N (1 - t^N) - \lambda w_N r_0^N (\delta_0 t^N - M_1(1 - t^N)) \\
=& w_N r_0^N [3K(t)(1 - t^N) - \lambda(\delta_0 t^N - M_1(1 - t^N))],
\end{aligned}
$$

which implies that $\varphi(\eta_t) < 0$ whenever $\lambda > \lambda_0$.

Step 4. We will check the C-condition in the following.

Let $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p(x)}(\Omega)$ be a sequence such that $\varphi(u_n) \to c$ and $(1 + \|u_n\|) m(u_n) \to 0$ as $n \to \infty$.

Moreover, since φ is coercive, it follows that $\{u_n\}_{n \geq 1}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Hence by passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$. Next we will prove that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$.

Since $W_0^{1,p(x)}(\Omega)$ is embedded compactly in $L^{p(x)}(\Omega)$, we obtain that $u_n \to u$ in $L^{p(x)}(\Omega)$. Moreover, since $\|u_n^*\|_* \to 0$, we get $|\langle u_n^*, u_n \rangle| \leq \varepsilon_n$.

Note that $u_n^* = A(u_n) - w_n$, we have

$$\langle A(u_n), u_n - u \rangle - \textstyle\int_\Omega w_n (u_n - u) dx \leq \varepsilon_n, \forall n \geq 1.$$

Moreover, $\int_\Omega w_n(u_n - u)dx \to 0$, since $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\{w_n\}_{n \geq 1}$ in $L^{p'(x)}(\Omega)$ are bounded, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Therefore,

$$\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0.$$

So using Proposition 2.2, we have $u_n \to u$ as $n \to \infty$. Thus φ satisfies the nonsmooth C-condition.

Step 5. We will show that there exists another nontrivial weak solution of problem (P).

From Lebourg's Mean Value Theorem, we obtain

$$F(x, t) - F(x, 0) = \langle w, t \rangle$$

for some $w \in \partial F(x, \vartheta t)$ and $0 < \vartheta < 1$. Thus, hypothesis $(\mathbf{f_2})$ implies that there exists $\beta \in (0, 1)$ such that

$$|F(x, t)| \leq |\langle w, t \rangle| \leq \mu(x) |t|^{\gamma(x)}, \quad \forall |t| < \beta \text{ and a.e. } x \in \Omega. \tag{3}$$

It follows from the conditions $(\mathbf{f_1})$ and $1 < \alpha^- \leq \alpha^+ < p^- \leq p^+ < \gamma(x) < p^*(x)$ that for all $|t| > \beta$ and a.e. $x \in \Omega$,

$$
\begin{aligned}
|F(x, t)| \leq& c_1 |t| + \frac{c_2}{\alpha(x)} |t|^{\alpha(x)} \\
\leq& c_1 |t| + c_2 |t|^{\alpha(x)} \\
\leq& \left(\frac{c_1}{\beta^{\gamma(x) - 1}} + \frac{c_2}{\beta^{\gamma(x) - \alpha(x)}} \right) |t|^{\gamma(x)} \\
\leq& \left(\frac{c_1}{\beta^{\gamma^+ - 1}} + \frac{c_2}{\beta^{\gamma^+ - \alpha^-}} \right) |t|^{\gamma(x)},
\end{aligned}
$$

this together with (3) yields that for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$,

$$|F(x,t)| \leq \left(\mu(x) + \frac{c_1}{\beta\gamma^+ - 1} + \frac{c_2}{\beta\gamma^+ - \alpha^-}\right)|t|^{\gamma(x)} \leq c_3|t|^{\gamma(x)},$$

for some positive constant c_3.

Note that $p^+ < \gamma(x) < p^*(x)$, then by Proposition 2.1 we have the continuous embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\gamma(x)}(\Omega)$. That is, there exists c_4 such that

$$|u|_{\gamma(x)} \leq c_4\|u\|, \forall u \in W_0^{1,p(x)}(\Omega).$$

For all $\lambda > \lambda_0$, $\|u\| < 1$ and $|u|_{\gamma(x)} < 1$, we have

$$\begin{aligned}
\varphi(u) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}}\right)dx - \lambda\int_\Omega F(x,u(x))dx \\
&\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \lambda c_3\int_\Omega |u(x)|^{\gamma(x)}dx \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - c_3c_4^{\gamma^-}\|u\|^{\gamma^-}.
\end{aligned}$$

Therefore, for $\rho > 0$ small enough, there exists a $\nu > 0$ such that

$$\varphi(u) > \nu, \text{ for } \|u\| = \rho$$

and $\|u_0\| > \rho$. So by the Nonsmooth Mountain Pass Theorem (cf. Lemma 2.1), we can get $u_1 \in W_0^{1,p(x)}(\Omega)$ satisfies

$$\varphi(u_1) = c > 0 \text{ and } m(u_1) = 0.$$

So, u_1 is another nontrivial critical point of φ.

Remark 3.1. The result in this paper is different from the one in [20] since the assumption on the nonsmooth potential function F is different. In fact, our conditions ($\mathbf{f_1}$)–($\mathbf{f_3}$) are weaker than conditions ($\mathbf{h_1}$)–($\mathbf{h_3}$) in [20]. For example, we can find a nonsmooth potential function satisfying the hypothesis of our Theorem 3.1. But the function does not satisfy conditions Theorem 3.1 of Zhou and Ge [20]. For more details, please see (2) in the Summary.

So far the results involved potential functions exhibiting $p(x)$-sublinearity. The next theorem concerns problems where the potential function is $p(x)$-superlinear.

Theorem 3.2. Let us suppose that H(F), ($\mathbf{f_1}$), ($\mathbf{f_2}$), ($\mathbf{f_3}$) hold $\alpha^- > p^+$ and the following condition ($\mathbf{f_4}$) hold,

($\mathbf{f_4}$) For almost all $x \in \Omega$ and all $t \in \mathbb{R}$, we have

$F(x,t) \leq \nu(x)$ with $\nu \in L^{\beta(x)}(\Omega)$, $1 \leq \beta(x) < p^-$.

Then there exists a $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, the problem (P) has at least two nontrivial solutions.

Proof: The steps are similar to those of Theorem 3.1. In fact, we only need to modify Step 1 and Step 4 as follows: $(1')$ Show that φ is coercive under the condition ($\mathbf{f_4}$); $(4')$ Show that there exists a second nontrivial solution under the conditions ($\mathbf{f_1}$) and ($\mathbf{f_2}$). Then from Steps $(1')$, 2, 3 and $(4')$ above, the problem (P) has at least two nontrivial solutions.

Step $1'$. Due to the assumption $(\mathbf{f_4})$, for all $u \in W_0^{1,p(x)}(\Omega)$, $\|u\| > 1$, we have

$$\begin{aligned}\varphi(u) =& \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x, u(x)) dx \\ \geq & \frac{1}{p^+} \|u\|^{p^-} - \lambda \int_\Omega \nu(x) dx \to \infty, \quad \text{as } \|u\| \to \infty.\end{aligned}$$

Step $4'$. By hypothesis $(\mathbf{f_1})$ and the mean value theorem for locally Lipschitz functions, we have

$$\begin{aligned}F(x,t) \leq & c_1 |t| + c_2 |t|^{\alpha(x)} \\ \leq & c_1 |\frac{t}{\beta}|^{\alpha(x)-1} |t| + c_2 |t|^{\alpha(x)} \\ = & c_1 |\frac{1}{\beta}|^{\alpha^+ -1} |t|^{\alpha(x)} + c_2 |t|^{\alpha(x)} \\ = & c_5 |t|^{\alpha(x)}\end{aligned} \tag{4}$$

for a.e. $x \in \Omega$, all $|t| \geq \beta$ with $c_5 > 0$.

Combining (3) and (4), it follows that

$$|F(x,t)| \leq \mu(x) |t|^{\gamma(x)} + c_5 |t|^{\alpha(x)}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

Thus, for all $\lambda > \lambda_0$, $\|u\| < 1$, $|u|_{\gamma(x)} < 1$ and $|u|_{\alpha(x)} < 1$, we have

$$\begin{aligned}\varphi(u) =& \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x, u(x)) dx \\ \geq & \frac{1}{p^+} \|u\|^{p^+} - \lambda \int_\Omega \mu(x) |u|^{\gamma(x)} dx - \lambda c_5 \int_\Omega |u|^{\alpha(x)} dx \\ \geq & \frac{1}{p^+} \|u\|^{p^+} - \lambda c_6 \|u\|^{\gamma^-} - \lambda c_7 \|u\|^{\alpha^-}\end{aligned}$$

So, for $\rho > 0$ small enough, there exists a $\nu > 0$ such that

$$\varphi(u) > \nu, \text{ for } \|u\| = \rho$$

and $\|u_0\| > \rho$. Arguing as in proof of Step 4 of Theorem 3.1, we conclude that φ satisfies the nonsmooth C-condition. Furthermore, by the Nonsmooth Mountain Pass Theorem (cf. Lemma 2.1), we can conclude that $u_1 \in W_0^{1,p(x)}(\Omega)$ satisfies

$$\varphi(u_1) = c > 0 \text{ and } m(u_1) = 0.$$

So, u_1 is second nontrivial critical point of φ.

Remark 3.2. We shall give an example in (3) in the Summary.

§4 Summary

(1) If $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, then $\partial F(x,t) = \{f(x,t)\}$. Therefore by Theorem 3.1 we can show the existence of two weak solutions of the following Dirichlet problem involving the $p(x)$-Laplacian-like

$$\begin{cases} -\operatorname{div}\left((1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}) |\nabla u|^{p(x)-2} \nabla u \right) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P_2}$$

In [24], Manuela Rodrigues was able to prove that, under suitable conditions, the problem (P_2) might have at least one solution, or have infinite number of solutions.

(2) We give an example in the following to illustate our viewpoint in Remark 3.1. Let $p^- >$max $\{\alpha^+, \theta^+\}$ and consider the following nonsmooth locally Lipschitz function:

$$F(x,t) = \begin{cases} t^{\gamma(x)}, & 0 \le t < 1, \\ -(-t)^{\gamma(x)}, & 0 \ge t > -1, \\ \max\{|t-1|^{\theta(x)}, |t-1|^{\alpha(x)}\} + 1, & t \ge 1, \\ \max\{|t+1|^{\theta(x)}, |t+1|^{\alpha(x)}\} - 1, & t \le -1, \end{cases}$$

where $\inf\limits_{x\in\Omega}(\alpha(x) - \theta(x)) > 0$, $\theta^- > 1$ and $\theta^+ < \alpha^-$.

We can choose $q(x) = \gamma(x)$, then $\limsup\limits_{t\uparrow 0} \dfrac{F(x,t)}{|t|^{q(x)}} = -1$ and $\limsup\limits_{t\downarrow 0} \dfrac{F(x,t)}{|t|^{q(x)}} = 1$ uniformly a.e. $x \in \Omega$.

Obviously, $t \mapsto F(x,t)$ is locally Lipschitz. Then

$$\partial F(x,t) = \begin{cases} \gamma(x)t^{\gamma(x)-1}, & 0 \le t < 1, \\ \gamma(x)(-t)^{\gamma(x)-1}, & -1 < t \le 0, \\ \theta(x)(t-1)^{\theta(x)-1}, & 1 < t < 2, \\ -\theta(x)(-t-1)^{\theta(x)-1}, & -1-1 < t < -1, \\ \alpha(x)(t-1)^{\alpha(x)-1}, & t > 2, \\ -\alpha(x)(-t-1)^{\alpha(x)-1}, & t < -2, \\ [0, \gamma(x)], & t = \pm 1, \\ [\theta(x), \alpha(x)], & t = 2, \\ [-\alpha(x), -\theta(x)], & t = -2, \end{cases}$$

Hence, for any $w \in \partial F(x,t)$, we have

$$|w| \le \begin{cases} \gamma(x)t^{\alpha(x)-1}t^{\gamma(x)-\alpha(x)} \le \gamma^+|t|^{\alpha(x)-1}, & 0 \le t < 1, \\ \gamma(x)(-t)^{\alpha(x)-1}(-t)^{\gamma(x)-\alpha(x)} \le \gamma^+|t|^{\alpha(x)-1}, & -1 < t \le 0, \\ \theta(x)(t-1)^{\theta(x)-1} < \theta^+ < \theta^+|t|^{\alpha(x)-1}, & 1 < t < 2, \\ \theta(x)(-t-1)^{\theta(x)-1} < \theta^+ < \theta^+|t|^{\alpha(x)-1}, & -2 < t < -1, \\ \alpha(x)(t-1)^{\alpha(x)-1} \le \alpha^+|t|^{\alpha(x)-1}, & t > 2, \\ \alpha(x)(-t-1)^{\alpha(x)-1} \le \alpha^+|t|^{\alpha(x)-1}, & t < -2, \\ [0, \gamma(x)] \le \gamma^+, & t = \pm 1, \\ [\theta(x), \alpha(x)] \le \alpha^+, & t = \pm 2. \end{cases}$$

Therefore,

$|w| \le (\gamma^+ + \alpha^+) + (\gamma^+ + \alpha^+ + \theta^+)|t|^{\alpha(x)-1}$, $\forall w \in \partial F(x,t)$,

$\limsup\limits_{t\downarrow 0} \frac{<w,t>}{|t|^{\gamma(x)}} = \lim\limits_{t\downarrow 0} \frac{\gamma(x)t^{\gamma(x)}}{t^{\gamma(x)}} = \gamma(x)$ and

$\limsup\limits_{t\uparrow 0} \frac{\gamma(x)(-t)^{\gamma(x)-1}t}{(-t)^{\gamma(x)}} = \lim\limits_{t\uparrow 0} \frac{-\gamma(x)(-t)^{\gamma(x)}}{(-t)^{\gamma(x)}} = -\gamma(x)$, uniformly for almost all $x \in \Omega$ and all $w \in \partial j(x,t)$.

(3) We can find the following nonsmooth, locally Lipschitz function satisfying the conditions stated in Theorem 3.2:

$$F(x,t) = \begin{cases} -\sin(\frac{\pi}{4}|t|^{\gamma(x)}), & |t| \leq 1, \\ \dfrac{1}{\sqrt{2|t|}}, & |t| > 1, \end{cases}$$

It is clear that $F(x,0) = 0$ for a.e. $x \in \Omega$, thus hypotheses H(F) is satisfied. A direct verification shows that conditions $(\mathbf{f_3})$ and $(\mathbf{f_4})$ are satisfied. Note that

$$\partial F(x,t) = \begin{cases} \{-\frac{\pi}{4}\gamma(x)t^{\gamma(x)-1}\cos(\frac{\pi}{4}t^{\gamma(x)})\}, & 0 \leq t < 1, \\ \{\frac{\pi}{4}\gamma(x)(-t)^{\gamma(x)-1}\cos(\frac{\pi}{4}(-t)^{\gamma(x)})\}, & -1 < t \leq 0, \\ [-2^{-\frac{3}{2}}, 0], & t = 1, \\ [0, 2^{-\frac{3}{2}}], & t = -1, \\ \{-(2t)^{-\frac{3}{2}}\}, & t > 1, \\ \{(-2t)^{-\frac{3}{2}}\}, & t < -1, \end{cases}$$

So, for any $w \in \partial F(x,t)$, we have

$$|w| \leq (\frac{\pi}{2}\gamma(x) + \frac{1}{2})|t|^{\gamma(x)-1},$$

$$\lim_{t\downarrow 0} \frac{-\frac{\pi}{4}\gamma(x)t^{\gamma(x)-1}t\cos(\frac{\pi}{4}t^{\gamma(x)})}{t^{\gamma(x)}} = -\frac{\pi}{4}\gamma(x),$$

$$\lim_{t\downarrow 0} \frac{\frac{\pi}{4}\gamma(x)(-t)^{\gamma(x)-1}t\cos(\frac{\pi}{4}(-t)^{\gamma(x)})}{(-t)^{\gamma(x)}} = -\frac{\pi}{4}\gamma(x),$$

which shows that assumptions $(\mathbf{f_1})$ and $(\mathbf{f_2})$ are fulfilled.

References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1993.

[2] E. Acerbi, G. Mingione, G. A. Seregin, Regularity results for parabolic systems related to a class of non-Newtonian fluids, Ann. Inst. H. Poincaré Anal. Non Linéaire, 21 (2004), 25–60.

[3] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer-Verlag, Berlin, Heidelberg, 2011.

[4] C. Zhang, S. Zhou, Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data, J. Differential Equations. 248 (2010) 1376–1400.

[5] J. Simsen, M. S. Simsen, PDE and ODE limit problems for $p(x)$-Laplacian parabolic equations, J. Math. Anal. Appl., 383 (2011) 71–81.

[6] P. Harjulehto, P. Hästö, V. Latvala and O. Toivanen, The strong minimum principle for quasisuperminimizers of non-standard growth, Ann. Inst. H. Poincaré Anal. Non Linéaire. 28 (2011), 731–742.

[7] N. Costea, G. Moroşanu, A multiplicity result for an elliptic anisotropic differential inclusion involving variable exponents, Set-Valued. Var. Anal., (21) 2012, 311–332.

[8] S. H. Liang, J. H. Zhang, Infinitely many small solutions for the $p(x)$-Laplacian operator with nonlinear boundary conditions, Ann. Mat. Pur. Appl., 192 (2013), 1–16.

[9] W. S. Niu, Long-time behavior for a nonlinear parabolic problem with variable exponents, J. Math. Anal. Appl., 393 (2012), 56–65.

[10] B. Ge, Q. M. Zhou, X. P. Xue, Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$ involving $p(x)$-Laplacian and oscillatory terms, Z. Angew. Math. Phys., (63) 2012, 691–711.

[11] B. Ge, Q. M. Zhou, X. P. Xue, Existence of at least five solutions for a differential inclusion problem involving the $p(x)$-Laplacian, Nonlinear Anal.-Real World Appl., (12) 2011, 2304–2318.

[12] J. X. Yin, J. K. Li, Y. Y. Ke, Existence of positive solutions for the $p(x)$-Laplacian equation, Rocky Mountain J. Math., (42) 2012, 1675–1758.

[13] C. Ji, F. Fang, Infinitely many solutions for the $p(x)$-Laplacian equations without (AR)-type growth condition, Ann. Pol. Math., (105) 2012, 89–99.

[14] X. Fan, Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form, J. Differential Equations. 235 (2007) 397–417.

[15] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory, in: Lecture Notes in Math.*, Springer, Berlin, 2000.

[16] K. Rajagopal, M. Růžička, Mathematical modelling of electro-rheological fluids, Contin. Mech. Thermodyn. 13 (2001) 59–78.

[17] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math., 66 (2006), 1383–1406.

[18] V. V. Zhikov, On the density of smooth functions in Sobolev–Orlicz spaces, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 310 (2004), 67–81.

[19] G. Fragnelli, Positive periodic solutions for a system of anisotropic parabolic equations, J. Math. Anal. Appl., 367 (2010), 204–228.

[20] Q. M. Zhou, B. Ge, Three solutions for inequalities Dirichlet problem Driven by $p(x)$-Laplacian-Like, Abstr. Appl. Anal., 2013 (2013) 6. Article ID 575328.

[21] X. L. Fan, Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problems. Nonlinear Anal., (52) 2003, 1843–1852.

[22] X. L. Fan, D. Zhao, On the generalized Orlicz–Sobolev spaces $W^{k,p(x)}(\Omega)$. J. Gansu Educ. College, (12) 1998, 1–6.

[23] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}$ and $W^{m,p(x)}$. J. Math. Anal. Appl., (263) 2001, 424–446.

[24] M. M. Rodrigues, Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators, Mediterr. J. Math., (9) 2012, 211–222.

[25] N. Kourogenic, N. S. Papageorgiou, Nonsmooth critical point theory and nonlinear elliptic equations at resonance, J. Aust. Math. Soc., 69 (A) (2000), 245–271.

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