Electronic Journal of Qualitative Theory of Differential Equations
2006, No. **3**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

AN APPLICATION OF THE ANTIMAXIMUM PRINCIPLE FOR A FOURTH ORDER PERIODIC PROBLEM

P. AMSTER AND P. DE NÁPOLI

Abstract. We study the existence of solutions for a periodic fourth order problem. We prove an associated uniform antimaximum principle and develop a method of upper and lower solutions in reversed order. Furthermore, by the quasilinearization method we construct an iterative sequence that converges quadratically to a solution.

1. Introduction

In this work, we study the fourth order problem

$$u^{(4)} + g(t, u, u'') = 0 \tag{1.1}$$

under periodic conditions

$$u^{(j)}(0) = u^{(j)}(T) \qquad \text{for } j = 0, 1, 2, 3. \tag{1.2}$$

In the last years there has been an increasing interest on higher order problems, both because of their intrinsic mathematical interest and their applications to different problems in Mathematical Physics (for example, in multi-ion electrodiffussion problems [17], beam theory [10] and quantum models for semiconductors [11]).

In order to study the existence of solutions of problem (1.1)-(1.2), we shall apply method of upper and lower solutions and the so-called quasilinearization technique.

The method of upper and lower solutions is one of the most extensively used tools in nonlinear analysis, both for ODE's and PDE's problems. There exists a vast literature on this subject (see for example [7] for a survey). It is worth to mention that this method has been applied mostly to second order equations, since it relies on the maximum principle associated to the problem (see however [9]). Recently, the case where the upper and lower solutions are in the reversed order

Key words and phrases. Fourth order periodic problems - Antimaximum principle - Quasilinearization method - Upper and lower solutions.

has also received some attention (see e.g. [2, 3, 4]). In this case, an antimaximum principle is required.

The antimaximum principle was originally derived by P. Clemént and L. Peletier [6], for the Dirichlet problem

$$\begin{cases} -\Delta u &= \lambda u + f(x) \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\quad \text{on} \quad \partial\Omega \end{cases} \tag{1.3}$$

in the following form: given $f \in L^\infty(\Omega)$ with $f \geqslant 0$ a.e. and $f \not\equiv 0$, there exists $\delta = \delta(f)$ such that if u is a solution of (1.3) with $\lambda_1 < \lambda < \lambda_1 + \delta$ then $u < 0$ in Ω. Here λ_1 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. After their pioneering work, the antimaximum principle was extended to other situations, like parabolic problems and problems involving the p-Laplacian.

For our purposes, we are interested in the case where the uniform antimaximum principle holds, i.e. where δ can be chosen independently of f. This situation is rather exceptional (see [8]). In our case, we derive a uniform antimaximum principle for fourth order equations with the aid of the Green's function (see Theorem 2.2). As an application, we develop the method of upper and lower solutions for (1.1)-(1.2) in the following "reversed order" cases:

(1) $\alpha'' + k\alpha \geq \beta'' + k\beta$ for some k such that $0 < k < \left(\frac{\pi}{2T}\right)^2$.
(2) $\alpha'' - k\alpha \leq \beta'' - k\beta$ for some k such that $0 < k < \left(\frac{\pi}{2T}\right)^2$.
(3) $\alpha'' + k\alpha \leq \beta'' + k\beta$ for some k such that $0 < k < \left(\frac{\pi}{2T}\right)^2$.

Here α and β are respectively a lower and an upper solution according to the definition given in section 3.

We remark that the proof of the antimaximum principle does not apply to linear differential operators involving odd order derivatives; for this reason we cannot allow the nonlinearity g in problem (1.1)-(1.2) to depend on u' and u'''.

Finally, assuming a convexity condition on g, we construct a sequence that converges quadratically to a solution of the problem by the quasilinearization method.

The quasilinearization method has been developed by Bellman and Kalaba [1], and generalized by Lakshmikantham [19, 20] (see also the monograph [16]). It has been applied to different nonlinear problems in the presence of an ordered couple of a lower and an upper solution. In the recent work [12] it has been successfully applied for a second order Neumann problem in the reversed order case. Quasilinearization method for first order periodic boundary problems has been considered in [14], [15]; for higher order periodic problems see [5], [18].

This paper is organized as follows. In section 2 we prove a version of the antimaximum principle fourth order problems. In section 3, we apply this antimaximum principle to obtain existence results for (1.1)-(1.2) by the method of upper and lower solutions. Finally, in section 4 we apply the method of quasilinearization to our problem, and obtain a quadratically convergent iterative scheme for constructing approximate solutions.

2. An antimaximum principle for fourth order problems

Throughout this work we shall denote by $H^k_{per}(0,T)$ the Sobolev space of T-periodic functions, namely

$$H^k_{per}(0,T) = \{u \in H^k(0,T) : u^{(j)}(0) = u^{(j)}(T) \text{ for } 0 \leqslant j \leqslant k-1\}.$$

In order to obtain an antimaximum principle for our problem, let us consider first the second order linear operator $L_\lambda : H^2_{per}(0,T) \to L^2(0,T)$ given by $L_\lambda u = u'' + \lambda u$. By standard results, if $\lambda \notin \left(\frac{2\pi}{T}\mathbb{N}\right)^2$ then for each $\varphi \in L^2(0,T)$ the equation $L_\lambda u = \varphi$ admits a unique T-periodic solution u, with

$$u(t) = \int_0^T G_\lambda(t,s)\varphi(s)ds.$$

Here G_λ denotes the Green's function given by

$$G_\lambda(t,s) = \begin{cases} \frac{1}{2\sqrt{|\lambda|}}\left(aC_\lambda(t-s) + S_\lambda(t-s)\right) & \text{if } t \geq s \\ \frac{1}{2\sqrt{|\lambda|}}\left(aC_\lambda(t-s) - S_\lambda(t-s)\right) & \text{if } t < s \end{cases} \tag{2.1}$$

where

$$a = \frac{S_\lambda(T)}{1 - C_\lambda(T)}$$

and the functions C_λ, S_λ are defined by

$$C_\lambda(t) = \begin{cases} \cos(\sqrt{\lambda}t) & \text{if } \lambda > 0 \\ \cosh(\sqrt{|\lambda|}t) & \text{if } \lambda < 0 \end{cases} \qquad S_\lambda(t) = \begin{cases} \sin(\sqrt{\lambda}t) & \text{if } \lambda > 0 \\ \sinh(\sqrt{|\lambda|}t) & \text{if } \lambda < 0. \end{cases}$$

If $\lambda < 0$, the classical maximum principle for the periodic problem says that if $L_\lambda u \geq 0$ then $u \leq 0$. On the other hand, if $0 < \lambda < \left(\frac{\pi}{2T}\right)^2$ it is straightforward to prove that $G_\lambda \geq 0$. As a consequence, we have the following antimaximum principle for the second order periodic problem:

Theorem 2.1. *Let* $0 < \lambda < \left(\frac{\pi}{2T}\right)^2$ *and assume that* $u \in H^2_{per}(0,T)$ *satisfies* $L_\lambda u \geq 0$ *a.e. in* $(0,T)$*. Then* $u \geq 0$.

Let us consider now the fourth order linear operator $L_{r,s} : H^4_{per}(0,T) \to L^2(0,T)$ given by

$$L_{r,s}u = u^{(4)} + ru'' + su.$$

We shall assume that $r^2 \geq 4s \neq 0$, then $L_{r,s}$ factorizes as $L_{r,s} = L_{\lambda^+}L_{\lambda^-}$, where

$$\lambda^\pm = \frac{r \pm \sqrt{r^2 - 4s}}{2}.$$

In particular, if $\lambda^\pm \notin \left(\frac{2\pi}{T}\mathbb{N}\right)^2$ then $L_{r,s}$ is invertible, with

$$L^{-1}_{r,s}(\varphi)(t) = \int_0^T G_{r,s}(t,\tau)\varphi(\tau)d\tau \tag{2.2}$$

and

$$G_{r,s}(t,\tau) = \int_0^T G_{\lambda^+}(t,\sigma)G_{\lambda^-}(\sigma,\tau)d\sigma.$$

If $r \leq 0 < s$, then $\lambda^- \leq \lambda^+ < 0$, and we obtain a "classical" maximum principle for the fourth order problem: if u is T-periodic and satisfies $L_{r,s}u \geq 0$ a.e. in $(0,T)$, then $u'' + \lambda^\pm u \leq 0$ and $u \geq 0$.

We are interested in the case in which this maximum principle does not hold, namely when $r > 0$ or $s < 0$ (and hence $\lambda^+ > 0$). A simple computation shows that if

$$\left(\frac{2T}{\pi}\right)^2 r < 1 + \min\left\{1, \left(\frac{2T}{\pi}\right)^4 s\right\}, \tag{2.3}$$

then $\lambda^- \leq \lambda^+ < \left(\frac{\pi}{2T}\right)^2$. Thus, we have the following antimaximum type result:

Theorem 2.2. *Assume that* $r^2 \geq 4s \neq 0$ *and that (2.3) holds. If* $u \in H^4_{per}(0,T)$ *satisfies* $L_{r,s}u \geq 0$ *a.e. in* $(0,T)$*, then:*

1. *If* $r, s > 0$*, then* $u'' + \lambda^\pm u \geq 0$ *and* $u \geq 0$.
2. *If* $r \leq 0$ *and* $s < 0$*, or if* $r \geq 0 > s$*, then* $u'' + \lambda^+ u \leq 0 \leq u'' + \lambda^- u$ *and* $u \leq 0$.

3. Upper and lower solutions in reversed order

In this section we prove some existence results based on the anti-maximum result given in Theorem 2.2. For completeness, let us recall the definition of upper and lower solutions for a fourth order periodic problem.

Definition: we say that $\alpha, \beta \in H^4_{per}(0,T)$ are respectively a *lower solution* and an *upper solution* for problem (1.1)-(1.2) if

$$\alpha^{(4)} + g(t,\alpha,\alpha'') \le 0 \le \beta^{(4)} + g(t,\beta,\beta'') \qquad \text{a.e. } in(0,T).$$

For simplicity we shall assume that g is continuous and twice continuously differentiable with respect to u and u''.

Then we have the following theorem:

Theorem 3.1. *Assume that α and β are respectively a lower and an upper solution as before, with*

$$\alpha'' + k\alpha \ge \beta'' + k\beta \qquad \textit{for some } k \textit{ such that } 0 < k < \left(\frac{\pi}{2T}\right)^2.$$

Furthermore, assume that

$$k\frac{\partial g}{\partial u''}(t,u,v) \le k^2 + \frac{\partial g}{\partial u}(t,u,v)$$

for $(t,u,v) \in \mathcal{C}$, where

$$\mathcal{C} = \left\{ (t,u,v) \in [0,T] \times \mathbb{R}^2 : \begin{array}{r} \alpha''(t) + k\alpha(t) \ge v + ku \ge \beta''(t) + k\beta(t) \\ \alpha(t) \ge u \ge \beta(t) \end{array} \right\}.$$

Then there exists u solution of (1.1)-(1.2) such that $(t,u(t),u''(t)) \in \mathcal{C}$ for $0 \le t \le T$.

Proof. Set $r < 0$ such that $r \le \frac{\partial g}{\partial u''}(t,u,v)$ for every $(t,u,v) \in \mathcal{C}$, and $s = k(r-k)$. Moreover, consider the convex subset $\mathcal{K} \subset C^2([0,T])$ defined by

$$\mathcal{K} = \{u \in C^2([0,T]) : (t,u(t),u''(t)) \in \mathcal{C} \text{ for } 0 \le t \le T\}.$$

Next, define a fixed point operator $T : \mathcal{K} \to C^2([0,T])$ in the following way: for $u \in \mathcal{K}$, let $v = Tu$ the unique solution of

$$v^{(4)} + rv'' + sv = ru'' + su - g(t,u,u'')$$

under periodic conditions

$$v^{(j)}(0) = v^{(j)}(T) \qquad \text{for } j = 0,1,2,3.$$

From (2.2), Tu is given by

$$Tu(t) = \int_0^T G_{r,s}(t,\tau)[ru''(\tau) + su(\tau) - g(\tau, u(\tau), u''(\tau))]d\tau.$$

By Arzelá-Ascoli Theorem it follows that $\overline{T(\mathcal{K})}$ is compact. Moreover, if $w = Tu - \beta$ then

$$w^{(4)} + rw'' + sw = r(u-\beta)'' + s(u-\beta) - g(t,u,u'') - \beta^{(4)}$$

$$\leq \left[r - \frac{\partial g}{\partial u''}(t,\xi)\right](u-\beta)'' + \left[s - \frac{\partial g}{\partial u}(t,\xi)\right](u-\beta)$$

for some mean value $\xi = (\xi_1(t), \xi_2(t))$. From the choice of r and s, and using the fact that $(u-\beta)'' \geq -k(u-\beta)$, we conclude that

$$w^{(4)} + rw'' + sw \leq \left(-k^2 - \frac{\partial g}{\partial u}(t,\xi) + k\frac{\partial g}{\partial u''}(t,\xi)\right)(u-\beta) \leq 0.$$

Hence, by Theorem 2.2 we deduce that $(Tu)'' + kTu \geq \beta'' + k\beta$, and $Tu \geq \beta$. In the same way, $(Tu)'' + kTu \leq \alpha'' + k\alpha$, and $Tu \leq \alpha$; thus, $T(\mathcal{K}) \subset \mathcal{K}$ and the proof follows from Schauder Theorem.

$\square$

The following two results can be proved in an analogous way:

Theorem 3.2. *Assume that α and β are respectively a lower and an upper solution as before, with*

$$\alpha'' - k\alpha \leq \beta'' - k\beta \qquad \text{for some } k \text{ such that } 0 < k < \left(\frac{\pi}{2T}\right)^2.$$

Furthermore, assume that

$$-k\frac{\partial g}{\partial u''}(t,u,v) \leq k^2 + \frac{\partial g}{\partial u}(t,u,v)$$

and $\frac{\partial g}{\partial u''}(t,u,v) \leq r < \left(\frac{\pi}{2T}\right)^2 - k$ for some nonnegative r and $(t,u,v) \in \mathcal{C}$, where

$$\mathcal{C} = \left\{(t,u,v) \in [0,T] \times \mathbb{R}^2 : \begin{array}{c} a''(t) - k\alpha(t) \leq v - ku \leq \beta''(t) - k\beta(t) \\ \alpha(t) \geq u \geq \beta(t) \end{array}\right\}.$$

Then there exists u solution of (1.1)-(1.2) such that $(t, u(t), u''(t)) \in \mathcal{C}$ for $0 \leq t \leq T$.

Theorem 3.3. *Assume that α and β are respectively a lower and an upper solution as before, with*

$$\alpha'' + k\alpha \leq \beta'' + k\beta \qquad \text{for some } k \text{ such that } 0 < k < \left(\frac{\pi}{2T}\right)^2.$$

Furthermore, assume that

$$k\frac{\partial g}{\partial u''}(t,u,v) \geq k^2 + \frac{\partial g}{\partial u}(t,u,v)$$

and $\frac{\partial g}{\partial u''}(t,u,v) \leq r < k + \left(\frac{\pi}{2T}\right)^2$ *for some* $r > k$ *and* $(t,u,v) \in \mathcal{C}$*, where*

$$\mathcal{C} = \left\{ (t,u,v) \in [0,T] \times \mathbb{R}^2 : \begin{array}{c} a''(t) + k\alpha(t) \leq v + ku \leq \beta''(t) + k\beta(t) \\ \alpha(t) \leq u \leq \beta(t) \end{array} \right\}.$$

Then there exists u solution of (1.1)-(1.2) such that $(t, u(t), u''(t)) \in \mathcal{C}$ for $0 \leq t \leq T$.

4. A quasilinearization method

In this section, we apply the quasilinearization method to problem (1.1)-(1.2). For simplicity, we consider only the situation of Theorem 3.1.

We shall define recursively a sequence of functions as follows. Let $u_0 = \beta$, and assume that u_n is known and verifies: $\alpha'' + k\alpha \geq u_n'' + ku_n \geq \beta'' + k\beta$. Next, define u_{n+1} as a T-periodic solution (not necessarily unique) of the quasilinear problem:

$$\begin{aligned} u^{(4)} + ru'' + su = rQ_n(t,u,u'') + sP_n(t,u) - [g(t,u_n,u_n'') \\ + \frac{\partial g}{\partial u}(t,u_n,u_n'')(P_n(t,u) - u_n) + \frac{\partial g}{\partial u''}(t,u_n,u_n'')(Q_n(t,u,u'') - u_n'')] \end{aligned} \tag{4.1}$$

where r and s are chosen as in the proof of Theorem 3.1, and

$$P_n(t,u) = \begin{cases} \alpha(t) & \text{if } u > \alpha(t) \\ u_n(t) & \text{if } u < u_n(t) \\ u & \text{otherwise,} \end{cases}$$

$$Q_n(t,u,v) = \begin{cases} \alpha''(t) + k(\alpha - P_n(t,u)) & \text{if } v + kP_n(t,u) > \alpha''(t) + k\alpha(t) \\ u_n''(t) + k(u_n - P_n(t,u)) & \text{if } v + kP_n(t,u) < u_n''(t) + ku_n(t) \\ v & \text{otherwise} \end{cases}$$

Note that the right-hand term of (4.1) is bounded, and hence the existence of u_{n+1} follows from Schauder Theorem by standard arguments.

In order to prove the convergence of $\{u_n\}$, we shall assume that the Hessian of g respect to u and u'' given by

$$Hg(t,u,v)(\eta_1,\eta_2)=\frac{\partial^2 g}{\partial u^2}(t,u,v)\eta_1^2+2\frac{\partial^2 g}{\partial u\partial u''}(t,u,v)\eta_1\eta_2+\frac{\partial^2 g}{\partial u''^2}(t,u,v)\eta_2^2$$

is nonnegative definite.

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold. Furthermore, assume that $Hg\geq 0$ on $\mathcal{C}$. Then the sequence given by the quasilinearization method (4.1) is well defined and converges to a solution u of problem (1.1)-(1.2) such that $\alpha\geq u\geq\beta$. Moreover, the convergence $u_n\to u$ is quadratic for the C^2-norm.*

Proof. Let us prove first that u_1 is an upper solution of the problem, with $\alpha''+k\alpha\geq u_1''+ku_1\geq u_0''+ku_0$. Indeed, we have that

$$u_1^{(4)}+ru_1''+su_1=rQ_0(t,u_1,u_1'')+sP_0(t,u_1)-g(t,u_0,u_0'')$$

$$-\frac{\partial g}{\partial u}(t,u_0,u_0'')(P_0(t,u_1)-u_0)-\frac{\partial g}{\partial u''}(t,u_0,u_0'')(Q_0(t,u_1,u_1'')-u_0'')$$

$$=ru_0''+su_0-g(t,u_0,u_0'')+\left[s-\frac{\partial g}{\partial u}(t,\xi)\right](P_0(t,u_1)-u_0)$$

$$+\left[r-\frac{\partial g}{\partial u''}(t,\xi)\right](Q_0(t,u_1,u_1'')-u_0'').$$

It is easy to see that $Q_0(t,u_1,u_1'')-u_0''\geq -k(P_0(t,u_1)-u_0)$, then as in Theorem 3.1 we conclude that

$$u_1^{(4)}+ru_1''+su_1\leq ru_0''+su_0-g(t,u_0,u_0'')\leq u_0^{(4)}+ru_0''+su_0.$$

It follows that $u_1''+ku_1\geq u_0''+ku_0$, and $u_1\geq u_0$. On the other hand, consider the Taylor expansion

$$g(t,u,v)=g(t,u_0,u_0'')+\frac{\partial g}{\partial u}(t,u_0,u_0'')(u-u_0)$$

$$+\frac{\partial g}{\partial u''}(t,u_0,u_0'')(v-u_0'')+R(t,u,v)$$

where $R(t,u,v)=\frac{1}{2}Hg(t,\xi)[u-u_0,v-u_0'']$ for some mean value $\xi=(\xi_1(t),\xi_2(t))$. By hypothesis, if $(t,u,v)\in\mathcal{C}$ the remainder R is nonnegative; hence, setting $u=P_0(t,u_1)$, $v=Q_0(t,u_1,u_1'')$ we obtain:

$$u_1^{(4)}+ru_1''+su_1\geq rQ_0(t,u_1,u_1'')+sP_0(t,u_1)-g(t,P_0(t,u_1),Q_0(t,u_1,u_1'')).$$

In the same way as before, for $w = u_1 - \alpha$, using the hypothesis and the fact that $Q_0(t, u_1, u_1'') - \alpha'' \leq -k(P_0(t, u_1) - \alpha)$ it follows that $w^{(4)} + rw'' + sw \geq 0$, and then $u_1'' + ku_1 \leq \alpha'' + k\alpha$, $u_1 \leq \alpha$.

Finally, as $(t, u_1, u_1'') \in \mathcal{C}$ we deduce that u_1 is a solution of the following linear equation:

$$u_1^{(4)} = -g(t, u_0, u_0'') - \frac{\partial g}{\partial u}(t, u_0, u_0'')(u_1 - u_0) - \frac{\partial g}{\partial u''}(t, u_0, u_0'')(u_1'' - u_0'').$$

Using again the Taylor expansion and the fact that Hg is nonnegative definite, we conclude that $u_1^{(4)} \geq -g(t, u_1, u_1'')$ and hence u_1 is an upper solution of the problem.

Repeating the procedure, we obtain a sequence $\{u_n\}$ such that

$$\beta'' + k\beta \leq u_n'' + ku_n \leq u_{n+1}'' + ku_{n+1} \leq \alpha'' + k\alpha$$

and

$$u_{n+1}^{(4)} = -g(t, u_n, u_n'') - \frac{\partial g}{\partial u}(t, u_n, u_n'')(u_{n+1} - u_n) - \frac{\partial g}{\partial u''}(t, u_n, u_n'')(u_{n+1}'' - u_n''). \tag{4.2}$$

Furthermore, $\beta \leq u_n \leq u_{n+1} \leq \alpha$, and u_n converges pointwise to some function u, $\beta \leq u \leq \alpha$ and hence u_n'' converges pointwise to some function v. As $\{u_n\}$ and $\{u_n''\}$ are clearly bounded, it follows that u_n is bounded in $H^4_{per}(0, T)$.

From the compact imbedding $H^4_{per}(0, T) \hookrightarrow C^3[0, T]$, there exists a subsequence u_{n_k} such that $u_{n_k} \to u$ in $C^3[0, T]$. We deduce that $v = u''$; moreover, if we consider any test function $\varphi \in C^4_{per}[0, T]$, then

$$\int_0^T u_{n+1}^{(3)} \varphi' \, dt =$$

$$\int_0^T [g(t, u_n, u_n'') + \frac{\partial g}{\partial u}(t, u_n, u_n'')(u_{n+1} - u_n) + \frac{\partial g}{\partial u''}(t, u_n, u_n'')(u_{n+1}'' - u_n'')]\varphi \, dt.$$

By dominated convergence, it follows that u is a weak solution of (1.1)-(1.2) and hence, by standard regularity results, a classical solution. Moreover, it is clear by construction that $\beta'' + k\beta \leq u'' + ku \leq \alpha'' + k\alpha$.

In order to prove that $u_n \to u$ quadratically for the C^2-norm, let us define the error term $\mathcal{E}_n := u - u_n$. Then $\mathcal{E}_n$ is nonincreasing, $\mathcal{E}_n \to 0$ and $\mathcal{E}_n'' \geq -k\mathcal{E}_n$. On the other hand,

$$\mathcal{E}_{n+1}^{(4)} = -g(t, u, u'') + g(t, u_n, u_n'') + \frac{\partial g}{\partial u}(t, u_n, u_n'')(u_{n+1} - u_n)$$

$$+\frac{\partial g}{\partial u''}(t,u_n,u_n'')(u_{n+1}''-u_n'')$$
$$=-\frac{1}{2}Hg(t,\xi)(\mathcal{E}_n,\mathcal{E}_n'')-\frac{\partial g}{\partial u}(t,u_n,u_n'')\mathcal{E}_{n+1}-\frac{\partial g}{\partial u''}(t,u_n,u_n'')\mathcal{E}_{n+1}''$$
for some $\xi=(\xi_1(t),\xi_2(t))$ between $(u_n(t),u_n''(t))$ and $(u(t),u''(t))$. As before, it follows that

$$\mathcal{E}_{n+1}^{(4)}+r\mathcal{E}_{n+1}''+s\mathcal{E}_{n+1}\le -\frac{1}{2}Hg(t,\xi)(\mathcal{E}_n,\mathcal{E}_n'').$$

Denote by ϕ the unique T-periodic solution of

$$\phi^{(4)}+r\phi''+s\phi=-\frac{1}{2}Hg(t,\xi)(\mathcal{E}_n,\mathcal{E}_n''). \tag{4.3}$$

By Theorem 2.2,

$$\mathcal{E}_{n+1}''+k\mathcal{E}_{n+1}\le\phi''+k\phi,\qquad \mathcal{E}_{n+1}\le\phi.$$

On the other hand,

$$\phi(t)=-\frac{1}{2}\int_0^T G_{r,s}(t,\tau)Hg(\tau,\xi)(\mathcal{E}_n,\mathcal{E}_n'')\,d\tau.$$

Then

$$\|\phi\|_{C([0,T])}\le c\left(\|\mathcal{E}_n\|_{C([0,T])}^2+\|\mathcal{E}_n''\|_{C([0,T])}^2\right)$$

for some constant c, and from (4.3) it follows that also

$$\|\phi''\|_{C([0,T])}\le c\left(\|\mathcal{E}_n\|_{C([0,T])}^2+\|\mathcal{E}_n''\|_{C([0,T])}^2\right)$$

for some constant c. Thus $\|\mathcal{E}_{n+1}\|_{C([0,T])}\le c\|\mathcal{E}_n\|_{C^2([0,T])}^2$ for some constant c. Moreover, as

$$-k\mathcal{E}_{n+1}\le\mathcal{E}_{n+1}''\le\phi''+k(\phi-\mathcal{E}_{n+1}),$$

we obtain that also

$$\|\mathcal{E}_{n+1}''\|_{C([0,T])}\le c\|\mathcal{E}_n\|_{C^2([0,T])}^2$$

for some constant c, and the proof is complete.

□

Remark 4.1. From equation (4.2) it turns out that each step of the quasilinearization method is indeed equivalent to a Newton iteration, once it is proved that $\alpha''+k\alpha\ge u_{n+1}''+ku_{n+1}\ge u_n''+ku_n$.

However, in general it is not possible to define directly a Newton iteration since the linear differential operator in (4.2) is not necessarily invertible.

ACKNOWLEDGEMENTS

The authors thank the anonymous referee for the careful reading of the manuscript and his/her fruitful comments.

This work was partially supported by Fundación Antorchas and CONICET.

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Pablo Amster (e-mail: pamster@dm.uba.ar)
Pablo De Nápoli (e-mail: pdenapo@dm.uba.ar)

Departamento de Matemática - FCEyN - Universidad de Buenos Aires,
Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina
CONICET

(Received May 16, 2005)