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System of singular second-order differential equations with integral condition on the positive half-line

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Abstract

In this work, we are concerned with the existence and the multiplicity of nontrivial positive solutions for a boundary value problem of a system of second-order differential equations subject to an integral boundary condition and posed on the positive half-line. The positive nonlinearities depend on the solution and their derivatives and may have space singularities. New existence results of single and multiple solutions are obtained by means of the fixed point index theory on special cones in some weighted Banach space. Examples with numerical computations are included to illustrate the obtained existence theorems. This paper surveys and generalizes previous works.

1 Introduction

In this paper, we are interested in the following nonlinear second-order boundary value system with an integral condition at positive infinity and posed on the positive half-line:

$$\begin{cases} -Y''(t) + k^2 Y(t) = F(t, Y(t), Y'(t)), & t \in I \\ Y(0) = 0, & \lim\limits_{t \to +\infty} Y(t)e^{-kt} = \int_0^{+\infty} g(s)Y(s)ds, \end{cases} \tag{1.1}$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}, \ Y' = \begin{pmatrix} y_1' \\ y_2' \\ \cdot \\ \cdot \\ \cdot \\ y_n' \end{pmatrix}, \ Y'' = \begin{pmatrix} y_1'' \\ y_2'' \\ \cdot \\ \cdot \\ \cdot \\ y_n'' \end{pmatrix},$$

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$$F(t,Y,Y') = \begin{pmatrix} \phi_1(t)f_1(t,y_1,\ldots,y_n,y'_1,\ldots,y'_n) \\ \phi_2(t)f_2(t,y_1,\ldots,y_n,y'_1,\ldots,y'_n) \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \phi_n(t)f_n(t,y_1,\ldots,y_n,y'_1,\ldots,y'_n) \end{pmatrix},$$

$g(t) = diag(g_1(t), g_2(t), \ldots, g_n(t))$, and $k > 0$. For $i = 1, 2, \ldots, n$, the nonnegative functions $\phi_i \in C(\mathbb{R}_+)$ are such that $\phi_i \not\equiv 0$ and $\int_0^{+\infty} e^{-ks}\phi_i(s)ds < \infty$. The functions $f_i = f_i(t, Y, Z) : \mathbb{R}_+ \times (\mathbb{R}_+^*)^n \times (\mathbb{R} \setminus \{0\})^n \longrightarrow \mathbb{R}_+$ are continuous and may be singular at $Y = 0_{\mathbb{R}^n}$ and $Z = 0_{\mathbb{R}^n}$. The scalar functions $g_i \in L^1(\mathbb{R}_+)$ (for $i \in \{1, \ldots, n\}$) satisfy

$$(\mathcal{H}_0) \qquad \int_0^{+\infty} (e^{ks} - e^{-ks}) g_i(s) ds < 1.$$

The interval $I := (0, +\infty)$ denotes the set of positive real numbers, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_+^* = (0, +\infty)$, and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. For brevity, $i \in \{1, \ldots, n\}$ will be written $i \in [1, n]$ throughout.

Throughout this paper, by a positive solution, it is meant a vector-function $Y = (y_1, y_2, \ldots, y_n) \in C^1([0, +\infty), \mathbb{R}_+^n)$ such that Y'' exists and Y satisfies (1.1) with $Y \geq 0_{\mathbb{R}^n}$ on $[0, +\infty)$. For $V = (v_1, \ldots, v_n), V' = (v'_1, \ldots, v'_n) \in \mathbb{R}_+^n$, $V \geq V'$ means that $v_i \geq v'_i$, for all $i \in [1, n]$ and $V > V'$ means that $v_i > v'_i$, $i \in [1, n]$, i.e. component-wise. Singular differential systems arise in many phenomena involved in applied mathematics and physics (gas dynamics, Newtonian fluid mechanics, nuclear physics,...). Boundary value problems (bvps for short) for such systems have been the subject of several research works during the last couple of years; many authors have been interested in investigating various questions relating to the existence as well as to the behavior of solutions (see, e.g., [2, 11, 20, 22, 23] and the references therein). Regarding the existence of positive solutions to systems of boundary value problems on finite intervals, we refer the reader to [18, 19, 27, 26, 32, 33] and related works. To deal with such problems, several methods have been employed so far; we quote the application of the fixed point theory in some special Banach spaces, the index fixed point theory on cones of special Banach spaces [6, 18, 19, 26], the upper and lower solutions method [27], as well as the monotone iterative techniques [32]. In 2001, Ma [21] studied the existence of positive solutions to the following second-order differential equation with an integral boundary condition at some end-point:

$$\begin{cases} y'' + a(t)f(y) = 0, \ 0 < t < 1, \\ y(0) = 0, \ y(1) = \int_\alpha^\beta h(t)y(t)dt, \end{cases}$$

where $[\alpha, \beta] \subset (0, 1)$ and the nonlinearity f has either superlinear or sublinear growth in terms of the variable y; the problem reduces to a three point bvp. The case of integral boundary conditions on a bounded interval is also considered in many recent papers (see, e.g., [5, 10, 30]). In [15, 16], Karakostas and Tsamatos weakened the restrictions on the nonlinear term f and considered boundary conditions given by a Riemann-Stieltjes integral, improving by the way some results obtained in [21]. This was further improved by Webb and Infante who used the index fixed point theory and gave a general method for solving problems with integral BCs of Riemann-Stieltjes type (see [24, 25]). In 2009, Xi, Jia, and Ji [28], using the Krasnosel'skii fixed point theorem, studied the existence of positive solutions to a boundary value problem for the following system of second-order differential equations with an integral boundary condition on the half-line:

$$\begin{cases} y_1''(t) + f_1(t, y_1(t), y_2(t)) = 0, \ t > 0, \\ y_2''(t) + f_2(t, y_1(t), y_2(t)) = 0, \ t > 0, \\ y_1(0) = y_2(0) = 0, \\ y_1'(+\infty) = \int_0^{+\infty} g_1(s)y_1(s)ds, \ y_2'(+\infty) = \int_0^{+\infty} g_2(s)y_2(s)ds. \end{cases}$$

Some of the results obtained were improved by the same authors in [29] where they employed a three-functional fixed point theorem in a cone due to Avery-Henderson and a fixed point theorem due to Avery-Peterson (see also [17] for such theory) in order to prove the existence of multiple positive solutions for n equations in the above system. The special cases regarding the following two equations

$$\begin{cases} -y'' + cy' + \lambda y = f(x, y), \ (c, \lambda > 0) \\ y(0) = y(+\infty) = 0 \end{cases}$$

and

$$\begin{cases} -x'' + k^2 x = m(t)f(t, x), \\ y(0) = y(+\infty) = 0 \end{cases}$$

are investigated in [8], [9].

In this work, the aim is to extend some of these works to the case of a system in which the positive nonlinearities do also depend on the first derivatives and are allowed to be singular at the space arguments; in addition the nonlinearities satisfy general growth conditions, including the polynomial one. We prove the existence and the multiplicity of nontrivial positive solutions in suitable cones of some weighted Banach space. The singularity involved in the nonlinearities is treated by approximating a fixed point operator with the help of some compactness arguments.

The proofs of our existence theorems rely on the Krasnosel'skii fixed theorem of cone expansion [1], a recent fixed point theorem of cone expansion and compression of functional type (see [3], [4]) and the Zima compactness criterion (see [34, 35]) adapted to our purpose. Recall that the fixed point theorem of cone expansion and compression of functional type is an extension of the fixed point theorem of cone expansion and compression of norm type which is usually referred to as Krasnosel'skii's fixed point theorem in cones (see [12, 13, 14]). It makes use of positive functionals instead of usual norms. More recently, Avery, Anderson, and Krueger [4] have used the convergence of Picard iterates to establish an extension of the fixed point theorem of cone expansion and compression of functional type by proving the convergence of sequences to the fixed point. This theorem will be used in proving existence of at least one solution.

Some preliminaries needed to transform System (1.1) into an abstract fixed point problem are presented in Section 2 together with some appropriate compactness criterion. In particular, important properties of the Green's function are given and the main assumptions are enunciated. Then, we construct a special cone in a weighted Banach space. The properties of a fixed point operator denoted A are studied in detail in the same Section. Section 3 is devoted to proving existence results of single and twin solutions when the nonlinearities are not singular. The cases when they are singular at $Y = 0_{\mathbb{R}^n}$ and $Z = 0_{\mathbb{R}^n}$ are studied in Section 4. Each example of application is illustrated with numerical computations.

2 Problem setting

2.1 Cones of solutions

First, we recall that a mapping in a Banach space is completely continuous if it is continuous and maps bounded sets into relatively compact sets. In the following, we give some definitions regarding cones and their properties. More details may be found in [7, 12, 31].

Definition 2.1. *A nonempty subset $\mathcal{P}$ of a Banach space X is called a cone if $\mathcal{P}$ is convex, closed, and satisfies the conditions:*

(i) $\alpha x \in \mathcal{P}$ for all $x \in \mathcal{P}$ and any real positive number α,

(ii) $x, -x \in \mathcal{P}$ imply $x = 0$.

Every cone $\mathcal{P} \subset X$ induces in X an ordering denoted $\leq$ and given by

$$x \leq y \text{ if and only if } y - x \in \mathcal{P}.$$

Definition 2.2. *A nonempty cone $\mathcal{P}$ of a real Banach space X is said to be normal if there exists a positive constant ξ such that $\|x + y\| \geq \xi$ for all $x, y \in \mathcal{P}$ with $\|x\| = \|y\| = 1$.*

The following result characterizes normal cones.

Proposition 2.1. *[12] The cone $\mathcal{P}$ is normal if and only if the norm of the Banach space X is semi-monotone; that is there exists a constant $N > 0$ such that $0 \leq x \leq y$ implies that $\|x\| \leq N\|y\|$.*

As for functions defined on cones, we have

Definition 2.3. *Let $\mathcal{P}$ be a cone in a real Banach space X and $\leq$ be the partial ordering defined by $\mathcal{P}$. Let D be a subset of X and $F: D \to X$ a mapping. Then the operator F is said to be increasing on D provided $x_1, x_2 \in D$ with $x_1 \leq x_2$ implies $Fx_1 \leq Fx_2$.*

Throughout this work, given some real parameter $\theta > k$, consider the weighted space:

$$\mathbb{X} = \left\{ \begin{array}{l} Y = (y_1, y_2, \ldots, y_n) : \ y_i \in C^1(\mathbb{R}_+, \mathbb{R}) \ \text{ and } \\ \sup\limits_{t \in \mathbb{R}_+} \left([\,|y_i(t)| + |y_i'(t)|\,]\, e^{-\theta t} \right) < \infty, \ \text{ for } \ i \in [1, n] \end{array} \right\}.$$

This is a Banach space with the norm

$$\|Y\|_\theta = \sum_{i=1}^{n} \|y_i\|_\theta, \ \text{ where } \|y_i\|_\theta = \sup_{t \in \mathbb{R}_+} (\,[\,|y_i(t)| + |y_i'(x)|\,]\, e^{-\theta t}).$$

Let $0 < \gamma < \delta$ be given positive numbers. The interval $[\gamma, \delta]$ will play a key role in estimating the solutions of System (1.1). Let

$$\left\{ \begin{array}{lcl} \Lambda_0 & = & \min(e^{-k\delta}, e^{k\gamma} - e^{-k\gamma}), \\ \Lambda_1 & = & \frac{k}{k+1}\, e^{-k\delta}, \\ \Lambda_2 & = & \min\left(\frac{1-k}{1+k}\, e^{-k\delta}, e^{k\gamma} + \frac{k-1}{k+1}\, e^{-k\gamma} \right). \end{array} \right. \tag{2.1}$$

Obviously, these constants are less than 1. Let $\mathcal{P}$ denote the positive cone defined in $\mathbb{X}$, for $k \geq 1$, by

$$\mathcal{P} = \left\{ Y \in \mathbb{X} : \ Y \geq 0_{\mathbb{R}^n} \text{ on } \mathbb{R}_+ \text{ and } \sum_{i=1}^{n} \min_{t \in [\gamma, \delta]} \left(2k y_i(t) + y_i'(t) \right) \geq \frac{\Lambda_1}{2} \|Y\|_\theta \right\}, \tag{2.2}$$

and, for $0 < k < 1$, by

$$\mathcal{P} = \left\{ Y \in \mathbb{X} : \ Y \geq 0_{\mathbb{R}^n} \text{ on } \mathbb{R}_+ \text{ and } \sum_{i=1}^{n} \min_{t \in [\gamma, \delta]} \left(y_i(t) + y_i'(t) \right) \geq \frac{\Lambda_2}{2} \|Y\|_\theta \right\}. \tag{2.3}$$

2.2 The Green's function

In this subsection, we study the linear problem associated with (1.1).

Lemma 2.1. *Assume that $(\mathcal{H}_0)$ holds. Let $V = (v_1, v_2, \ldots, v_n) \in C(\mathbb{R}_+, \mathbb{R}_+^n)$ be such that*

$$\int_0^{+\infty} e^{-ks} v_i(s) ds < \infty, \ i \in [1, n].$$

Then $Y \in C^1(\mathbb{R}_+, \mathbb{R}_+^n)$ is the unique solution of

$$\begin{cases} -Y'' + k^2 Y = V(t), \quad t \in I, \\ Y(0) = 0, \quad \lim_{t\to+\infty} Y(t)e^{-kt} = \int_0^{+\infty} g(s)Y(s)ds, \end{cases} \tag{2.4}$$

if and only if

$$Y(t) = \int_0^{+\infty} H(t,s)V(s)ds, \quad t \in \mathbb{R}_+, \tag{2.5}$$

where $H(t,s) = diag(H_1(t,s), H_2(t,s), \cdots, H_n(t,s))$ and the positive functions H_i ($i \in [1,n]$) are defined on $\mathbb{R}_+ \times \mathbb{R}_+$ by

$$H_i(t,s) = G(t,s) + \frac{(e^{kt} - e^{-kt}) \int_0^{+\infty} g_i(\tau)G(s,\tau)d\tau}{1 - \int_0^{+\infty} (e^{ks} - e^{-ks})g_i(s)ds}$$

and

$$G(t,s) = \frac{1}{2k} \begin{cases} e^{-ks}(e^{kt} - e^{-kt}), & 0 \le t \le s < +\infty, \\ e^{-kt}(e^{ks} - e^{-ks}), & 0 \le s \le t < +\infty, \end{cases} \tag{2.6}$$

with partial derivative with respect to t

$$G_t(t,s) = \frac{1}{2} \begin{cases} e^{-ks}(e^{kt} + e^{-kt}), & 0 \le t < s < +\infty, \\ -e^{-kt}(e^{ks} - e^{-ks}), & 0 \le s < t < +\infty. \end{cases} \tag{2.7}$$

Proof. Let $y_i \in C^1(\mathbb{R}_+)$ be an ith component of a solution of (2.4) and

$$u_i(s) = y_i'(s) - k y_i(s), \quad s \in \mathbb{R}_+. \tag{2.8}$$

Then

$$u_i'(s) + k u_i(s) = -v_i(s), \quad s \in \mathbb{R}_+. \tag{2.9}$$

Multiplying (2.9) by e^{ks} and integrating over $[0,t]$ yield

$$u_i(t) = e^{-kt}\left(u_i(0) - \int_0^t e^{ks} v_i(s)ds\right), \quad t \in I. \tag{2.10}$$

Similarly, multiplying (2.8) by e^{-ks} and integrating over $[0,t]$ guarantee that

$$y_i(t) = e^{kt}\left(y_i(0) + \int_0^t e^{-ks} u_i(s)ds\right), \quad t \in I. \tag{2.11}$$

From (2.10) and (2.11), we obtain that for $t \in I$

$$y_i(t) = \frac{1}{2k}\left(C_1 e^{kt} + C_2 e^{-kt} + \int_0^t \left(e^{-k(t-s)} - e^{k(t-s)}\right) v_i(s)ds\right), \tag{2.12}$$

where $C_1 = y_i'(0) + k y_i(0)$ and $C_2 = k y_i(0) - y_i'(0)$. In addition (2.4) yields

$$0 = y_i(0) = \frac{1}{2k}(C_1 + C_2) \Longrightarrow C_1 = -C_2.$$

Moreover, (2.11) gives

$$\frac{y_i(t)}{e^{kt}} = \frac{1}{2k}\left(C_1 + C_2 e^{-2kt} + e^{-2kt}\int_0^t e^{ks} v_i(s)ds - \int_0^t e^{-ks} v_i(s)ds\right).$$

We claim that

$$\lim_{t\to+\infty} e^{-2kt}\int_0^t e^{ks} v_i(s)ds = 0. \tag{2.13}$$

Indeed, if $\int_0^{+\infty} e^{ks} v_i(s)ds < \infty$, then (2.13) holds. If $\int_0^{+\infty} e^{ks} v_i(s)ds = \infty$, then

$$\lim_{t\to+\infty} e^{-2kt}\int_0^t e^{ks} v_i(s)ds = \lim_{t\to+\infty}\frac{\int_0^t e^{ks} v_i(s)ds}{e^{2kt}}.$$

Hence, from L'Hospital's rule, we get

$$\lim_{t\to+\infty}\frac{\int_0^t e^{ks} v_i(s)ds}{e^{2kt}} = \lim_{t\to+\infty}\frac{e^{kt} v_i(t)}{2k e^{2kt}} = \lim_{t\to+\infty}\frac{1}{2k}\, e^{-kt} v_i(t) = 0.$$

From (2.13) and the boundary conditions, we obtain the values

$$\begin{array}{rcl} C_1 & = & 2k\left(\int_0^{+\infty} g_i(s)\, y_i(s)ds + \int_0^{+\infty} e^{-ks} v_i(s)ds\right) \\ C_2 & = & -2k\left(\int_0^{+\infty} g_i(s)\, y_i(s)ds + \int_0^{+\infty} e^{-ks} v_i(s)ds\right). \end{array}$$

A substitution in (2.12) gives

$$\begin{array}{rcl} y_i(t) & = & e^{kt}\left(\int_0^{+\infty} g_i(s)\, y_i(s)ds + \int_0^{+\infty} e^{-ks} v_i(s)ds\right) \\ & & -e^{-kt}\left(\int_0^{+\infty} g_i(s)\, y_i(s)ds + \int_0^{+\infty} e^{-ks} v_i(s)ds\right) \\ & & +\frac{1}{2k}\int_0^t (e^{-k(t-s)} - e^{k(t-s)}) v_i(s)ds \\ & = & \left(e^{kt} - e^{-kt}\right)\int_0^{+\infty} g_i(s) y_i(s)ds + \int_0^{+\infty} e^{k(t-s)} v_i(s)ds \\ & & +\int_0^{+\infty} e^{-k(t+s)} v_i(s)ds + \frac{1}{2k}\int_0^t (e^{-k(t-s)} - e^{k(t-s)}) v_i(s)ds. \end{array}$$

Hence

$$y_i(t) = \left(e^{kt} - e^{-kt}\right)\int_0^{+\infty} g_i(s) y_i(s)ds + \int_0^{+\infty} G(t,s)\, v_i(s)ds, \tag{2.14}$$

where

$$G(t,s) = \frac{1}{2k}\begin{cases} e^{-ks}(e^{kt} - e^{-kt}), & 0 \le t \le s < +\infty, \\ e^{-kt}(e^{ks} - e^{-ks}), & 0 \le s \le t < +\infty. \end{cases}$$

Multiplying (2.14) by $g_i(.)$ and integrating over $[0, +\infty)$ yield

$$\begin{array}{rcl} \int_0^{+\infty} g_i(s) y_i(s)ds & = & \int_0^{+\infty}\left(g_i(s)\left(e^{ks} - e^{-ks}\right)\int_0^{+\infty} g_i(\tau) y_i(\tau)d\tau\right)ds \\ & & +\int_0^{+\infty}\left(g_i(s)\int_0^{+\infty} G(s,\tau)\, v_i(\tau)d\tau\right)ds \\ & = & \left(\int_0^{+\infty} g_i(\tau) y_i(\tau)d\tau\right)\left(\int_0^{+\infty} g_i(s)\left(e^{ks} - e^{-ks}\right)ds\right) \\ & & +\int_0^{+\infty}\left(\int_0^{+\infty} g_i(s) G(s,\tau)\, v_i(\tau)ds\right)d\tau. \end{array}$$

Then

$$\begin{aligned}
& \int_0^{+\infty} g_i(s)y_i(s)ds \left(1 - \int_0^{+\infty} g_i(s)\left(e^{ks} - e^{-ks}\right) ds\right) \\
= & \int_0^{+\infty} \left(\int_0^{+\infty} g_i(s)G(s,\tau)ds\right) v_i(\tau)d\tau \\
= & \int_0^{+\infty} \left(\int_0^{+\infty} g_i(\tau)G(\tau,s)d\tau\right) v_i(s)ds.
\end{aligned}$$

Hence

$$\int_0^{+\infty} g_i(s)y_i(s)ds = \int_0^{+\infty} \frac{\int_0^{+\infty} g_i(\tau)G(s,\tau)d\tau}{1-\int_0^{+\infty}(e^{ks}-e^{-ks})g_i(s)ds} v_i(s)ds.$$

By substitution in (2.14), we arrive at the formula

$$y_i(t) = \int_0^{+\infty} \left(\frac{\left(e^{kt}-e^{-kt}\right)\int_0^{+\infty} g_i(\tau)G(s,\tau)d\tau}{1-\int_0^{+\infty}(e^{ks}-e^{-ks})g_i(s)ds} + G(t,s)\right) v_i(s)ds,$$

i.e.

$$y_i(t) = \int_0^{+\infty} H_i(t,s)\, v_i(s)ds, \ i \in [1,n],$$

where

$$H_i(t,s) = G(t,s) + \frac{\left(e^{kt}-e^{-kt}\right)\int_0^{+\infty} g_i(\tau)G(s,\tau)d\tau}{1-\int_0^{+\infty}(e^{ks}-e^{-ks})g_i(s)ds}\cdot$$

Consequently,

$$Y(t) = \int_0^{\infty} H(t,s)V(s)ds, \quad t \in \mathbb{R}_+,$$

with $H(t,s) = diag(H_1(t,s), H_2(t,s), \cdots, H_n(t,s))$.

Conversely, let $y_i \in C^1(\mathbb{R}_+)$ be defined by (2.5). A direct differentiation of (2.5) gives for $i \in [1,n]$, and $t \geq 0$

$$\begin{aligned}
y_i'(t) &= \int_0^{\infty} \frac{\partial H_i}{\partial t}(t,s)v_i(s)ds, \\
&= \int_0^{\infty} \left(\frac{k\left(e^{kt}+e^{-kt}\right)\int_0^{+\infty} g_i(\tau)G(s,\tau)d\tau}{1-\int_0^{+\infty}(e^{ks}-e^{-ks})g_i(s)ds} + G_t(t,s)\right) v_i(s)ds,
\end{aligned} \tag{2.15}$$

where

$$G_t(t,s) = \frac{1}{2}\begin{cases} e^{-ks}(e^{kt}+e^{-kt}), & 0 \leq t < s < +\infty, \\ -e^{-kt}(e^{ks}-e^{-ks}), & 0 \leq s < t < +\infty. \end{cases}$$

Differentiating once again (2.15) leads to

$$\begin{aligned}
Y''(t) &= -V(s) + k^2 \int_0^{\infty} G(t,s)V(s)ds \\
&= -V(t) + k^2 Y(t), \ \ t \in \mathbb{R}_+.
\end{aligned}$$

Hence $Y \in C^1(\mathbb{R}_+)$ and Y satisfies (2.4). $\square$

Some fundamental properties of the function G are given hereafter. We omit the proofs.

Lemma 2.2. *The function G satisfies*

(a) $G(t,s) \geq 0, \quad \forall\, t,s \in \mathbb{R}_+$

(b) $G(t,s) \leq e^{\mu t} e^{-ks} G(s,s), \ \forall\, t,s \in \mathbb{R}_+; \ \forall\, \mu \geq k.$

(c) $G(x,s) \geq \Lambda_0 G(s,s) e^{-ks}, \ \forall\, t \in [\gamma,\delta]; \ \forall\, s \in \mathbb{R}_+.$

Denote by $G_t(s+0,s)$ the right-hand side derivative of (2.6) at (s,s) and $G_t(s-0,s)$ the left-hand side derivative at this point. The first partial derivative of G then satisfies

Lemma 2.3.

(a) $|G_t(t,s)| \leq e^{\mu t} e^{-ks}, \quad \forall\, t,s \in \mathbb{R}_+$ *and* $\forall\, \mu \geq k.$

(b) *Assume that* $k \geq 1$. *Then for every* $t \in [\gamma,\delta], s < t, \ s \in \mathbb{R}_+$ *and* $\mu \geq k$

$$\begin{array}{rcl} 2kG(t,s) + G_t(t,s) & \geq & \Lambda_1 \left[G(s,s) + |G_t(s+0,s)| \right] e^{-ks} \\ & \geq & \frac{\Lambda_1}{2} \left[G(t,s) + |G_t(t,s)| \right] e^{-\mu t}. \end{array}$$

(c) *Assume that* $0 < k < 1$. *Then for every* $t \in [\gamma,\delta], s < t, \ s \in \mathbb{R}_+$ *and* $\mu \geq k$

$$\begin{array}{rcl} G(t,s) + G_t(t,s) & \geq & \Lambda_2 \left[G(s,s) + |G_t(s+0,s)| \right] e^{-ks} \\ & \geq & \frac{\Lambda_2}{2} \left[G(t,s) + |G_t(t,s)| \right] e^{-\mu t}. \end{array}$$

Moreover, the first inequalities in (b), (c) remain valid if we take $G_t(s-0,s)$ *and* $s > t$ *instead.*

Let

$$\nu_i = \left(1 - \int_0^{+\infty} (e^{ks} - e^{-ks}) g_i(s) ds \right)^{-1} \ \text{and} \ \Theta_i(s) = \int_0^{+\infty} g_i(\tau) G(s,\tau) d\tau.$$

From [Lemma 2.2, (b)] and [Lemma 2.3, (a)], we get the following properties of any function $H = (H_1, \dots, H_n)$.

Lemma 2.4.

(a) $e^{\mu t} H_i(t,s) \geq 0, \quad \forall\, t,s \in \mathbb{R}_+.$

(b) *Assume that* $k \geq 1$. *Then for every* $t,s \in \mathbb{R}_+$ *and* $\mu \geq k$,

$$e^{-\mu t} \left(H_i(t,s) + |\frac{\partial}{\partial t} H_i(t,s)| \right) \leq e^{-ks} \left(G(s,s) + 1 \right) + 2 \max(k,1) \nu_i \Theta_i(s).$$

Proof. We prove **(b)**. For any $t, s \in \mathbb{R}_+$ and $\mu \geq k$, we have the estimates:

$$\begin{array}{rl} & e^{-\mu t}\left(H_i(t,s) + |\frac{\partial}{\partial t}H_i(t,s)|\right) \\ = & e^{-\mu t}\left(G(t,s) + |G_t(t,s)|\right) + e^{-\mu t}\left((e^{kt} - e^{-kt}) + k(e^{kt} + e^{-kt})\right)\nu_i\Theta_i(s), \end{array}$$

with, for $k \geq 1$

$$\begin{array}{rl} & e^{-\mu t}\left(G(t,s) + |G_t(t,s)|\right) + e^{-\mu t}\left((e^{kt} - e^{-kt}) + k(e^{kt} + e^{-kt})\right)\nu_i\Theta_i(s) \\ \leq & \left(G(s,s) + 1\right)e^{-ks} + \left((k+1)e^{(k-\mu)t} + (k-1)e^{-(k+\mu)t}\right)\nu_i\Theta_i(s), \end{array}$$

and for $0 < k < 1$

$$\begin{array}{rl} & e^{-\mu t}\left(G(t,s) + |G_t(t,s)|\right) + e^{-\mu t}\left((e^{kt} - e^{-kt}) + k(e^{kt} + e^{-kt})\right)\nu_i\Theta_i(s) \\ \leq & \left(G(s,s) + 1\right)e^{-ks} + 2e^{(k-\mu)t}\nu_i\Theta_i(s). \end{array}$$

□

2.3 A compactness criterion

Let $p : \mathbb{R}_+ \longrightarrow (0, +\infty)$ be a continuous function. Denote by $\mathbb{X}$ the space of all weighted functions $Y = (y_1, y_2, \ldots, y_n)$, where for all $i \in [1, n]$, y_i is continuously differentiable on $\mathbb{R}_+$ and satisfies

$$\sup_{t \in \mathbb{R}_+}\left(\left[\,|y_i(t)| + |y_i'(t)|\,\right]p(t)\right) < \infty,\ i \in [1, n].$$

Equipped with the Bielecki's type norm

$$\|Y\|_p = \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+}\left(\left[\,|y_i(t)| + |y_i'(t)|\,\right]p(t)\right),$$

$\mathbb{X}$ is a Banach space. Recall that a set of functions $Y \in \Omega \subset \mathbb{X}$ is said to be almost equicontinuous if it is equicontinuous on each interval $[0, T]$, $0 \leq T < +\infty$. The following compactness result involves the boundedness of solutions with respect to a dominant weight. It is an adaptation of Zima's compactness criterion [34, 35] to the case of systems.

Proposition 2.2. *Let $\Omega \subset \mathbb{X}$ and assume that the functions $Y \in \Omega$ and their derivatives are almost equicontinuous on $\mathbb{R}_+$ and uniformly bounded in the sense of the norm*

$$\|Y\|_q = \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+}\left(\left[\,|y_i(t)| + |y_i'(t)|\,\right]q(t)\right),$$

where the function q is positive, continuous on $\mathbb{R}_+$ and satisfies $\lim\limits_{t \to +\infty} \frac{p(t)}{q(t)} = 0$. Then Ω is relatively compact in $\mathbb{X}$.

Proof. Let $(Y_m)_{m\in\mathbb{N}} = (y_{1,m}, y_{2,m}, \ldots, y_{n,m})_{m\in\mathbb{N}}$ be a sequence in Ω, uniformly bounded with respect to the norm $\|.\|_q$. Then there exists some $K > 0$ such that for all $m \in \mathbb{N}$, $\|Y_m\|_q \le K$; thus

$$\sup_{t\in\mathbb{R}_+} (\,[\,|y_{i,m}(t)| + |y'_{i,m}(t)|\,]\,q(t)) \le K,\ i = 1, 2, \ldots n.$$

Hence

$$\forall\, m \in \mathbb{N},\ \forall\, t \in \mathbb{R}_+,\ |y_{i,m}(t)| + |y'_{i,m}(t)| \le K/q(t).$$

For $i \in [1, n]$, the functions $(y_{i,m})_{m\in\mathbb{N}}$ and $(y'_{i,m})_{m\in\mathbb{N}}$ are uniformly bounded on any subinterval of $\mathbb{R}_+$. In addition, these functions are, by assumption, equicontinuous on subintervals of $\mathbb{R}_+$. By the Ascoli-Arzela Lemma and a diagonal procedure, for each $i \in [1, n]$, there exists some subsequence $(\xi^{(m)}_{i,m})_{m\in\mathbb{N}}$ of $(y_{i,m})_{m\in\mathbb{N}}$ converging almost uniformly to some limit function y_i and the sequence $((\xi^{(m)}_{i,m})')_{m\in\mathbb{N}}$ is almost uniformly convergent to the derivative y'_i in the interval $[0, +\infty)$; moreover

$$|y_i(t)| + |y'_i(t)| \le K/q(t).$$

We prove that the sequence $(\xi^{(m)}_m)_{m\in\mathbb{N}} = \left(\xi^{(m)}_{1,m}, \xi^{(m)}_{2,m}, \ldots, \xi^{(m)}_{n,m}\right)_{m\in\mathbb{N}}$, converges in $\mathbb{X}$ for the p-weighted norm. Indeed, for all $T > 0$ and $i \in [1, n]$, we have

$$\begin{aligned}
&\sup_{t\in\mathbb{R}_+} (\,[\,|\xi^{(m)}_{i,m}(t) - y_i(t)| + |(\xi^{(m)}_{i,m})'(t) - y'_i(t)|\,]\,p(x)) \\
\le\ &\sup_{t\in[0,T]} (\,[\,|\xi^{(m)}_{i,m}(t) - y_i(t)| + |(\xi^{(m)}_{i,m})'(t) - y'_i(t)|\,]\,p(t)) \\
&+\sup_{t>T}(\,[\,|\xi^{(m)}_{i,m}(t) - y_i(t)| + |(\xi^{(m)}_{i,m})'(t) - y'_i(t)|\,]\,p(t)).
\end{aligned}$$

Then

$$\begin{aligned}
\|\xi^{(m)}_m - y_i\|_p \ \le\ &\sum_{i=1}^{n} \sup_{t\in[0,T]} (\,[\,|\xi^{(m)}_{i,m}(t) - y_i(t)| + |(\xi^{(m)}_{i,m})'(t) - y'_i(t)|\,]\,p(t)) \\
&+2nK \sup_{t>T} \tfrac{p(t)}{q(t)}.
\end{aligned}$$

Since, for any $i \in [1, n]$, the sequence $(\xi^{(m)}_{i,m})_{m\in\mathbb{N}}$ converges almost uniformly to y_i, the sequence $((\xi^{(m)}_{i,m})')_{m\in\mathbb{N}}$ is almost uniformly convergent to y'_i in $[0, +\infty)$, and $\sup_{t>T} \frac{p(t)}{q(t)} \to 0$, as $T \to +\infty$, we deduce that $\lim_{n\to\infty} \|\xi^{(m)}_m - y_i\|_p = 0$, proving our claim. □

3 The regular problem

This section deals with Problem (1.1) when no singularity is assumed on the nonlinearities which first satisfy the following hypothesis:

($\mathcal{H}_1$) The functions $f_i : \mathbb{R}_+ \times (\mathbb{R}_+)^n \times \mathbb{R}^n \to \mathbb{R}_+$ are continuous and when $y_1, \dots, y_n, z_1, \dots, z_n$ are bounded, $f_i(t, e^{\theta t}y_1, \dots, e^{\theta t}y_n, e^{\theta t}z_1, \dots, e^{\theta t}z_n)$ are bounded on $[0, +\infty)$. In addition for $i \in [1, n]$, the integrals

$$B_i = \int_0^{+\infty} \phi_i(s) \Big((G(s,s)+1)\, e^{-ks} + 2\max(k,1)\nu_i \Theta_i(s) \Big)\, ds$$

are convergent.

3.1 A fixed point operator

Let $\Omega \subset \mathbb{X}$ be a bounded subset and $Y = (y_1, \dots, y_n) \in \Omega$. Then, there exists $M > 0$ such that $\|Y\|_\theta \leq M$. From Assumption ($\mathcal{H}_1$), let

$$S_M^{(i)} = \sup \left\{ \begin{array}{c} f_i(t, e^{\theta t}y_1, \dots, e^{\theta t}y_n, e^{\theta t}z_1, \dots, e^{\theta t}z_n),\ t \in \mathbb{R}_+ \\ (y_1, \dots, y_n) \in [0, M]^n,\ (|z_1|, \dots, |z_n|) \in [0, M]^n \end{array} \right\}.$$

Hence for any $t \geq 0$, $0 \leq y_i(t)e^{-\theta t} \leq M$ and $|y_i'(t)|\, e^{-\theta t} \leq M$ $(i \in [1, n])$ we have

$$\begin{array}{rl} & \int_0^{+\infty} e^{-ks}\phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds \\ = & \int_0^{+\infty} e^{-ks}\phi_i(s) f_i(s, e^{\theta s}e^{-\theta s}y_1(s), \dots, e^{-\theta s}e^{\theta s}y_n(s), \\ & e^{-\theta s}e^{\theta s}y_1'(s), \dots, e^{-\theta s}e^{\theta s}y_n'(s)) ds \\ = & S_M^{(i)} \int_0^{+\infty} e^{-ks}\phi_i(s) ds < \infty,\ i \in [1, n]. \end{array}$$

So for all $i \in [1, n]$, the integrals

$$\int_0^{+\infty} e^{-ks}\phi_i(s)) f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds$$

are convergent. From Lemma 2.1, we deduce that the boundary value problem (1.1) is equivalent to the integral equation

$$Y(t) = \int_0^{+\infty} H(t,s) F(s, Y(s), Y'(s)) ds.$$

For $i \in [1, n]$, define the integral operators $A_i : \overline{\Omega} \cap \mathcal{P} \longrightarrow C^1(\mathbb{R}_+, \mathbb{R}_+)$ by

$$(A_i Y)(t) = \int_0^{+\infty} H_i(t,s)\phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds$$

and let $(AY)(t) = (A_1Y(t), A_2Y(t), \dots, A_nY(t))^T$. We have

$$\begin{array}{rcl} A : \overline{\Omega} \cap \mathcal{P} & \longrightarrow & C^1(\mathbb{R}_+, \mathbb{R}_+^n) \\ Y & \longmapsto & (AY)(t) = \int_0^{+\infty} H(t,s) F(s, Y(s), Y'(s)) ds. \end{array} \tag{3.1}$$

Next, we study the compactness of the operator A.

Lemma 3.1. *Under Assumptions* $(\mathcal{H}_0)$ *and* $(\mathcal{H}_1)$, *A maps the set* $\overline{\Omega} \cap \mathcal{P}$ *into* $\mathcal{P}$.

Proof.
Claim 1. $A(\overline{\Omega} \cap \mathcal{P}) \subset \mathbb{X}$. Indeed, by $(\mathcal{H}_0), (\mathcal{H}_1)$, and [Lemma 2.4, (b) with $\mu = \theta$], we obtain the following estimates, for all $i \in [1, n]$, $Y \in \overline{\Omega} \cap \mathcal{P}$ and $t \in \mathbb{R}_+$:

$$\begin{aligned}
&e^{-\theta t}[\,|(A_iY)(t)| + |(A_iY)'(t)|\,]\\
&= \int_0^\infty \left(H_i(t,s) + |\tfrac{\partial}{\partial t}H_i(t,s)|\right) \phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds\\
&\le S_M^{(i)} \int_0^{+\infty} \left((G(s,s)+1))\, e^{-ks} + 2\max(k,\,1)\nu_i\Theta_i(s)\right) \phi_i(s) ds\\
&= S_M^{(i)}\, B_i < \infty, \ \forall\, i \in [1, n].
\end{aligned}$$

Claim 2. $A(\overline{\Omega} \cap \mathcal{P}) \subset \mathcal{P}$. Clearly $AY(t) \geq 0\ \forall\, t \in \mathbb{R}_+$. Using the inequalities in part (b) of Lemma 2.3 with $\mu = \theta$, we obtain for $t \in [\gamma, \delta]$ and $\tau \in \mathbb{R}_+$ the successive estimates:

$$\begin{aligned}
& 2k(A_iY)(t) + (A_iY)'(t)\\
= \;& \int_0^{+\infty} \left(2kH_i(t,s) + \tfrac{\partial}{\partial t}H_i(t,s)\right) \phi_i(s)\\
& f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds\\
\geq \;& \int_0^{+\infty} \left(2kG(t,s) + G_t(t,s) + k(3e^{kt} - e^{-kt})\nu_i\Theta_i(s)\right) \phi_i(s)\\
& \times f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds\\
\geq \;& \Lambda_1 \int_0^t \left(G(s,s) + |G_t(s+0,s)|\right) e^{-ks}\phi_i(s)\\
& \times f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds\\
& + \Lambda_1 \int_t^{+\infty} \left(G(s,s) + |G_t(s-0,s)|\right) e^{-ks}\phi_i(s)\\
& \times f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds\\
\geq \;& \tfrac{1}{2}\Lambda_1 e^{-\theta\tau} \int_0^t \left(G(\tau,s) + |G_t(\tau,s)|\right) \phi_i(s)\\
& \times f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds\\
& + \tfrac{1}{2}\Lambda_1 e^{-\theta\tau} \int_t^{+\infty} \left(G(\tau,s) + |G_t(\tau,s)|\right) \phi_i(s)\\
& \times f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s)) ds\\
\geq \;& \tfrac{1}{2}\Lambda_1 e^{-\theta\tau} \left(|(A_iY)(\tau)| + |(A_iY)'(\tau)|\right).
\end{aligned}$$

Passing to the infimum respectively over t and then over $\tau \in \mathbb{R}_+$ guarantee that for all $\tau \in \mathbb{R}_+$

$$\begin{aligned}
\min_{t\in[\gamma,\delta]} \left(2kA_iY(t) + (A_iY)'(t)\right) &\geq \tfrac{1}{2}\Lambda_1 e^{-\theta\tau} \left(|(A_iY)(\tau)| + |(A_iY)'(\tau)|\right),\\
\min_{t\in[\gamma,\delta]} \left(2kA_iY(t) + (A_iY)'(t)\right) &\geq \tfrac{1}{2}\Lambda_1 \|A_iY\|_\theta\,,\ \forall\, i \in [1, n]\\
\sum_{i=1}^n \min_{t\in[\gamma,\delta]} \left(2kA_iY(t) + (A_iY)'(t)\right) &\geq \tfrac{1}{2}\Lambda_1 \|AY\|_\theta\,,
\end{aligned}$$

ending the proof of the lemma. □

Next, we prove a compactness result.

Lemma 3.2. *Under Assumptions* $(\mathcal{H}_0)$ *and* $(\mathcal{H}_1)$, *the map* $A : \overline{\Omega} \cap \mathcal{P} \to \mathcal{P}$ *is completely continuous.*

Proof.
Claim 1. A is continuous on $\overline{\Omega} \cap \mathcal{P}$. Let the convergent sequence $Y_m = (y_{1,m}, \dots y_{n,m}) \to Y = (y_1, \dots, y_n)$ in $\overline{\Omega} \cap \mathcal{P}$, as $m \to +\infty$. Then there exists $N > 0$ independent of n such that $\max\{\|Y\|_\theta, \sup\limits_{m \geq 1} \|Y_m\|_\theta\} \leq N$. Let

$$S_N^{(i)} = \sup \left\{ \begin{array}{c} f_i \left(t, e^{\theta t} y_{1,m}, \dots, e^{\theta t} y_{n,m}, e^{\theta t} z_{1,m}, \dots, e^{\theta t} z_{n,m}\right), \ t \in [0, +\infty), \\ (y_{1,m}, \dots, y_{n,m}) \in [0, N]^n, \ (|z_{1,m}|, \dots, |z_{n,m}|) \in [0, N]^n \end{array} \right\}.$$

So for $i \in [1, n]$, we have

$$\left| f_i \left(t, y_{1,m}, \dots, y_{n,m}, y'_{1,m}, \dots, y'_{n,m}\right) - f_i \left(t, y_1, \dots, y_n, y'_1, \dots, y'_n\right) \right| \leq 2 S_N^{(i)}.$$

By continuity of the functions $f_i, \ i \in [1, n]$, we have as $m \to +\infty$

$$\left| f_i \left(t, y_{1,m}, \dots, y_{n,m}, y'_{1,m}, \dots, y'_{n,m}\right) - f_i \left(t, y_1, \dots, y_n, y'_1, \dots, y'_n\right) \right| \to 0.$$

Hence, for each $i \in [1, n]$, the Lebesgue dominated convergence theorem implies that

$$\begin{array}{ll} & \|A_i Y_m - A_i Y\|_\theta \\ = & \sup\limits_{t \in \mathbb{R}_+} \left(|(A_i Y_m)(t) - (A_i Y)(t)| \, e^{-\theta t} + |(A_i Y_m)'(t) - (A_i Y)'(t)| \, e^{-\theta t} \right) \\ \leq & \sup\limits_{t \in \mathbb{R}_+} e^{-\theta t} \int_0^{+\infty} \left(H_i(t, s) + |\frac{\partial}{\partial t} H_i(t, s)| \right) \phi_i(s) \\ & \times |f_i(s, y_{1,m}(s), \dots, y_{n,m}(s), y'_{1,m}(s), \dots, y'_{n,m}(s)) \\ & - f_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s))| ds \\ \leq & \int_0^{+\infty} \left((G(s, s) + 1) \, e^{-ks} + 2 \max(k, 1) \nu_i \Theta_i(s) \right) \phi_i(s) \\ & \times |f_i \left(s, y_{1,m}(s), \dots, y_{n,m}(s), y'_{1,m}(s), \dots, y'_{n,m}(s)\right) \\ & - f_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s))| ds, \end{array}$$

where the right-hand side tends to 0, as $m \to +\infty$. Consequently,

$$\|A Y_m - A Y\|_\theta = \sum_{i=1}^{n} \|A_i Y_m - A_i Y\|_\theta \longrightarrow 0, \ \text{ as } \ m \to +\infty,$$

proving our claim.
Claim 2. A is completely continuous. Let Ω be some bounded subset of $\mathbb{X}$; then there exists $M > 0$ such that $\|Y\|_\theta \leq M$, for all $Y \in \overline{\Omega} \cap \mathcal{P}$.
(a) The functions $\{AY, \ Y \in \overline{\Omega} \cap \mathcal{P}\}$ are almost equicontinuous on $\mathbb{R}_+$. Indeed, for any $Y \in \overline{\Omega} \cap \mathcal{P}$, $T > 0$, and $t_1, t_2 \in [0, T]$ $(t_1 < t_2)$, we have for $i \in [1, n]$ the estimates

$$\begin{array}{l} |(A_i Y)(t_1) - (A_i Y)(t_2)| \\ = \int_0^\infty |H_i(t_1, s) - H_i(t_2, s)| \, \phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s)) ds \\ = \int_0^{t_1} |H_i(t_1, s) - H_i(t_2, s)| \, \phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s)) ds \\ + \int_{t_1}^{t_2} |H_i(t_1, s) - H_i(t_2, s)| \, \phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s)) ds \\ + \int_{t_2}^{\infty} |H_i(t_1, s) - H_i(t_2, s)| \, \phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y'_1(s), \dots, y'_n(s)) ds. \end{array}$$

Now, we estimate each of the sums in the right-hand side:

$$\begin{aligned}
&\int_0^{t_1} |H_i(t_1,s)-H_i(t_2,s)|\,\phi_i(s)f_i(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s))ds\\
\leq\ & S_M^{(i)}\int_0^{t_1}\phi_i(s)[\tfrac{1}{2k}\left|e^{-kt_1}(e^{ks}-e^{-ks})-e^{-kt_2}(e^{ks}-e^{-ks})\right|\\
&+\left|(e^{kt_1}-e^{-kt_1})-(e^{kt_2}-e^{-kt_2})\right|\nu_i\Theta_i(s)]ds\\
=\ & S_M^{(i)}\left|e^{-kt_1}-e^{-kt_2}\right|\int_0^{t_1}\tfrac{1}{2k}\phi_i(s)\left(\left|e^{ks}-e^{-ks}\right|+2k\nu_i\Theta_i(s)ds\right)ds\\
&+S_M^{(i)}\left|e^{kt_1}-e^{kt_2}\right|\int_0^{t_1}\phi_i(s)\nu_i\Theta_i(s)ds\\
&\longrightarrow 0,\ \text{ as } |t_1-t_2|\to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_1}^{t_2} |H_i(t_1,s)-H_i(t_2,s)|\,\phi_i(s)f_i(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s))ds\\
&\leq S_M^{(i)}\int_{t_1}^{t_2}\phi_i(s)[\tfrac{1}{2k}\left|e^{-ks_1}(e^{kt_1}-e^{-kt_1})-e^{-kt_2}(e^{ks}-e^{-ks})\right|\\
&+\left|(e^{kt_1}-e^{-kt_1})-(e^{kt_2}-e^{-kt_2})\right|\nu_i\Theta_i(s)]ds\\
&=\tfrac{1}{2k}S_M^{(i)}\left|e^{kt_1}-e^{-kt_1}\right|\int_{t_1}^{t_2}\phi_i(s)e^{-ks}ds\\
&+\tfrac{1}{2k}S_M^{(i)}e^{-kt_2}\int_{t_1}^{t_2}\phi_i(s)\left|e^{ks}-e^{-ks}\right|ds\\
&+S_M^{(i)}\left|e^{kt_1}-e^{-kt_1}-e^{kt_2}-e^{-kt_2}\right|\int_{t_1}^{t_2}\phi_i(s)\nu_i\Theta_i(s)ds\\
&\longrightarrow 0,\ \text{ as } |t_1-t_2|\to 0.
\end{aligned}$$

Finally

$$\begin{aligned}
&\int_{t_2}^{+\infty} |H_i(t_1,s)-H_i(t_2,s)|\,\phi_i(s)f_i(s,y_1(s),\dots,y_n(s),y_1'(s),\dots,y_n'(s))ds,\\
&\leq S_M^{(i)}\int_{t_2}^{+\infty}\phi_i(s)[\tfrac{1}{2k}\left|e^{-ks_1}(e^{kt_1}-e^{-kt_1})-e^{-ks_1}(e^{kt_2}-e^{-kt_2})\right|\\
&+\left|(e^{kt_1}-e^{-kt_1})-(e^{kt_2}-e^{-kt_2})\right|\nu_i\Theta_i(s)]ds\\
&=\tfrac{1}{2k}S_M^{(i)}\left|e^{kt_1}-e^{-kt_1}-e^{kt_2}-e^{-kt_2}\right|\int_{t_2}^{+\infty}\phi_i(s)\left(e^{-ks}+2k\nu_i\Theta_i(s)ds\right)ds\\
&\longrightarrow 0,\ \text{ as } |t_1-t_2|\to 0.
\end{aligned}$$

Similarly, we obtain, for any $i\in[1,n]$ and for all $Y\in\overline{\Omega}\cap\mathcal{P}$, that the difference $|(A_iY)'(t_1)-(A_iY)'(t_2)|$ tends to 0, as $|t_1-t_2|\to 0$. Then

$$\begin{aligned}
\|(AY)(t_1)-(AY)(t_2)\| &= \sum_{i=1}^{n}|(A_iY)(t_1)-(A_iY)(t_2)|\longrightarrow 0,\\
\|(AY)'(t_1)-(AY)'(t_2)\| &= \sum_{i=1}^{n}|(A_iY)'(t_1)-(A_iY)'(t_2)|\longrightarrow 0.
\end{aligned}$$

This shows that $F(\overline{\Omega}\cap\mathcal{P})$ is equicontinuous.

(b) Consider the open ball $\Omega=\{y\in X:\|y\|_{\theta^*}<R\}$ with some positive real number $k<\theta^*<\theta$. The family $\{AY:\ Y\in\overline{\Omega}\cap\mathcal{P}\}$ is uniformly bounded with respect to the norm $\|.\|_{\theta^*}$ because, as in Lemma 3.1, claim 1, we have

$$\begin{aligned}
\|AY\|_{\theta^*} &= \sum_{i=1}^{n}\|A_iY\|_{\theta^*}\\
&= \sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\left([\,|(A_iY)(t)|+|(A_iY)'(t)|\,]\,e^{-\theta^* t}\right)\\
&\leq \sum_{i=1}^{n}S_M^{(i)}B_i<\infty,\ \forall Y\in\mathcal{P}\cap\overline{\Omega}.
\end{aligned}$$

(c) Taking the dominant weight $q(t) = e^{-\theta^* t} > e^{-\theta t} = p(t)$ in Proposition 2.2, we conclude that the operator A is completely continuous on $\mathcal{P} \cap \bar{\Omega}$. $\square$

3.2 Existence of at least one solution

In this subsection, we shall apply a functional type fixed point Theorem in order to establish the existence of at least one positive solution of System (1.1). Let α and β be nonnegative continuous functionals on $\mathcal{P}$ and, for positive real numbers r and R, define the sets:

$$\begin{aligned} \mathcal{P}(\beta, R) &= \{x \in \mathcal{P} : \ \beta(x) < R\}, \\ \mathcal{P}(\beta, \alpha, r, R) &= \{x \in \mathcal{P} : \ \beta(x) < R \text{ and } \alpha(x) > r\}. \end{aligned}$$

If α and β are usual norms in the space $\mathbb{X}$, the sets $\mathcal{P}(\beta, R)$ and $\mathcal{P}(\beta, \alpha, r, R)$ are respectively the open ball and the annulus. The following result is the extension of the fixed point theorem of cone expansion and compression of functional type and provides solutions in the conical shell $\overline{\mathcal{P}(\beta, \alpha, r, R)}$.

Theorem 3.1. *[3] Let $\mathcal{P}$ be a cone in a real Banach space $(X, \|.\|)$ and let α and β be nonnegative continuous functionals on $\mathcal{P}$. Let $\mathcal{P}(\beta, \alpha, r, R)$ be a nonempty bounded subset of $\mathcal{P}$ and*

$$\overline{\mathcal{P}(\alpha, r)} \subseteq \mathcal{P}(\beta, R).$$

Let the mapping

$$F : \ \overline{\mathcal{P}(\beta, \alpha, r, R)} \to \mathcal{P}$$

be completely continuous. Assume that either one of the following two conditions hold true:

(H1) *$\alpha(Fy) \leq r, \ \forall\, y \in \partial\mathcal{P}(\alpha, r)$, $\beta(Fy) \geq R, \ \forall\, y \in \partial\mathcal{P}(\beta, R)$, and*

$$\inf_{y \in \partial\mathcal{P}(\beta, R)} \|Fy\| > 0,$$

and for all $y \in \partial\mathcal{P}(\alpha, r)$, $z \in \partial\mathcal{P}(\beta, R)$, $\lambda \geq 1$, and $\mu \in (0, 1]$, the functionals satisfy the properties

$$\alpha(\lambda y) \geq \lambda\alpha(y), \ \beta(\mu z) \leq \mu\beta(z), \ \text{and} \ \alpha(0) = 0,$$

or

(H2) *$\alpha(Fy) \geq r, \ \forall\, y \in \partial\mathcal{P}(\alpha, r)$, $\beta(Fy) \leq R, \ \forall\, y \in \partial\mathcal{P}(\beta, R)$, and*

$$\inf_{y \in \partial\mathcal{P}(\alpha, r)} \|Fy\| > 0$$

and for all $y \in \partial\mathcal{P}(\alpha, r)$, $z \in \partial\mathcal{P}(\beta, R)$, $\mu \geq 1$, and $\lambda \in (0, 1]$, the functionals satisfy the properties

$$\alpha(\lambda y) \leq \lambda\alpha(y), \ \beta(\mu z) \geq \mu\beta(z), \ \text{and} \ \beta(0) = 0.$$

Then, F has at least one positive fixed point $y^* \in \overline{\mathcal{P}(\beta,\alpha,r,R)}$.

The following theorem complements the results of Theorem 3.1 when the cone $\mathcal{P}$ is normal. It is concerned with the estimates of some iterates which converge to the fixed point y^*. We denote U^n the n-time composition $U^n = U \circ U \circ \ldots \circ U$.

Theorem 3.2. *[4, Theorem 2.1, p.19] Further to the assumptions in Theorem 3.1, let $\mathcal{P}$ be a normal cone and suppose that there exist $y_l, y_u \in \mathcal{P}$ such that $\overline{\mathcal{P}(\beta,\alpha,r,R)} \subset [y_l, y_u]$. Then the following statements hold:*

(E1) *If there exists an increasing completely continuous operator $U : [y_l, y_u] \to \mathcal{P}$ such that $Fy \leq Uy$ for all $y \in [y_l, y_u]$ and $U^2 y_u \leq U y_u$, then*

$$y^* \leq y_u^* \leq U^n y_u, \ \forall\, n \in \mathbb{N},$$

where $y_u^* = \lim\limits_{n\to+\infty} U^n y_u$.

(E2) *If there exists an increasing completely continuous operator $L : [y_l, y_u] \to \mathcal{P}$ such that $Ly \leq Fy$ for all $y \in [y_l, y_u]$ and $Ly_l \leq L^2 y_l$, then*

$$L^n y_l \leq y_l^* \leq y^*, \ \forall\, n \in \mathbb{N},$$

where $y_l^* = \lim\limits_{n\to+\infty} L^n y_l$.

To prove our first existence result, we distinguish between the cases $k \geq 1$ and $0 < k < 1$. For $k \geq 1$, consider the subset of the half-space

$$\Delta_1 = \left\{ \begin{array}{c} (y_1,\ldots,y_n,z_1,\ldots,z_n) \in (\mathbb{R}_+^*)^n \times (\mathbb{R}^*)^n : \\ \sum\limits_{i=1}^n (2ky_i + z_i) \geq 0 \text{ and } y_i + |z_i| \leq Re^{\theta\delta}, \ i \in [1,n] \end{array} \right\}. \tag{3.2}$$

Clearly, this set is nonempty. In the sequel, we will denote

$$y_1(t),\ldots,y_n(t),z_1(t),\ldots,z_n(t)$$

by $y_1,\ldots,y_n,z_1,\ldots,z_n$, respectively. We need the following hypothesis.

($\mathcal{H}_2$) The functions $f_i : \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}^n \to \mathbb{R}_+$ are continuous and there exist continuous functions $h, a_i \in C(\mathbb{R}_+,\mathbb{R}_+)$, $b_i \in C(\mathbb{R}_+^n,\mathbb{R}_+)$ and $c_i \in C(\mathbb{R}^n,\mathbb{R}_+)$ such that for all $t \in \mathbb{R}_+$, $y_i \in \mathbb{R}_+$, and $z_i \in \mathbb{R}, (i \in [1,n])$ we have

$$\begin{aligned} 0 &\leq f_i(t,y_1,\ldots,y_n,z_1,\ldots,z_n) \\ &\leq a_i\left(\sum_{i=1}^n \min_{t\in[\gamma,\delta]} (h(t) + 2ky_i + z_i)\right) \\ &\quad \times \left(b_i\left(e^{-\theta t}y_1,\ldots,e^{-\theta t}y_n\right) + c_i\left(e^{-\theta t}z_1,\ldots,e^{-\theta t}z_n\right)\right), \end{aligned}$$

where a_i is nonincreasing and b_i, c_i are nondecreasing functions.

($\mathcal{H}_3$) There exits $R > 0$ such that

$$\sum_{i=1}^{n} B_i a_i(\frac{\Lambda_1}{2}R)\,[b_i(R,\ldots,R)+c_i(R,\ldots,R)] < R, \tag{3.3}$$

where, for $i \in [1,n]$,

$$B_i = \int_0^{+\infty} \phi_i(s)\left((G(s,s)+1)\,e^{-ks} + 2k\nu_i\Theta_i(s)\right) ds,$$

$$\nu_i = (1 - \int_0^{+\infty}(e^{ks}-e^{-ks})g_i(s)ds)^{-1},$$

and

$$\Theta_i(t) = \int_0^{+\infty} g_i(\tau)G(t,\tau)d\tau.$$

Finally, the following notations will be used throughout ($i \in [1,n]$)

$$\begin{cases} C_i = \min\limits_{t\in[\gamma,\delta]} \int_\gamma^\delta \phi_i(s)\left(2kH_i(t,s) + \frac{\partial}{\partial t}H_i(t,s)\right) ds, \\ D_i = \min\limits_{t\in[\gamma,\delta]} \int_\gamma^\delta \phi_i(s)\left(H_i(t,s) + \frac{\partial}{\partial t}H_i(t,s)\right) ds. \end{cases} \tag{3.4}$$

Theorem 3.3. *Let $k \geq 1$ and assume that Assumptions ($\mathcal{H}_0$), ($\mathcal{H}_2$) and ($\mathcal{H}_3$) hold together with*

($\mathcal{H}_4$) $\quad N_i = \min\limits_{t\in[\gamma,\delta],\,(y_1,\ldots,y_n,z_1,\ldots,z_n)\in\Delta_1} f_i(t,y_1,\ldots,y_n,z_1,\ldots,z_n) > 0,\ i \in [1,n].$

Then System (1.1) has at least one positive solution $Y^* = (y_1^*, y_2^*, \ldots, y_n^*)$ satisfying

$$\begin{aligned} \sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+}\left(e^{-\theta x}\,[\,|y_i^*(t)| + |(y_i^*)'(t)|\,]\right) &< R, \\ \sum_{i=1}^{n} \min_{t\in[\gamma,\delta]}(2ky_i^*(t) + (y_i^*)'(t)) &\geq \max\left(\tfrac{\Lambda_1}{2}\|Y^*\|_\theta, \sum_{i=1}^{n} C_iN_i\right) > 0. \end{aligned} \tag{3.5}$$

In addition, under Assumption

($\mathbb{H}$) $\Theta_i(t) - \dfrac{1}{2k\nu_i}\left(e^{-kt} - e^{-3kt}\right) \geq 0,$ *for* $t \in [0,T]$ $(T > 0)$, $i \in [1,n]$,

we have that $(Y^*)' \geq 0_{\mathbb{R}^n}$ on $\mathbb{R}_+$.

Proof.
(a) First part. Let R be as defined by Assumption ($\mathcal{H}_3$), consider the open set

$$\Omega_R := \{Y \in \mathbb{X} : \|Y\|_\theta < R\},$$

and let r be any real number such that

$$0 < r < \min\left(\frac{\Lambda_1}{2}R, \sum_{i=1}^{n} C_i N_i\right). \tag{3.6}$$

On the cone $\mathcal{P}$, define the positive functionals

$$\begin{aligned}
\alpha(Y) &= \sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} \left(2ky_i(t) + y_i'(t)\right), \\
\beta(Y) &= \sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+} \left(e^{-\theta t}\left[|y_i(t)| + |y_i'(t)|\right]\right) = \|Y\|_\theta.
\end{aligned} \tag{3.7}$$

We show that Assumption $(\mathcal{H}_2)$ in Theorem 3.1 is satisfied.

Claim 1. $\overline{\mathcal{P}(\alpha, r)} \subset \mathcal{P}(\beta, R)$. Indeed, if $Y = (y_1, \ldots, y_n) \in \overline{\mathcal{P}(\alpha, r)}$, then

$$r \geq \sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} \left(2ky_i(t) + y_i'(t)\right) \geq \frac{1}{2}\Lambda_1 \|Y\|_\theta.$$

With (4.4), we infer that $\|Y\|_\theta \leq 2\frac{r}{\Lambda_1} < R$, and so $Y \in \mathcal{P}(\beta, R)$. Also, for all $Y \in \partial\mathcal{P}(\alpha, r)$, $Z \in \partial\mathcal{P}(\beta, R)$, $\lambda \in (0, 1]$, and $\mu \geq 1$, the functionals α and β satisfy

$$\alpha(\lambda Y) = \lambda\alpha(Y),\ \beta(\mu Z) = \mu\beta(Z) \text{ and } \beta(0_{\mathbb{R}^n}) = 0.$$

Claim 2. $\alpha(AY) \geq r$ for all $Y \in \partial\mathcal{P}(\alpha, r)$. Indeed, Let $Y = (y_1, \ldots, y_n) \in \partial\mathcal{P}(\alpha, r)$, that is $\alpha(Y) = \sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} (2ky_i(t) + y_i'(t)) = r$. As checked in Claim 1, $\|Y\|_\theta \leq R$. Hence, for every $t \in [\gamma, \delta]$, $(y_1(t), \ldots, y_n(t), y_1'(t), \ldots, y_n'(t)) \in \Delta_1$, where Δ_1 is given by (3.2). By Assumption $(\mathcal{H}_4)$, we have the estimates:

$$\begin{aligned}
\alpha(AY) &= \sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} \left(2kA_iY(t) + (A_iY)'(t)\right) \\
&\geq \sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} \int_\gamma^\delta \phi_i(s)\left(2kH_i(t,s)\right. \\
&\quad \left.+\tfrac{\partial}{\partial t}H_i(t,s)\right) f_i\left(y_1(s), \ldots, y_n(s), y_1'(s), \ldots, y_n'(s)\right) ds \\
&\geq \sum_{i=1}^{n} C_i N_i > r.
\end{aligned}$$

Claim 3. $\beta(AY) \leq R$, for all $Y \in \partial\mathcal{P}(\beta, R)$. Let $Y \in \partial\mathcal{P}(\beta, R)$. By Assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, for all $t \in \mathbb{R}_+$ and $i \in [1, n]$, we get successively the following estimates:

$$\begin{aligned}
\beta(AY) &= \sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+} \left(e^{-\theta t}\left[(A_iY)(t) + |(A_iY)'(t)|\right]\right)\\
&= \sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+} \int_0^{+\infty} e^{-\theta t}\left(H_i(t,s) + |\tfrac{\partial}{\partial t}H_i(t,s)|\right)\phi_i(s)\\
&\quad f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s))ds\\
&\le \sum_{i=1}^{n} \int_0^{+\infty} \phi_i(s)\left((G(s,s)+1)\,e^{-ks} + 2k\nu_i\Theta_i(s)\right)\\
&\quad \times a_i(\sum_{i=1}^{n} \min_{s\in[\gamma,\delta]} (h(s) + 2ky_i(s) + y_i'(s)))\\
&\quad \times\left[b_i\left(e^{-\theta t}y_1(s), \dots, e^{-\theta t}y_n(s)\right) + c_i\left(e^{-\theta t}|y_1'(s)|, \dots, e^{-\theta t}|y_n'(s)|\right)\right] ds\\
&\le \sum_{i=1}^{n} B_i a_i(\tfrac{\Lambda_1}{2}R)\left[b_i(R,\dots,R) + c_i(R,\dots,R)\right] < R.
\end{aligned}$$

Claim 4. $\inf\limits_{Y\in\partial\mathcal{P}(\alpha,r)} \|AY\|_\theta > 0$. For every $Y \in \partial\mathcal{P}(\alpha, r)$, and for some $t_0 \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+} \left(e^{-\theta t}\left[(A_iY)(t) + |(A_iY)'(t)|\right]\right)\\
= &\sum_{i=1}^{n} \sup_{t\inf\mathbb{R}_+} \int_0^{+\infty} e^{-\theta t}\left(H_i(t,s) + |\tfrac{\partial}{\partial t}H_i(t,s)|\right)\phi_i(s)\\
&f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s))ds\\
\ge &\sum_{i=1}^{n} N_i \int_0^{+\infty} e^{-\theta t_0}\left(H_i(t_0,s) + |\tfrac{\partial}{\partial t}H_i(t_0,s)|\right)\phi_i(s)ds = K_0.
\end{aligned}$$

Passing to the infimum, we get $\inf\limits_{Y\in\partial\mathcal{P}(\alpha,r)} \|AY\|_\theta \ge K_0 > 0$. Therefore Hypothesis (**H2**) in Theorem 3.1 is satisfied. Using Lemmas 3.1 and 3.2, we find some $Y^* = (y_1^*, \dots, y_n^*)$ lying in the conical shell $\overline{\mathcal{P}(\alpha,\beta,r,R)}$ such that $AY^* = Y^*$ satisfies

$$\begin{aligned}
&0 < \|Y^*\|_\theta = \sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+} \left(e^{-\theta t}\left[\,|y_i^*(t)| + |(y_i^*)'(t)|\,\right]\right) \le R\\
&\sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} (2ky_i^*(t) + (y_i^*)'(t)) \ge \tfrac{\Lambda_1}{2}\|Y^*\|_\theta \qquad (3.8)\\
&\sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} (2ky_i^*(t) + (y_i^*)'(t)) \ge \sum_{i=1}^{n} C_iN_i > r.
\end{aligned}$$

Since r satisfying (3.6) is arbitrary, the last estimate in (3.8) follows from Claim 2. Moreover $\|Y^*\|_\theta < R$. Indeed, if $\|Y^*\|_\theta = R$ then the definition of the function β and the condition (3.3) will lead to a contradiction.

(b) Second part: We will prove that $(y_i^*)'(t) \ge 0,\ \forall\, t \in \mathbb{R}_+,\ i \in [1, n]$. In fact, we have

$$
\begin{aligned}
(y_i^*)'(t) &= (A_iY^*)'(t) \\
&= \int_0^\infty \frac{\partial}{\partial t} H_i(t,s)\phi_i(s) f_i(s, y_1^*(s), \ldots, y_n^*(s), (y_1^*)'(s), \ldots, (y_n^*)'(s))ds \\
&= \int_0^{+\infty} \left(k\nu_i \left(e^{kt} + e^{-kt}\right) \Theta_i(s) + G_t(t,s)\right) \phi_i(s) \\
&\quad \times f_i(s, y_1^*(s), \ldots, y_n^*(s), (y_1^*)'(s), \ldots, (y_n^*)'(s))ds, \\
&\geq \int_0^t \left(k\nu_i \left(e^{kt} + e^{-kt}\right) \Theta_i(s) - \tfrac{1}{2} e^{-kt}(e^{ks} - e^{-ks})\right) \phi_i(s) \\
&\quad f_i(s, y_1^*(s), \ldots, y_n^*(s), (y_1^*)'(s), \ldots, (y_n^*)'(s))ds \\
&\geq k\nu_i \left(e^{kt} + e^{-kt}\right) \int_0^t (\Theta_i(s) - \tfrac{1}{2k\nu_i} e^{-kt} \tfrac{e^{ks} - e^{-ks}}{e^{kt}})\phi_i(s) \\
&\quad \times f_i(s, y_1^*(s), \ldots, y_n^*(s), (y_1^*)'(s), \ldots, (y_n^*)'(s))ds \\
&\geq k\nu_i \left(e^{kt} + e^{-kt}\right) \int_0^t (\Theta_i(s) - \tfrac{1}{2k\nu_i} \left(e^{-kt} - e^{-3kt}\right))\phi_i(s) \\
&\quad \times f_i(s, y_1^*(s), \ldots, y_n^*(s), (y_1^*)'(s), \ldots, (y_n^*)'(s))ds \\
&\geq 0.
\end{aligned}
$$

□

Regarding the other case $0 < k < 1$, we can prove a similar existence result to Theorem 3.3. It suffices to use the inequalities (a), (b) in Lemma 2.2 and part (b) in Lemma 2.3. We omit the details.

Theorem 3.4. *Let* $0 < k < 1$ *and*

$$
\Delta_2 = \left\{ \begin{array}{c} (y_1, \ldots, y_n, z_1, \ldots, z_n) \in (\mathbb{R}_+^*)^n \times (\mathbb{R}^*)^n : \\ \sum\limits_{i=1}^{n} (y_i + z_i) \geq 0 \text{ and } y_i + |z_i| \leq Re^{\theta\delta},\ i \in [1, n] \end{array} \right\}.
$$

Further to Assumptions $(\mathcal{H}_0)$*, assume that the following hypotheses hold:*

$(\mathcal{H}_2')$ $f_i : \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}^n \to \mathbb{R}_+$ *are continuous and there exist continuous functions* $h, a_i \in C(\mathbb{R}_+, \mathbb{R}_+)$*,* $b_i \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ *and* $c_i \in C(\mathbb{R}^n, \mathbb{R}_+)$*, such that for all* $t \in \mathbb{R}_+, y_i \in \mathbb{R}_+, z_i \in \mathbb{R}$ $(i \in [1, n])$*, we have*

$$
\begin{aligned}
& 0 \leq f_i(t, y_1, \ldots, y_n, z_1, \ldots, z_n) \\
\leq\ & a_i(\sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} (h(t) + y_i + z_i)) \left[b_i\left(e^{-\theta t} y_1, \ldots, e^{-\theta t} y_n\right)\right. \\
& \left. + c_i\left(e^{-\theta t} z_1, \ldots, e^{-\theta t} z_n\right)\right],
\end{aligned}
$$

where a_i *is nonincreasing and* b_i, c_i *are nondecreasing functions.*

$(\mathcal{H}_3')$ *There exits* $R > 0$ *such that*

$$
\sum_{i=1}^{n} B_i a_i(\frac{\Lambda_2}{2} R) \left[b_i(R, \ldots, R) + c_i(R, \ldots, R)\right] < R,
$$

where for $i \in [1, n]$

$$
B_i = \int_0^{+\infty} \phi_i(s) \left((G(s,s) + 1)\, e^{-ks} + 2\nu_i \Theta_i(s) \right) ds
$$

together with

$(\mathcal{H}_4')$ $N_i' = \min\limits_{t\in[\gamma,\delta],\,(y_1,\dots,y_n,z_1,\dots,z_n)\in\Delta_2} f_i(t,y_1,\dots,y_n,z_1,\dots,z_n) > 0,\ i\in[1,n].$

Then System (1.1) has at least one positive solution $Y^* = (y_1^*, y_2^*, \dots, y_n^*)$ *satisfying*

$$\begin{aligned}\sum_{i=1}^{n}\sup_{t\in\mathbb{R}_+}\left(e^{-\theta x}\left[\,|y_i^*(t)| + |(y_i^*)'(t)|\,\right]\right) &< R\\ \sum_{i=1}^{n}\min_{t\in[\gamma,\delta]}\left(y_i^*(t) + (y_i^*)'(t)\right) &\geq \max\left(\frac{\Lambda_2}{2}\|Y^*\|_\theta\,, \sum_{i=1}^{n} D_i N_i'\right) > 0.\end{aligned}$$

Moreover, if Assumption $(\mathbb{H})$ *holds, then* $(Y^*)' \geq 0_{\mathbb{R}^n}$ *on* $\mathbb{R}_+$.

Remark 3.1. *Assumptions* $(\mathcal{H}_4)$ *and* $(\mathcal{H}_4')$ *imply that* $f(t,0,\dots,0)\not\equiv 0_{\mathbb{R}^n}$ *preventing Problem (1.1) from having the trivial solution. Moreover, it is clear that* $(\mathcal{H}_2)$ *and* $(\mathcal{H}_2')$ *imply* $(\mathcal{H}_1)$.

Remark 3.2. *The cone* $P = \{Y\in\mathbb{X}:\ Y\geq 0_{\mathbb{R}^n}\ \text{on}\ \mathbb{R}_+\}$ *is not normal in* $\mathbb{X}$ *endowed with the norm* $\|Y\|_\theta = \sum\limits_{i=1}^{n}\sup\limits_{t\in\mathbb{R}_+}\left(\left[\,|y_i(t)| + |y_i'(t)|\,\right]e^{-\theta t}\right)$. *In fact, if P was normal, then, by Proposition 2.1, there would exist some constant* $N>0$ *such that* $0_{\mathbb{R}^n}\leq X\leq Y$ *implies* $\|X\|\leq N\|Y\|$. *For* $n=2$, *let* $X_m = ((1-\cos(mt))e^{\theta t}, e^{\theta t})$, $Y_m = (2e^{\theta t}, 2e^{\theta t})$. *Then* $0_{\mathbb{R}^2}\leq X_m\leq Y_m$, $\|X_m\|_\theta = 3(\theta+1)+m$ *and* $\|Y_m\|_\theta = 4(\theta+1)$. *Consequently,* $m\leq(\theta+1)N$, $\forall\, m\in\mathbb{N}$, *which is impossible.*

Now, consider the normal cone

$$\mathbb{P} = \left\{Y\in\mathbb{X}:\ Y\geq 0_{\mathbb{R}^n}\ \text{ and } Y'\geq 0_{\mathbb{R}^n}\ \text{on}\ \mathbb{R}_+\right\}.$$

We denote the partial order induced by $\mathbb{P}$ on $\mathbb{X}$ by $\leq$. Regarding the solutions obtained in Theorems 3.3 and 3.4, we can state a more precise result:

Corollary 3.1. *Let* $\theta>1$. *Further to Assumptions* $(\mathcal{H}_0)$, $(\mathcal{H}_2)$, *and* $(\mathcal{H}_3)$ *for* $k\geq 1$ *(respectively* $(\mathcal{H}_3')$ *for* $0<k<1$*) and* $(\mathbb{H})$, *suppose that*

$$(\mathcal{H}_3)''\ \begin{cases}\exists\,\rho>R>0:\ \mathcal{B}_i a_i(n\min\limits_{s\in[\gamma,\delta]}h(s))\left[b_i(\rho,\dots,\rho)+c_i(\theta\rho,\dots,\theta\rho)\right]<\rho,\\ \text{where for } i\in[1,n]\\ \mathcal{B}_i = \int_0^{+\infty}\left((G(s,s)+1)e^{-ks} + \max(2k,1)\,\nu_i\Theta_i(s)\right)\phi_i(s)ds,\end{cases}$$

$$(\mathcal{H}_4)''\ \begin{cases}\text{For } i\in[1,n],\ \ f_i(t,y_1,\dots,y_n,z_1,\dots,z_n)\geq\psi_i(t,y_1,\dots,y_n),\\ \forall\,(t,y_1,\dots,y_n,z_1,\dots,z_n)\in[\gamma,\delta]\times\mathbb{R}_+^n\times\mathbb{R}^n,\\ \text{where}\\ \psi_i\in C([\gamma,\delta]\times\mathbb{R}_+^n,\mathbb{R}_+)\ \text{ satisfies}\\ \mathbf{K}_i = \min\limits_{t\in[\gamma,\delta],\,(y_1,\dots,y_n)\in\left[0,\rho e^{\theta\delta}\right]^n}\psi_i(t,y_1,\dots,y_n)>0.\end{cases}$$

Then Y^* satisfies:

$$Y_l(t) \le Y^*(t) \le Y_u^*(t) \le U^j Y_u(t),\ \forall\, j \in \mathbb{N},$$

where the mapping $U = (U_1, \dots, U_n)^T$ is defined on $\mathbb{X}$ by

$$\begin{aligned} U_i Y(t) = \int_0^{+\infty} & H_i(t,s)\phi_i(s) a_i(n \min_{s\in[\gamma,\delta]} h(s)) \\ \times \Big[b_i \left(e^{-\theta s} y_1(s), \dots, e^{-\theta s} y_n(s) \right) &+ c_i \left(e^{-\theta s} y_1'(s), \dots, e^{-\theta s} y_n'(s) \right) \Big]\, ds, \end{aligned} \tag{3.9}$$

$Y_u^*(t) = \lim\limits_{n\to\infty} U^n Y_u, Y_u(t) = (\rho e^{\theta t}, \dots, \rho e^{\theta t}), Y_l(t) = (\mathbf{K}_1 \mathcal{G}_1(t), \dots, \mathbf{K}_n \mathcal{G}_n(t))$ *and* $\mathcal{G}_i(t) = \int_\gamma^\delta \phi_i(s) \left(G(t,s) + \left(e^{kt} - e^{-kt} \right) \nu_i \Theta_i(s) \right) ds,\ i \in [1,n]$.

Remark 3.3. *The convergence of the integrals $\mathcal{B}_i$ follows from the convergence of the integrals B_i and clearly $(\mathcal{H}_4)''$ implies $(\mathcal{H}_4)$ and $(\mathcal{H}'_4)$.*

Proof. To study the convergence of some iterates of the solution Y^* obtained in the previous theorem, we define an increasing operator $U : \mathbb{P} \to \mathbb{P}$ by (3.9) and then check Assumptions **(E1)** and **(E2)** in Theorem 3.2. In the proof of Theorem 3.2, we can observe that we only need to assume $Y^* \in [Y_l, Y_u]$ instead of the stronger condition $\overline{\mathbb{P}(\beta, \alpha, r, R)} \subset [Y_l, Y_u]$.

Claim 1. $Y^* \in [Y_l, Y_u]$. Since by (3.7), $\beta(Y^*) \le R$, we get

$$y_i^*(t) \le R e^{\theta t} \le \rho e^{\theta t} \text{ and } (y_i^*)'(t) \le \rho e^{\theta t} \le \theta \rho e^{\theta t},\quad \forall\, t \in \mathbb{R}_+,\ \forall\, i \in [1,n].$$

So $(Y_u - Y^*)(t) \ge 0_{\mathbb{R}^n}$ and $(Y_u - Y^*)'(t) \ge 0_{\mathbb{R}^n},\ \forall\, t \in \mathbb{R}_+$. Then $Y^* \le Y_u$. In addition, for all $t \in \mathbb{R}_+$ and $i \in [1,n]$,

$$\begin{array}{rcl} y_i^*(t) &=& \int_0^{+\infty} H_i(t,s)\phi_i(s) f_i(s, y_1^*(s), \dots, y_n^*(s), (y_1^*)'(s), \dots, (y_n^*)'(s)) ds \\ &\ge& \int_\gamma^\delta \phi_i(s) \left(G(t,s) + \left(e^{kt} - e^{-kt} \right) \nu_i \Theta_i(s) \right) \psi_i(s, y_1^*(s), \dots, y_n^*(s)) ds \\ &\ge& \mathcal{G}_i(t) \min\limits_{t\in[\gamma,\delta],\ y_i \in \left[0, \rho e^{\theta\delta}\right]} \psi_i(s, y_1(s), \dots, y_n(s)) = \mathbf{K}_i \mathcal{G}_i(t) = y_{l,i}(t), \end{array}$$

and also $(y_i^*)'(t) \ge \mathbf{K}_i \mathcal{G}_i'(t) = (y_{l,i})'(t)$; hence $Y_l \le Y^*$. Therefore, $Y_l \le Y^* \le Y_u$.

Claim 2. $Y_l \le AY \le UY$. From the growth assumption $(\mathcal{H}_3)$, we have that for all $Y \in \mathbb{P}, t \in \mathbb{R}_+$ and $i \in [1, n]$

$$\begin{aligned}
A_iY(t) &= \int_0^{+\infty} H_i(t,s)\phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s))ds \\
&\leq \int_0^{+\infty} H_i(t,s)\phi_i(s) a_i(n \min_{s\in[\gamma,\delta]} h(s)) \\
&\quad \times \left(b_i\left(e^{-\theta t}y_1(s), \dots, e^{-\theta t}y_n(s)\right)\right. \\
&\quad \left.+ c_i\left(e^{-\theta t}y_1'(s), \dots, e^{-\theta t}y_n'(s)\right)\right) ds \\
&= U_iY(t). \\
(A_iY)'(t) &= \int_0^{+\infty} \frac{\partial}{\partial t} H_i(t,s)\phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s))ds \\
&\leq (U_iY)'(t).
\end{aligned}$$

In addition, if $Y \in [Y_l, Y_u]$ then, for every $t \in [\gamma, \delta]$ and $i \in [1, n]$ $y_i(t)$ lies in $\left[0, \rho e^{\theta\delta}\right]$. As in Claim 1, we can find that

$$\begin{aligned}
&A_iY(t) = \textstyle\int_0^{+\infty} H_i(t,s)\phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s))ds \\
&\geq y_{l,i}(t),\ \forall\, t \in \mathbb{R}_+ \\
&(A_iY)'(t) = \textstyle\int_0^{+\infty} \frac{\partial}{\partial t} H_i(t,s)\phi_i(s) f_i(s, y_1(s), \dots, y_n(s), y_1'(s), \dots, y_n'(s))ds \\
&\geq (y_{l,i})'(t),\ \forall\, t \in \mathbb{R}_+.
\end{aligned}$$

Claim 3. $Uy_u \leq y_u$. Since $\theta > \max(k, 1)$, we get, by Lemma 2.2(b) with $\mu = \theta$ together with Assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}''_3)$ the estimates:

$$\begin{aligned}
&U_iY_u(t) \\
&= e^{\theta t} \textstyle\int_0^{+\infty} e^{-\theta t} H_i(t,s)\phi_i(s) a_i(n \min_{s\in[\gamma,\delta]} h(s)) \\
&\times \left[b_i\left(e^{-\theta s}y_{u,1}(s), \dots, e^{-\theta s}y_{u,n}(s)\right) + c_i\left(e^{-\theta s}y_{u,1}'(s), \dots, e^{-\theta s}y_{u,n}'(s)\right)\right] ds \\
&\leq e^{\theta t}\mathcal{B}_i a_i(n \min_{s\in[\gamma,\delta]} h(s)) \left[b_i(\rho, \dots, \rho) + c_i(\theta\rho, \dots, \theta\rho)\right] \\
&\leq \rho e^{\theta t} = y_{u,i}(t),\ \forall\, t \in \mathbb{R}_+.
\end{aligned}$$

Also we have

$$\begin{aligned}
&(U_iY_u)'(t) \\
&= e^{\theta t} \textstyle\int_0^{+\infty} e^{-\theta t} \frac{\partial}{\partial t} H_i(t,s)\phi_i(s) a_i(n \min_{s\in[\gamma,\delta]} h(s)) \\
&\times \left[b_i\left(e^{-\theta s}y_{u,1}(s), \dots, e^{-\theta s}y_{u,n}(s)\right) + c_i\left(e^{-\theta s}y_{u,1}'(s), \dots, e^{-\theta s}y_{u,n}'(s)\right)\right] ds \\
&\leq e^{\theta t}\mathcal{B}_i a_i(n \min_{s\in[\gamma,\delta]} h(s)) \left[b_i(\rho, \dots, \rho) + c_i(\theta\rho, \dots, \theta\rho)\right] \\
&\leq \rho e^{\theta t} \leq \theta\rho e^{\theta t} = y_{u,i}'(t),\ \forall\, t \in \mathbb{R}_+.
\end{aligned}$$

Since U is increasing, $U^2 y_u \leq U y_u$. As in Lemma 3.2, we can see that U is completely continuous. To sum up, let the constant operator L on $\mathbb{X}$ be defined by $LY = Y_l$. Then, for any $j \in \mathbb{N}$, $L^j Y_l = Y_l$ and $LY \leq AY$

follow from Claim 2. Therefore, by Theorem 3.2, the solution Y^* satisfies the estimates

$$\forall\, j \in \mathbb{N},\, \forall\, t \in \mathbb{R}_+,\; Y_l(t) = LY_l(t) = L^j Y_l(t) \le Y^*(t) \le Y_u^*(t) \le U^j Y_u(t),$$

where $Y_u^* = (y_{u,1}, \ldots, y_{u,n}) = (\lim_{j\to+\infty} U_1^j Y_u, \ldots, \lim_{j\to+\infty} U_n^j Y_u) = \lim_{j\to+\infty} U^j Y_u$, ending the proof of the theorem. $\square$

3.3 Existence of at least two solutions

Theorem 3.5. *[1, Theorem 7.9] Let $\mathcal{P}$ be a cone in a Banach space $(E, \|.\|)$, $0 < r < R < L$ be three real constants and let $\|.\|$ be increasing with respect to the cone $\mathcal{P}$. Let $A : \overline{B_L} \cap \mathcal{P} \to \mathcal{P}$ be a completely continuous operator such that the following conditions hold:*

(a) *$x \neq Ax$ for all $x \in \mathcal{P} \cap \partial B_R$.*

(b) *$\|Ax\| > \|x\|$, for all $x \in \mathcal{P} \cap \partial B_L$.*

(c) *$\|Ax\| \le \|x\|$, for all $x \in \mathcal{P} \cap \partial B_R$.*

(d) *$\|Ax\| > \|x\|$, for all $x \in \mathcal{P} \cap \partial B_r$.*

Then A has at least two fixed points x_1 and x_2 with $x_1 \in \mathcal{P} \cap (B_R \backslash B_r)$ and $x_2 \in \mathcal{P} \cap (B_L \backslash \overline{B_R})$.

As a consequence, we easily derive

Corollary 3.2. *Let $\mathcal{P}$ be a cone in a Banach space $(E, \|.\|)$, $0 < r < R < L$ be three real constants and let $\|.\|$ be increasing with respect to the cone $\mathcal{P}$. Let $A : \overline{B_L} \cap \mathcal{P} \to \mathcal{P}$ be a completely continuous operator and assume that the following conditions hold*

(a) *$\|Ax\| \ge \|x\|$, for all $x \in \mathcal{P} \cap \partial B_r$.*

(b) *$\|Ax\| < \|x\|$, for all $x \in \mathcal{P} \cap \partial B_R$.*

(c) *$\|Ax\| \ge \|x\|$, for all $x \in \mathcal{P} \cap \partial B_L$.*

Then A has at least two fixed points x_1 and x_2 with $x_1 \in \mathcal{P} \cap (B_R \backslash B_r)$ and $x_2 \in \mathcal{P} \cap (\overline{B_L} \backslash \overline{B_R})$.

In this subsection, we prove the existence of two distinct nontrivial positive solutions to System (1.1).

Theorem 3.6. *Let $k \geq 1$ and assume that hypotheses $(\mathcal{H}_0), (\mathcal{H}_2)$, and $(\mathcal{H}_3)$ hold together with*

$$(\mathcal{H}_5) \qquad \liminf_{\sum_{i=1}^{n}(y_i+|z_i|)\to 0} \min_{t\in[\gamma,\delta]} \frac{f_i(t,y_1,\ldots,y_n,z_1,\ldots,z_n)}{\sum_{i=1}^{n}(y_i+|z_i|)} \geq \frac{4k}{\Lambda_1}M_0$$

$$\text{and} \qquad \liminf_{\sum_{i=1}^{n}(2ky_i+z_i)\to+\infty} \min_{t\in[\gamma,\delta]} \frac{f_i(t,y_1,\ldots,y_n,z_1,\ldots,z_n)}{\sum_{i=1}^{n}(2ky_i+z_i)} \geq \frac{2}{\Lambda_1}M_0,$$

where $M_0 = \left(\sum_{i=1}^{n} \max_{t\in[\gamma,\delta]} \int_\gamma^\delta e^{-\theta t}\left(H_i(t,s)+|\frac{\partial}{\partial t}H_i(t,s)|\right)\phi_i(s)ds\right)^{-1}$.

Then System (1.1) has at least two positive solutions Y_1, Y_2 such that

$$0 < \|Y_1\|_\theta < R < \|Y_2\|_\theta.$$

Proof. Define the operator A by (3.1) and consider the open set $\Omega_R = \{Y \in \mathbb{X} : \|Y\|_\theta < R\}$ where R is as introduced in Assumption $(\mathcal{H}_3)$. Lemmas 3.1 and 3.2 guarantee that $A : \overline{\Omega_L} \cap \mathcal{P} \longrightarrow \mathcal{P}$ is completely continuous. So, we only have to verify the conditions of Corollary 3.2.

Claim 1. As in the proof of Theorem 3.3, Claim 3, we can check that

$$\|AY\|_\theta < \|Y\|_\theta, \quad \forall Y \in \partial\Omega_R \cap \mathcal{P}. \tag{3.10}$$

So the condition (b) of Corollary 3.2 is satisfied.

Claim 2. Since $\liminf_{\sum_{i=1}^{n}(y_i+|z_i|)\to 0} \min_{t\in[\gamma,\delta]} \frac{f_i(t,y_1,\ldots,y_n,z_1,\ldots,z_n)}{\sum_{i=1}^{n}(y_i+|z_i|)} \geq \frac{4k}{\Lambda_1}M_0$, then there exists a positive number r_0 such that for $t \in [\gamma,\delta]$ and $\sum_{i=1}^{n}(y_i+|y_i'|) \leq r_0$, we have

$$f_i(t,y_1,\ldots,y_n,y_1',\ldots,y_n') \geq \frac{4k}{\Lambda_1}M_0 \sum_{i=1}^{n}(y_i+|y_i'|).$$

Consider the open set $\Omega_r = \{Y \in \mathbb{X} : \|Y\|_\theta < r\}$, where $r < \min(R, \frac{r_0}{e^{\theta\delta}})$. Then, for all $Y \in \partial\mathcal{P}_r$, $\sum_{i=1}^{n}\left(e^{-\theta t}(y_i(t)+|y_i'(t)|)\right) \leq r, \forall t \in \mathbb{R}_+$. Hence

$\sum\limits_{i=1}^{n}(y_i(t)+|y_i'(t)|)\leq re^{\theta\delta}\leq r_0,\ \forall\, t\in[\gamma,\delta].$

$$
\begin{array}{rcl}
\|AY\|_\theta & = & \sum\limits_{i=1}^{n}\sup\limits_{t\in\mathbb{R}_+}\left(e^{-\theta t}\left[(A_iY)(t)+|(A_iY)'(t)|\right]\right)\\
& \geq & \sum\limits_{i=1}^{n}\max\limits_{t\in[\gamma,\delta]}\int_\gamma^\delta e^{-\theta t}\left(H_i(t,s)+|\frac{\partial}{\partial t}H_i(t,s)|\right)\phi_i(s)\\
& & \times f_i\left(s,y_1(s),\ldots,y_n(s),y_1'(s),\ldots,y_n'(s)\right)ds\\
& \geq & \sum\limits_{i=1}^{n}\max\limits_{t\in[\gamma,\delta]}\int_\gamma^\delta e^{-\theta t}\left(H_i(t,s)+|\frac{\partial}{\partial t}H_i(t,s)|\right)\phi_i(s)ds\\
& & \times\frac{4k}{\Lambda_1}M_0\sum\limits_{i=1}^{n}(y_i(s)+|y_i'(s)|)\\
& \geq & \frac{4k}{\Lambda_1}\sum\limits_{i=1}^{n}\left(y_i(s)+\frac{1}{2k}|y_i'(s)|\right)\\
& \geq & \frac{2}{\Lambda_1}\sum\limits_{i=1}^{n}\min\limits_{s\in[\gamma,\delta]}(2ky_i(s)+y_i'(s))\\
& \geq & \frac{2}{\Lambda_1}\frac{\Lambda_1}{2}\|Y\|_\theta=\|Y\|_\theta.
\end{array}
$$

Therefore

$$\|AY\|_\theta\geq\|Y\|_\theta,\quad \forall\, Y\in\partial\Omega_r\cap\mathcal{P}.$$

So the condition (a) of Corollary 3.2 is satisfied.

Claim 3. Since $\liminf\limits_{\sum_{i=1}^{n}(2ky_i+z_i)\to+\infty}\ \min\limits_{t\in[\gamma,\delta]}\frac{f_i(t,y_1,\ldots,y_n,z_1,\ldots,z_n)}{\sum_{i=1}^{n}(2ky_i+z_i)}\geq\frac{2}{\Lambda_1}M_0$, there exists $R_0>0$ such that for $t\in[\gamma,\delta]$ and $\sum\limits_{i=1}^{n}(2ky_i+y_i')\geq R_0$, we have

$$f_i(t,y_1,\ldots,y_n,y_1',\ldots,y_n')\geq\frac{2}{\Lambda_1}M_0\sum_{i=1}^{n}(2ky_i+y_i')\,. \tag{3.11}$$

Consider the open set $\Omega_L=\{Y\in\mathbb{X}:\ \|Y\|_\theta<L\}$, where $L>\max(R,\frac{2}{\Lambda_1}R_0)$. $Y\in\partial\mathcal{P}_L$ implies that

$$
\begin{array}{rcl}
\sum\limits_{i=1}^{n}(2ky_i(t)+y_i'(t)) & \geq & \sum\limits_{i=1}^{n}\min\limits_{t\in[\gamma,\delta]}(2ky_i(t)+y_i'(t))\\
& \geq & \frac{\Lambda_1}{2}\|Y\|_\theta=\frac{\Lambda_1}{2}L\geq R_0,\ \forall\, t\in[\gamma,\delta].
\end{array}
$$

Using the inequality (3.11) and arguing as in Claim 2, we can prove that

$$\|AY\|_\theta\geq\|Y\|_\theta,\quad \forall\, Y\in\partial\Omega_L\cap\mathcal{P}.$$

As a consequence, the condition (c) of Corollary 3.2 is satisfied. According to Corollary 3.2, we infer that A has at least two positive fixed points $Y_1,Y_2\in\mathcal{P}$ such that $r\leq\|Y_1\|_\theta<R<\|Y_2\|_\theta\leq L$. Consequently, Y_1 and Y_2 are two distinct positive solutions of System (1.1) and satisfy

$$r\leq\|Y_1\|_\theta<R<\|Y_2\|_\theta\leq L.$$

□

The following result deals with the case $0 < k < 1$; the proof is analogous and is omitted.

Theorem 3.7. *Let $0 < k < 1$ and assume Assumptions $(\mathcal{H}_0), (\mathcal{H}_2)$ and $(\mathcal{H}'_2), (\mathcal{H}'_3)$ hold, together with*

$$(\mathcal{H}'_5) \qquad \liminf_{\sum\limits_{i=1}^{n}(y_i+|z_i|)\to 0} \min_{t\in[\gamma,\delta]} \frac{f_i(t, y_1, \dots, y_n, z_1, \dots, z_n)}{\sum\limits_{i=1}^{n}(y_i + |z_i|)} \geq \frac{2}{\Lambda_2} M_0,$$

$$\text{and} \quad \liminf_{\sum\limits_{i=1}^{n}(y_i+z_i)\to +\infty} \min_{t\in[\gamma,\delta]} \frac{f_i(t, y_1, \dots, y_n, z_1, \dots, z_n)}{\sum\limits_{i=1}^{n}(y_i + z_i)} \geq \frac{2}{\Lambda_2} M_0.$$

Then System (1.1) has at least two positive solutions Y_1, Y_2 such that

$$0 < \|Y_1\|_\theta < R < \|Y_2\|_\theta.$$

3.4 Example

Let $g_1(t) = g_2(t) = e^{-3t}$, $\phi_1(t) = e^{-\frac{t}{10}}, \phi_2(t) = e^{-\frac{t}{5}}$ and consider the nonlinearities

$$f_i(t, y_1, y_2, z_1, z_2) =$$
$$a_i(\sum_{i=1}^{2} \min_{t\in[\gamma,\delta]} (h(t) + 2ky_i + z_i)) \left[b_i\left(e^{-\theta t}y_1, e^{-\theta t}y_2\right) + c_i\left(e^{-\theta t}z_1, e^{-\theta t}z_2\right)\right]$$

for $(t, y_1, y_2, z_1, z_2) \in \mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}^2$ and $i = 1, 2$, where

$$h(t) = t, \quad a_1(u) = \frac{1}{50u + 1} + 1, \quad a_2(u) = \frac{1}{10u + 1} + 1, \ u \in \mathbb{R}_+$$

$$b_1(u, v) = \frac{10^{-5}}{4}\sqrt{(u + v)^3} + \frac{1}{100}, \ b_2(u, v) = \frac{10^{-5}}{8}\sqrt{(u + v)^3} + \frac{1}{100}, \ u, v \in \mathbb{R}_+$$

$$c_1(u, v) = \frac{10^{-10}}{8} e^{u+v}, \ c_2(u, v) = \frac{10^{-10}}{9} e^{u+v}, \ u, v \in \mathbb{R}.$$

In order to check the inequality (3.3) in Assumption $(\mathcal{H}_3)$, we choose $k = 2 > 1$, $\gamma = \frac{1}{5}$, and $\delta = 50$; then $\theta = \frac{21}{10}$ and $R = 5$. Moreover

$$\Lambda_1 = \frac{2}{3}e^{-100}, \ (B_1, B_2) = \left(\frac{2210}{1891}, \frac{1495}{1364}\right)$$

and

$$1 - \int_0^{+\infty} (e^{ks} - e^{-ks}) g_i(s) ds = \frac{1}{5}, \ i = 1, 2.$$

Hence

$$\sum_{i=1}^{2} B_i a_i(\frac{\Lambda_1}{2} R) \left[b_i(R, R) + c_i(R, R)\right] \approx 0.0456 < R\cdot$$

Moreover, we have that $f_i(t,y_1,y_2,z_1,z_2) > \frac{1}{100}$, for any $(t,y_1,y_2,z_1,z_2) \in \mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}^2$ and $i = 1,2$. Therefore Assumptions $(\mathcal{H}_0)$-$(\mathcal{H}_4)$ are satisfied. As a consequence, the boundary value problem

$$\begin{cases} -y_1''(t) + 4y_1(t) = f_1(t, y_1(t), y_2(t), y_1'(t), y_2'(t)), \ t > 0, \\ -y_2''(t) + 4y_2(t) = f_2(t, y_1(t), y_2(t), y_1'(t), y_2'(t)), \ t > 0, \\ y_1(0) = 0, \quad \lim\limits_{t\to+\infty} y_1(t)e^{-2t} = \int_0^{+\infty} g_1(s)y_1(s)ds, \\ y_2(0) = 0, \quad \lim\limits_{t\to+\infty} y_2(t)e^{-2t} = \int_0^{+\infty} g_2(s)y_2(s)ds \end{cases} \tag{3.12}$$

has at least one positive solution $Y = (y_1, y_2)$. In addition, for $i = 1,2$ Assumption $(\mathbb{H})$ in Theorem 3.3 is obvious. Indeed

$$\Theta_i(t) - \frac{1}{2k\nu_i}\left(e^{-kt} - e^{-3kt}\right) = -\frac{1}{5}e^{-3t} + \frac{3}{20}e^{-2t} + \frac{1}{20}e^{-6t} \geq 0, \ t \in \mathbb{R}_+.$$

Also, we have that $(\mathcal{B}_1, \mathcal{B}_2) = (\frac{2210}{1891}, \frac{1495}{1364})$, $(\mathbf{K}_1, \mathbf{K}_2) = (\frac{1}{100}, \frac{1}{100})$ and then the inequality in Assumptions $(\mathcal{H}_3)''$ is obvious for $\rho = 6$. Finally Assumptions $(\mathcal{H}_4)''$ in Theorem 3.1 is fulfilled. Therefore, the solution also verifies the conclusion of this theorem. Some computations of the bounds of the functions $Y_l(x)$ and $U^nY_u(x)$ are shown in the following Table:

t	$Y_l(t) = (y_{l,1}, y_{l,2})$	$UY_u(t) = (Uy_{u,1}(t), Uy_{u,2}(t))$
$t = \frac{1}{10}$	$(0.0008,\ 0.0008)$	$(0.1218,\ 0.1241)$
$t = \frac{1}{2}$	$(\ 0.0047,\ 0.0043)$	$(\ 0.5896\ ,0.6009)$
$t = 1$	$(0.0120,\ 0.0110)$	$(1.5192,\ 1.5485)$
$t = 2$	$(0.0781,\ \ 0.0715)$	$(\ 9.9776, 10.1701)$
$t = 5$	$(0.0307 \times 10^3, 0.0282 \times 10^3)$	$(3.9327 \times 10^3, 4.0086 \times 10^3)$
$t = 10$	$(0.0676 \times 10^7, 0.0621 \times 10^7)$	$(8.6620 \times 10^7, 8.8290 \times 10^7)$
$t = 20$	$(0.0328 \times 10^{16}, 0.0301 \times 10^{16})$	$(4.2025 \times 10^{16}, 4.2835 \times 10^{16})$
$t = 50$	$(0.0375 \times 10^{42}, 0.0344 \times 10^{42})$	$(4.7993 \times 10^{42}, 4.8918 \times 10^{42})$
$t = 100$	$(0.0101 \times 10^{86}, 0.0092 \times 10^{86})$	$(1.2901 \times 10^{86}, 1.3150 \times 10^{86})$

We further can check that Assumption $(\mathcal{H}_5)$ in Theorem 3.6 is fulfilled. Indeed, for all $t \in [\gamma, \delta]$, $(y_1, y_2) \in [0, +\infty)^2$, $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\frac{f_i(t, y_1, y_2, z_1, z_2)}{\sum\limits_{i=1}^{2}(y_i + |z_i|)} \geq \frac{10^{-2}}{y_1 + y_2 + |z_1| + |z_2|},$$

and if $\sum\limits_{i=1}^{2}(2ky_i + z_i) > 0$, then

$$\frac{f_i(t, y_1, y_2, z_1, z_2)}{\sum\limits_{i=1}^{2}(2ky_i + z_i)} \geq 10^{-11}\frac{\sqrt{e^{-7t}(y_1+y_2)^3} + e^{e^{-3t}(z_1+z_2)}}{2k(y_1+y_2) + z_1 + z_2}.$$

Therefore, Problem (3.12) has at least two positive solutions.

4 The singular problem

In this section, we study the existence of one or two positive solutions to (1.1) when the nonlinearity F has possible singularities at $Y = 0_{\mathbb{R}^n}$ and $Z = 0_{\mathbb{R}^n}$.

4.1 Existence of at least one solution

For $k \geq 1$, consider the subset of the half-space

$$\Gamma_1 = \left\{ \begin{array}{c} (y_1, \ldots, y_n, z_1, \ldots, z_n) \in (\mathbb{R}_+^*)^n \times (\mathbb{R}^*)^n : \\ \sum\limits_{i=1}^{n} (2ky_i + z_i) \geq 0 \text{ and } y_i + |z_i| \leq R(e^{\theta\delta} + 2),\ i \in [1, n] \end{array} \right\}.$$

Clearly, this set is nonempty. We will consider the following hypothesis:

($\mathbf{H}_2$) The functions $f_i : \mathbb{R}_+ \times (\mathbb{R}_+^*)^n \times (\mathbb{R}^*)^n \to \mathbb{R}_+$ are continuous and there exist continuous functions $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $a_i \in C(\mathbb{R}_+^*, \mathbb{R}_+)$, $b_i \in C((\mathbb{R}_+^*)^n, \mathbb{R}_+)$, and $c_i \in C((\mathbb{R}^*)^n, \mathbb{R}_+)$ such that for all $t \in \mathbb{R}_+$, $y_i \in \mathbb{R}_+^*$, $z_i \in \mathbb{R}^*$, ($i \in [1, n]$) we have

$$\begin{aligned} & f_i(t, y_1, \ldots, y_n, z_1, \ldots, z_n) \\ \leq\ & a_i \left(\sum_{i=1}^{n} \min_{t \in [\gamma, \delta]} (h(t) + 2ky_i + z_i) \right) \left(b_i \left(e^{-\theta t} y_1, \ldots, e^{-\theta t} y_n \right) \right. \\ & \left. + c_i \left(e^{-\theta t} z_1, \ldots, e^{-\theta t} z_n \right) \right), \end{aligned}$$

where a_i is nonincreasing and b_i, c_i are nondecreasing functions.

($\mathbf{H}_3$) There exits $R > 0$ such that

$$\sum_{i=1}^{n} B_i a_i(\frac{\Lambda_1}{2} R) \left[b_i(2R, \ldots, 2R) + c_i(2R, \ldots, 2R) \right] < R, \tag{4.1}$$

where $B_i = \int_0^{+\infty} \phi_i(s) \left((G(s, s) + 1) e^{-ks} + 2k\nu_i \Theta_i(s) \right) ds, i \in [1, n]$.

Theorem 4.1. *Let $k \geq 1$ and assume that Assumptions ($\mathcal{H}_0$), ($\mathbf{H}_2$) – ($\mathbf{H}_3$) hold together with*

($\mathbf{H}_4$) $\mathcal{N}_i = \min\limits_{t \in [\gamma, \delta],\, (y_1, \ldots, y_n, z_1, \ldots, z_n) \in \Gamma_1} f_i(t, y_1, \ldots, y_n, z_1, \ldots, z_n) > 0,\ i \in [1, n].$

Then System (1.1) has at least one positive solution $Y = (y_1, \ldots, y_n)$ satisfying

$$\begin{aligned} \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+} \left(e^{-\theta x} \left[|y_i(t)| + |(y_i)'(t)| \right] \right) &< R, \\ \sum_{i=1}^{n} \min_{t \in [\gamma, \delta]} (2ky_i(t) + (y_i)'(t)) &\geq \max \left(\tfrac{\Lambda_1}{2} \|Y\|_\theta, \sum_{i=1}^{n} C_i \mathcal{N}_i \right), \end{aligned} \tag{4.2}$$

where the constants C_i, $i \in [1, n]$ are given by (3.4).

Proof. Let R be as defined by Assumption $(\mathbf{H}_3)$ and consider the open set

$$\Omega_R := \{Y \in \mathbb{X} : \|Y\|_\theta < R\}.$$

For each $m \in \{1, 2, \ldots\}$, define a sequence of functions by

$$f_i^{(m)}(t, Y, Z) = f_i\big(t, y_1 + \tfrac{1}{m}, \ldots, y_n + \tfrac{1}{m}, z_1 + \tfrac{1}{m}, \ldots, z_n + \tfrac{1}{m}\big),\ i \in [1, n].$$

Then, for $Y \in \overline{\Omega}_R \cap \mathcal{P}$, define a sequence of operators by

$$(A^{(m)}Y)(t) = \int_0^\infty H(t, s)F^{(m)}(s, Y(s), Y'(s))ds, \tag{4.3}$$

where

$$F^{(m)}(t, Y, Z) = \begin{pmatrix} \phi_1(t) f_1^{(m)}(t, y_1, \ldots, y_n, z_1, \ldots, z_n) \\ \vdots \\ \phi_n(t) f_n^{(m)}(t, y_1, \ldots, y_n, z_1, \ldots, z_n) \end{pmatrix}.$$

Let R be given by Assumption $(\mathbf{H}_3)$ and r be any real number such that

$$0 < r < \min\left(\frac{\Lambda_1}{2}R, \sum_{i=1}^n C_i \mathcal{N}_i\right). \tag{4.4}$$

On the cone $\mathcal{P}$, we introduce the positive functionals

$$\begin{aligned} \alpha(Y) &= \sum_{i=1}^n \min_{t\in[\gamma,\delta]} (2ky_i(t) + y_i'(t)), \\ \beta(Y) &= \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \left(e^{-\theta t}\left[|y_i(t)| + |y_i'(t)|\right]\right) = \|Y\|_\theta. \end{aligned} \tag{4.5}$$

We show that Assumption $(\mathbf{H}_2)$ in Theorem 3.1 is satisfied.

Claim 1. $\overline{\mathcal{P}(\alpha, r)} \subset \mathcal{P}(\beta, R)$. Indeed, if $Y = (y_1, \ldots, y_n) \in \overline{\mathcal{P}(\alpha, r)}$, then

$$r \ \geq \ \sum_{i=1}^n \min_{t\in[\gamma,\delta]} (2ky_i(t) + y_i'(t)) \geq \tfrac{1}{2}\Lambda_1 \|Y\|_\theta.$$

Using (4.4), we infer that $\|Y\|_\theta \leq 2\frac{r}{\Lambda_1} < R$, and so $Y \in \mathcal{P}(\beta, R)$. Also, for all $Y \in \partial\mathcal{P}(\alpha, r),\ Z \in \partial\mathcal{P}(\beta, R),\ \lambda \in (0, 1]$ and $\mu \geq 1$, the functionals α and β satisfy

$$\alpha(\lambda Y) = \lambda\alpha(Y),\ \beta(\mu Z) = \mu\beta(Z) \text{ and } \beta(0_{\mathbb{R}^n}) = 0.$$

Claim 2. $\alpha(A^{(m)}Y) \geq r$, for all $Y \in \partial\mathcal{P}(\alpha, r)$ and $m \geq m_0 > \frac{1}{R}$. Indeed, let $Y = (y_1, \ldots, y_n) \in \partial\mathcal{P}(\alpha, r)$, that is $\alpha(Y) = \sum_{i=1}^n \min_{t\in[\gamma,\delta]} (2ky_i(t) + y_i'(t)) = r$. As checked in Claim 1, $\|Y\|_\theta \leq R$. Hence, for every $t \in [\gamma, \delta]$,

$$\left(y_1(t) + \frac{1}{m}, \ldots, y_n(t) + \frac{1}{m}, y_1'(t) + \frac{1}{m}, \ldots, y_n'(t) + \frac{1}{m}\right) \in \Gamma_1.$$

By Assumption $(\mathbf{H}_4)$, we deduce the estimates:

$$\begin{aligned}
\alpha(A^{(m)}Y) &= \sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} \int_0^{+\infty} \left(2kH_i(t,s) + \tfrac{\partial}{\partial t}H_i(t,s)\right)\phi_i(s) \\
&\quad \times f_i^{(m)}(s, y_1(s), \ldots, y_n(s), y_1'(s), \ldots, y_n'(s))ds \\
&\geq \sum_{i=1}^{n} \min_{t\in[\gamma,\delta]} \int_\gamma^\delta \phi_i(s)\left(2kH_i(t,s) + \tfrac{\partial}{\partial t}H_i(t,s)\right) \\
&\quad \times f_i\left(s, y_1(s)+\tfrac{1}{m}, \ldots, y_n(s)+\tfrac{1}{m}, y_1'(s)+\tfrac{1}{m}, \ldots, y_n'(s)+\tfrac{1}{m}\right)ds \\
&\geq \sum_{i=1}^{n} C_i\mathcal{N}_i > r.
\end{aligned}$$

Claim 3. $\beta(A^{(m)}Y) \leq R$, for all $Y \in \partial\mathcal{P}(\beta, R)$ and $m \geq m_0 > \frac{1}{R}$. Let $Y \in \partial\mathcal{P}(\beta, R)$. By Assumptions $(\mathbf{H_2})$ and $(\mathbf{H_3})$, for all $t \in \mathbb{R}_+$ and $i \in [1, n]$, we have the estimates

$$\begin{aligned}
\beta(A^{(m)}Y) &= \sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+} \left(e^{-\theta t}\left[(A_i^{(m)}Y)(t) + |(A_i^{(m)}Y)'(t)|\right]\right) \\
&\leq \sum_{i=1}^{n} \int_0^{+\infty} \phi_i(s)\left((G(s,s)+1)\,e^{-ks} + 2k\nu_i\Theta_i(s)\right) \\
&\quad \times f_i\left(s, y_1(s)+\tfrac{1}{m}, \ldots, y_n(s)+\tfrac{1}{m}, y_1'(s)+\tfrac{1}{m}, \ldots, y_n'(s)+\tfrac{1}{m}\right)ds \\
&\leq \sum_{i=1}^{n} \int_0^{+\infty} \phi_i(s)\left((G(s,s)+1)\,e^{-ks} + 2k\nu_i\Theta_i(s)\right) \\
&\quad \times a_i\Big(\sum_{i=1}^{n} \min_{s\in[\gamma,\delta]} \left(h(s) + 2k(y_i(s)+\tfrac{1}{m}) + y_i'(s)+\tfrac{1}{m}\right)\Big) \\
&\quad \times [b_i(e^{-\theta t}y_1(s) + \tfrac{e^{-\theta s}}{m}, \ldots, e^{-\theta t}y_n(s) + \tfrac{e^{-\theta s}}{m}) \\
&\quad + c_i(e^{-\theta t}y_1'(s) + \tfrac{e^{-\theta s}}{m}, \ldots, e^{-\theta t}y_n'(s) + \tfrac{e^{-\theta s}}{m})]ds \\
&\leq \sum_{i=1}^{n} \int_0^{+\infty} \phi_i(s)\left((G(s,s)+1)\,e^{-ks} + 2k\nu_i\Theta_i(s)\right) \\
&\quad \times a_i\Big(\sum_{i=1}^{n} \min_{s\in[\gamma,\delta]} (2k(y_i(s) + y_i'(s)))\Big) \\
&\quad \times [b_i(e^{-\theta t}y_1(s) + \tfrac{e^{-\theta s}}{m}, \ldots, e^{-\theta t}y_n(s) + \tfrac{e^{-\theta s}}{m}) \\
&\quad + c_i(e^{-\theta t}|y_1'(s)| + \tfrac{e^{-\theta s}}{m}, \ldots, e^{-\theta t}|y_n'(s)| + \tfrac{e^{-\theta s}}{m})]ds.
\end{aligned}$$

Then

$$\beta(A^{(m)}Y) \leq \sum_{i=1}^{n} B_i a_i(\frac{\Lambda_1}{2}R)\,[b_i(2R, \ldots, 2R) + c_i(2R, \ldots, 2R)] < R.$$

Claim 4. $\inf_{Y\in\partial\mathcal{P}(\alpha,r)} \|A^{(m)}Y\|_\theta > 0$. Indeed, for every $Y \in \partial\mathcal{P}(\alpha, r)$, for $m \geq m_0 > \frac{1}{R}$ and some $t_0 \in \mathbb{R}_+$, we have that

$$\begin{aligned}
&\sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+} (e^{-\theta t}[(A_i^{(m)}Y)(t) + |(A_i^{(m)}Y)'(t)|]) \\
\geq\ & \sum_{i=1}^{n} \int_0^{+\infty} e^{-\theta t_0}\left(H_i(t_0,s) + |\tfrac{\partial}{\partial t}H_i(t_0,s)|\right) \\
& \times f_i\left(s, y_1(s)+\tfrac{1}{m}, \ldots, y_n(s)+\tfrac{1}{m}, y_1'(s)+\tfrac{1}{m}, \ldots, y_n'(s)+\tfrac{1}{m}\right)ds \\
\geq\ & \sum_{i=1}^{n} \mathcal{N}_i \int_0^{+\infty} e^{-\theta t_0}\left(H_i(t_0,s) + |\tfrac{\partial}{\partial t}H_i(t_0,s)|\right)ds = K_1.
\end{aligned}$$

Passing to the infimum yields $\inf\limits_{Y\in\partial\mathcal{P}(\alpha,r)} \|A^{(m)}Y\|_\theta \geq K_1 > 0$. Therefore Hypothesis (**H2**) in Theorem 3.1 is satisfied. With Lemmas 3.1 and 3.2, we conclude that, for each $m \geq m_0$, there exists some $Y_m = (y_{1,m}, \dots, y_{n,m})$ lying in the conical shell $\overline{\mathcal{P}(\alpha,\beta,r,R)}$ such that $A^{(m)}Y_m = Y_m$ with

$$\begin{array}{l}
0 < \|Y_m\|_\theta = \sum\limits_{i=1}^{n} \sup\limits_{t\in\mathbb{R}_+} \left(e^{-\theta t}\left[\,|y_{i,m}(t)| + |(y_{i,m})'(t)|\,\right]\right) < R \\
\sum\limits_{i=1}^{n} \min\limits_{t\in[\gamma,\delta]} (2ky_{i,m}(t) + (y_{i,m})'(t)) \geq \frac{\Lambda_1}{2}\|Y_m\|_\theta \\
\sum\limits_{i=1}^{n} \min\limits_{t\in[\gamma,\delta]} (2ky_{i,m}(t) + (y_{i,m})'(t)) \geq \sum\limits_{i=1}^{n} C_iN_i > 0.
\end{array} \tag{4.6}$$

Since r satisfying (4.4) is arbitrary, the last estimate in (4.6) follows from Claim 2. Now consider the sequence of functions $(y_{1,m}, \dots, y_{n,m})_{m\geq m_0}$.

(i) Let $k < \theta^* < \theta$ be some positive real number. The sequence of functions $\{Y_m = (y_{1,m}, \dots, y_{n,m})\}_{m\geq m_0}$ is uniformly bounded with respect to the norm $\|.\|_{\theta^*}$. As in Lemma 3.1, Claim 1, we have

$$\begin{array}{l}
\|Y_m\| = \|A^{(m)}Y_m\|_{\theta^*} = \sum\limits_{i=1}^{n} \|A_i^{(m)}Y_m\|_{\theta^*} \\
= \sum\limits_{i=1}^{n} \sup\limits_{t\in\mathbb{R}_+} (e^{-\theta^* t}[|(A_i^{(m)}Y_m)(t)| + |(A_i^{(m)}Y_m)'(t)|]) \\
\leq \sum\limits_{i=1}^{n} S_R^{(i)}B_i < \infty,\ \forall\, m \geq m_0,
\end{array}$$

where

$$S_R^{(i)} = \sup\left\{\begin{array}{c} f_i(t, e^{\theta t}y_{1,m}, \dots, e^{\theta t}y_{n,m}, e^{\theta t}z_{1,m}, \dots, e^{\theta t}z_{n,m}),\ t \geq 0, \\ (y_{1,m}, \dots, y_{1,m}) \in [0,R]^n,\ (|z_{1,m}|, \dots, |z_{1,m}|) \in [0,R]^n \end{array}\right\}.$$

(ii) The sequence of functions $\{Y_m = (y_{1,m}, \dots, y_{n,m})\}_{m\geq m_0}$ is almost equicontinuous on $\mathbb{R}_+$. The proof is identical to that in Lemma 3.2, Claim 2. We have that for all $Y_m,\ m \geq m_0$

$$\begin{array}{l}
\|(Y_m)(t_1) - (Y_m)(t_2)\| = \sum\limits_{i=1}^{n} \left|(A_i^{(m)}Y_m)(t_1) - (A_i^{(m)}Y_m)(t_2)\right| \\
\|(Y_m)'(t_1) - (Y_m)'(t_2)\| = \sum\limits_{i=1}^{n} \left|(A_i^{(m)}Y_m)'(t_1) - (A_i^{(m)}Y_m)'(t_2)\right|.
\end{array}$$

The right-hand side tends to 0, as $|t_1 - t_2| \to 0$.

Consequently, Proposition 2.2 guarantees the existence of a convergent subsequence $\{Y_{m_j}\}_{j\geq1} = (y_{1,m_j}, \dots, y_{n,m_j})$ of $\{Y_m\}_{m\geq m_0} = (y_{1,m}, \dots, y_{n,m})$

such that $\lim\limits_{j\to+\infty} Y_{m_j} = Y = (y_1, \dots, y_n)$ strongly in $\mathbb{X}$. Moreover the continuity of the functions $f_i,\ i \in [1, n]$ implies

$$\begin{aligned} 0 &< \lim_{j\to+\infty} f_i^{(m_j)}(t, y_1, \dots, y_n, y_1', \dots, y_n') \\ &= \lim_{j\to+\infty} f_i(t, y_1 + \tfrac{1}{m_j}, \dots, y_n + \tfrac{1}{m_j}, y_1' + \tfrac{1}{m_j}, \dots, y_n' + \tfrac{1}{m_j}) \\ &= f_i(t, y_1, \dots, y_n, y_1', \dots, y_n'),\ i \in [1, n]. \end{aligned}$$

Moreover, $\lim\limits_{j\to+\infty} Y_{m_j} = (\lim\limits_{j\to+\infty} y_{1,m_j}, \dots, \lim\limits_{j\to+\infty} y_{n,m_j})$. The dominated convergence theorem guarantees that $Y = (y_1, \dots, y_n)$ with

$$\begin{aligned} y_i(t) &= \lim_{j\to+\infty} y_{i,m_j}(t) \\ &= \lim_{j\to+\infty} \int_0^{+\infty} H_i(t, s)\phi_i(s) \\ &\quad \times f_i\left(s, y_1 + \frac{1}{m_j}, \dots, y_n + \frac{1}{m_j}, y_1' + \frac{1}{m_j}, \dots, y_n' + \frac{1}{m_j}\right) ds \\ &= \int_0^{+\infty} H_i(t, s)\phi_i(s) f_i\left(s, y_1, \dots, y_n, y_1', \dots, y_n'\right) ds, \quad t \in \mathbb{R}_+. \end{aligned}$$

Finally, $0 < \|Y_{m_j}\|_\theta < R, \forall j \geq 1$ implies that $0 \leq \|Y\|_\theta \leq R$. Also (4.1) guarantees that $\|Y\|_\theta \neq R$. Hence

$$\begin{aligned} \sum_{i=1}^n \sup_{t\in\mathbb{R}_+} \left(e^{-\theta x}\left[\,|y_i(x)| + |(y_i)'(x)|\,\right]\right) &< R \\ \sum_{i=1}^n \min_{t\in[\gamma,\delta]} \left(2ky_i(x) + (y_i)'(x)\right) &\geq \tfrac{\Lambda_1}{2} \|Y\|_\theta \\ \sum_{i=1}^n \min_{t\in[\gamma,\delta]} \left(2ky_i(x) + (y_i)'(x)\right) &\geq \sum_{i=1}^n C_i \mathcal{N}_i > 0, \end{aligned}$$

as claimed. □

For the other case $0 < k < 1$, we can prove an existence result similar to Theorem 4.1. It is sufficient to use the inequalities (a), (b) in Lemma 2.2 and part (b) in Lemma 2.3. The constant R is as given by (4.7). We state without proof the result.

Theorem 4.2. *Let $0 < k < 1$ and*

$$\Gamma_2 = \left\{ \begin{array}{c} (y_1, \dots, y_n, z_1, \dots, z_n) \in (\mathbb{R}_+^*)^n \times (\mathbb{R}^*)^n : \\ \sum\limits_{i=1}^n (y_i + z_i) \geq 0 \ \text{and}\ y_i + |z_i| \leq R(e^{\theta\delta} + 2),\ i \in [1, n] \end{array} \right\}.$$

Assume that Assumptions $(\mathcal{H}_0), (\mathcal{H}_1)$ hold and

($\mathbf{H_2'}$) *$f_i : \mathbb{R}_+ \times (\mathbb{R}_+^*)^n \times (\mathbb{R}^*)^n \to \mathbb{R}_+$ are continuous and there exist continuous functions $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $a_i \in C(\mathbb{R}_+^*, \mathbb{R}_+)$, $b_i \in C((\mathbb{R}_+^*)^n, \mathbb{R}_+)$ and $c_i \in C((\mathbb{R}^*)^n, \mathbb{R}_+)$ such that for all $t \in \mathbb{R}_+$, $y_i \in \mathbb{R}_+^*$, $z_i \in \mathbb{R}^*$, $(i \in [1, n])$, we have*

$$\begin{aligned} & f_i(t, y_1, \ldots, y_n, z_1, \ldots, z_n) \\ \leq \; & a_i \left(\sum_{i=1}^{n} \min_{t \in [\gamma, \delta]} (h(t) + y_i + z_i) \right) \left(b_i \left(e^{-\theta t} y_1, \ldots, e^{-\theta t} y_n \right) \right. \\ & \left. + c_i \left(e^{-\theta t} z_1, \ldots, e^{-\theta t} z_n \right) \right), \end{aligned}$$

where a_i is nonincreasing and b_i, c_i are nondecreasing functions.

($\mathbf{H_3'}$) *There exits $R > 0$ such that*

$$\sum_{i=1}^{n} B_i a_i \left(\frac{\Lambda_2}{2} R\right) \left[b_i(2R, \ldots, 2R) + c_i(2R, \ldots, 2R) \right] < R, \tag{4.7}$$

where $B_i = \int_0^{+\infty} \left((G(s, s) + 1) e^{-ks} + 2\nu_i \Theta_i(s) \right) \phi_i(s) ds, i \in [1, n]$

together with

$$N_i' = \min_{t \in [\gamma, \delta], \, (y_1, \ldots, y_n, z_1, \ldots, z_n) \in \Gamma_2} f_i(t, y_1, \ldots, y_n, z_1, \ldots, z_n) > 0, \; i \in [1, n].$$

Then System (1.1) has at least one positive solution $Y = (y_1, y_2, \ldots, y_n)$ satisfying

$$\begin{aligned} \sum_{i=1}^{n} \sup_{t \in \mathbb{R}_+} \left(e^{-\theta x} \left[|y_i(t)| + |(y_i)'(t)| \right] \right) & < R \\ \sum_{i=1}^{n} \min_{t \in [\gamma, \delta]} (y_i(t) + (y_i)'(t)) & \geq \max \left(\tfrac{\Lambda_2}{2} \|Y\|_\theta, \sum_{i=1}^{n} D_i \mathcal{N}_i' \right), \end{aligned}$$

where the constants D_i, $i \in [1, n]$ are given by (3.4).

4.2 Existence of at least two solutions

In this subsection, using Corollary 3.2, we prove an existence theorem of two distinct nontrivial positive solutions to System (1.1).

Theorem 4.3. *Let $k \geq 1$ and assume that Assumptions $(\mathcal{H}_0)$ and $(\mathbf{H}_2) - (\mathbf{H}_3)$ hold together with*

$(\mathbf{H}_5)$:

$$\liminf_{\sum_{i=1}^{n} (y_i + |z_i|) \to 0} \min_{t \in [\gamma, \delta]} \frac{f_i(t, y_1 + \mu, \ldots, y_n + \mu, z_1 + \mu, \ldots, z_n + \mu)}{\sum_{i=1}^{n} (y_i + |z_i|)} \geq \frac{4k}{\Lambda_1} M_0$$

and

$$\liminf_{\sum_{i=1}^{n} (2ky_i + z_i) \to +\infty} \min_{t \in [\gamma, \delta]} \frac{f_i(t, y_1 + \mu, \ldots, y_n + \mu, z_1 + \mu, \ldots, z_n + \mu)}{\sum_{i=1}^{n} (2ky_i + z_i)} \geq \frac{2}{\Lambda_1} M_0,$$

where μ is an arbitrary real positive constant and

$$M_0 = \left(\sum_{i=1}^{n} \max_{t\in[\gamma,\delta]} \int_\gamma^\delta e^{-\theta t} \left(H_i(t,s) + |\frac{\partial}{\partial t} H_i(t,s)| \right) \phi_i(s) ds \right)^{-1}.$$

Then System (1.1) has at least two positive solutions Y_1, Y_2 such that

$$0 < \|Y_1\|_\theta < R < \|Y_2\|_\theta.$$

Proof. Define a sequence of operators $A^{(m)}$ by (4.3) and consider the open set $\Omega_R = \{Y \in \mathbb{X} : \|Y\|_\theta < R\}$ where R is as introduced in Assumption $(\mathbf{H}_3)$. Lemmas 3.1 and 3.2 guarantee that $A^{(m)} : \overline{\Omega_L} \cap \mathcal{P} \longrightarrow \mathcal{P}$ is completely continuous.

Claim 1. As in the proof of Theorem 3.3, Claim 3, we can check that

$$\|A^{(m)}Y\|_\theta < \|Y\|_\theta, \quad \forall\, Y \in \partial\Omega_R \cap \mathcal{P},\ \forall\, m \in \{m_0, m_0+1, \ldots\}. \tag{4.8}$$

So the condition (b) of Corollary 3.2 is satisfied.

Claim 2. Since $\liminf\limits_{\sum_{i=1}^{n}(y_i+|z_i|)\to 0} \min\limits_{t\in[\gamma,\delta]} \frac{f_i(t,y_1+\mu,\ldots,y_n+\mu,z_1+\mu,\ldots,z_n+\mu)}{\sum_{i=1}^{n}(y_i+|z_i|)} \geq \frac{4k}{\Lambda_1} M_0$, then there exists $r_0 > 0$ such that for all $m > m_0 > 0$, $t \in [\gamma,\delta]$, and $\sum_{i=1}^{n}(y_i+|y_i'|) \leq r_0$,

$$f_i(t, y_1 + \frac{1}{m}, \ldots, y_n + \frac{1}{m}, y_1' + \frac{1}{m}, \ldots, y_n' + \frac{1}{m}) \geq \frac{4k}{\Lambda_1} M_0 \sum_{i=1}^{n} \left(y_i + |y_i'| \right).$$

Consider the open set $\Omega_r = \{Y \in \mathbb{X} : \|Y\|_\theta < r\}$, where $r < \min(R, \frac{r_0}{e^{\theta\delta}})$. Then, for all $Y \in \partial\mathcal{P}_r$,

$$\sum_{i=1}^{n} \left(e^{-\theta t}(y_i(t) + |y_i'(t)|) \right) \leq r, \forall\, t \geq 0.$$

Then $\sum_{i=1}^{n} (y_i(t) + |y_i'(t)|) \leq r e^{\theta\delta} \leq r_0$, $\forall\, t \in [\gamma,\delta]$. In addition

$$\begin{aligned}
& \|A^{(m)}Y\|_\theta \\
= \ & \sum_{i=1}^{n} \sup_{t\in\mathbb{R}_+} \left(e^{-\theta t} \left[(A_i^{(m)}Y)(t) + |(A_i^{(m)}Y)'(t)| \right] \right) \\
\geq \ & \sum_{i=1}^{n} \max_{t\in[\gamma,\delta]} \textstyle\int_\gamma^\delta e^{-\theta t} \left(H_i(t,s) + |\frac{\partial}{\partial t} H_i(t,s)| \right) \phi_i(s) \\
& \times f_i\left(s, y_1(s) + \frac{1}{m}, \ldots, y_n(s) + \frac{1}{m}, y_1'(s) + \frac{1}{m}, \ldots, y_n'(s) + \frac{1}{m}\right) ds \\
\geq \ & \frac{4k}{\Lambda_1} \sum_{i=1}^{n} \left(y_i(s) + \frac{1}{2k}|y_i'(s)| \right) \\
\geq \ & \frac{2}{\Lambda_1} \sum_{i=1}^{n} \min_{s\in[\gamma,\delta]} (2k y_i(s) + y_i'(s)) \\
\geq \ & \frac{2}{\Lambda_1} \frac{\Lambda_1}{2} \|Y\|_\theta = \|Y\|_\theta.
\end{aligned}$$

As a consequence,

$$\|A^{(m)}Y\|_\theta \geq \|Y\|_\theta, \quad \forall\, Y \in \partial\Omega_r \cap \mathcal{P},\ \forall\, m \in \{m_0, m_0+1, \ldots\}.$$

So the condition (a) of Corollary 3.2 is satisfied.

Claim 3. Since $\liminf\limits_{\sum_{i=1}^{n}(2ky_i+z_i)\to+\infty} \min\limits_{t\in[\gamma,\delta]} \frac{f_i(t,y_1+\mu,\ldots,y_n+\mu,z_1+\mu,\ldots,z_n+\mu)}{\sum_{i=1}^{n}(2ky_i+z_i)} > \frac{2}{\Lambda_1}M_0$, there exists $R_0 > 0$ such that for $t \in [\gamma,\delta]$ and $\sum\limits_{i=1}^{n}(2ky_i + y_i') \geq R_0$, we have for $m > m_0 > 0$

$$f_i(t, y_1+\frac{1}{m}, \ldots, y_n+\frac{1}{m}, y_1'+\frac{1}{m}, \ldots, y_n'+\frac{1}{m}) \geq \frac{2}{\Lambda_1}M_0 \sum_{i=1}^{n}\left(2ky_i + y_i'\right). \quad (4.9)$$

Now, consider the open set $\Omega_L = \{Y \in \mathbb{X}\ \|Y\|_\theta < L\}$, where $L > \max(R, \frac{2}{\Lambda_1}R_0)$. Notice that $Y \in \partial\mathcal{P}_L$ implies that for all $t \in [\gamma,\delta]$

$$\begin{aligned}\sum_{i=1}^{n}(2ky_i(t) + y_i'(t)) &\geq \sum_{i=1}^{n}\min_{t\in[\gamma,\delta]}(2ky_i(t) + y_i'(t))\\ &\geq \tfrac{\Lambda_1}{2}\|Y\|_\theta = \tfrac{\Lambda_1}{2}L \geq R_0.\end{aligned}$$

Using the inequality (4.9) and arguing as in Claim 2, we can prove that

$$\|A^{(m)}Y\|_\theta \geq \|Y\|_\theta, \quad \forall\, Y \in \partial\Omega_L \cap \mathcal{P},\ \forall\, m \in \{m_0, m_0+1, \ldots\}.$$

So the condition (c) of Corollary 3.2 is satisfied. According to Corollary 3.2, we infer that, for each $m \in \{m_0, m_0+1, \ldots\}$, the operator $A^{(m)}$ has at least two positive fixed points $Y_{m,1}, Y_{m,2} \in \mathcal{P}$ such that $r \leq \|Y_{m,1}\|_\theta < R < \|Y_{m,2}\|_\theta \leq L$. Consider the sequence of functions $\{Y_{m,i}\}_{m\geq m_0}$, $i = 1,2$. Then the same argument used for $\{Y_m\}_{m\geq m_0}$ in Theorem 3.3 shows that $\{Y_{m,i}\}_{m\geq m_0}$, $i = 1,2$ has a convergent subsequence $\{Y_{m_j,i}\}_{j\geq 1}$ such that $\lim\limits_{j\to+\infty} Y_{m_j,i} = Y_i$, $i = 1,2$ for the norm topology of $\mathbb{X}$. Consequently, Y_1 and Y_2 are two positive solutions of System (1.1) and satisfy

$$r \leq \|Y_1\|_\theta \leq R \leq \|Y_2\|_\theta \leq L.$$

In addition, (4.1) guarantees that $\|Y_1\|_\theta \neq R$ and $\|Y_2\|_\theta \neq R$. Finally, System (1.1) has at least two positive solutions Y_1, Y_2 with $0 < \|Y_1\|_\theta < R < \|Y_2\|_\theta$. □

The following result deals with the case $0 < k < 1$ and the proof is identical.

Theorem 4.4. *Let $0 < k < 1$ and assume that Assumptions $(\mathcal{H}_0)$ and $(\mathbf{H'}_2) - (\mathbf{H'}_3)$ hold together with*

$$(\mathbf{H'}_5):$$
$$\liminf_{\sum_{i=1}^{n}(y_i+|z_i|)\to 0} \min_{t\in[\gamma,\delta]} \frac{f_i(t,y_1+\mu,\ldots,y_n+\mu,z_1+\mu,\ldots,z_n+\mu)}{\sum_{i=1}^{n}(y_i+|z_i|)} \geq \frac{2}{\Lambda_2}M_0,$$
and
$$\liminf_{\sum_{i=1}^{n}(y_i+z_i)\to +\infty} \min_{t\in[\gamma,\delta]} \frac{f_i(t,y_1+\mu,\ldots,y_n+\mu,z_1+\mu,\ldots,z_n+\mu)}{\sum_{i=1}^{n}(y_i+z_i)} \geq \frac{2}{\Lambda_2}M_0,$$

where μ is arbitrary real positive constant. Then System (1.1) has at least two positive solutions Y_1, Y_2 satisfying

$$0 < \|Y_1\|_\theta < R < \|Y_2\|_\theta.$$

4.3 Example

Let $g_1(t) = g_2(t) = e^{-\frac{3}{2}t}$, $\phi_1(t) = e^{-t}, \phi_2(t) = e^{-2t}$, and consider the nonlinearities

$$f_i(t,y_1,y_2,z_1,z_2) =$$
$$a_i\left(\sum_{i=1}^{2}\min_{t\in[\gamma,\delta]}(h(t)+y_i+z_i)\right)\left(b_i\left(e^{-\theta t}y_1, e^{-\theta t}y_2\right) + c_i\left(e^{-\theta t}z_1, e^{-\theta t}z_2\right)\right)$$

for $(t,y_1,y_2,z_1,z_2) \in \mathbb{R}_+ \times (\mathbb{R}_+^*)^2 \times (\mathbb{R}^*)^2$ and $i = 1,2$, where

$$h(t) = (\gamma - t)^2,\ a_1(u) = \frac{1}{50u} + 1,\ a_2(u) = \frac{1}{10u} + 1,\ u \in \mathbb{R}_+,$$

$$b_1(u,v) = \frac{10^{-3}}{4}(u+v)^2 + \frac{1}{10},\ b_2(u,v) = \frac{10^{-3}}{8}(u+v)^2 + \frac{1}{50},\ u,v \in \mathbb{R}_+,$$

and
$$c_1(u,v) = 10^{-6}\,e^{u+v},\ c_2(u,v) = 10^{-8}\,e^{u+v},\ u,v \in \mathbb{R}.$$

In order to check the inequality (4.7) in Assumption $(\mathbf{H}'_3)$, let $k = \frac{2}{3} < 1$, $\gamma = \frac{1}{5}$, and $\delta = 1$ so that we can choose $\theta = 3$, and $R = 2$. Moreover

$$\Lambda_2 = \tfrac{1}{5}e^{-\frac{2}{3}},\ (B_1,B_2) = (\tfrac{28}{17}, \tfrac{3225}{3808}),$$
$$\text{and } 1 - \textstyle\int_0^{+\infty}(e^{ks} - e^{-ks})g_i(s)ds = \tfrac{17}{65},\ i = 1,2.$$

Then
$$\sum_{i=1}^{2} B_i a_i(\frac{\Lambda_2}{2}R)\,[b_i(2R,2R) + c_i(2R,2R)] \approx 0.2810 < R\cdot$$

Indeed, we have that $f_1(t,y_1,y_2,z_1,z_2) > \frac{1}{10}$ and $f_2(t,y_1,y_2,z_1,z_2) > \frac{1}{50}$, for all $(t,y_1,y_2,z_1,z_2) \in \mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}^2$. Therefore Assumptions $(\mathcal{H}_0)$-$(\mathcal{H}_1)$

and $(\mathbf{H}_2')$-$(\mathbf{H}_3')$ in Theorem 4.2 are satisfied. As a consequence, the singular boundary value problem

$$\begin{cases} -y_1''(t) + \frac{9}{4}y_1(t) = f_1(t, y_1(t), y_2(t), y_1'(t), y_2'(t)), \ t > 0, \\ -y_2''(t) + \frac{9}{4}y_2(t) = f_2(t, y_1(t), y_2(t), y_1'(t), y_2'(t)), \ t > 0, \\ y_1(0) = 0, \quad \lim\limits_{t \to +\infty} y_1(t) e^{-\frac{3}{2}t} = \int_0^{+\infty} g_1(s) y_1(s) ds, \\ y_2(0) = 0, \quad \lim\limits_{t \to +\infty} y_2(t) e^{-\frac{3}{2}t} = \int_0^{+\infty} g_2(s) y_2(s) ds \end{cases} \tag{4.10}$$

has at least one nontrivial positive solution $Y = (y_1, y_2)$. Finally, Assumption $(\mathbf{H}_5')$ in Theorem 4.4 is satisfied. Indeed, for any positive real constant μ and $t \in [\gamma, \delta]$, $(y_1, y_2) \in \mathbb{R}_+^2$, $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\frac{f_i(t, y_1 + \mu, y_2 + \mu, z_1 + \mu, z_2 + \mu)}{\sum\limits_{i=1}^{2} (y_i + |z_i|)} \geq \frac{\frac{1}{50}}{(y_1 + y_2 + |z_1| + |z_2|)},$$

and in the other hand, if $\sum\limits_{i=1}^{2} (y_i + z_i) > 0$ then

$$\frac{f_i(t, y_1 + \mu, y_2 + \mu, z_1 + \mu, z_2 + \mu)}{\sum\limits_{i=1}^{2} (y_i + z_i)} \geq 10^{-8} \frac{e^{-6t}(y_1 + y_2)^2 + e^{e^{-3t}(z_1 + z_2)}}{y_1 + y_2 + z_1 + z_2}.$$

Therefore Problem (4.10) has at least two nontrivial positive solutions.

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