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# ON THE OSCILLATORY BEHAVIOR OF EVEN ORDER NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME-SCALES

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Abstract. We establish some new criteria for the oscillation of the even order neutral dynamic equation

$$\left(a(t)\left(\left(x(t)-p(t)x(\tau(t))\right)^{\Delta^{n-1}}\right)^{\alpha}\right)^{\Delta}+q(t)\left(x^{\sigma}(g(t))\right)^{\lambda}=0$$

on a time scale $\mathbb{T}$, where $n \geq 2$ is even, $\alpha$ and $\lambda$ are ratios of odd positive integers, $a$, $p$ and $q$ are real valued positive rd-continuous functions defined on $\mathbb{T}$, and $g$ and $\tau$ are real valued rd-continuous functions on $\mathbb{T}$. Examples illustrating the results are included.

## 1. Introduction

This paper is concerned with the oscillatory behavior of all solutions of the even order neutral delay dynamic equation

$$\left(a(t)\left(\left(x(t)-p(t)x(\tau(t))\right)^{\Delta^{n-1}}\right)^{\alpha}\right)^{\Delta}+q(t)\left(x^{\sigma}(g(t))\right)^{\lambda}=0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T} \subseteq \mathbb{R}$ with $\sup \mathbb{T} = \infty$ and $n \geq 2$ an even integer. Whenever we write $t \geq t_1$ we mean $t \in [t_1, \infty) \cap \mathbb{T} = [t_1, \infty)_{\mathbb{T}}$. We will use the basic concepts and notation for the time scale calculus; we refer the reader to the monograph of Bohner and Peterson [3] for additional details.

We shall assume that:

(i) $\alpha$ and $\lambda$ are ratio of positive odd integers;

(ii) $a$, $p$, and $q : \mathbb{T} \to R^{+} = (0, \infty)$ are real-valued rd-continuous functions, $a^{\Delta}(t) \geq 0$ for $t \geq t_0$, and

$$\int^{\infty} a^{-1/\alpha}(s)\Delta s = \infty; \tag{1.2}$$

(iii) $g$, $\tau : \mathbb{T} \to \mathbb{T}$ are rd-continuous functions such that $g(t) \leq t$, $\tau(t) \leq t$, $g^{\Delta} \geq 0$, $\tau^{\Delta} > 0$, $\lim_{t\to\infty} g(t) = \infty$, and $\lim_{t\to\infty} \tau(t) = \infty$;

(iv) $\xi(t) := (\tau^{-1} \circ g)(t) \leq t$, $\xi^{\Delta}(t) \geq 0$, $\lim_{t\to\infty} \xi(t) = \infty$.

We recall that a solution $x$ of equation (1.1) is said to be *nonoscillatory* if there exists a $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$; otherwise, it is said to be *oscillatory*. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The study of dynamic equations on time-scales goes back to its founder Hilger [16] and has received a lot of attention in the last ten years. Recently, there has been an increasing

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interest in studying the oscillatory behavior of first and second order dynamic equations on time-scales; for example see [1, 9, 11] and the references contained therein.

As to the oscillation of neutral delay dynamic equations on time-scales, Mathsen et al. [19] considered the first order equation

$$(x(t)-p(t)x(\tau(t)))^{\Delta}+q(t)x(g(t))=0, \quad t\in\mathbb{T}, \tag{1.3}$$

and established oscillation criteria that included some results for first order neutral delay ordinary differential equations as special cases. Han et al. [15] established some results on the oscillatory and asymptotic behavior of solutions of equation (1.1) with $n=3$ and $0<p(t)<1$. There are few results on the oscillation of solutions of higher order nonlinear neutral delay differential equations on time-scales (see [2, 4, 5, 6, 7, 8, 17, 18]). The purpose of this paper is to establish some new criteria for the oscillation of equation (1.1). In so doing, we present conditions under which all bounded solutions of the equation

$$\left(a(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}+q(t)x^{\lambda}(g(t))=0 \tag{1.4}$$

with $n$ even are oscillatory.

This paper is organized as follows. In Section 2, we study the oscillatory properties of equation (1.1) with $p(t)=0$, while Section 3 is devoted to the study of the oscillatory behavior of equation (1.1) with $-1<p(t)<0$. In Section 4, we establish oscillation results for (1.1) in case $0<p(t)<1$. Applications to the time scales $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$ are given to illustrate our results.

## 2. Oscillation of Equation (1.1) with $p(t)=0$

In this section, we consider the equation

$$\left(a(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}+q(t)\left(x^{\sigma}(g(t))\right)^{\lambda}=0, \ \ n \text{ is even.} \tag{2.1}$$

Since $a^{\Delta}(t)\ge 0$ for $t\ge t_0$, if $x$ is a positive solution of equation (2.1) with $x^{\Delta^{n-1}}(t)>0$ for $t\ge t_0$, we have

$$0\ge\left(a(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}=a^{\Delta}(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}+a^{\sigma}(t)\left(\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}.$$

This implies

$$\left(\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}\le 0 \quad \text{for} \quad t\in[t_0,\infty)_{\mathbb{T}}.$$

Set $z=x^{\Delta^{n-1}}$ on $[t_0,\infty)_{\mathbb{T}}$. From [3, Theorem 1.90], we see that

$$0\ge\left(\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}=(z^{\alpha})^{\Delta}=\alpha z^{\Delta}\int_0^1[z+h\mu z^{\Delta}]^{\alpha-1}dh\ge\alpha z^{\Delta}\int_0^1 z^{\alpha-1}dh=\alpha z^{\alpha-1}z^{\Delta},$$

which implies

$$z^{\Delta}=x^{\Delta^{n}}\le 0 \quad \text{on} \quad [t_0,\infty)_{\mathbb{T}}.$$

We will make use of the following Kiguradze's type lemma.

**Lemma 2.1.** *Let $x(t) \in C_{rd}^{n}([t_0,\infty), R^{+})$. If $x^{\Delta^n}(t)$ is of one sign on $[t_0,\infty)_{\mathbb{T}}$ and not identically zero on $[t_1,\infty)_{\mathbb{T}}$ for any $t_1 \geq t_0$, then there exist $t_x \geq t_0$ and an integer $m$, $0 \leq m \leq n$, with $n+m$ even if $x^{\Delta^n} \geq 0$ or $m+n$ odd if $x^{\Delta^n} \leq 0$ such that*

$$m > 0 \quad \textit{implies} \quad x^{\Delta^k} > 0 \quad \textit{for} \quad t \geq t_x \quad \textit{and} \quad k \in \{1,2,...,m-1\} \tag{2.2}$$

*and*

$$m \leq n-1 \quad \textit{implies} \quad (-1)^{m+k} x^{\Delta^k} > 0 \quad \textit{for} \quad t \geq t_x \quad \textit{and} \quad k \in \{m, m+1, ..., n-1\}. \tag{2.3}$$

**Lemma 2.2.** ([9]) *Suppose $|x|^{\lambda} > 0$ on $[t_0,\infty)_{\mathbb{T}}$, $\lambda > 0$, and $\lambda \neq 1$. Then*

$$\frac{|x|^{\Delta}}{(|x|^{\sigma})^{\lambda}} \leq \frac{\left(|x|^{1-\lambda}\right)^{\Delta}}{1-\lambda} \leq \frac{|x|^{\Delta}}{\left(|x|^{\lambda}\right)} \quad \textit{on} \quad [t_0,\infty)_{\mathbb{T}}. \tag{2.4}$$

It will be convenient to employ the Taylor monomials (see [3, Sec. 1.6]) $\{h_n(t,s)\}_{n=0}^{\infty}$ which are defined recursively by

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau, \quad t,s \in \mathbb{T} \quad \text{and} \quad n \geq 1.$$

Now $h_1(t,s) = t-s$ for any time scale, but there are no general formulas for $n \geq 2$.

We now present our main results in this section.

**Theorem 2.1.** *Let $t_0 \in \mathbb{T}$. Suppose conditions (i)-(iii) and (1.2) hold. Equation* (2.1) *is oscillatory if for every integer $m \in \{1,3,...,n-1\}$ and $t \geq t_0$:*

$$\int_{t_0}^{\infty} g^{\Delta}(s)(h_{m-1}(g(s),t_0)h_{n-m-1}(s,g(s))\left(\frac{1}{a(s)}\int_s^{\infty} q(u)\Delta u\right)^{1/\alpha} \Delta s = \infty \quad \textit{if } \lambda > \alpha; \tag{2.5}$$

$$\limsup_{t\to\infty}(h_m(g(t),t_0)h_{n-m-1}(t,g(t)))\left(\frac{1}{a(t)}\int_t^{\infty} q(s)\Delta s\right)^{1/\alpha} > 1 \quad \textit{if } \lambda = \alpha;, \tag{2.6}$$

$$\int_{t_0}^{\infty} a^{-\lambda/\alpha}(s)(h_m(g(s),t_0)h_{n-m-1}(s,g(s)))^{\lambda} q(s)\Delta s = \infty \quad \textit{if } \lambda < \alpha. \tag{2.7}$$

*Proof.* Let $x(t)$ be a nonoscillatory solution of equation (2.1), say $x(t) > 0$ for $t \geq t_0 \in \mathbb{T}$. Since $\lim_{t\to\infty} g(t) = \infty$, we can choose $t_1 \geq t_0$ such that $g(t) \geq t_0$ for all $t \geq t_1$. Notice that (1.2) implies $x^{\Delta^{n-1}}(t) \geq 0$ for $t \geq t_1$. Hence, $\left(a(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} \leq 0$ and so $x^{\Delta^n}(t) \leq 0$ for all $t \geq t_1$, and $x^{\Delta^n}(t)$ is not identically zero for all large $t$. Using Lemma 2.1, there exists an integer $m \in \{1,3,...,n-1\}$ such that (2.2) and (2.3) hold for all $t \geq t_1$. From (2.2), we see that

$$x^{\Delta^{m-1}}(t) > 0, \quad x^{\Delta^m}(t) > 0, \quad \text{and} \quad x^{\Delta^{m+1}}(t) < 0 \tag{2.8}$$

for $t \geq t_1$. Thus,

$$x^{\Delta^{m-1}}(t) = x^{\Delta^{m-1}}(t_1) + \int_{t_1}^{t} x^{\Delta^m}(s)\Delta s \geq h_1(t,t_1)x^{\Delta^m}(t) \quad for \quad t \geq t_1.$$

Integrating this inequality $(m-1)$-times from $t_1$ to $t \geq t_1$ and using the fact that $x^{\Delta^m}(t)$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, we have

$$x^{\Delta}(t) \geq h_{m-1}(t,t_1)x^{\Delta^m}(t) \quad \text{and} \quad x(t) \geq h_m(t,t_1)x^{\Delta^m}(t) \quad \text{for} \quad t \geq t_1.$$

Replacing $t$ by $g(t)$ in the above inequality, we obtain

$$x^{\Delta}(g(t)) \geq h_{m-1}(g(t),t_1)x^{\Delta^m}(g(t)) \quad \text{for} \quad t \geq t_2 \tag{2.9}$$

where $g(t) \geq t_1$ for $t \geq t_2$. It follows that

$$x(g(t)) \geq h_m(g(t),t_1)x^{\Delta^m}(g(t)) \quad \text{for} \quad t \geq t_2. \tag{2.10}$$

From (2.3) and applying Taylor's formula (see [3, Theorem 1.111]) there exists $v \geq u \geq t_1$ such that

$$x^{\Delta^m}(u) \geq h_{n-m-1}(v,u)x^{\Delta^{n-1}}(v).$$

Setting $v = t$ and $u = g(t)$ gives

$$x^{\Delta^m}(g(t)) \geq h_{n-m-1}(t,g(t))x^{\Delta^{n-1}}(g(t)) \quad \text{for} \quad t \geq t_2. \tag{2.11}$$

Combining the inequalities (2.9), (2.10), and (2.11), we have

$$x^{\Delta}(g(t)) \geq h_{m-1}(g(t),t_1)h_{n-m-1}(t,g(t))x^{\Delta^{n-1}}(t) \quad \text{for} \quad t \geq t_2, \tag{2.12}$$

and so

$$x(g(t)) \geq h_m(g(t),t_1)h_{n-m-1}(t,g(t))x^{\Delta^{n-1}}(t) \quad \text{for} \quad t \geq t_2. \tag{2.13}$$

Now, integrating equation (2.1) for $u \geq t \geq t_2$ and letting $u \to \infty$, we obtain

$$x^{\Delta^{n-1}}(t) \ \geq \left(\frac{1}{a(t)}\int_t^\infty q(s)\left(x^\sigma(g(s))\right)^\lambda \Delta s\right)^{1/\alpha},$$

or

$$x^{\Delta^{n-1}}(t) \geq \left(\frac{1}{a(t)}\int_t^\infty q(s)\Delta s\right)^{1/\alpha} \left(x^\sigma(g(t))\right)^{\lambda/\alpha} \quad \text{for} \quad t \geq t_2. \tag{2.14}$$

If $\lambda > \alpha$, we substitute (2.14) into (2.12) to obtain

$$\begin{aligned} x^{\Delta}(g(t)) &\geq h_{m-1}(g(t),t_1)h_{n-m-1}(t,g(t))x^{\Delta^{n-1}}(t) \\ &\geq \left(h_{m-1}(g(t),t_1)h_{n-m-1}(t,g(t))\right)\left(\frac{1}{a(t)}\int_t^\infty q(s)\Delta s\right)^{1/\alpha} \left(x^\sigma(g(t))\right)^{\lambda/\alpha}, \end{aligned}$$

or

$$x^{\Delta}(g(t))\left(x^{\sigma}(g(t))\right)^{-\lambda/\alpha}g^{\Delta}(t)\geq(h_{m-1}(g(t),t_1)h_{n-m-1}(t,g(t)))g^{\Delta}(t)\left(\frac{1}{a(t)}\int_t^{\infty}q(s)\Delta s\right)^{1/\alpha}.$$

Applying the first inequality in (2.4) and then integrating from $t_2$ to $t$ gives a contradiction to (2.5).

In case $\lambda=\alpha$, substituting (2.14) into (2.13) gives

$$x(g(t))\geq(h_m(g(t),t_1)h_{n-m-1}(t,g(t)))\left(\frac{1}{a(t)}\int_t^{\infty}q(s)\Delta s\right)^{1/\alpha}x^{\lambda/\alpha}(g(t)),$$

or

$$x^{1-\lambda/\alpha}(g(t))\geq(h_m(g(t),t_1)h_{n-m-1}(t,g(t)))\left(\frac{1}{a(t)}\int_t^{\infty}q(s)\Delta s\right)^{1/\alpha}\quad\text{for}\quad t\geq t_2.\tag{2.15}$$

Taking the lim sup of both sides of inequality (2.15) as $t\to\infty$ gives a contradiction to condition (2.6).

Finally, if $\lambda<\alpha$, using (2.13) in (2.1), we have

$$\begin{aligned}-\left(a(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta}&=q(t)\left(x^{\sigma}(g(t))\right)^{\lambda}\\&\geq q(t)(h_m(g(t),t_1)h_{n-m-1}(t,g(t)))^{\lambda}\left(x^{\Delta^{n-1}}(t)\right)^{\lambda}\end{aligned}$$

for $t\geq t_2$. Setting $w(t)=a(t)\left(x^{\Delta^{n-1}}(t)\right)^{\alpha}$, we have

$$-w^{\Delta}(t)\geq q(t)a^{-\lambda/\alpha}(t)(h_m(g(t),t_1)h_{n-m-1}(t,g(t)))^{\lambda}w^{\lambda/\alpha}\quad\text{for}\quad t\geq t_2,$$

so

$$-w^{\Delta}(t)w^{-\lambda/\alpha}(t)\geq q(t)a^{-\lambda/\alpha}(t)(h_m(g(t),t_1)h_{n-m-1}(t,g(t)))^{\lambda}\quad\text{for}\quad t\geq t_2.$$

Applying the second inequality in (2.4), and integrating from $t_2$ to $t$ yields a contradiction to condition (2.7). This completes the proof of the theorem. □

The following result is immediate.

**Theorem 2.2.** *Let $t_0\in\mathbb{T}$. Suppose conditions (i)-(iii) and (1.2) hold. If for every integer $m\in\{1,3,5,...,n-1\}$ and $t\geq t_0\in T$,*

$$\limsup_{t\to\infty}(h_m(g(t),t_0)h_{n-m-1}(t,g(t)))\left((a(t))^{-1}\int_t^{\infty}q(s)\Delta s\right)^{1/\alpha}=\infty,\tag{2.16}$$

*then every bounded solution of equation (2.1) is oscillatory.*

*Proof.* The conclusion follows from applying (2.16) to inequality (2.15). □

As an example, we let $\mathbb{T}=\mathbb{R}$, i.e., the continuous case. Here equation (2.1) becomes

$$\left(a(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)x^{\lambda}(g(t)) = 0, \tag{2.17}$$

where $\int^{\infty} a^{-1/\alpha}(s)ds = \infty$, and Theorem 2.1 takes the following form.

**Theorem 2.3.** *Let conditions (i)-(iii) hold. Equation* (2.17) *is oscillatory if for every integer* $m \in \{1,3,...,n-1\}$ *and* $t \geq t_0$*:*

$$\int_{t_0}^{\infty} g'(t)\left(\frac{(g(t)-t_0)^{m-1}}{(m-1)!}\frac{(t-g(t))^{n-m-1}}{(n-m-1)!}\right)\left(\frac{1}{a(t)}\int_t^{\infty} q(s)ds\right)^{1/\alpha} dt = \infty \quad \text{if } \lambda > \alpha;$$

$$\limsup_{t\to\infty}\left(\frac{(g(t)-t_0)^{m}}{m!}\frac{(t-g(t))^{n-m-1}}{(n-m-1)!}\right)\left(\frac{1}{a(t)}\int_t^{\infty} q(s)ds\right)^{1/\alpha} > 1 \quad \text{if } \lambda = \alpha;$$

*and*

$$\int_{t_0}^{\infty}\left(\frac{(g(t)-t_0)^{m}}{m!}\frac{(t-g(t))^{n-m-1}}{(n-m-1)!}\right)^{\lambda} a^{-\lambda/\alpha}(t)q(t)dt = \infty \quad \text{if } \lambda < \alpha.$$

Next, we take $\mathbb{T}=\mathbb{Z}$, i.e., the discrete case. In this case, equation (2.1) takes the form

$$\Delta\left(a(t)(\Delta^{n-1}x(t))^{\alpha}\right) + q(t)(x^{\sigma}(g(t)))^{\lambda} = 0, \tag{2.18}$$

where $\sum^{\infty} a^{-1/\alpha}(t) = \infty$. Theorem 2.1 becomes the following.

**Theorem 2.4.** *Let conditions (i)-(iii) hold. Assume that for every integer* $m \in \{1,3,5,...,$ $n-1\}$ *and* $t \geq t_0 \in N_0$*, we have:*

$$\sum_{t=t_0}^{\infty}(\Delta g(t))\left(\frac{(g(t)-t_0)^{(m-1)}}{(m-1)!}\frac{(t-g(t))^{(n-m-1)}}{(n-m-1)!}\right)\left(\frac{1}{a(t)}\sum_{s=t}^{\infty} q(s)\right)^{1/\alpha} = \infty \quad \text{if } \lambda > \alpha;$$

$$\limsup_{t\to\infty}\left(\frac{(g(t)-t_0)^{(m)}}{m!}\frac{(t-g(t))^{(n-m-1)}}{(n-m-1)!}\right)\left(\frac{1}{a(t)}\sum_{s=t}^{\infty} q(s)\right)^{1/\alpha} > 1 \quad \text{if } \lambda = \alpha;$$

$$\sum_{t=t_0}^{\infty}\left(\frac{(g(t)-t_0)^{(m)}}{m!}\frac{(t-g(t))^{(n-m-1)}}{(n-m-1)!}\right)^{\lambda} a^{-\lambda/\alpha}(t)q(t) = \infty \quad \text{if } \lambda < \alpha.$$

*Then equation* (2.18) *is oscillatory.*

## 3. Oscillation of Equation (1.1) with $-1 < p(t) < 0$

In this section we consider equation (1.1) with $-1 < p(t) < 0$ on $\mathbb{T}$. Here, we let $p^*(t) = -p(t)$ so equation (1.1) becomes

$$\left(a(t)\left(\left(x(t)+p^*(t)x(\tau(t))\right)^{\Delta^{n-1}}\right)^{\alpha}\right)^{\Delta} + q(t)(x^{\sigma}(g(t)))^{\lambda} = 0, \tag{3.1}$$

where $n$ is even and $0 < p^*(t) < 1$. We establish the following oscillation criterion for equation (3.1).

**Theorem 3.1.** *Let $t_0 \in \mathbb{T}$ and assume that conditions (i)-(iii) and* (1.2) *hold. If for every integer $m \in \{1, 3, 5, ..., n-1\}$ and $t \geq t_0 \in \mathbb{T}$, conditions* (2.5)–(2.7) *hold with $q(t)$ replaced by $q(t)(1 - p^*(\sigma(g(t))))^{\lambda}$, then equation* (3.1) *is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of equation (3.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(g(t)) > 0$ for $t \geq t_0 \in \mathbb{T}$. Set

$$y(t) = x(t) + p^*(t)x(\tau(t)) \quad \text{for } t \geq t_0.$$

Then equation (3.1) takes the form

$$\left(a(t)\left(y^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} + q(t)(x^{\sigma}(g(t)))^{\lambda} = 0, \quad t \geq t_0. \tag{3.2}$$

Clearly, $y(t) > 0$ and $\left(a(t)\left(y^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} \leq 0$; hence $y^{\Delta^n} \leq 0$ for $t \geq t_0$. By Lemma 2.1, we see that $y^{\Delta}(t) > 0$ for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$. Thus,

$$\begin{aligned} x(t) &= y(t) - p^*(t)x(\tau(t)) \\ &= y(t) - p^*(t)[y(\tau(t) - p^*(\tau(t))x(\tau \circ \tau(t))] \\ &\geq y(t) - p^*(t)y(\tau(t)) \geq (1 - p^*(t))y(t) \quad \text{for } t \geq t_1. \end{aligned} \tag{3.3}$$

Using (3.3) in equation (3.2), we obtain

$$\left(a(t)\left(y^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} + q(t)(1 - p^*(\sigma(g(t))))^{\lambda}\left(y^{\sigma}(g(t))\right)^{\lambda} \leq 0 \quad \text{for} \quad t \geq t_1.$$

The remainder of the proof is exactly the same as that of Theorem 2.1 and hence is omitted. ☐

## 4. Oscillation of equation (1.1) with $0 < p(t) < 1$

In this section, we consider equation (1.1) with $0 < p(t) < 1$ and establish the following result.

**Theorem 4.1.** *Let $t_0 \in \mathbb{T}$. Suppose conditions (i)-(iv) and* (1.2) *hold and assume that for every integer $m \in \{1, 3, 5, ..., n-1\}$ and $t \geq t_0 \in \mathbb{T}$, either:*

$$\begin{cases} \limsup\limits_{t\to\infty}(h_m(g(t), t_0)h_{n-m-1}(t, g(t)))\left((a(t))^{-1}\int\limits_t^{\infty} q(s)\Delta s\right)^{1/\alpha} > 1 \\ \text{and} & \text{if } \lambda = \alpha; \\ \limsup\limits_{t\to\infty}(a(\xi(t)))^{-1}\int\limits_{\xi(t)}^{t} q(s)h^{\lambda}_{n-1}(\xi(t), \xi(s))\Delta s > 1 \end{cases} \tag{4.1}$$

*or*

$$\begin{cases} \int\limits_{t_0}^{\infty} (h_m(g(t),t_0) h_{n-m-1}^{\lambda}(t,g(t)) a^{-\lambda/\alpha}(t) q(s) \Delta s = \infty \\ \quad \textit{and} & \textit{if } \lambda < \alpha. \\ \int\limits_{t_0}^{\infty} q(s) a^{-\lambda/\alpha}(s) h_{n-1}^{\lambda}(t,\xi(s)) \Delta s = \infty \end{cases} \tag{4.2}$$

*Then equation* (1.1) *is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of (1.1) with $x(t) > 0$, $x(\tau(t)) > 0$, and $x(g(t)) > 0$ for $t \geq t_0 \in \mathbb{T}$. Set

$$z(t) = x(t) - p(t)x(\tau(t)) \quad \text{for } t \geq t_0. \tag{4.3}$$

Then,

$$\left(a(t)\left(z^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} + q(t)(x^{\sigma}(g(t)))^{\lambda} = 0 \quad \text{for } t \geq t_0. \tag{4.4}$$

It is easy to see that $z^{\Delta^n}(t) \leq 0$ is of one sign on $[t_0, \infty)_{\mathbb{T}}$. Now, we distinguish between two cases: (I) $z(t) > 0$ or (II) $z(t) < 0$ for $t \geq t_0$.

Case (I). Assume that $z(t) > 0$ for $t \geq t_0$. Then $x(t) \geq z(t)$ for $t \geq t_0$ and equation (4.4) becomes

$$\left(a(t)\left(z^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} + q(t)(z^{\sigma}(g(t)))^{\lambda} \leq 0 \quad \text{for } t \geq t_0.$$

Proceeding as in the proof of Theorem 2.1, we arrive at the desired contradiction.

Case (II). Assume that $z(t) < 0$ for $t \geq t_0$. Then

$$y(t) := -z(t) = p(t)x(\tau(t)) - x(t) \leq p(t)x(\tau(t)) \leq x(\tau(t)) \quad \text{for } t \geq t_0,$$

so

$$x(g(t)) \ \geq y(\tau^{-1} \circ g(t)) = y(\xi(t)) \quad \text{for } t \geq t_1 \in [t_0, \infty)_{\mathbb{T}} \, . \tag{4.5}$$

Using (4.5) in equation (4.4), we have

$$\left(a(t)\left(y^{\Delta^{n-1}}(t)\right)^{\alpha}\right)^{\Delta} \geq q(t)(y^{\sigma}(\xi(t)))^{\lambda} \quad \text{for } t \geq t_1. \tag{4.6}$$

From the above, we also see that $x(t) \leq p(t)x(\tau(t)) \leq x(\tau(t))$ for $t \geq t_0$.

Thus, $x(t)$ and hence $y(t)$ are bounded functions for $t \geq t_1$. By Lemma 2.1, we see that $y(t)$ satisfies

$$(-1)^k y^{\Delta^k}(t) > 0 \quad \text{for } t \geq t_1,\ k = 1, 2, ..., n. \tag{4.7}$$

As in the proof of Theorem 2.1, for $v \geq u \geq t_1$, we have

$$y(u) \geq h_{n-1}(v,u)\left(-y^{\Delta^{n-1}}(v)\right). \tag{4.8}$$

For $t \geq s \geq t_1$, letting $u = \xi(s)$ and $v = \xi(t)$ in (4.8) gives

$$y(\xi(s)) \geq h_{n-1}(\xi(t),\xi(s))\left(-y^{\Delta^{n-1}}(\xi(t))\right) \quad \text{for } t \geq t_2 \geq t_1. \tag{4.9}$$

Also, letting $u = \xi(t)$ and $v = t$ in (4.8), we have

$$y(\xi(t)) \geq h_{n-1}(t,\xi(t))\left(-y^{\Delta^{n-1}}(t)\right) \quad \text{for } t \geq t_2 \geq t_1. \tag{4.10}$$

Integrating (4.6) from $\xi(t)$ to $t$ and using (4.9), we have

$$\left(-y^{\Delta^{n-1}}(\xi(t))\right)^{\alpha} \geq (a(\xi(t)))^{-1} \int_{\xi(t)}^{t} q(s)y^{\lambda}(\xi(s))\Delta s$$
$$\geq (a(\xi(t)))^{-1} \left( \int_{\xi(t)}^{t} q(s)h_{n-1}^{\lambda}(\xi(t),\xi(s))\Delta s \right) \left(-y^{\Delta^{n-1}}(\xi(t))\right)^{\lambda}$$

or

$$\left(-y^{\Delta^{n-1}}(\xi(t))\right)^{\alpha-\lambda} \geq (a(\xi(t)))^{-1} \left( \int_{\xi(t)}^{t} q(s)h_{n-1}^{\lambda}(\xi(t),\xi(s))\Delta s \right).$$

Taking the lim sup of both sides of the above inequality as $t \to \infty$, we arrive at the desired contradiction if $\lambda = \alpha$.

Setting $0 < w(t) = -a(t)\left(y^{\Delta^{n-1}}(t)\right)^{\alpha}$ in (4.6) and using (4.10) yields

$$-w^{\Delta}(t) \geq q(t)a^{-\lambda/\alpha}(t)h_{n-1}^{\lambda}(t,\xi(s))w^{\lambda/\alpha}(t) \quad \text{for } t \geq t_2.$$

The rest of the proof is similar to that of Theorem 2.1 for the case $\lambda < \alpha$. This completes the proof of the theorem. $\square$

To illustrate this result, consider the case $\mathbb{T} = \mathbb{R}$. Then equation (1.1) takes the form

$$\left(a(t)\left((x(t)-p(t)x(\tau(t)))^{(n-1)}\right)^{\alpha}\right)' + q(t)x^{\lambda}(g(t)) = 0 \tag{4.11}$$

and Theorem 4.1 becomes the following result.

**Theorem 4.2.** *Let conditions (i)-(iv) and* (1.2) *hold and assume that for every integer* $m \in \{1,3,5,...,n-1\}$ *and* $t \geq t_0 \in \mathbb{T} = \mathbb{R}$*, either*

$$\begin{cases} \limsup\limits_{t\to\infty} \left(\frac{(g(t)-t_0)^m}{m!}\frac{(t-g(t))^{n-m-1}}{(n-m-1)!}\right)\left((a(t))^{-1}\int_t^{\infty} q(s)ds\right)^{1/\alpha} > 1 & \\ \text{and} & \text{if } \lambda = \alpha; \\ \limsup\limits_{t\to\infty} (a(\xi(t)))^{-1}\int_{\xi(t)}^{t} \frac{(\xi(t),\xi(s))^{n-1}}{(n-1)!}q(s)ds > 1 & \end{cases}$$

*or*

$$\begin{cases} \int_{t_0}^{\infty} \left(\frac{(g(t)-t_0)^m}{m!}\frac{(t-g(t))^{n-m-1}}{(n-m-1)!}\right)^{\lambda} (a(t))^{-\lambda/\alpha}q(t)dt = \infty & \\ \text{and} & \text{if } \lambda < \alpha; \\ \int_{t_0}^{\infty} \left(\frac{(t-\xi(s))^{n-1}}{(n-1)!}\right)^{\lambda} (a(s))^{-\lambda/\alpha}q(s)ds = \infty. & \end{cases}$$

*Then equation* (4.11) *is oscillatory.*

Now if $\mathbb{T}=\mathbb{Z}$, equation (1.1) becomes

$$\Delta\left(a(t)\left(\Delta^{n-1}(x(t)-p(t)x(\tau(t)))\right)^{\alpha}\right)+q(t)(x^{\sigma}(g(t)))^{\lambda}=0, \tag{4.12}$$

and Theorem 4.1 has the following formulation.

**Theorem 4.3.** *Let conditions (i)-(iv) and* (1.2) *hold and assume that for every integer* $m\in\{1,3,5,...,n-1\}$ *and* $t\geq t_0\in\mathbb{T}=\mathbb{Z}$, *either*

$$\begin{cases}\limsup\limits_{t\to\infty}\left(\frac{(g(t)-t_0)^{(m)}}{m!}\frac{(t-g(t))^{(n-m-1)}}{(n-m-1)!}\right)\left((a(t))^{-1}\sum\limits_{s=t}^{\infty}q(s)\right)^{1/\alpha}>1 & \\ \text{and} & \text{if } \lambda=\alpha; \\ \limsup\limits_{t\to\infty}(a(\xi(t)))^{-1}\sum\limits_{s=\xi(t)}^{\infty}\frac{(\xi(t)-\xi(s))^{(n-1)}}{(n-1)!}q(s)>1 & \end{cases}$$

*or*

$$\begin{cases}\sum\limits_{t=t_0}^{\infty}\left(\frac{(g(t)-t_0)^{(m)}}{m!}\frac{(t-g(t))^{(n-m-1)}}{(n-m-1)!}\right)^{\lambda}(a(t))^{-\lambda/\alpha}q(t)=\infty & \\ \text{and} & \text{if } \lambda<\alpha; \\ \sum\limits_{t=t_0}^{\infty}\left(\frac{(t-\xi(t))^{(n-1)}}{(n-1)!}\right)^{\lambda}(a(s))^{-\lambda/\alpha}q(t)=\infty. & \end{cases}$$

*Then equation* (4.12) *is oscillatory.*

From the proof of Theorem 4.1, we extract the following result that is concerned with the oscillatory behavior of all bounded solutions of equation (1.4).

**Theorem 4.4.** *Let* $t_0\in\mathbb{T}$ *and let* $p(t)\equiv 0$. *Suppose conditions (i)-(iv) and* (1.2) *hold. If*

$$\limsup_{t\to\infty}\int_{g(t)}^{t}q(s)h_{n-1}^{\lambda}(g(t),g(s))\Delta s>1 \quad \text{if } \lambda=\alpha,$$

*or*

$$\int_{t_0}^{t}q(s)(a(s))^{-\lambda/\alpha}h_{n-1}^{\lambda}(g(t),g(s))\Delta s=\infty \quad \text{if } \lambda<\alpha.$$

*Then every bounded solution of equation* (1.4) *oscillates.*

*Proof.* The proof follows from the proof of Case (II) of Theorem 4.1 and hence is omitted. □

*Remark* 4.5. Notice that Theorems 2.1 and 3.1 cover both super-linear and sub-linear delay dynamic equations. The results here can easily be extended to dynamic equations of the form

$$\left(a(t)\left((x(t)-p(t)x(\tau(t)))^{\Delta^{n-1}}\right)^{\alpha}\right)^{\Delta}+f(t,x^{\sigma}(g(t)))=0,$$

where the functions $a$, $p$, $g$ and $\tau$ are as in equation (1.1) and $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is continuous, $xf(t,x)>0$ for $x\neq 0$ and $t\in\mathbb{T}$ and $f$ satisfies a super-linear or sub-linear growth condition. The details are left to the reader. We applied our results to the continuous and discrete cases but they clearly apply to other types of time-scales such as $\mathbb{T}=h\mathbb{Z}$ with $h>0$, $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, $\mathbb{T}=\mathbb{N}_0^2$, etc. An interesting open problem is to find similar results for the cases where

$p(t) \geq 1$ and $p(t) \leq -1$. The oscillatory character of equation (1.1) is different for these cases and we refer the reader to the papers [14] and [21] for a discussion in the continuous and discrete cases.

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