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A class of BVPS for first order impulsive functional integro-differential equations with a parameter

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Abstract This paper is concerned with a class of boundary value problems for the nonlinear impulsive functional integro-differential equations with a parameter by establishing new comparison principles and using the method of upper and lower solutions together with monotone iterative technique. Sufficient conditions are established for the existence of extremal system of solutions for the given problem. Finally, we give an example that illustrates our results.

Keywords: nonlinear impulsive integro-differential equations; parameter; upper and lower solutions; monotone iterative technique

MSC(2010): 34B37, 34K10, 34L15, 34L30, 34A45

1 Introduction

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, biotechnology and economics. There has been a significant development in impulse theory ([1][2]).

The differential equations with parameters play important roles and tools not only in mathematics but also in physics, population dynamics, control systems, dynamical systems and engineering to create the

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mathematical modelling of many physical phenomena. It is more accurate than the average differential equations to describe the objective world. And the existence of solutions for the BVPS of these equations have been studied by many authors([11]-[13]).

Especially, there is an increasing interest in the study of nonlinear mixed integro-differential equations with deviating arguments and multipiont BVPS([4]-[10]) for impulsive differential equations. And theorems about existence, uniqueness of differential and impulsive functional differential abstract evolution Cauchy problem with nonlocal conditions have been studied by Byszewski and Lakshmikantham [21], by G.Infants [22], by Chang et al.[20][25], by Anguraj et al.[19], and by Akca et al.[24] and the references therein.

In this paper, we are concerned with the following BVPS for the nonlinear mixed impulsive functional integro-differential equations with a parameter:

$$\begin{cases} u'(t) = f(t, u(t), u(\alpha(t)), Tu, Su, \varrho) \quad t \neq t_k, \quad t \in J = [0, T] \\ \Delta u(t_k) = I_k(u(t_k), \varrho) \qquad k = 1, 2, \cdots, m \\ u(0) = \lambda_1 u(T) + \lambda_2 \int_0^T w(s, u(s))ds + \Sigma_{i=1}^p a_i u(\eta_i) + \zeta \\ Q(u(T), \varrho) = 0, \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T, f \in C(J \times R^5, R)$, $I_k \in C(R \times R, R)$, $(Tu)(t) = \int_0^{\beta(t)} k(t,s)u(\gamma(s))ds$, $(Su)(t) = \int_0^T h(t,s)u(\delta(s))ds$, and $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $w \in C(J \times R, R)$, $Q \in C(R \times R, R)$, $0 \leq \lambda_1 \leq 1$, $0 \leq \lambda_2$, $0 \leq a_i$, $\zeta, \varrho \in R$, and $0 \leq \eta_i \leq T$. The assumption about $\alpha, \beta, \gamma, \delta, k$ and h will appear latter.

Special cases

(i) If $\lambda_1 = 1$, $a_i = \lambda_2 = k = 0$, $(i = 1, 2, \cdots p)$ then the Eq.(1.1)reduces to the periodic boundary value problem which has been studied in ([14]-[16] [18]).

(ii) If $a_j = 1 + \lambda_1$, $\eta_i = 0$, $\lambda_2 = k = 0$, $a_i = 0$, $(i = 1, 2, j-1, j+1, \cdots p)$ then the Eq.(1.1) reduces to the anti-periodic boundary value problem which has been studied in ([3] [17] [19]).

(iii) If $\lambda_2 \neq 0$, $a_i = k = 0$, $(i = 1, 2, \cdots p)$ then the Eq.(1.1) reduces to the integral boundary value problem which has been studied in [23].

(iv) If $a_i \neq 0$, $0 < \eta_i < T$, $\lambda_2 = k = 0$, $(i = 1, 2, \cdots p)$ then the Eq.(1.1) can be regarded as the nonlocal Cauchy problem.

The article is organized as follow. In section 2, we establish new comparison principles. In section 3, by using of the monotone iterative technique and the method of upper and lower solutions, we obtain the existence result for the extremal solutions of BVPS(1.1). In section 4, we give an example that illustrates our results.

2 Preliminaries and lemmas

Let $PC(J) = \{x : J \to R; x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k) = x(t_k^-),\ k = 1, 2, \cdots, m\}$; $PC^1(J) = \{x \in PC(J) : x'(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k) = x'(t_k^-),\ k = 1, 2, \cdots, m\}$. Let $J^- = J \setminus \{t_k,\ k = 1, 2, \cdots, m\}$, $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms $\| x \|_{PC} = sup\{|x(t)| :\ t \in J\}$ and $\| x \|_{PC^1} = max\{\| x \|_{PC},\quad \| x' \|_{PC}\}$. $(x, \tau) \in PC^1(J) \times R$ is called a solution of BVPS (1.1) if it satisfies $Eq.(1.1)$.

Let $(x_i(t), \tau_i) \in PC^1(J) \times R (i = 1, 2)$, $\tau_1 \le \tau_2$, $x_1(t) \le x_2(t)$ denote that $(x_1(t), \tau_1) \le (x_2(t), \tau_2)$. The interval $[x_1, x_2] \times [\tau_1, \tau_2]$ denote that $\{(x(t), \tau) \in PC(J) \times R \mid \tau_1 \le \tau \le \tau_2,\ x_1(t) \le x(t) \le x_2(t)\}$.

For conveniences, we set

$$\begin{cases} N^*(t) = N(t)e^{\int_0^t M(s)ds}e^{-\int_0^{\alpha(t)} M(s)ds}, \qquad K^*(t) = K(t)e^{\int_0^t M(s)ds}, \\ H^*(t) = H(t)e^{\int_0^t M(s)ds}, \qquad k^*(t,s) = k(t,s)e^{-\int_0^{\gamma(s)} M(\tau)d\tau}, \\ h^*(t,s) = h(t,s)e^{-\int_0^{\delta(s)} M(\tau)d\tau}, r^* = re^{-\int_0^T M(s)ds}, \end{cases} \tag{2.1}$$

$\theta^*(t) = N^*(t) + K^*(t)\int_0^{\beta(t)} k^*(t,s)ds + H^*(t)\int_0^T h^*(t,s)ds \not\equiv 0$ for $t \in J$, $\mu^* = \int_0^T \theta^*(t)dt$.

(H_2) $[\mu^* + \sum_{k=1}^{m} L_k] \le r^*$.

Lemma 2.1 Assume that $(H_1)(H_2)$ hold and $q \in PC^1(J)$ such that

$$\begin{cases} q'(t) \le -M(t)q(t) - (\mathscr{H}q)(t) \quad t \ne t_k, \quad t \in J = [0, T] \\ \Delta q(t_k) \le -L_k(q(t_k)) \qquad k = 1, 2, \cdots, m \\ q(0) \le rq(T), \end{cases} \tag{2.2}$$

where the operator $\mathscr{H}$ is defined as

$$(\mathscr{H}q)(t) = N(t)q(\alpha(t)) + K(t)\int_0^{\beta(t)} k(t,s)q(\gamma(s))ds + H(t)\int_0^T h(t,s)q(\delta(s))ds.$$

Then $q(t) \le 0$ for $t \in J$.

Proof : Let $p(t) = q(t)e^{\int_0^t M(s)ds}$. Obviously $p(t)$ and $q(t)$ have the same sign on J. In view of (2.2), we have

$$\begin{cases} p'(t) \le -(\mathscr{H}^*p)(t) \quad t \ne t_k, \quad t \in J = [0, T] \\ \Delta p(t_k) \le -L_k(p(t_k)) \qquad k = 1, 2, \cdots, m \\ p(0) \le r^*p(T), \end{cases} \tag{2.3}$$

where $(\mathscr{H}^*p)(t) = N^*(t)p(\alpha(t)) + K^*(t)\int_0^{\beta(t)} k^*(t,s)p(\gamma(s))ds + H^*(t)\int_0^T h^*(t,s)p(\delta(s))ds$.

Next, we will show $p(t) \le 0$.

Suppose, to the contrary, that $p(t) > 0$ for some $t \in J$.

(i)If $p(t) \geq 0$, $p(t) \not\equiv 0$ for $t \in J$, we get $p'(t) \leq 0$, in view of the first inequality of (2.3). By the second one in (2.3), we obtain that $p(t)$ is non-increasing in J. Then $0 \leq p(T) \leq p(t) \leq p(0)$. On the other hand, by the third inequality in (2.3), if $r^* = 1$, then $p(T) \leq p(t) \leq p(0) \leq p(T)$, we get $p(t) \equiv C > 0$. Hence $p'(t) \equiv 0$. By the first inequality in (2.3) again, we have

$$0 \quad \leq \quad -C\theta^*(t) \qquad\qquad \forall t \in J.$$

By (H_1) we get that $C \leq 0$ which is a contradiction.

If $0 < r^* < 1$, then $p(T) \leq p(0) \leq r^* p(T)$, so $p(T)(1 - r^*) \leq 0$. we have $0 \leq p(T) \leq 0$. Since p is non-increasing in J, we infer $p(t) \equiv 0$. It is a contradiction.

(ii) If $p(t^*) = \sup_{t\in J} p(t) > 0$, $p(t_*) = \inf_{t \in J} p(t) = -\lambda < 0$, then $\lambda > 0$.

Case 1 If $t_* < t^*$, integrating from t_* to t^*, we get from (2.3)

$$\begin{aligned} 0 < p(t^*) &= p(t_*) + \int_{t_*}^{t^*} p'(s)ds + \sum_{t_* \leq t_k < t^*} \Delta p(t_k) \\ &\leq -\lambda + \int_{t_*}^{t^*} -(\mathscr{H}^* p)(s)ds - \sum_{t_* \leq t_k < t^*} L_k p(t_k) \\ &\leq -\lambda + \mu^*\lambda + \lambda \sum_{k=1}^{m} L_k. \end{aligned}$$

Hence

$$1 < \mu^* + \sum_{k=1}^{m} L_k$$

which is in contradiction to (H_2).

Case 2 If $t^* < t_*$, we have

$$\begin{aligned} 0 < p(t^*) &= p(0) + \int_0^{t^*} p'(s)ds + \sum_{0 < t_k < t^*} \Delta p(t_k) \\ &\leq p(0) + \int_0^{t^*} -(\mathscr{H}^* p)(s)ds + \lambda \sum_{0 < t_k < t^*} L_k \\ &\leq p(0) + \lambda \int_0^{t^*} \theta^*(s)ds + \lambda \sum_{0 < t_k < t^*} L_k, \end{aligned}$$

$$\begin{aligned} p(T) &= p(t_*) + \int_{t_*}^{T} p'(s)ds + \sum_{t_* \leq t_k < T} \Delta p(t_k) \\ &\leq -\lambda + \int_{t_*}^{T} -(\mathscr{H}^* p)(s)ds + \lambda \sum_{t_* \leq t_k < T} L_k \\ &\leq -\lambda + \lambda \int_{t_*}^{T} \theta^*(s)ds + \lambda \sum_{t_* \leq t_k < T} L_k. \end{aligned}$$

By the two inequalities above, we obtain

$$\begin{aligned}
& -\lambda + \frac{1}{r^*}\lambda \int_{t_*}^{T} \theta^*(s)ds + \frac{1}{r^*}\lambda \sum_{t_* \le t_k < T} L_k \\
\ge\ & -\lambda + \lambda \int_{t_*}^{T} \theta^*(s)ds + \lambda \sum_{t_* \le t_k < T} L_k \\
\ge\ & p(T) \ge \frac{1}{r^*} p(0) \\
>\ & -\frac{1}{r^*}\lambda \int_0^{t^*} \theta^*(s)ds - \frac{1}{r^*}\lambda \sum_{0 < t_k < t^*} L_k \\
\ge\ & -\frac{1}{r^*}\lambda \int_0^{t_*} \theta^*(s)ds - \frac{1}{r^*}\lambda \sum_{0 < t_k < t_*} L_k .
\end{aligned}$$

Therefore, we get that $(\mu^* + \sum_{k=1}^m L_k) > r^*$, which is in contradiction to (H_2). Hence $p(t) \le 0$, $q(t) \le 0$. We complete the proof.

Lemma 2.2 Assume that (H_1), (H_2) and $\int_0^T M(s)ds > 0$ as $r = 1$ are satisfied. Let $C_k, d \in R, \sigma \in PC(J)$. Then the linear problem

$$\begin{cases} u'(t) = -M(t)u(t) - (\mathscr{H}u)(t) + \sigma(t), \quad t \ne t_k, \quad t \in J = [0, T], \\ \Delta u(t_k) = -L_k(u(t_k)) + C_k, \qquad\qquad k = 1, 2, \cdots, m, \\ u(0) = ru(T) + d, \end{cases} \tag{2.4}$$

has a unique solution $x \in PC^1(J, E)$ and it is represented by:

$$\begin{aligned}
u(t) \ =\ & \frac{d e^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} + \int_0^T G(t,s)(\sigma(s) - (\mathscr{H}u)(s))ds \\
& + \frac{r e^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} \sum_{k=1}^m e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k) \\
& + \sum_{0 < t_k < t} e^{-\int_0^t M(\tau)d\tau} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k),
\end{aligned} \tag{2.5}$$

where

$$G(t,s) = \begin{cases} \dfrac{e^{\int_t^T M(\tau)d\tau} e^{\int_0^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r}, & 0 \le s \le t \le T, \\ \\ \dfrac{r e^{\int_t^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r}, & 0 \le t \le s \le T. \end{cases} \tag{2.6}$$

Proof : First, differentiating (2.5), we have

$$\begin{aligned}
u'(t) &= \frac{d}{dt}\Big[\frac{de^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}+\int_0^T G(t,s)(\sigma(s)-(\mathscr{H}u)(s))ds \\
&\quad +\frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^{m} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k))+C_k) \\
&\quad +\sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau}e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k))+C_k)\Big] \\
&= -M(t)\Big[\frac{de^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}+\int_0^T G(t,s)(\sigma(s)-(\mathscr{H}u)(s))ds \\
&\quad +\frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^{m} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k))+C_k) \\
&\quad +\sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau}e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k))+C_k)\Big] \\
&\quad +\Big(\frac{-r}{e^{\int_0^T M(\tau)d\tau}-r}+\frac{e^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}\Big)(\sigma(t)-(\mathscr{H}u)(t)) \\
&= -M(t)u(t)-(\mathscr{H}u)(t)+\sigma(t) \qquad \forall t\in J^-,
\end{aligned}$$

$$\begin{aligned}
\Delta u(t_k) &= u(t_k^+)-u(t_k^-) \\
&= \sum_{0<t_j\le t_k}\Delta u(t_j)-\sum_{0<t_j<t_k}\Delta u(t_j) \\
&= \sum_{j=1}^{k}(-L_j(u(t_j))+C_j)-\sum_{j=1}^{k-1}(-L_j(u(t_j))+C_j) \\
&= -L_k(u(t_k))+C_k.
\end{aligned}$$

Also

$$\begin{aligned}
u(0) &= \frac{r}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^{m} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k))+C_k) \\
&\quad +\int_0^T \frac{re^{\int_0^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}(\sigma(s)-(\mathscr{H}u)(s))ds+\frac{de^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}, \\
u(T) &= \frac{1}{e^{\int_0^T M(\tau)d\tau}-r}\sum_{k=1}^{m} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k))+C_k) \\
&\quad +\int_0^T \frac{e^{\int_0^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-r}(\sigma(s)-(\mathscr{H}u)(s))ds+\frac{d}{e^{\int_0^T M(\tau)d\tau}-r}.
\end{aligned}$$

It is easy to check that $u(0)=ru(T)+d$. Hence, we know that (2.5) is a solution of (2.4).

Next we show that the solution of (2.4) is unique. Let u_1, u_2 are the solutions of (2.4) and set $p=u_1-u_2$, we get

$$\begin{aligned}
p' &= u_1'-u_2' \\
&= -M(t)u_1(t)-(\mathscr{H}u_1)(t)+\sigma(t) \\
&\quad -(-M(t)u_2(t)-(\mathscr{H}u_2)(t)+\sigma(t)) \\
&= -Mp-(\mathscr{H}p)(t),
\end{aligned}$$

$$\begin{aligned}\Delta p(t_k) &= \Delta u_1 - \Delta u_2 \\ &= -L_k u_1(t_k) + C_k - (-L_k u_2(t_k) + C_k) \\ &= -L_k p(t_k),\end{aligned}$$

$$\begin{aligned}p(0) &= u_1(0) - u_2(0) \\ &= r u_1(T) + d - (r u_2(T) + d) \\ &= r p(T).\end{aligned}$$

In view of Lemma 2.1, we get $p \leq 0$ which implies $u_1 \leq u_2$. Similarly, we have $u_1 \geq u_2$. Hence $u_1 = u_2$. The proof is complete.

Lemma 2.3 Let $\sigma \in PC(J)$, and $L_k \geq 0$, $M \in C(J,R)$, $0 < r \leq 1$, $\int_0^T M(s)ds > 0$ as $r = 1$. If (H_3) holds

$$\varpi = e^{\int_0^T |M(\tau)|d\tau}\left(1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r}\right)\left(\mu + \sum_{k=1}^{m} L_k\right) < 1, \tag{2.7}$$

where $\mu = \int_0^T [N(t) + K(t)\int_0^{\beta(t)} k(t,s)ds + H(t)\int_0^T h(t,s)ds]dt$, then Eq.(2.5) has a unique solution u in $PC(J)$.

Proof : Define an operator F by

$$\begin{aligned}(Fu)(t) &= \frac{de^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} + \int_0^T G(t,s)(\sigma(s) - (\mathscr{H}u)(s))ds \\ &\quad + \frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r}\sum_{k=1}^{m} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k) \\ &\quad + \sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(u(t_k)) + C_k).\end{aligned}$$

If $0 \leq s \leq t \leq T$,

$$\begin{aligned}\frac{e^{\int_t^T M(\tau)d\tau} e^{\int_0^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} &\leq \frac{e^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} \\ &= \frac{e^{\int_0^T M(\tau)d\tau} - r + r}{e^{\int_0^T M(\tau)d\tau} - r} \\ &= 1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r},\end{aligned}$$

if $0 \leq t \leq s \leq T$,

$$\begin{aligned}\frac{re^{\int_t^s M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} &\leq \frac{re^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} \\ &\leq \frac{e^{\int_0^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} \\ &= 1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r},\end{aligned}$$

it is easy to see that

$$\max\{G(t,s),\ (t,s)\in J^2\} = 1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r}.$$

Now, for $x, y \in PC(J)$, we have

$$\begin{aligned}
& \| (Fx)(t) - (Fy)(t) \|_{PC} \\
= & \| \int_0^T G(t,s)(-(\mathscr{H}x)(s) + (\mathscr{H}y)(s))ds \\
& + \frac{re^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau} - r} \sum_{k=1}^{m} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(x(t_k)) + L_k(y(t_k))) \\
& + \sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k(x(t_k)) + L_k(y(t_k))) \|_{PC} \\
\le & (1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r}) \int_0^T \mid (-\mathscr{H}x(s) + \mathscr{H}y(s))ds \mid \\
& + (1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r}) \max\{ \sum_{0<t_k<t} e^{\int_{t_k}^t |M(\tau)|d\tau} \mid (-L_k(x(t_k)) + L_k(y(t_k))) \mid \\
& + \sum_{t\le t_k<T} e^{\int_t^{t_k} |M(\tau)|d\tau} \mid (-L_k(x(t_k)) + L_k(y(t_k))) \mid \} \\
\le & (1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r}) \int_0^T \mid (-\mathscr{H}x(s) + \mathscr{H}y(s))ds \mid \\
& + (1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r}) \max\{ \sum_{0<t_k<t} e^{\int_0^T |M(\tau)|d\tau} \mid (-L_k(x(t_k)) + L_k(y(t_k))) \mid \\
& + \sum_{t\le t_k<T} e^{\int_0^T |M(\tau)|d\tau} \mid (-L_k(x(t_k)) + L_k(y(t_k))) \mid \} \\
\le & e^{\int_0^T |M(\tau)|d\tau} (1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r})(\mu + \sum_{k=1}^{m} L_k) \| x - y \|_{PC} \\
= & \varpi \| x - y \|_{PC},
\end{aligned}$$

Consequently, the Banach fixed point theorem implies that F has a unique fixed point u in $PC(J)$, and the lemma is proved.

3 Main Results

For convenience, let us list the following conditions:

(H_4) There exist $(u_0, \alpha_0), (v_0, \beta_0) \in PC^1(J) \times R$ satisfying

$$\begin{cases} u_0'(t) \le f(t, u_0(t), u_0(\alpha(t)), Tu_0, Su_0, \alpha_0) \quad t \ne t_k, \quad t \in J = [0, T] \\ \Delta u_0(t_k) \le I_k(u_0(t_k), \alpha_0) \qquad\qquad k = 1, 2, \cdots, m \\ u_0(0) \le \lambda_1 u_0(T) + \lambda_2 \int_0^T w(s, u_0(s))ds + \sum_{i=1}^p a_i u_0(\eta_i) + \zeta \\ 0 \le Q(u_0(T), \alpha_0), \\ v_0'(t) \ge f(t, v_0(t), v_0(\alpha(t)), Tv_0, Sv_0, \beta_0) \quad t \ne t_k, \quad t \in J = [0, T] \\ \Delta v_0(t_k) \ge I_k(v_0(t_k), \beta_0) \qquad\qquad k = 1, 2, \cdots, m \\ v_0(0) \ge \lambda_1 v_0(T) + \lambda_2 \int_0^T w(s, v_0(s))ds + \sum_{i=1}^p a_i v_0(\eta_i) + \zeta \\ 0 \ge Q(v_0(T), \beta_0). \end{cases} \tag{3.1}$$

(H_5) f and I_k are nondecreasing with respect to the last variable.

(H_6)

$$\begin{aligned} &f(t, \overline{u}, \overline{u}(\alpha(t)), T\overline{u}, S\overline{u}, \varrho) - f(t, u, u(\alpha(t)), Tu, Su, \varrho) \\ &\ge -M(t)(\overline{u} - u) - N(t)(\overline{u}(\alpha(t)) - u(\alpha(t))) - K(t)T(\overline{u} - u) - H(t)S(\overline{u} - u), \end{aligned} \tag{3.2}$$

$$I_k(\overline{u}, \varrho) - I_k(u, \varrho) \ge -L_k(\overline{u} - u), \tag{3.3}$$

where $u_0 \le u \le \overline{u} \le v_0$.

(H_7) There exist $0 \le M_1$, $0 < M_2$ satisfying

$$Q(\overline{u}, \overline{\varrho}) - Q(u, \varrho) \ge M_1(\overline{u} - u) - M_2(\overline{\varrho} - \varrho), \tag{3.4}$$

where $u_0 \le u \le \overline{u} \le v_0$, $\alpha_0 \le \varrho \le \overline{\varrho} \le \beta_0$.

(H_8) Assume that $a(t)$ is non-negative integrabe function, such that

$$w(t, \overline{u}) - w(t, u) \ge a(t)(\overline{u} - u), \tag{3.5}$$

where $u_0 \le u \le \overline{u} \le v_0$.

Theorem 3.1 Assume the hypotheses $(H_1) - (H_8)$ hold. Suppose in addition that $\int_0^T M(s)ds > 0$ as $\lambda_1 = 1$, and $(u_0, \alpha_0), (v_0, \beta_0) \in PC^1(J) \times R$ such that $u_0 \le v_0$, $\alpha_0 \le \beta_0$. Then Eq.(1.1) has the extremal solutions $(u^*(t), \alpha^*), (v^*(t), \beta^*) \in [u_0, v_0] \times [\alpha_0, \beta_0]$. And there exist two sequences $\{(u_n, \alpha_n)\}$ and $\{(v_n, \beta_n)\}$ satisfying

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0, \tag{3.6}$$

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \cdots \le \beta_n \le \cdots \le \beta_1 \le \beta_0, \tag{3.7}$$

such that $\{u_n\}$,$\{v_n\}$ uniformly converge to $u^*(t)$, $v^*(t)$ on J, respectively, and $\{\alpha_n\}$,$\{\beta_n\}$ converge to α^*, β^* on J, respectively. Where $\{u_n\}$, $\{v_n\}$ are defined as :

$$\begin{aligned} u_n \;=\; & \frac{e^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-\lambda_1}\Big(\sum_{i=1}^{p} a_i u_{n-1}(\eta_i)+\lambda_2\int_0^T w(s,u_{n-1}(s))ds+\zeta\Big) \\ & +\int_0^T G^*(t,s)\{f(s,u_{n-1},u_{n-1}(\alpha(s)),Tu_{n-1},Su_{n-1},\alpha_{n-1}) \\ & +M(s)u_{n-1}-(\mathscr{H}(u_n-u_{n-1}))(s)\}ds \\ & +\frac{\lambda_1 e^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-\lambda_1}\sum_{k=1}^{m} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k u_n(t_k)+I_k(u_{n-1}(t_k),\alpha_{n-1})+L_k u_{n-1}(t_k)) \\ & +\sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau}e^{\int_0^{t_k} M(\tau)d\tau}(-L_k u_n(t_k)+I_k(u_{n-1}(t_k),\alpha_{n-1})+L_k u_{n-1}(t_k)) \\ & \forall t\in J, n=1,2,\cdots \end{aligned} \tag{3.8}$$

$$\begin{aligned} v_n \;=\; & \frac{e^{\int_t^T M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-\lambda_1}\Big(\sum_{i=1}^{p} a_i v_{n-1}(\eta_i)+\lambda_2\int_0^T w(s,v_{n-1}(s))ds+\zeta\Big) \\ & +\int_0^T G^*(t,s)\{f(s,v_{n-1},v_{n-1}(\alpha(s)),Tv_{n-1},Sv_{n-1},\beta_{n-1}) \\ & +M(s)v_{n-1}-(\mathscr{H}(v_n-v_{n-1}))(s)\}ds \\ & +\frac{\lambda_1 e^{-\int_0^t M(\tau)d\tau}}{e^{\int_0^T M(\tau)d\tau}-\lambda_1}\sum_{k=1}^{m} e^{\int_0^{t_k} M(\tau)d\tau}(-L_k v_n(t_k)+I_k(v_{n-1}(t_k),\beta_{n-1})+L_k v_{n-1}(t_k)) \\ & +\sum_{0<t_k<t} e^{-\int_0^t M(\tau)d\tau}e^{\int_0^{t_k} M(\tau)d\tau}(-L_k v_n(t_k)+I_k(v_{n-1}(t_k),\beta_{n-1})+L_k v_{n-1}(t_k)) \\ & \forall t\in J, n=1,2,\cdots \end{aligned} \tag{3.9}$$

Proof : For $(\xi,e)\in[u_0,v_0]\times[\alpha_0,\beta_0]$, considering the following problem

$$\begin{cases} u'(t)=-M(t)u(t)-(\mathscr{H}u)(t)+M(t)\xi(t) \\ \qquad\qquad +(\mathscr{H}\xi)(t)+f(t,\xi(t),\xi(\alpha(t)),T\xi,S\xi,e), \quad t\neq t_k, \quad t\in J=[0,T], \\ \Delta u(t_k)=-L_k(u(t_k))+I_k(\xi(t_k),e)+L_k\xi(t_k), \qquad k=1,2,\cdots,m, \\ u(0)=\lambda_1 u(T)+\lambda_2\int_0^T w(s,\xi(s))ds+\Sigma_{i=1}^{p}a_i\xi(\eta_i)+\zeta, \end{cases} \tag{3.10}$$

$$Q(\xi(T),e)+M_1(u(T)-\xi(T))-M_2(\varrho-e)=0. \tag{3.11}$$

By Lemma 2.2 and Lemma 2.3, the BVPS has a unique solution $(u,\varrho)\in[u_0,v_0]\times[\alpha_0,\beta_0]$.

We define an operator φ by $(u,\varrho)=\varphi(\xi,e)$, then φ is an operator from $[u_0,v_0]\times[\alpha_0,\beta_0]$ to $PC(J)\times R$.

We claim that

(a) $(u_0,\alpha_0)\leq\varphi(u_0,\alpha_0)$, $\quad\varphi(v_0,\beta_0)\leq(v_0,\beta_0)$,

(b) φ is nondecreasing on $[u_0,v_0]\times[\alpha_0,\beta_0]$.

We prove (a), let $(u_1, \alpha_1) = \varphi(u_0, \alpha_0)$, $p(t) = u_0(t) - u_1(t)$, $q = \alpha_0 - \alpha_1$,

$$\begin{aligned} p' &= u_0' - u_1' \\ &\leq f(t, u_0(t), u_0(\alpha(t)), Tu_0, Su_0, \alpha_0) - [f(t, u_0(t), u_0(\alpha(t)), Tu_0, Su_0, \alpha_0) \\ &\quad +M(t)u_0(t) + (\mathscr{H}u_0)(t) - Mu_1(t) - (\mathscr{H}u_1)(t)] \\ &= -Mp(t) - (\mathscr{H}p)(t), \end{aligned}$$

$$\begin{aligned} \Delta p(t_k) &= \Delta u_0(t_k) - \Delta u_1(t_k) \\ &\leq I_k(u_0(t_k), \alpha_0) - [I_k(u_0(t_k), \alpha_0) - L_k(u_1 - u_0)] \\ &= -L_k p(t_k), \end{aligned}$$

$$\begin{aligned} p(0) &= u_0(0) - u_1(0) \\ &\leq \lambda_1 u_0(T) + \lambda_2 \textstyle\int_0^T w(s, u_0(s))ds + \sum_{i=1}^p a_i u_0(\eta_i) + \zeta \\ &\quad -(\lambda_1 u_1(T) + \lambda_2 \textstyle\int_0^T w(s, u_0(s))ds + \sum_{i=1}^p a_i u_0(\eta_i) + \zeta) \\ &= \lambda_1 p(T). \end{aligned}$$

By Lemma 2.1, we have $p \leq 0$. That is $u_0 \leq u_1$.

And $0 = Q(u_0(t), \alpha_0) + M_1(u_1(t) - u_0(t)) - M_2(\alpha_1 - \alpha_0) \geq -M_2(\alpha_1 - \alpha_0) = M_2 q$, which implies $q \leq 0$. Then $\alpha_0 \leq \alpha_1$. Hence we have $(u_0, \alpha_0) \leq (u_1, \alpha_1)$. Similarly, we can prove $(v_1, \beta_1) \leq (v_0, \beta_0)$.

To prove (b), let $(\gamma_1, \varrho_1), (\gamma_2, \varrho_2) \in [u_0, v_0] \times [\alpha_0, \beta_0]$, and $\gamma_1 \leq \gamma_2$, $\varrho_1 \leq \varrho_2$, $(\gamma_1^*, \varrho_1^*) = \varphi(\gamma_1, \varrho_1)$, $(\gamma_2^*, \varrho_2^*) = \varphi(\gamma_2, \varrho_2)$, $p = \gamma_1^* - \gamma_2^*$, $q = \varrho_1^* - \varrho_2^*$ then

$$\begin{aligned} p'(t) &= \gamma_1'^* - \gamma_2'^* \\ &= f(t, \gamma_1(t), \gamma_1(\alpha(t)), T\gamma_1, S\gamma_1, \varrho_1) \\ &\quad +M\gamma_1(t) + (\mathscr{H}\gamma_1)(t) - M\gamma_1^*(t) - (\mathscr{H}\gamma_1^*)(t) \\ &\quad -[f(t, \gamma_2(t), \gamma_2(\alpha(t)), T\gamma_2, S\gamma_2, \varrho_2) \\ &\quad +M\gamma_2(t) + (\mathscr{H}\gamma_2)(t) - M\gamma_2^*(t) - (\mathscr{H}\gamma_2^*)(t)] \\ &\leq f(t, \gamma_1(t), \gamma_1(\alpha(t)), T\gamma_1, S\gamma_1, \varrho_2) \\ &\quad +M\gamma_1(t) + (\mathscr{H}\gamma_1)(t) - M\gamma_1^*(t) - (\mathscr{H}\gamma_1^*)(t) \\ &\quad -[f(t, \gamma_2(t), \gamma_2(\alpha(t)), T\gamma_2, S\gamma_2, \varrho_2) \\ &\quad +M\gamma_2(t) + (\mathscr{H}\gamma_2)(t) - M\gamma_2^*(t) - (\mathscr{H}\gamma_2^*)(t)] \\ &\leq -Mp - (\mathscr{H}p)(t), \end{aligned}$$

$$
\begin{aligned}
\Delta p(t_k) &= \Delta\gamma_1^*(t_k)-\Delta\gamma_2^*(t_k)\\
&= I_k(\gamma_1(t_k),\varrho_1)-L_k(\gamma_1^*(t_k)-\gamma_1(t_k))\\
&\quad -(I_k(\gamma_2(t_k),\varrho_2)-L_k(\gamma_2^*(t_k)-\gamma_2(t_k)))\\
&= I_k(\gamma_1(t_k),\varrho_1)-I_k(\gamma_2(t_k),\varrho_2)+L_k(\gamma_1-\gamma_2)-L_k(\gamma_1^*-\gamma_2^*)\\
&\le I_k(\gamma_1(t_k),\varrho_2)-I_k(\gamma_2(t_k),\varrho_2)+L_k(\gamma_1-\gamma_2)-L_k(\gamma_1^*-\gamma_2^*)\\
&\le -L_k p(t_k),
\end{aligned}
$$

$$
\begin{aligned}
p(0) &= \gamma_1^*(0)-\gamma_2^*(0)\\
&\le \lambda_1\gamma_1^*(T)+\lambda_2\int_0^T w(s,\gamma_1(s))ds+\sum_{i=1}^p a_i\gamma_1(\eta_i)+\zeta\\
&\quad -(\lambda_1\gamma_2^*(T)+\lambda_2\int_0^T w(s,\gamma_2(s))ds+\sum_{i=1}^p a_i\gamma_2(\eta_i)+\zeta)\\
&= \lambda_1 p(T)+\lambda_2\int_0^T a(s)(\gamma_1(s)-\gamma_2(s))ds+\sum_{i=1}^p a_i(\gamma_1(\eta_i)-\gamma_2(\eta_i))\\
&\le \lambda_1 p(T).
\end{aligned}
$$

In view of Lemma 2.1, we know $\gamma_1^*\le\gamma_2^*$.

And

$$
\begin{aligned}
0 &= Q(\gamma_1(t),\varrho_1)+M_1(\gamma_1^*(t)-\gamma_1(t))-M_2(\varrho_1^*-\varrho_1)\\
&\quad -Q(\gamma_2(t),\varrho_2)-M_1(\gamma_2^*(t)-\gamma_2(t))+M_2(\varrho_2^*-\varrho_2)\\
&= Q(\gamma_1(t),\varrho_1)-Q(\gamma_2(t),\varrho_2)+M_2(\varrho_1-\varrho_2)\\
&\quad -M_1(\gamma_1(t)-\gamma_2(t))-M_2q+M_1(\gamma_1^*(t)-\gamma_2^*(t))\\
&\le -M_2q,
\end{aligned}
$$

which implies $q\le 0$. We get $\varrho_1^*\le\varrho_2^*$. Hence (b) holds.

We define two sequences $\{(u_n,\alpha_n)\}$ and $\{(v_n,\beta_n)\}$ in $PC^1(J)\times R$

$$(u_{n+1},\alpha_{n+1})=\varphi(u_n,\alpha_n),\ (v_{n+1},\beta_{n+1})=\varphi(v_n,\beta_n)\quad (n=0,1,2,\cdots).$$

By (a) and (b), we know that (3.6)(3.7) hold.

And each $\{(u_n,\alpha_n)\}$, $\{(v_n,\beta_n)\}$ in $PC^1(J)\times R$ satisfies

$$
\begin{cases}
u_n'(t)=f(t,u_{n-1}(t),,u_{n-1}(\alpha(t)),Tu_{n-1},Su_{n-1},\alpha_{n-1})-M(t)(u_n(t)-u_{n-1}(t))\\
-(\mathscr{H}(u_n-u_{n-1}))(t),\quad t\ne t_k,,\quad t\in J=[0,T],\\
\Delta u_n(t_k)=-L_ku_n(t_k)+I_k(u_{n-1}(t_k),\alpha_{n-1})+L_ku_{n-1}(t_k),\qquad k=1,2,\cdots,m,\\
u_n(0)=\lambda_1u_n(T)+\lambda_2\int_0^T w(s,u_{n-1}(s))ds+\sum_{i=1}^p a_iu_{n-1}(\eta_i)+\zeta,\\
Q(u_{n-1}(T),\alpha_{n-1})+M_1(u_n(T)-u_{n-1}(T))-M_2(\alpha_n-\alpha_{n-1})=0,
\end{cases}
$$

$$\begin{cases} v_n'(t) = f(t, v_{n-1}(t), v_{n-1}(\alpha(t)), Tv_{n-1}, Sv_{n-1}, \beta_{n-1}) - M(t)(v_n - v_{n-1}) \\ -(\mathscr{H}(v_n - v_{n-1}))(t), \quad t \neq t_k, \quad t \in J = [0, T], \\ \Delta v_n(t_k) = -L_k v_n(t_k) + I_k(v_{n-1}(t_k), \beta_{n-1}) + L_k v_{n-1}(t_k), \qquad k = 1, 2, \cdots, m, \\ v_n(0) = \lambda_1 v_n(T) + \lambda_2 \int_0^T w(s, v_{n-1}(s))ds + \sum_{i=1}^p a_i v_{n-1}(\eta_i) + \zeta, \\ Q(v_{n-1}(T), \beta_{n-1}) + M_1(v_n(T) - v_{n-1}(T)) - M_2(\beta_n - \beta_{n-1}) = 0. \end{cases}$$

Therefore, we have that $\{u_n\}$, $\{v_n\}$ are monotonically and uniformly convergent to $u^*(t)$ and $v^*(t)$ on J, respectively, and $\{\alpha_n\}$, $\{\beta_n\}$ converge to α^*, β^* on J, respectively. By the Ascoli-Arzela theorem, this implies that $(u^*(t), \alpha^*), (v^*(t), \beta^*)$ are solutions of Eq.(1.1).

Finally, we assert that if $(u, \varrho) \in [u_0, v_0] \times [\alpha_0, \beta_0]$ is any solution of Eq.(1.1), then $u^*(t) \leq u(t) \leq v^*(t)$, $\alpha^* \leq \varrho \leq \beta^*$ on J. We will prove that if $u_n \leq u \leq v_n$, $\alpha_n \leq \varrho \leq \beta_n$, for $n = 0, 1, 2, \cdots$, then $u_{n+1}(t) \leq u(t) \leq v_{n+1}(t)$, $\alpha_{n+1} \leq \varrho \leq \beta_{n+1}$.

Letting $p(t) = u_{n+1}(t) - u(t)$, $q = \alpha_{n+1} - \varrho$ then

$$\begin{aligned} p'(t) &= u_{n+1}' - u'(t) \\ &= f(t, u_n(t), u_n(\alpha(t)), Tu_n, Su_n, \alpha_n) + Mu_n(t) + (\mathscr{H} u_n)(t) \\ &\quad -Mu_{n+1}(t) - (\mathscr{H} u_{n+1})(t) - f(t, u(t), u(\alpha(t)), Tu, Su, \varrho) \\ &\leq f(t, u_n(t), u_n(\alpha(t)), Tu_n, Su_n, \varrho) + Mu_n(t) + (\mathscr{H} u_n)(t) \\ &\quad -Mu_{n+1}(t) - (\mathscr{H} u_{n+1})(t) - f(t, u(t), u(\alpha(t)), Tu, Su, \varrho) \\ &\leq -M(u_{n+1}(t) - u(t)) - (\mathscr{H}(u_{n+1} - u))(t) \\ &\leq -Mp - (\mathscr{H} p), \end{aligned}$$

$$\begin{aligned} \Delta p(t_k) &= \Delta u_{n+1}(t_k) - \Delta u(t_k) \\ &= I_k(u_n(t_k), \alpha_n) - L_k(u_{n+1}(t_k) - u_n(t_k)) - I_k(u(t_k), \varrho) \\ &\leq I_k(u_n(t_k), \varrho) - L_k(u_{n+1}(t_k) - u_n(t_k)) - I_k(u(t_k), \varrho) \\ &\leq -L_k(u_n(t_k) - u(t_k)) - L_k(u_{n+1}(t_k) - u_n(t_k)) \\ &= -L_k(u_{n+1}(t_k) - u(t_k)) \\ &= -L_k p(t_k), \end{aligned}$$

$$\begin{aligned} p(0) &= u_{n+1}(0) - u(0) \\ &\leq \lambda_1 u_{n+1}(T) + \lambda_2 \int_0^T w(s, u_n(s))ds + \sum_{i=1}^p a_i u_n(\eta_i) + \zeta \\ &\quad -(\lambda_1 u(T) + \lambda_2 \int_0^T w(s, u(s))ds + \sum_{i=1}^p a_i u(\eta_i) + \zeta) \\ &= \lambda_1 p(T) + \lambda_2 \int_0^T a(s)(u_n(s) - u(s))ds + \sum_{i=1}^p a_i(u_n(\eta_i) - u(\eta_i)) \\ &\leq \lambda_1 p(T). \end{aligned}$$

By Lemma 2.1, we have $p(t) \leq 0$ for all $t \in J$, that is $u_{n+1}(t) \leq u(t)$.

And

$$\begin{aligned} 0 &= Q(u_n(t),\alpha_n)+M_1(u_{n+1}(t)-u_n(t))-M_2(\alpha_{n+1}-\alpha_n)-Q(u(t),\varrho) \\ &\le M_1(u_{n+1}(t)-u(t))-M_2(\alpha_{n+1}-\varrho) \\ &\le -M_2 q. \end{aligned}$$

We have $\alpha_{n+1}\le\varrho$. Hence $(u_{n+1},\alpha_{n+1})\le(u,\varrho)$. Similarly, we can prove $(u,\varrho)\le(v_{n+1},\beta_{n+1})$, which implies $(u(t),\varrho)\in[u^*(t),v^*]\times[\alpha^*,\beta^*]$. The proof is complete.

Remark In (1.1), if $w(s,x(s))=a(s)x(s)$, where $a(t)$ is non-negative integral function, then (H_8) is not required in Theorem 3.1, and we have the following theorem.

Theorem 3.2 Suppose that conditions $(H_1)-(H_7)$are satisfied. Let $\int_0^T M(s)ds>0$ as $\lambda_1=1$, and $(u_0,\alpha_0),(v_0,\beta_0)\in PC^1(J)\times R$ such that $u_0\le v_0$, $\alpha_0\le\beta_0$. Then the conclusion of Theorem 3.1 holds. The proof is almost similar to theorem 3.1, so we omit it.

4 Example

Consider the following problems

$$\begin{cases} u'(t)=\dfrac{t^4u(t)}{100}-\dfrac{t}{600}\sin(u(\dfrac{t}{2}))-\dfrac{t}{100}\displaystyle\int_0^t su(s)ds-\dfrac{t^3}{1000}\int_0^1 u(s)ds+\varrho, \quad t\ne\dfrac{1}{2}, t\in J=[0,1], \\ \Delta u(\dfrac{1}{2})=-\dfrac{27}{160}u^3(\dfrac{1}{2})+\varrho \\ u(0)=\dfrac{1}{2}u(1)+\dfrac{1}{100}u(\eta)+\dfrac{1}{100}\displaystyle\int_0^1(u(s)-s)ds+\dfrac{1}{150} \qquad \eta\in[0,1], \\ -3u(1)+\varrho^2=0. \end{cases} \tag{4.1}$$

Let $f(t,x,y,z,w,\varrho)=\dfrac{t^4x}{100}-\dfrac{t}{600}y-\dfrac{1}{100}z-t^3w+\varrho$, $\varrho\in R$, $M(t)=0$, $N(t)=\dfrac{t}{600}$, $K(t)=\dfrac{1}{100}$, $H(t)=t^3$, $k(t,s)=ts$, $h(t,s)=\dfrac{1}{1000}$, $Tu(t)=t\displaystyle\int_0^t su(s)ds$, $Su(t)=\displaystyle\int_0^1\dfrac{1}{1000}u(s)ds$, $\alpha(t)=\dfrac{t}{2}$, $\beta(t)=t$, $\gamma(s)=s$, $\delta(s)=s$, $w(s,u(s))=u(s)-s$.

We can easily verify that (4.1) admits the lower solution $(u_0(t)=0,\alpha_0=0)$ and the upper solution $(v_0(t),\beta_0=2)$, where

$$v_0(t)=\begin{cases} \dfrac{2}{3}t+1, & t\in[0,\dfrac{1}{2}], \\ \dfrac{2}{3}t+\dfrac{2}{3}, & t\in(\dfrac{1}{2},1], \end{cases}$$

and $u_0(t)\le v_0(t)$, $\alpha_0\le\beta_0$. It is easy to see that

$$\begin{aligned} I_k(x(t_k),\varrho)-I_k(y(t_k,\varrho)) &= -\frac{27}{160}(x^3(t_k)-y^3(t_k)) \\ &\ge -\frac{3}{10}(x(t_k)-y(t_k)) \\ &= -L_1(x(t_k)-y(t_k)), \end{aligned}$$

where $u_0(t_k) \le y(t_k) \le x(t_k) \le v_0(t_k)$, $L_1 = \frac{3}{10}$.
Obviously,

$$\begin{aligned} &f(t,\overline{u},\overline{u}(\alpha(t)),T\overline{u},S\overline{u},\varrho) - f(t,u,u(\alpha(t)),Tu,Su,\varrho) \\ &\ge -M(t)(\overline{u}-u) - N(t)(\overline{u}-u)(\alpha(t)) - K(t)T(\overline{u}-u) - H(t)S(\overline{u}-u), \end{aligned}$$

$$W(t,\overline{u}(t)) - W(t,u(t)) = \overline{u}(t) - u(t) \ge \frac{t}{3}(\overline{u}(t)-u(t)),$$

for all $u_0(t) \le u(t) \le \overline{u}(t) \le v_0(t)$ in J.
And it is obvious that (H_5) (H_7) hold. And we can check that $r^* = r = \frac{1}{2}$, $[\mu^* + \sum_{k=1}^{m} L_k] < r^*$, $e^{\int_0^T |M(\tau)|d\tau}(1 + \frac{r}{e^{\int_0^T M(\tau)d\tau} - r})(\mu + \sum_{k=1}^{m} L_k) < 0.96 < 1$, then all conditions of Theorem 3.1 are satisfied. Therefore, the conclusion of Theorem 3.1 holds for the problem (4.1).

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