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Mild and classical solutions to a fractional singular second order evolution problem

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Abstract

Existence and uniqueness of mild and classical solutions are discussed for an abstract second-order evolution problem. The nonlinearity contains a local term and a non-local term. The non-local term is an integral in the form of a convolution of a singular kernel and a regular function involving fractional derivatives. This term may be regarded also as a fractional integral of that regular function. In addition the initial conditions are nonlocal and involve fractional derivatives too.

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1 Introduction

It is by now well-known that problems with non-local conditions arise naturally in applications and that they play an important role in many fields, see [1-3,5,7-9] (to cite but a few) for the case of abstract second order differential equations. In contrast, one cannot find many papers in the literature dealing with non-local conditions involving fractional derivatives. Nevertheless, we may find few papers treating well-posedness and asymptotic behavior of solutions for some problems with boundary conditions containing fractional derivatives (see [10-17]).

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The type of equations we discuss here is of the following nature

$$u''(t) = Au(t) + f\left(t, u(t), t^{\lambda_\alpha} D^\alpha u(t)\right) + \int_0^t l\left(t, s, u(s), s^{\lambda_\nu} D^\nu u(s)\right) ds.$$

The operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \geq 0$ of bounded linear operators in the Banach space X and f, l are nonlinear functions from $\mathbf{R}^+ \times X \times X$ to X and $\mathbf{R}^+ \times \mathbf{R}^+ \times X \times X$ to X, respectively. D^α and D^ν denote "fractional" time differentiation (in the sense of Riemann-Liouville) of order α and ν, respectively. In many situations in applications the integral term involves a singular kernel of the form $(t-s)^{\sigma-1}/\Gamma(\sigma)$, $\sigma > 0$ and becomes the convolution of this kernel with another function, say g. The resulting expression is then identified right away as a "fractional integral", that is

$$\begin{aligned} \int_0^t l\left(t, s, u(s), s^{\lambda_\nu} D^\nu u(s)\right) ds &= \tfrac{t^{\sigma-1}}{\Gamma(\sigma)} * g\left(t, u(t), t^{\lambda_\nu} D^\nu u(t)\right) \\ &= I^\sigma g\left(t, u(t), t^{\lambda_\nu} D^\nu u(t)\right). \end{aligned}$$

This kind of nonlinearities appear in many hereditary phenomena with fading memory like in viscoelasticity and in heat conduction. Mathematically speaking the problem becomes more challenging because of the difficulties that may be caused by the singular kernel.

In [23] the present author has introduced the following nonlocal conditions involving fractional derivatives and called them "nonlocal conditions of fractional type"

$$\begin{aligned} u(0) &= u^0 + p\left(u, t^{\lambda_\beta} D^\beta u(t)\right), \\ u'(0) &= u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u(t)\right). \end{aligned}$$

Here u^0 and u^1 are given initial data in X and the functions $p, q : [C(I;X)]^2 \to X$ are given continuous functions. This type of nonlocal initial data covers and extends many existing ones in the literature like

$$\begin{aligned} u(0) &= u^0 + p(u, u'), \\ u'(0) &= u^1 + q(u, u') \end{aligned}$$

and the discrete cases where u and u' are specified at some finite number of values of t. In fact it will be shown here that these nonlocal conditions will not bring any new considerable difficulties in the study of the well-posedness of our problem provided that the fractional derivatives in there are well-defined and continuous in the underlying space. The main contribution of the present work is rather related to the type of the considered nonlinearities. As a matter of fact, in a previous work [22] we have studied the case where the fractional derivatives (in the nonlinearities and here in the nonlocal conditions as well as) were between 0 and 1. The case where the derivatives are of order between 1 and 2 was left open as some regularity difficulties arose in the proofs.

In the present paper we consider the case where the orders are between 0 and 2 with a particular emphasis on orders between 1 and 2. The difficulties encountered earlier have been overcome by the use of some kind of integration by parts for fractional integrals, some new lemmas, some appropriately chosen spaces and of course some reasonable regularity assumptions on the functions encompassed in the nonlinearities. Therefore, this paper may be regarded as a continuation of the paper [22,23]. In the integer order case (derivatives of order 1), the underlying space where to look for mild solutions is the space of continuously differentiable functions whereas in case the derivatives are of order between 1 and 2, more regularity is needed on the initial data and also on the nonlinearities. Our results are new even in the local initial data case, that is $p \equiv q \equiv 0$.

The problem we will treat here is therefore

$$\left\{ \begin{array}{l} u''(t) = Au(t) + f\left(t, u(t), t^{\lambda_\alpha} D^\alpha u(t)\right) + I^\sigma g\left(t, u(t), t^{\lambda_\nu} D^\nu u(t)\right), \ t \in I \\ u(0) = u^0 + p\left(u, t^{\lambda_\beta} D^\beta u(t)\right), \\ u'(0) = u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u(t)\right), \end{array} \right. \tag{1}$$

where $I = [0, T]$, $\lambda_\alpha, \lambda_\beta, \lambda_\gamma, \lambda_\nu \geq 0$, $0 < \alpha, \beta, \gamma, \nu < 2$ and $\sigma > 0$. Similar results may be derived for more general problems like

$$\left\{ \begin{array}{l} u''(t) \;=\; Au(t) + f\left(t, u(t), t^{\lambda_{\alpha_1}} D^{\alpha_1} u(t), ..., t^{\lambda_{\alpha_n}} D^{\alpha_n} u(t)\right) \\ \qquad\quad +\; I^\sigma g\left(t, u(t), t^{\lambda_{\nu_1}} D^{\nu_1} u(t), ..., t^{\lambda_{\nu_n}} D^{\nu_n} u(t)\right), \ t \in I = [0, T] \\ u(0) = u^0 + p\left(u, t^{\lambda_{\beta_1}} D^{\beta_1} u(t), ..., t^{\lambda_{\beta_m}} D^{\beta_m} u(t)\right), \\ u'(0) = u^1 + q\left(u, t^{\lambda_{\gamma_1}} D^{\gamma_1} u(t), ..., t^{\lambda_{\gamma_r}} D^{\gamma_r} u(t)\right). \end{array} \right.$$

This problem has been studied in case $\sigma, \alpha, \beta, \gamma$ are 0 or 1 (and $\lambda_\alpha, \lambda_\beta, \lambda_\gamma = 0$) (see [1-3,5,7,8,26] and references therein) and in [23] for $\sigma > 1$ and fractional derivatives. Well-posedness has been proved using different methods such as fixed point theorems and the theory of strongly continuous cosine families in Banach spaces. We refer the reader to [6,24,25] for a good account on the theory of the cosine family. Several results on the existence of classical solutions and mild solutions have been established under different conditions on the nonlinearities and the initial data.

The next section of this paper contains some notation and preliminary results needed in our proofs. Section 3 treats the existence and uniqueness of a mild solution in an appropriate space. Section 4 is devoted to the existence and uniqueness of a classical solution.

2 Preliminaries

In this section we present some notation, assumptions and results needed in our proofs later. We start by some definitions which maybe found for instance in [10,19-21].

Definition 1. *The integral*

$$(I_{a+}^{\alpha}h)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{h(t)dt}{(x-t)^{1-\alpha}},\ x > a$$

is called the Riemann-Liouville fractional integral of h of order $\alpha > 0$ when the right side exists.

Here Γ is the usual Gamma function

$$\Gamma(z) := \int_0^{\infty} e^{-s}s^{z-1}ds,\ z > 0.$$

Definition 2. *The left hand Riemann-Liouville fractional derivative of h defined in the interval $[a,b]$ of order $\alpha > 0$ is defined by*

$$(D_{a+}^{\alpha}h)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{h(t)dt}{(x-t)^{\alpha-n+1}},\ n = [\alpha]+1$$

whenever the right side is pointwise defined.

In particular

$$(D_{a+}^{\gamma}h)(x) = \frac{1}{\Gamma(2-\gamma)}\left(\frac{d}{dx}\right)^2 \int_a^x \frac{h(t)dt}{(x-t)^{\gamma-1}},\ 1 < \gamma < 2.$$

Definition 3. *The right hand Riemann-Liouville fractional derivative of h defined in the interval $[a,b]$ of order $\alpha > 0$ is defined by*

$$(D_{b-}^{\alpha}h)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_x^b \frac{h(t)dt}{(t-x)^{\alpha-n+1}},\ n = [\alpha]+1$$

whenever the right side is pointwise defined.

See [10,18-21] for more on fractional derivatives and fractional integrals.

To lighten the notation we will write I^α for I^α_{0+}, D^γ for D^γ_{0+} and D^α_- for D^α_{b-}.

We will assume that

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbf{R}$, of bounded linear operators in the Banach space X.

The associated sine family $S(t)$, $t \in \mathbf{R}$ is defined by

$$S(t)x := \int_0^t C(s)x ds,\ t \in \mathbf{R},\ x \in X.$$

It is known (see [24-26]) that there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$|C(t)| \leq Me^{\omega|t|},\ t \in \mathbf{R}\ and\ |S(t) - S(t_0)| \leq M \left| \int_{t_0}^t e^{\omega|s|} ds \right|,\ t, t_0 \in \mathbf{R}.$$

We will denote by $\tilde{M}$ and $\tilde{N}$ some respective bounds for $C(t)$ and $S(t)$ ($\tilde{M} \geq 1$ and $\tilde{N} \geq 1$ depend on T).

If we define

$$E := \{x \in X : C(t)x\ is\ once\ continuously\ differentiable\ on\ \mathbf{R}\}$$

then we have

Lemma 1. *(see [24-26])*

*Assume that **(H1)** is satisfied. Then*

(i) $S(t)X \subset E$, $t \in \mathbf{R}$,

(ii) $S(t)E \subset D(A)$, $t \in \mathbf{R}$,

(iii) $\frac{d}{dt}C(t)x = AS(t)x$, $x \in E$, $t \in \mathbf{R}$,

(iv) $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$, $x \in D(A)$, $t \in \mathbf{R}$.

Lemma 2. *(see [24-26])*

*Suppose that **(H1)** holds,* $v : \mathbf{R} \to X$ *a continuously differentiable function and* $q(t) = \int_0^t S(t-s)v(s)ds$. *Then,* $q(t) \in D(A)$, $q'(t) = \int_0^t C(t-s)v(s)ds$ *and* $q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t)$.

Below we make clear what we mean by a classical solution and a mild solution.

Definition 4. *A function $u(.) \in C^2(I,X)$, such that $t^{\lambda_\eta} D^\eta u \in C(I,X)$, $\eta = \alpha, \beta, \gamma, \nu$, is called a classical solution of (1) if $u(.) \in D(A)$ satisfies the equation in (1) and the initial conditions are verified.*

Definition 5. *A continuous solution u, such that $t^{\lambda_\eta} D^\eta u \in C(I,X)$, $\eta = \alpha, \beta, \gamma, \nu$, of the integro-differential equation*

$$\begin{array}{c} u(t) = C(t)\left[u^0 + p\left(u, t^{\lambda_\beta} D^\beta u(t)\right)\right] + S(t)\left[u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u\left(t\right)\right)\right] \\ + \int_0^t S(t-s)\left[f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right) + I^\sigma g\left(s, u\left(s\right), s^{\lambda_\nu} D^\nu u\left(s\right)\right)\right] ds \end{array} \tag{2}$$

is called a mild solution of problem (1).

It is known from [26] that, in case of continuity of the nonlinearities, solutions of (1) are solutions of the more general problem (2).

The following lemmas will be very useful later. The first three can be found in [10,21]

Lemma 3. *If $\varphi(x) \in AC^n[a,b] := \{\phi : [a,b] \to R$ and $(D^{n-1}\phi)(x) \in AC[a,b]\}$, $\alpha > 0$ and $n = [\alpha] + 1$, then*

$$(D_a^\alpha \varphi)(x) = \sum\nolimits_{k=0}^{n-1} \frac{\varphi^{(k)}(a)(x-a)^{k-\alpha}}{\Gamma(1+k-\alpha)} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\varphi^{(n)}(t)dt}{(x-t)^{\alpha-n+1}},\ x > a.$$

Lemma 4. *Let $\alpha > 0$, $\beta < 0$ and $\varphi \in L^1(a,b)$ be such that $I^{n+\beta}\varphi \in AC^n([a,b])$. Then*

$$I_{a+}^\alpha I_{a+}^\beta \varphi(x) = I_{a+}^{\alpha+\beta}\varphi(x) - \sum\nolimits_{k=0}^{n-1} \frac{\varphi_{n+\beta}^{(n-k-1)}(a)}{\Gamma(\alpha-k)}(x-a)^{\alpha-k-1}$$

where $\varphi_{n+\beta}(x) = I_{a+}^{n+\beta}\varphi(x)$ and $n = [-\beta] + 1$.

Lemma 5. *Let $f(x) \in C[a,b]$ and $g(x) \in C[a,b]$ be such that $(D_{a+}^\alpha g)(x)$ and $(D_{b-}^\alpha f)(x)$, $0 < \alpha < 1$ exist and are continuous at every $x \in [a,b]$. Then, we have $\int_a^b f(x)(D_{a+}^\alpha g)(x)dx = \int_a^b g(x)(D_{b-}^\alpha f)(x)dx$.*

The lemma to follow may be found in [4].

Lemma 6. *Let $\alpha > 0$ and $\beta > 0$ be such that $n-1 < \alpha \leq n$, $m-1 < \beta \leq m$ ($n, m \in \mathbf{N}$). If $D_{a+}^\beta f$ exists and is finite on $[a,b]$ and is such that $D_{a+}^\alpha(D_{a+}^\beta f)$ exists also and is finite, then*

$$D_{a+}^\alpha D_{a+}^\beta f(x) = D_{a+}^{\alpha+\beta} f(x) - \sum\nolimits_{k=1}^{m} \frac{A_k}{\Gamma(1-\alpha-k)}(x-a)^{-\alpha-k},\ x \in [a,b]$$

where $A_k = \lim_{x\to a+} D_{a+}^{m-k} I_{a+}^{m-\beta} f(x)$*,* $k = 1, 2, ..., m$*, provided that*

(i) $n + m - \alpha - \beta \geq 1$ *or*

(ii) $n + m - \alpha - \beta < 1$ *and* f *is such that* $|x-a|^{\lambda} f(x)$ *is continuous on* $[a, b]$ *for some* $\lambda \in [0, 1-\gamma)$*, with* $1 - n - m + \alpha + \beta \leq \gamma \leq 1$*.*

Lemma 7. *([10], Property 2.2)*

If $\alpha > \beta > 0$ *and* $f \in L^1(\mathbf{R})$*, then*

$$(D^{\beta} I^{\alpha} f)(x) = (I^{\alpha-\beta} f)(x).$$

3 Existence of mild solutions

In this section we prove existence and uniqueness of a mild solution in the space

$$C_{\eta}^{RL}(I;X) := \{v \in C(I;X) : t^{\eta} D^{\eta} v \in C(I;X),\ \eta = \alpha, \beta, \gamma, \nu\} \tag{3}$$

equipped with the norm $\|v\|_{\kappa} := \|v\|_C + \sum_{\eta} \|t^{\eta} D^{\eta} v\|_C$ where $\|.\|_C$ is the sup norm in $C(I;X)$. Here $0 < \alpha, \beta, \gamma, \nu < 2$, and to make the problem more interesting we assume that at least one of them is between 1 and 2 and $\sigma > 0$. As a matter of fact, the most difficult and important cases are when $1 < \alpha, \nu < 2$ and $0 < \sigma < 1$. Indeed, in Section 4, we will have to differentiate $D^{\alpha} u$ and $D^{\nu} u$. This situation is delicate when $1 < \alpha, \nu < 2$ as it will force us to narrow the underlying space to functions which are more regular than C^2. To prevent such an unpleasant situation, the derivative of $D^{\alpha} u$ is avoided by taking E-valued functions f and the undesirable effect of the derivative of $D^{\nu} u$ is overcome by using a "fractional" integration by parts. The case $\sigma \geq 1$ is relatively simple in as much as the fractional integration I^{σ} will absorb the differentiation operator and prevents its effect on the integrand. We also define, for $\eta > 0$, the space

$$\tilde{E}_{\eta} := \{x \in X : t^{\eta} D^{\eta} C(t) x \text{ is continuous on } \mathbf{R}^+,\ \eta = \alpha, \beta, \gamma, \nu\}. \tag{4}$$

The assumptions on f, g, p and q are

(H2) The function g is such that $g(0, u, v) = 0$ for all $u, v \in C([0, T])$. Moreover, the functions f, $t^{\eta_f} D^{\eta-1} f$ $(0 < \eta_f < 1)$, g, $D^{\eta-1-\sigma} g$ (when $\sigma < \eta - 1, \eta = \alpha, \nu$): $\mathbf{R}^+ \times X \times X \to X$ are continuous and satisfy the Lipschitz conditions

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq A_f \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right),$$

$$\left\| t^{\eta_f} D^{\eta-1} f(t,x_1,y_1) - t^{\eta_f} D^{\eta-1} f(t,x_2,y_2) \right\| \le \tilde{A}_{\eta f} \left(\|x_1-x_2\| + \|y_1-y_2\| \right)$$

for $0<\eta_f<1$,

$$\|g(t,x_1,y_1) - g(t,x_2,y_2)\| \le A_g \left(\|x_1-x_2\| + \|y_1-y_2\| \right),$$

$$\left\| D^{\eta-1-\sigma} g(t,x_1,y_1) - D^{\eta-1-\sigma} g(t,x_2,y_2) \right\| \le \tilde{A}_{\eta g} \left(\|x_1-x_2\| + \|y_1-y_2\| \right),$$

for $x_1,y_1,x_2,y_2 \in X$, $t\in I$ and some positive constants A_f, $\tilde{A}_{\eta f}$, A_g and $\tilde{A}_{\eta g}$.

(H3) $p : [C(I;X)]^2 \to X$ and $q : [C(I;X)]^2 \to X$ are continuous and satisfy

$$\|p(x_1,y_1) - p(x_2,y_2)\| \le A_p \left(\|x_1-x_2\|_C + \|y_1-y_2\|_C \right),$$

$$\|q(x_1,y_1) - q(x_2,y_2)\| \le A_q \left(\|x_1-x_2\|_C + \|y_1-y_2\|_C \right),$$

for $x_1,y_1,x_2,y_2 \in C(I;X)$ and some positive constants A_p and A_q.

Lemma 8. *If $I^{1-\nu}R(t)x \in C^1([0,T])$, $T>0$ ($R(t)$ is a bounded linear operator), then, for $0<\nu<1$, we have*

$$D^\nu \int_0^t R(t-s)x ds = \int_0^t D^\nu R(t-s)x ds + \lim_{t\to 0^+} I^{1-\nu} R(t)x,\ x\in X,\ t\in[0,T].$$

Proof. By Definition 2 and the assumption $I^{1-\nu}R(t)x \in C^1([0,T])$, we have

$$\begin{aligned} &D^\nu \int_0^t R(t-s)x ds = \tfrac{1}{\Gamma(1-\nu)} \tfrac{d}{dt} \int_0^t \tfrac{d\tau}{(t-\tau)^\nu} \int_0^\tau R(\tau-s)x ds \\ &= \tfrac{1}{\Gamma(1-\nu)} \tfrac{d}{dt} \int_0^t ds \int_s^t \tfrac{R(\tau-s)x}{(t-\tau)^\nu} d\tau \\ &= \tfrac{1}{\Gamma(1-\nu)} \int_0^t ds \tfrac{\partial}{\partial t} \int_s^t \tfrac{R(\tau-s)x}{(t-\tau)^\nu} d\tau + \tfrac{1}{\Gamma(1-\nu)} \lim_{s\to t^-} \int_s^t \tfrac{R(\tau-s)x}{(t-\tau)^\nu} d\tau. \end{aligned}$$

Moreover, a change of variable $\sigma=\tau-s$ leads to

$$\begin{aligned} D^\nu \int_0^t R(t-s)x ds = \tfrac{1}{\Gamma(1-\nu)} \int_0^t ds \tfrac{\partial}{\partial t} \int_0^{t-s} \tfrac{R(\sigma)x}{(t-s-\sigma)^\nu} d\sigma \\ + \tfrac{1}{\Gamma(1-\nu)} \lim_{t\to 0^+} \int_0^t \tfrac{R(\sigma)x}{(t-\sigma)^\nu} d\sigma \end{aligned}$$

and the proof is complete. □

We prove here its counterpart for $1<\nu<2$.

Lemma 9. *Let $R(t)$ be a continuously differentiable bounded linear operator on I and g be a continuous function on I such that $I^{2-\nu}g \in C^1([0,T])$. Then, for $1<\nu<2$, we have*

$$\begin{aligned} D^\nu \int_0^t R(t-s)g(s)ds = \int_0^t R'(s) D^{\nu-1} g(t-s) ds + R(0^+) D^{\nu-1} g(t) \\ + R'(t) \lim_{t\to 0^+} I^{2-\nu} g(t),\ t\in[0,T]. \end{aligned}$$

Proof. By the assumption $I^{2-\nu}g \in C^1([0,T])$ we infer that $D^{\nu-1}g(s)$ exists and is continuous on I. Moreover, Lemma 6 or formula (2.122) in [20]

$$D^\rho\left(\frac{d^n h(t)}{dt^n}\right) = D^{\rho+n}h(t) - \sum_{j=0}^{n-1}\frac{h^{(j)}(0)t^{j-\rho-n}}{\Gamma(1+j-\rho-n)}$$

gives us

$$\begin{aligned} &D^\nu \int_0^t R(t-s)g(s)ds = D^{\nu-1+1}\int_0^t R(t-s)g(s)ds = D^{\nu-1}\tfrac{d}{dt}\int_0^t R(t-s)g(s)ds \\ &= D^{\nu-1}\left[\int_0^t R'(t-s)g(s)ds + R(0^+)g(t)\right] \\ &= D^{\nu-1}\left[\int_0^t R'(s)g(t-s)ds + R(0^+)g(t)\right]. \end{aligned}$$

Our assumption $I^{2-\nu}g \in C^1([0,T])$ allows us also to adopt a similar argument as in the proof of Lemma 8 to obtain

$$\begin{aligned} D^\nu \int_0^t R(t-s)g(s)ds = \int_0^t R'(s)D^{\nu-1}g(t-s)ds + R(0^+)D^{\nu-1}g(t) \\ +R'(t)\lim_{t\to 0^+} I^{2-\nu}g(t),\ t \in [0,T]. \end{aligned}$$

This completes the proof. □

Corollary 1. *Let $S(t)$ be the associated sine family with the cosine family $C(t)$ and g be such that $I^{2-\nu}g \in C^1([0,T])$, $t \in [0,T]$ and $1 < \nu < 2$. Then, we have, for $t \in [0,T]$,*

$$D^\nu \int_0^t S(t-s)g(s)ds = \int_0^t C(s)D^{\nu-1}g(t-s)ds + C(t)\lim_{t\to 0^+} I^{2-\nu}g(t).$$

Proof. This is an immediate consequence of the fact that $S(t)$ is continuously differentiable, $S(0) = 0$, the previous lemma and

$$\begin{aligned} D^\nu \int_0^t S(t-s)g(s)ds = \int_0^t C(s)D^{\nu-1}g(t-s)ds + S(0^+)D^{\nu-1}g(t) \\ +C(t)\lim_{t\to 0^+} I^{2-\nu}g(t) \\ = \int_0^t C(s)D^{\nu-1}g(t-s)ds + C(t)\lim_{t\to 0^+} I^{2-\nu}g(t),\ t \in [0,T]. \end{aligned}$$

□

We are now ready to state and prove our first main result.

Theorem 1. *Assume that **(H1)-(H3)** hold. Let $1 < \alpha, \beta, \gamma, \nu < 2$, $0 < \sigma < 1$, $u^0 + p\left(u, t^{\lambda_\beta}D^\beta u(t)\right) \in \tilde{E}_\eta$, $u^1 + q\left(u, t^{\lambda_\gamma}D^\gamma u\,(t)\right) \in \tilde{E}_{\eta-1}$ and $\lambda_\eta \geq \eta$, $\eta = \alpha, \beta, \gamma, \nu$. If $\tilde{M}$, $\tilde{N}$, R, $\tilde{R}$ (bounds for $t^\eta D^\eta C(t)$ and $t^\eta D^\eta C(t)$ on I, respectively) and T are sufficiently small then problem (1) admits a unique mild solution $u \in C_\eta^{RL}([0,T])$.*

Proof. For $u \in C_{\eta}^{RL}([0,T])$ we consider the function

$$\begin{array}{c} \Phi(u)(t) := C(t)\left[u^0 + p(u, t^{\lambda_\beta} D^\beta u(t))\right] + S(t)\left[u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u\left(t\right)\right)\right] \\ + \int_0^t S(t-s) f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right) ds \\ + \int_0^t S(t-s) I^\sigma g\left(s, u\left(s\right), s^{\lambda_\nu} D^\nu u\left(s\right)\right) ds,\ t \in [0,T]. \end{array} \tag{5}$$

Its η-th derivative is (see Lemma 6 or (2.122) in [20], proof of Lemma 9)

$$\begin{array}{c} t^\eta D^\eta \Phi(u)(t) = t^\eta D^\eta C(t)\left[u^0 + p(u, t^{\lambda_\beta} D^\beta u(t))\right] + t^\eta D^{\eta-1} C(t) \\ \times\left[u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u\left(t\right)\right)\right] + t^\eta D^\eta \int_0^t S(t-s) f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right) ds \\ + t^\eta D^\eta \int_0^t S(t-s) I^\sigma g\left(s, u\left(s\right), s^{\lambda_\nu} D^\nu u\left(s\right)\right) ds,\ t \in [0,T]. \end{array} \tag{6}$$

Note that, under our assumptions, both expressions in (5) and (6) are well-defined. In fact, in virtue of Corollary 1, we can rewrite (6) as

$$\begin{array}{c} t^\eta D^\eta \Phi(u)(t) = t^\eta D^\eta C(t)\left[u^0 + p(u, t^{\lambda_\beta} D^\beta u(t))\right] \\ + t^\eta D^{\eta-1} C(t)\left[u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u\left(t\right)\right)\right] \\ + t^\eta \int_0^t C(t-s) D^{\eta-1} f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right) ds \\ + t^\eta C(t) \lim_{t\to 0^+} I^{2-\eta} f\left(t, u\left(t\right), t^{\lambda_\alpha} D^\alpha u\left(t\right)\right) \\ + t^\eta \int_0^t C(t-s) D^{\eta-1} I^\sigma g\left(s, u\left(s\right), s^{\lambda_\nu} D^\nu u\left(s\right)\right) ds \\ + t^\eta C(t) \lim_{t\to 0^+} I^{2-\eta} I^\sigma g\left(t, u\left(t\right), t^{\lambda_\nu} D^\nu u\left(t\right)\right) \end{array}$$

or, because of the continuity of the functions $f\left(t, u\left(t\right), t^{\lambda_\alpha} D^\alpha u\left(t\right)\right)$ and $g\left(t, u\left(t\right), t^{\lambda_\nu} D^\nu u\left(t\right)\right)$

$$\begin{array}{c} t^\eta D^\eta \Phi(u)(t) = t^\eta D^\eta C(t)\left[u^0 + p(u, t^{\lambda_\beta} D^\beta u(t))\right] \\ + t^\eta D^{\eta-1} C(t)\left[u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u\left(t\right)\right)\right] \\ + t^\eta \int_0^t C(t-s) D^{\eta-1} f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right) ds \\ + t^\eta \int_0^t C(t-s) D^{\eta-1} I^\sigma g\left(s, u\left(s\right), s^{\lambda_\nu} D^\nu u\left(s\right)\right) ds,\ t \in [0,T]. \end{array} \tag{7}$$

As $t^{\eta_f} D^{\eta-1} f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right)$ is continuous with $0 < \eta_f < 1$ we see that

$$\begin{array}{l} \left\| \int_0^t C(t-s) D^{\eta-1} f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right) ds \right\| \\ \leq \tilde{M} \sup_{t\in[0,T]} \left\| t^{\eta_f} D^{\eta-1} f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right) \right\| \int_0^t s^{-\eta_f} ds \\ \leq \frac{\tilde{M} t^{1-\eta_f}}{1-\eta_f} \sup_{t\in[0,T]} \left\| t^{\eta_f} D^{\eta-1} f\left(s, u\left(s\right), s^{\lambda_\alpha} D^\alpha u\left(s\right)\right) \right\|. \end{array} \tag{8}$$

Case $\sigma \geq \eta - 1,\ \eta = \alpha, \beta, \gamma, \nu:$

If $\sigma \geq \eta - 1$, then it is clear (by Lemma 7, $D^{\eta-1}I^{\sigma}g = I^{\sigma-(\eta-1)}g$) that $\Phi, t^{\eta}D^{\eta}\Phi : C_{\eta}^{RL}([0,T]) \to C([0,T])$. In addition to that, by assumptions (H2) and (H3), for $u, v \in C_{\eta}^{RL}([0,T])$, we see from (5), that

$$\begin{gathered}
\|\Phi(u) - \Phi(v)\| \leq \tilde{M}A_p \left(\|u-v\|_C + \left\|t^{\lambda_\beta}D^{\beta}u - t^{\lambda_\beta}D^{\beta}v\right\|_C\right) \\
+\tilde{N}A_q \left(\|u-v\|_C + \left\|t^{\lambda_\gamma}D^{\gamma}u - t^{\lambda_\gamma}D^{\gamma}v\right\|_C\right) \\
+\tilde{N}A_f \int_0^t \left(\|u-v\| + \left\|s^{\lambda_\alpha}D^{\alpha}u - s^{\lambda_\alpha}D^{\alpha}v\right\|\right) ds \\
+\frac{\tilde{N}A_g}{\Gamma(\sigma+1)} \int_0^t s^{\sigma} \sup_{0\leq\tau\leq s} \left(\|u-v\| + \left\|\tau^{\lambda_\nu}D^{\nu}u - \tau^{\lambda_\nu}D^{\nu}v\right\|\right) ds.
\end{gathered}$$

That is,

$$\begin{gathered}
\|\Phi(u) - \Phi(v)\| \leq \left(\tilde{M}A_p + \tilde{N}A_q + \tilde{N}A_f T + \frac{\tilde{N}A_g T^{\sigma+1}}{\Gamma(\sigma+1)}\right) \|u-v\|_C \\
+\tilde{M}A_p \left\|t^{\lambda_\beta}D^{\beta}u - t^{\lambda_\beta}D^{\beta}v\right\|_C + \tilde{N}A_q \left\|t^{\lambda_\gamma}D^{\gamma}u - t^{\lambda_\gamma}D^{\gamma}v\right\|_C \\
+\tilde{N}A_f T \left\|t^{\lambda_\alpha}D^{\alpha}u - t^{\lambda_\alpha}D^{\alpha}v\right\|_C + \frac{\tilde{N}A_g T^{\sigma+1}}{\Gamma(\sigma+1)} \left\|t^{\lambda_\nu}D^{\nu}u - t^{\lambda_\nu}D^{\nu}v\right\|_C .
\end{gathered} \tag{9}$$

In short, (9) can be written as

$$\|\Phi(u) - \Phi(v)\| \leq C_1 \left(\|u-v\|_C + \sum\nolimits_{\eta} \left\|t^{\lambda_\eta}D^{\eta}u - t^{\lambda_\eta}D^{\eta}v\right\|_C\right), \tag{10}$$

where C_1 is the max of all the coefficients in the right hand side of (9) and η takes the values α, β, γ and ν. Moreover, by Lemma 7 for $\sigma \geq \eta - 1$, we have

$$\begin{aligned}
&\left\|\int_0^t C(t-s)D^{\eta-1}I^{\sigma}\left[g\left(s,u\left(s\right),w\left(s\right)\right) - g\left(s,v\left(s\right),z\left(s\right)\right)\right] ds\right\| \\
&= \left\|\int_0^t C(t-s)I^{\sigma-\eta+1}\left[g\left(s,u\left(s\right),w\left(s\right)\right) - g\left(s,v\left(s\right),z\left(s\right)\right)\right] ds\right\| \\
&\leq \frac{\tilde{M}A_g T^{\sigma-\eta+1}}{\Gamma(\sigma-\eta+2)} \int_0^t \sup_{0\leq\tau\leq s} \left(\|u-v\| + \|w-z\|\right) ds.
\end{aligned} \tag{11}$$

This fact, together with (7) and (8), implies

$$\begin{gathered}
\|t^{\eta}D^{\eta}\Phi(u) - t^{\eta}D^{\eta}\Phi(v)\| \leq RA_p \left(\|u-v\|_C + \left\|t^{\lambda_\beta}D^{\beta}u - t^{\lambda_\beta}D^{\beta}v\right\|_C\right) \\
+\tilde{R}A_q \left(\|u-v\|_C + \left\|t^{\lambda_\gamma}D^{\gamma}u - t^{\lambda_\gamma}D^{\gamma}v\right\|_C\right) \\
+\frac{\tilde{M}\tilde{A}_{\eta f}T^{\eta+1-\eta_f}}{1-\eta_f} \left(\|u-v\| + \left\|t^{\lambda_\alpha}D^{\alpha}u - t^{\lambda_\alpha}D^{\alpha}v\right\|_C\right) ds \\
+\frac{\tilde{M}A_g T^{\sigma+1}}{\Gamma(\sigma-\eta+2)} \int_0^t \sup_{0\leq\tau\leq s} \left(\|u-v\| + \left\|\tau^{\lambda_\nu}D^{\nu}u - \tau^{\lambda_\nu}D^{\nu}v\right\|\right) ds.
\end{gathered}$$

Here R is a bound for $t^{\eta}D^{\eta}C(t)$ on I and $\tilde{R}$ is a bound for $t^{\eta}D^{\eta-1}C(t)$ on I.

Therefore,

$$
\begin{aligned}
&\|t^\eta D^\eta \Phi(u) - t^\eta D^\eta \Phi(v)\| \\
&\leq \left(RA_p + \tilde{R}A_q + \tfrac{\tilde{M}\tilde{A}_{\eta f} T^{\eta+1-\eta_f}}{1-\eta_f} + \tfrac{\tilde{M}A_g T^{\sigma+2}}{\Gamma(\sigma-\eta+2)} \right) \|u-v\|_C \\
&+RA_p \left\| t^{\lambda_\beta} D^\beta u - t^{\lambda_\beta} D^\beta v \right\|_C + \tilde{R}A_q \left\| t^{\lambda_\gamma} D^\gamma u - t^{\lambda_\gamma} D^\gamma v \right\|_C \\
&+\tfrac{\tilde{M}\tilde{A}_{\eta f} T^{\eta+1-\eta_f}}{1-\eta_f} \left\| t^{\lambda_\alpha} D^\alpha u - t^{\lambda_\alpha} D^\alpha v \right\|_C + \tfrac{\tilde{M}A_g T^{\sigma+2}}{\Gamma(\sigma-\eta+2)} \left\| t^{\lambda_\nu} D^\nu u - t^{\lambda_\nu} D^\nu v \right\|_C
\end{aligned} \tag{12}
$$

or

$$
\|t^\eta D^\eta \Phi(u) - t^\eta D^\eta \Phi(v)\| \leq C_2 \left(\|u-v\|_C + \sum\nolimits_\eta \left\| t^{\lambda_\eta} D^\eta u - t^{\lambda_\eta} D^\eta v \right\|_C \right), \tag{13}
$$

where C_2 is the max of all the coefficients in the right hand side of (12). Selecting the different parameters in the coefficients C_i, $i = 1, 2$ in the relations (10) and (13) sufficiently small, the contraction principle ensures the existence and uniqueness of a mild solution on I.

Case $0 < \sigma < \eta - 1$ for one of α, β, γ or ν

If $0 < \sigma < \eta - 1$ with η equals α, β, γ or ν, then $D^{\eta-1} I^\sigma g = D I^{2-\eta} I^\sigma g = D I^{2-\eta+\sigma} g = D^{\eta-1-\sigma} g$. Assuming that $D^{\eta-1-\sigma} g$ is Lipschitz in its second and third variables, the previous argument applies with the Lipschitz constant $\tilde{A}_{\eta g}$ of $D^{\eta-1-\sigma} g$ in (11).

□

Remark 1. *In fact, instead of* $u'(0) \in \tilde{E}_{\eta-1}$ *we only need that* $t^\eta D^{\eta-1} C(t) u'(0)$ *exist and be continuous on* I.

Remark 2. *In the statement of the theorem we have assumed* $1 < \alpha, \beta, \gamma, \nu < 2$ *and* $0 < \sigma < 1$. *This is just to fix ideas and treat the most interesting cases. Our results hold for* $0 < \alpha, \beta, \gamma, \nu < 2$ *and* $\sigma > 0$.

4 Classical solutions

In this section we prove the existence and uniqueness of classical solutions to problem (1). In case of some extra Lipschitz conditions on f and g and the initial data are a little bit more regular then mild solutions are more regular as well. This is what is proved next. First we need the extra assumptions

(H4) The functions g, $t^{\eta_f} D^{\eta-1} f$, $\eta = \alpha, \nu$ and $D^{\eta-1-\sigma} g$ (when $\sigma < \eta - 1$, $\eta = \alpha, \nu$) are Lipschitzian in their first variables on I, that is

$$
\|g(t,x,y) - g(s,x,y)\| \leq B_g |t-s|, \; t, s \in I, \; x, y \in X,
$$

$$\left\| t^{\eta_f} D^{\eta-1} f(t,x,y) - t^{\eta_f} D^{\eta-1} f(s,x,y) \right\| \leq B_f \left| t-s \right|, \ t,s \in I, \ x,y \in X$$
$$\left\| D^{\eta-1-\sigma} g(t,x,y) - D^{\eta-1-\sigma} g(s,x,y) \right\| \leq \tilde{B}_{\eta g} \left| t-s \right|, \ t,s \in I, \ x,y \in X$$

for some positive constants B_g, B_f and $\tilde{B}_{\eta g}$,

We will shorten the notation as follows

$$\begin{array}{l} \tilde{f}(t) := f\left(t, u\left(t\right), t^{\lambda_\alpha} D^\alpha u\left(t\right)\right) \\ \tilde{g}(t) := g\left(t, u\left(t\right), t^{\lambda_\nu} D^\nu u\left(t\right)\right) \end{array}$$

and write $u(0)$ instead of $u^0 + p\left(u, t^{\lambda_\beta} D^\beta u(t)\right)$ and $u'(0)$ instead of $u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u\left(t\right)\right)$.

Proposition 1. *Assume that* ***(H1)-(H4)*** *hold. Let* $u(0) \in \tilde{E}_\eta$, $u'(0) \in \tilde{E}_{\eta-1}$, $t^{\lambda_\eta} D^\eta C(t) u(0)$ *and* $t^{\lambda_\eta} D^\eta S(t) u'(0)$, $\eta = \alpha, \nu$ *are Lipschitz continuous on* I, $\lambda_\eta \geq \eta$. *Consider the mild solution* u *of (1). If* u *is Lipschitzian on* I, *then* $t^{\lambda_\eta} D^\eta u$, $\eta = \alpha, \nu$ *are Hölder continuous on* I.

Proof. From (4) and Corollary 1, we have

$$\begin{array}{c} t^{\lambda_\eta} D^\eta u\left(t\right) = t^{\lambda_\eta} D^\eta C(t) u(0) + t^{\lambda_\eta} D^\eta S(t) u'(0) \\ + t^{\lambda_\eta} \int_0^t C(s) D^{\eta-1} \tilde{f}\left(t-s\right) ds + t^{\lambda_\eta} C(t) \lim_{t\to 0^+} I^{2-\eta} \tilde{f}(t) \\ + t^{\lambda_\eta} \int_0^t C(s) D^{\eta-1} I^\sigma \tilde{g}\left(t-s\right) ds + t^{\lambda_\eta} C(t) \lim_{t\to 0^+} I^{2-\eta} I^\sigma \tilde{g}(t), \ t \in [0,T] \end{array}$$

or, by the continuity of f and $I^\sigma g$ and Lemma 6)

$$\begin{array}{c} t^{\lambda_\eta} D^\eta u\left(t\right) = t^{\lambda_\eta} D^\eta C(t) u(0) + t^{\lambda_\eta} D^{\eta-1} C(t) u'(0) \\ + t^{\lambda_\eta} \int_0^t C(s) D^{\eta-1} \tilde{f}\left(t-s\right) ds + t^{\lambda_\eta} \int_0^t C(s) D^{\eta-1} I^\sigma \tilde{g}\left(t-s\right) ds, \ t \in [0,T]. \end{array} \tag{14}$$

By the hypothesis in (H2) on f we see that $t^{\eta_f} D^{\eta-1} f$ exists and is continuous on $[0,T]$. Therefore the third term in the right hand side of (14) is well-defined (see (8)). As for the fourth term in the right hand side of (14), if $\sigma \geq \eta - 1$ then $D^{\eta-1} I^\sigma \tilde{g} = I^{\sigma-(\eta-1)} \tilde{g}$ and hence it is clearly well-defined. In case $\sigma < \eta - 1$ (for one of the values of η) we use the continuity of $D^{\eta-1-\sigma} \tilde{g}$ in **(H2)**.

Case $\sigma \geq \eta - 1$, $\eta = \alpha, \beta, \gamma, \nu$

For $t \in I$ and h such that $t + h \in I$, we can write

$$\begin{array}{c} \left(t+h\right)^{\lambda_\eta} D^\eta u\left(t+h\right) - t^{\lambda_\eta} D^\eta u\left(t\right) = \left(\left(t+h\right)^{\lambda_\eta} D^\eta C(t+h) - t^{\lambda_\eta} D^\eta C(t)\right) u(0) \\ + \left(\left(t+h\right)^{\lambda_\eta} D^{\eta-1} C(t+h) - t^{\lambda_\eta} D^{\eta-1} C(t)\right) u'(0) \\ + \left(t+h\right)^{\lambda_\eta} \int_0^{t+h} C(t+h-s) D^{\eta-1} \tilde{f}\left(s\right) ds - t^{\lambda_\eta} \int_0^t C(t-s) D^{\eta-1} \tilde{f}\left(s\right) ds \\ + \left(t+h\right)^{\lambda_\eta} \int_0^{t+h} C(t+h-s) I^{\sigma-(\eta-1)} \tilde{g}\left(s\right) ds - t^{\lambda_\eta} \int_0^t C(t-s) I^{\sigma-(\eta-1)} \tilde{g}\left(s\right) ds. \end{array}$$

Taking the norm of both sides we find

$$\begin{aligned}
&\left\|(t+h)^{\lambda_\eta} D^\eta u(t+h) - t^{\lambda_\eta} D^\eta u(t)\right\| \le Kh \\
&+\Big\|(t+h)^{\lambda_\eta} \textstyle\int_0^t C(t-s) D^{\eta-1}\tilde{f}(s+h)\, ds \\
&+(t+h)^{\lambda_\eta} \textstyle\int_0^h C(t+h-s) D^{\eta-1}\tilde{f}(s)\, ds - t^{\lambda_\eta} \int_0^t C(t-s) D^{\eta-1}\tilde{f}(s)\, ds \\
&+(t+h)^{\lambda_\eta} \textstyle\int_0^t C(t-s) I^{\sigma-(\eta-1)}\tilde{g}(s+h)\, ds \\
&+(t+h)^{\lambda_\eta} \textstyle\int_0^h C(t+h-s) I^{\sigma-(\eta-1)}\tilde{g}(s)\, ds - t^{\lambda_\eta} \int_0^t C(t-s) I^{\sigma-(\eta-1)}\tilde{g}(s)\, ds\Big\|
\end{aligned} \tag{15}$$

where K is a positive constant (sum of two Lipschitz constants). Then, by adding and subtracting some appropriate terms in (15) we get

$$\begin{aligned}
&\left\|(t+h)^{\lambda_\eta} D^\eta u(t+h) - t^{\lambda_\eta} D^\eta u(t)\right\| \le Kh \\
&+\left\|(t+h)^{\lambda_\eta} \textstyle\int_0^t C(t-s) D^{\eta-1}\tilde{f}(s+h)\, ds - (t+h)^{\lambda_\eta} \int_0^t C(t-s) D^{\eta-1}\tilde{f}(s)\, ds\right\| \\
&+\Big\|(t+h)^{\lambda_\eta} \textstyle\int_0^t C(t-s) D^{\eta-1}\tilde{f}(s)\, ds + (t+h)^{\lambda_\eta} \int_0^h C(t+h-s) D^{\eta-1}\tilde{f}(s)\, ds \\
&-t^{\lambda_\eta} \textstyle\int_0^t C(t-s) D^{\eta-1}\tilde{f}(s)\, ds\Big\| + \Big\|(t+h)^{\lambda_\eta} \int_0^t C(t-s) I^{\sigma-(\eta-1)}\tilde{g}(s+h)\, ds \\
&-(t+h)^{\lambda_\eta} \textstyle\int_0^t C(t-s) I^{\sigma-(\eta-1)}\tilde{g}(s)\, ds\Big\| + \Big\|(t+h)^{\lambda_\eta} \int_0^t C(t-s) I^{\sigma-(\eta-1)}\tilde{g}(s)\, ds \\
&+(t+h)^{\lambda_\eta} \textstyle\int_0^h C(t+h-s) I^{\sigma-(\eta-1)}\tilde{g}(s)\, ds - t^{\lambda_\eta} \int_0^t C(t-s) I^{\sigma-(\eta-1)}\tilde{g}(s)\, ds\Big\|.
\end{aligned} \tag{16}$$

The second term in the right hand side of (16) is treated in the following way

$$\begin{aligned}
&\left\|\textstyle\int_0^t C(t-s)\left[(s+h)^{-\eta_f}.(s+h)^{\eta_f} D^{\eta-1}\tilde{f}(s+h) - s^{-\eta_f}.s^{\eta_f} D^{\eta-1}\tilde{f}(s)\right] ds\right\| \\
&\le \tilde{M} \textstyle\int_0^t \left\{(s+h)^{-\eta_f}\left\|(s+h)^{\eta_f} D^{\eta-1}\tilde{f}(s+h) - s^{\eta_f} D^{\eta-1}\tilde{f}(s)\right\|\right\} ds \\
&+\tilde{M} \textstyle\int_0^t \left\{\left|(s+h)^{-\eta_f} - s^{-\eta_f}\right| \left\|s^{\eta_f} D^{\eta-1}\tilde{f}(s)\right\|\right\} ds.
\end{aligned}$$

By the Lipschitz continuity of $s^{\eta_f} D^{\eta-1}\tilde{f}(s)$ we may write

$$\begin{aligned}
&\left\|\textstyle\int_0^t C(t-s)\left[(s+h)^{-\eta_f}.(s+h)^{\eta_f} D^{\eta-1}\tilde{f}(s+h) - s^{-\eta_f}.s^{\eta_f} D^{\eta-1}\tilde{f}(s)\right] ds\right\| \\
&\le \tilde{M} \textstyle\int_0^t (s+h)^{-\eta_f} \{B_f h \\
&+\tilde{A}_{\eta f}\left(\|u(s+h) - u(s)\| + \left\|(s+h)^{\lambda_\alpha} D^\alpha u(s+h) - s^{\lambda_\alpha} D^\alpha u(s)\right\|\right)\Big\} ds \\
&+\tilde{M} \sup_{s\in[0T]} \left\|s^{\eta_f} D^{\eta-1}\tilde{f}(s)\right\| \textstyle\int_0^t \left|(s+h)^{-\eta_f} - s^{-\eta_f}\right| ds.
\end{aligned} \tag{17}$$

For the last term in (17) we have for $h \le t$ (the case $h > t$ will then be clear afterwards)

$$\begin{array}{l} \int_0^t \left|(s+h)^{-\eta_f} - s^{-\eta_f}\right| ds \le \int_0^h \frac{\left|(s+h)^{\eta_f} - s^{\eta_f}\right|}{(s+h)^{\eta_f}.s^{\eta_f}} ds + \int_h^t \frac{s^{\eta_f}\left[\left(1+\frac{h}{s}\right)^{\eta_f} - 1\right]}{(s+h)^{\eta_f}.s^{\eta_f}} ds \\ \le (1+2^{\eta_f}) \int_0^h \frac{h^{\eta_f}}{h^{\eta_f}.s^{\eta_f}} ds + h \int_h^t \frac{\eta_f}{s(s+h)^{\eta_f}} ds \le L\left(h^{1-\eta_f} + h\left|h^{-\eta_f} - t^{-\eta_f}\right|\right) \\ \le L h^{1-\eta_f} \end{array} \tag{18}$$

where we have used the inequality $(1+t)^\delta \le 1 + \delta t$ and L is a positive constant which may change from line to line. Taking into account (17) and (18) in (16) we find

$$\begin{array}{c} \left\|(t+h)^{\lambda_\eta} D^\eta u(t+h) - t^{\lambda_\eta} D^\eta u(t)\right\| \le l(h) + (t+h)^{\lambda_\eta} \tilde{M} \int_0^t (s+h)^{-\eta_f} \\ \times \left\{\tilde{A}_{\eta f}\left(\|u(s+h) - u(s)\| + \left\|(s+h)^{\lambda_\alpha} D^\alpha u(s+h) - s^{\lambda_\alpha} D^\alpha u(s)\right\|\right)\right\} ds \\ + \tilde{M}\left|(t+h)^{\lambda_\eta} - t^{\lambda_\eta}\right| \int_0^t s^{-\eta_f}\left\|s^{\eta_f} D^{\eta-1} \tilde{f}(s)\right\| ds + (t+h)^{\lambda_\eta} \tilde{M} \\ \times \int_0^h s^{-\eta_f}\left\|s^{\eta_f} D^{\eta-1} \tilde{f}(s)\right\| ds + (t+h)^{\lambda_\eta} \tilde{M} \int_0^t I^{\sigma-(\eta-1)} \|\tilde{g}(s+h) - \tilde{g}(s)\| ds \\ + (t+h)^{\lambda_\eta} \tilde{M} \int_0^h I^{\sigma-(\eta-1)} \|\tilde{g}(s)\| ds + \tilde{M}\left|(t+h)^{\lambda_\eta} - t^{\lambda_\eta}\right| \int_0^t I^{\sigma-(\eta-1)} \|\tilde{g}(s)\| ds \end{array}$$

where $l(h)$ is function of h which may vary at different occurrences.

As u and g are Lipschitz on I and $\lambda_\eta \ge \eta > 1$, we see that

$$\begin{array}{c} \left\|(t+h)^{\lambda_\eta} D^\eta u(t+h) - t^{\lambda_\eta} D^\eta u(t)\right\| \le l(h) + (t+h)^{\lambda_\eta} \tilde{M}\tilde{A}_{\eta f} \int_0^t (s+h)^{-\eta_f} \\ \times \left\|(s+h)^{\lambda_\alpha} D^\alpha u(s+h) - s^{\lambda_\alpha} D^\alpha u(s)\right\| ds + (t+h)^{\lambda_\eta} \tilde{M} \int_0^t I^{\sigma-(\eta-1)} \{B_g h \\ + A_g\left(\|u(s+h) - u(s)\| + \left\|(s+h)^{\lambda_\nu} D^\nu u(s+h) - s^{\lambda_\nu} D^\nu u(s)\right\|\right)\} ds \end{array}$$

or, for $\eta = \alpha, \nu$, we obtain

$$\begin{array}{c} \left\|(t+h)^{\lambda_\eta} D^\eta u(t+h) - t^{\lambda_\eta} D^\eta u(t)\right\| \\ \le l(h) + L \int_0^t s^{-\eta_f}\left\|(s+h)^{\lambda_\alpha} D^\alpha u(s+h) - s^{\lambda_\alpha} D^\alpha u(s)\right\| ds \\ + L \int_0^t \sup_{0\le\tau\le s}\left\|(\tau+h)^{\lambda_\nu} D^\nu u(\tau+h) - \tau^{\lambda_\nu} D^\nu u(\tau)\right\| ds \\ \le l(h) + L \int_0^t (s^{-\eta_f} + 1) \sup_{0\le\tau\le s}\left\{\left\|(\tau+h)^{\lambda_\alpha} D^\alpha u(\tau+h) - \tau^{\lambda_\alpha} D^\alpha u(\tau)\right\|\right. \\ \left. + \left\|(\tau+h)^{\lambda_\nu} D^\nu u(\tau+h) - \tau^{\lambda_\nu} D^\nu u(\tau)\right\|\right\} ds. \end{array}$$

Since $t^{-\eta_f} + 1$ is a summable function on I we may apply the Gronwall inequality (for summable functions) to reach the conclusion that $t^{\lambda_\eta} D^\eta u$, $\eta = \alpha, \nu$ are Hölder continuous on I.

Case $\sigma < \alpha - 1$ or $\sigma < \nu - 1$

In this case we recall the fact that $D^{\alpha-1-\sigma}g$ (if $\sigma < \alpha - 1$) or $D^{\nu-1-\sigma}g$ (if $\sigma < \nu - 1$) is Lipschitzian on I (in all their variables) by hypotheses (H2) and (H4). Using this property in (14) and proceeding as in the first case we obtain the Hölder continuity of $t^{\lambda_\eta}D^\eta u$, $\eta = \alpha, \nu$ on I.

□

Lemma 10. *Assume that* $\psi \in C^1([0, T])$ *and* $1 < \eta < 2$, *then*

$$\partial_h \left(t^{\lambda_\eta} D^\eta \psi(t)\right) = \tfrac{\psi(0)}{\Gamma(1-\eta)}\partial_h \left(t^{\lambda_\eta - \eta}\right) + (t+h)^{\lambda_\eta} D^{\eta-1}\partial_h \psi'(t)$$
$$+\tfrac{\psi'(0)(t+h)^{\lambda_\eta}}{\Gamma(2-\eta)}\partial_h \left(t^{1-\eta}\right) + \tfrac{(t+h)^{\lambda_\eta}}{h\Gamma(1-\eta)}\int_0^h \tfrac{\psi'(s)-\psi'(0)}{(t+h-s)^\eta} ds + \partial_h \left(t^{\lambda_\eta}\right) D^{\eta-1}\psi'(t)$$

where $\partial_h v(t) := [v(t+h) - v(t)]/h$, $t \in I$ *and* h *is such that* $t + h \in I$.

Proof. This result is proved by using Lemma 6 (or formula (2.122) in [20], see proof of Lemma 9), Definition 2 and performing some manipulations. Indeed, it is clear, from Lemma 6, that

$$\partial_h \left(t^{\lambda_\eta} D^\eta \psi(t)\right) = \partial_h \left(\frac{\psi(0)\, t^{\lambda_\eta - \eta}}{\Gamma(1-\eta)} + t^{\lambda_\eta} D^{\eta-1}\psi'(t)\right) \tag{19}$$

and for $t \in I$ and h such that $t + h \in I$, we have

$$\partial_h \left(t^{\lambda_\eta} D^{\eta-1}\psi'(t)\right) = \tfrac{1}{h}\left[(t+h)^{\lambda_\eta} D^{\eta-1}\psi'(t+h) - t^{\lambda_\eta} D^{\eta-1}\psi'(t)\right]$$
$$= \tfrac{(t+h)^{\lambda_\eta}}{h\Gamma(2-\eta)}\tfrac{d}{dt}\left[\int_0^{t+h} \tfrac{\psi'(s)ds}{(t+h-s)^{\eta-1}} - \int_0^t \tfrac{\psi'(s)ds}{(t-s)^{\eta-1}}\right]$$
$$+\tfrac{1}{h\Gamma(2-\eta)}\left[(t+h)^{\lambda_\eta} - t^{\lambda_\eta}\right]\tfrac{d}{dt}\int_0^t \tfrac{\psi'(s)ds}{(t-s)^{\eta-1}}$$

or

$$\partial_h \left(t^{\lambda_\eta} D^{\eta-1}\psi'(t)\right) = \tfrac{(t+h)^{\lambda_\eta}}{h\Gamma(2-\eta)}\tfrac{d}{dt}\left[\int_0^t \tfrac{\psi'(s+h)ds}{(t-s)^{\eta-1}} - \int_0^t \tfrac{\psi'(s)ds}{(t-s)^{\eta-1}} + \int_0^h \tfrac{\psi'(s)ds}{(t+h-s)^{\eta-1}}\right]$$
$$+\tfrac{\partial_h\left(t^{\lambda_\eta}\right)}{\Gamma(2-\eta)}\tfrac{d}{dt}\int_0^t \tfrac{\psi'(s)ds}{(t-s)^{\eta-1}}$$
$$= (t+h)^{\lambda_\eta}\tfrac{d}{dt}I^{2-\eta}\partial_h\psi'(t) + \tfrac{(t+h)^{\lambda_\eta}}{h\Gamma(2-\eta)}\tfrac{d}{dt}\int_0^h \tfrac{\psi'(s)ds}{(t+h-s)^{\eta-1}} + \tfrac{\partial_h\left(t^{\lambda_\eta}\right)}{\Gamma(2-\eta)}\tfrac{d}{dt}\int_0^t \tfrac{\psi'(s)ds}{(t-s)^{\eta-1}}.$$

Therefore

$$\begin{aligned}\partial_h \left(t^{\lambda_\eta} D^{\eta-1}\psi'(t)\right) &= (t+h)^{\lambda_\eta} D^{\eta-1}\partial_h\psi'(t) + \tfrac{(t+h)^{\lambda_\eta}}{h\Gamma(1-\eta)}\int_0^h \tfrac{\psi'(s)-\psi'(0)}{(t+h-s)^\eta}ds \\ &+\tfrac{\psi'(0)(t+h)^{\lambda_\eta}}{\Gamma(2-\eta)}\partial_h t^{1-\eta} + \partial_h\left(t^{\lambda_\eta}\right) D^{\eta-1}\psi'(t),\ t, t+h \in [0, T].\end{aligned} \tag{20}$$

A combination of (19) and (20) completes the proof. □

We are now ready to prove our result on the existence and uniqueness of a classical solution.

Theorem 2. *Suppose that **(H1)-(H4)** hold. Assume further that*

$u(0)\in D(A)$, $u'(0)\in E$, $t^{\lambda_\eta}D^\eta C(t)u(0)$ and $t^{\lambda_\eta}D^\eta S(t)u'(0)$, $\eta=\alpha,\eta$ are Lipschitz continuous on I. If $\lambda_\nu\geq\nu+1$, $\lambda_\eta\geq\eta$, $\eta=\alpha,\beta,\gamma$, g is a continuously differentiable function such that $D_-^{\nu-1}\left(t^{\lambda_\nu}\tilde{g}_3(t)\right)$ is continuous for $t\in[0,T]$ ($\tilde{g}_3$ is the partial derivative of g with respect to its third variable) and f is an E-valued function Lipschitz on I in its first variable, then the mild solution $u(t)$ of problem (1) is twice continuously differentiable and satisfies (1) on $[0,T]$ for some $T>0$.

Proof. Observe first that $u(0)\in D(A)$ implies that $\frac{d}{dt}C(t)u(0)$ is continuous on I. This in turn implies that u' is continuous and we have

$$\begin{array}{c} u'(t)=AS(t)u(0)+C(t)u'(0)+\int_0^t C(t-s)\tilde{f}(s)\,ds \\ +\frac{1}{\Gamma(\sigma)}\int_0^t C(t-s)\int_0^s(s-z)^{\sigma-1}\tilde{g}(z)\,dzds,\ t\in[0,T]. \end{array} \tag{21}$$

Hence u is Lipschitzian on I. Second, observe also that $u(0)\in\tilde{E}_\eta$, $u'(0)\in\tilde{E}_{\eta-1}$ and therefore Theorem 1 gives us a mild solution in $C_\eta^{RL}([0,T])$.

Let us consider the problem

$$\begin{array}{c} \varphi(t)=C(t)Au(0)+AS(t)u'(0)+\int_0^t AS(t-s)\tilde{f}(s)\,ds+\tilde{f}(t) \\ +\int_0^t C(t-s)I^\sigma\left\{\tilde{g}_1(s)+\tilde{g}_2(s)u'(s)\right. \\ \left.+\tilde{g}_3(s)\left[\lambda_\nu s^{\lambda_\nu-1}D^\nu u(s)+s^{\lambda_\nu}\left(\frac{u'(0)t^{-\nu}}{\Gamma(1-\nu)}+\frac{u(0)t^{-1-\nu}}{\Gamma(-\nu)}\right)\right]\right\}ds \\ +\int_0^t C(t-s)I^\sigma\left[D_-^{\nu-1}\left(s^{\lambda_\nu}\tilde{g}_3(s)\right)\varphi(s)\right]ds \end{array} \tag{22}$$

for $t\in[0,T]$, where $\tilde{g}_i$, $i=1,2,3$ denotes the partial derivative of the function $\tilde{g}$ with respect to its i-th variable. Clearly, the first two terms in the right hand side of (22) are well-defined and (22) admits a unique solution $\varphi\in C([0,T])$. We claim that $u''=\varphi$ on I. To this end we will show that $\lim_{h\to0}\|\partial_h u'(t)-\varphi(t)\|=0$ where ∂_h and h are as in Lemma 10. The relations (21) and (22) imply that

$$\begin{array}{l} \partial_h u'(t)-\varphi(t)=\partial_h AS(t)u(0)-C(t)Au(0)+\partial_h C(t)u'(0)-AS(t)u'(0) \\ +\partial_h\int_0^t C(t-s)\tilde{f}(s)\,ds-\int_0^t AS(t-s)\tilde{f}(s)\,ds-\tilde{f}(t) \\ +\partial_h\int_0^t C(t-s)I^\sigma\tilde{g}(s)\,ds-\int_0^t C(t-s)I^\sigma\left\{\tilde{g}_1(s)+\tilde{g}_2(s)u'(s)\right. \\ \left.+\tilde{g}_3(s)\left[\lambda_\nu s^{\lambda_\nu-1}D^\nu u(s)+s^{\lambda_\nu}\left(\frac{u'(0)t^{-\nu}}{\Gamma(1-\nu)}+\frac{u(0)t^{-1-\nu}}{\Gamma(-\nu)}\right)\right]\right\}ds \\ -\int_0^t C(t-s)I^\sigma\left[D_-^{\nu-1}\left(s^{\lambda_\nu}\tilde{g}_3(s)\right)\varphi(s)\right]ds \end{array}$$

or by definition of ∂_h

$$\begin{aligned}
&\partial_h u'(t)-\varphi(t)=\partial_h AS(t)u(0)-C(t)Au(0)+\partial_h C(t)u'(0)-AS(t)u'(0)\\
&+\tfrac{1}{h}\left(\int_0^t C(t+h-s)\tilde{f}\,(s)\,ds-\int_0^t C(t-s)\tilde{f}\,(s)\,ds\right)\\
&+\tfrac{1}{h}\int_t^{t+h} C(t+h-s)\tilde{f}\,(s)\,ds-\int_0^t AS(t-s)\tilde{f}\,(s)\,ds-\tilde{f}\,(t)\\
&+\tfrac{1}{h}\int_0^t C(s)I^\sigma\tilde{g}\,(t+h-s)\,ds-\tfrac{1}{h}\int_0^t C(s)I^\sigma\tilde{g}\,(t-s)\,ds\\
&+\tfrac{1}{h}\int_t^{t+h} C(s)I^\sigma\tilde{g}\,(t+h-s)\,ds-\int_0^t C(t-s)I^\sigma\left\{\tilde{g}_1\,(s)+\tilde{g}_2\,(s)\,u'(s)\right.\\
&\left.+\tilde{g}_3\,(s)\left[\lambda_\nu s^{\lambda_\nu-1}D^\nu u\,(s)+s^{\lambda_\nu}\left(\tfrac{u'(0)t^{-\nu}}{\Gamma(1-\nu)}+\tfrac{u(0)t^{-1-\nu}}{\Gamma(-\nu)}\right)\right]\right\}ds\\
&-\int_0^t C(t-s)I^\sigma\left[D_-^{\nu-1}\left(s^{\lambda_\nu}\tilde{g}_3\,(s)\right)\varphi(s)\right]ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\partial_h u'(t)-\varphi(t)=\left[\partial_h AS(t)-C(t)A\right]u(0)+\left[\partial_h C(t)-AS(t)\right]u'(0)\\
&+\int_0^t\left[\partial_h C(t-s)-AS(t-s)\right]\tilde{f}\,(s)\,ds+\tfrac{1}{h}\int_t^{t+h} C(t+h-s)\left[\tilde{f}\,(s)-\tilde{f}\,(t)\right]ds\\
&+\left[\tfrac{S(h)-S(0)}{h}-C(0)\right]\tilde{f}\,(t)+\int_0^t C(s)\partial_h I^\sigma\tilde{g}\,(t-s)\,ds+\tfrac{1}{h}\int_t^{t+h} C(s)I^\sigma\tilde{g}\,(t+h-s)\,ds\\
&-\int_0^t C(t-s)I^\sigma\left\{\tilde{g}_1\,(s)+\tilde{g}_2\,(s)\,u'(s)\right.\\
&\left.+\tilde{g}_3\,(s)\left[\lambda_\nu s^{\lambda_\nu-1}D^\nu u\,(s)+s^{\lambda_\nu}\left(\tfrac{u'(0)s^{-\nu}}{\Gamma(1-\nu)}+\tfrac{u(0)s^{-1-\nu}}{\Gamma(-\nu)}\right)\right]\right\}ds\\
&-\int_0^t C(t-s)I^\sigma\left[D_-^{\nu-1}\left(s^{\lambda_\nu}\tilde{g}_3\,(s)\right)\varphi(s)\right]ds.
\end{aligned} \tag{23}$$

Since f is Lipschitz in all its variables and g is continuously differentiable in addition to

$$\frac{1}{h}\int_t^{t+h} C(s)I^\sigma\tilde{g}\,(t+h-s)\,ds\le Lh^\sigma$$

for some positive constant L (by definition of I^σ), it follows from (23) that

$$\begin{aligned}
&\|\partial_h u'(t)-\varphi(t)\|\le l(h)+\left\|\int_0^t C(s)\partial_h I^\sigma\tilde{g}\,(t-s)\,ds\right.\\
&-\int_0^t C(t-s)I^\sigma\left\{\tilde{g}_1\,(s)+\tilde{g}_2\,(s)\,u'(s)\right.\\
&\left.+\tilde{g}_3\,(s)\left[\lambda_\nu s^{\lambda_\nu-1}D^\nu u\,(s)+s^{\lambda_\nu}\left(\tfrac{u'(0)s^{-\nu}}{\Gamma(1-\nu)}+\tfrac{u(0)s^{-1-\nu}}{\Gamma(-\nu)}\right)\right]\right\}ds\\
&\left.-\int_0^t C(t-s)I^\sigma\left[D_-^{\nu-1}\left(s^{\lambda_\nu}\tilde{g}_3\,(s)\right)\varphi(s)\right]ds\right\|
\end{aligned} \tag{24}$$

where $l(h)$ is a generic function which satisfies $l(h)\to 0$ as $h\to 0$ and may differ from one place to another. Furthermore, as g is continuously differentiable we may write

$$\begin{aligned}&I^{\sigma}\tilde{g}\left(s+h\right)-I^{\sigma}\tilde{g}\left(s\right)=I^{\sigma}\left[\tilde{g}_{1}\left(s\right)h\right]+I^{\sigma}\left[\tilde{g}_{2}\left(s\right)\left(u\left(s+h\right)-u\left(s\right)\right)\right]\\&+I^{\sigma}\left[\tilde{g}_{3}\left(s\right)\left(\left(s+h\right)^{\lambda_{\nu}}D^{\nu}u\left(s+h\right)-s^{\lambda_{\nu}}D^{\nu}u\left(s\right)\right)\right]+\int_{0}^{h}\frac{\tilde{g}(\tau)d\tau}{\left(s+h-\tau\right)^{1-\sigma}}\\&+I^{\sigma}\left\|\left(h,u\left(s+h\right)-u\left(s\right),\left(s+h\right)^{\lambda_{\nu}}D^{\nu}u\left(s+h\right)-s^{\lambda_{\nu}}D^{\nu}u\left(s\right)\right)\right\|_{I\times X^{2}}\\&\times\left\|\omega\left(\tilde{g}\left(s\right),h\partial_{h}u\left(s\right),h\partial_{h}\left(s^{\lambda_{\nu}}D^{\nu}u\left(s\right)\right)\right)\right\|\end{aligned}\tag{25}$$

where $\left\|\omega\left(\tilde{g}\left(s\right),h\partial_{h}u\left(s\right),h\partial_{h}\left(s^{\lambda_{\nu}}D^{\nu}u\left(s\right)\right)\right)\right\|\to 0$ when

$$\begin{aligned}&\left\|\left(h,u\left(s+h\right)-u\left(s\right),\left(s+h\right)^{\lambda_{\nu}}D^{\nu}u\left(s+h\right)-s^{\lambda_{\nu}}D^{\nu}u\left(s\right)\right)\right\|_{I\times X^{2}}\\&=\left|h\right|+\left\|u\left(s+h\right)-u\left(s\right)\right\|+\left\|\left(s+h\right)^{\lambda_{\nu}}D^{\nu}u\left(s+h\right)-s^{\lambda_{\nu}}D^{\nu}u\left(s\right)\right\|\to 0.\end{aligned}$$

The relations (24) and (25) imply that

$$\begin{aligned}&\left\|\partial_{h}u'(t)-\varphi(t)\right\|\leq l(h)+\left\|\int_{0}^{t}C(t-s)I^{\sigma}\left\{\tilde{g}_{3}\left(s\right)\left[\partial_{h}\left(s^{\lambda_{\nu}}D^{\nu}u\left(s\right)\right)\right.\right.\right.\\&\left.\left.-\lambda_{\nu}s^{\lambda_{\nu}-1}D^{\nu}u\left(s\right)-s^{\lambda_{\nu}}\left(\frac{u'(0)s^{-\nu}}{\Gamma(1-\nu)}+\frac{u(0)s^{-1-\nu}}{\Gamma(-\nu)}\right)\right]\right\}ds\\&\left.-\int_{0}^{t}C(t-s)I^{\sigma}\left[D_{-}^{\nu-1}\left(s^{\lambda_{\nu}}\tilde{g}_{3}\left(s\right)\right)\varphi(s)\right]ds\right\|,\ t\in[0,T].\end{aligned}$$

Next, in virtue of Lemma 6, we have

$$\begin{aligned}&\left\|\partial_{h}u'(t)-\varphi(t)\right\|\leq l(h)+\tilde{M}\int_{0}^{t}\left\|I^{\sigma}\left\{\tilde{g}_{3}\left(s\right)\left[\frac{u'(0)(s+h)^{\lambda_{\nu}}}{\Gamma(2-\nu)}\partial_{h}\left(s^{1-\nu}\right)\right.\right.\right.\\&+\frac{u(0)\partial_{h}\left(s^{\lambda_{\nu}-\nu}\right)}{\Gamma(1-\nu)}+\left(s+h\right)^{\lambda_{\nu}}D^{\nu-1}\partial_{h}u'\left(s\right)+\frac{(s+h)^{\lambda_{\nu}}}{h\Gamma(1-\nu)}\int_{0}^{h}\frac{u'(\tau)-u'(0)}{(s+h-\tau)^{\nu}}d\tau-\frac{u(0)s^{-\nu}\partial_{h}\left(s^{\lambda_{\nu}}\right)}{\Gamma(1-\nu)}\\&\left.\left.\left.-s^{\lambda_{\nu}}\left(\frac{u'(0)s^{-\nu}}{\Gamma(1-\nu)}+\frac{u(0)s^{-1-\nu}}{\Gamma(-\nu)}\right)\right]\right\}-I^{\sigma}\left[D_{-}^{\nu-1}\left(s^{\lambda_{\nu}-1}\tilde{g}_{3}\left(s\right)\right)\varphi(s)\right]ds\right\|\end{aligned}$$

or

$$\begin{aligned}&\left\|\partial_{h}u'(t)-\varphi(t)\right\|\leq l(h)+\tilde{M}\int_{0}^{t}\left|\frac{(s+h)^{\lambda_{\nu}}}{\Gamma(2-\nu)}\partial_{h}\left(s^{1-\nu}\right)-\frac{s^{\lambda_{\nu}-\nu}}{\Gamma(1-\nu)}\right|\left\|u'(0)\right\|ds\\&+\tilde{M}\int_{0}^{t}\left|-\frac{u(0)s^{-\nu}\partial_{h}\left(s^{\lambda_{\nu}}\right)}{\Gamma(1-\nu)}+\frac{\partial_{h}s^{\lambda_{\nu}-\nu}}{\Gamma(1-\nu)}-\frac{s^{\lambda_{\nu}-1-\nu}}{\Gamma(-\nu)}\right|\left\|u(0)\right\|ds\\&+\tilde{M}\int_{0}^{t}\left\|I^{\sigma}\left\{\tilde{g}_{3}\left(s\right)\left[\left(s+h\right)^{\lambda_{\nu}}D^{\nu-1}\partial_{h}u'\left(s\right)-s^{\lambda_{\nu}}D^{\nu-1}\partial_{h}u'\left(s\right)\right.\right.\right.\\&\left.\left.+s^{\lambda_{\nu}}D^{\nu-1}\partial_{h}u'\left(s\right)+\frac{(s+h)^{\lambda_{\nu}}}{h\Gamma(1-\nu)}\int_{0}^{h}\frac{u'(\tau)-u'(0)}{(\tau+h-s)^{\nu}}d\tau\right]\right\}ds\\&\left.-\int_{0}^{t}C(t-s)I^{\sigma}\left[D_{-}^{\nu-1}\left(s^{\lambda_{\nu}}\tilde{g}_{3}\left(s\right)\right)\varphi(s)\right]\right\|ds.\end{aligned}$$

Observe that by continuity of u' we have

$$\lim_{h\to 0}\frac{1}{h}\int_{0}^{h}\frac{\left\|u'\left(\tau\right)-u'\left(0\right)\right\|}{\left(s+h-\tau\right)^{\nu}}d\sigma=0.$$

Therefore, this last property and an integration by parts (Lemma 5) yield

$$\|\partial_h u'(t)-\varphi(t)\| \le l(h)$$
$$+\tilde{M}\int_0^t I^\sigma \left|D_-^{\nu-1}\left(s^{\lambda_\nu}\tilde{g}_3(s)\right)\right| \sup_{0\le\tau\le s}\|\partial_h u'(\tau)-\varphi(\tau)\|\,ds.$$

By our hypotheses we derive

$$\|\partial_h u'(t)-\varphi(t)\| \le l(h)+L\int_0^t \sup_{0\le\tau\le s}\|\partial_h u'(\tau)-\varphi(\tau)\|\,ds,\ t\in[0,T]$$

for some positive constant L. By Gronwall inequality we deduce that

$$\lim_{h\to 0}\|\partial_h u'(t)-\varphi(t)\|=0.$$

This, with Proposition 2.4 in [26], implies that $u(t)$ is a classical solution. The proof is complete. □

Example: As an example we may consider the following problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,x)=u_{xx}(t,x)+F\left(t,x,u(t,x),t^{\lambda_\alpha}D^\alpha u(t,x)\right) \\ \qquad\qquad +I^\sigma G\left(t,x,u(t,x),t^{\lambda_\nu}D^\nu u(t,x)\right),\ t\in I=[0,T],\ x\in[a,b] \\ u(t,a)=u(t,b)=0,\ t\in I \\ u(0,x)=u^0(x)+\int_0^T P\left(u(s),s^{\lambda_\beta}D^\beta u(s)\right)(x)ds,\ x\in[a,b] \\ u'(0)=u^1(x)+\int_0^T Q\left(u(s),s^{\lambda_\gamma}D^\gamma u(s)\right)(x)ds,\ x\in[a,b] \end{cases} \tag{26}$$

in the space $X=L^2([0,\pi])$. This problem can be reformulated in the abstract setting (1). To this end we define the operator $Ay=y''$ with domain

$$D(A):=\{y\in H^2([0,\pi]):y(0)=y(\pi)=0\}.$$

The operator A has a discrete spectrum with $-n^2$, $n=1,2,...$ as eigenvalues and $z_n(s)=\sqrt{2/\pi}\sin(ns)$, $n=1,2,...$ as their corresponding normalized eigenvectors. So we may write

$$Ay=-\sum_{n=1}^\infty n^2(y,z_n)z_n,\ y\in D(A).$$

Since $-A$ is positive and self-adjoint in $L^2([0,\pi])$, the operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in\mathbf{R}$ which has the form

$$C(t)y=\sum_{n=1}^\infty \cos(nt)(y,z_n)z_n,\ y\in X.$$

The associated sine family is found to be

$$C(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}(y, z_n)z_n, \; y \in X.$$

One can also consider more general non-local conditions by allowing the Lebesgue measure ds to be of the form $d\mu(s)$ and $d\eta(s)$ for non-decreasing functions μ and η (or even more general: μ and η of bounded variation), that is

$$\begin{aligned} u(0,x) &= u^0(x) + \int_0^T P\left(u(s), s^{\lambda_\beta} D^\beta u(s)\right)(x) d\mu(s), \\ u_t(0,x) &= u^1(x) + \int_0^T Q\left(u(s), s^{\lambda_\gamma} D^\gamma u(s)\right)(x) d\eta(s). \end{aligned}$$

These (continuous) non-local conditions cover, of course, the discrete cases

$$\begin{aligned} u(0,x) &= u^0(x) + \sum_{i=1}^{n} \alpha_i u(t_i, x) + \sum_{i=1}^{m} \beta_i t_i^{\lambda_\beta} D^\beta u(t_i, x), \\ u_t(0,x) &= u^1(x) + \sum_{i=1}^{r} \gamma_i u(t_i, x) + \sum_{i=1}^{k} \lambda_i t_i^{\lambda_\gamma} D^\gamma u(t_i, x) \end{aligned}$$

which have been extensively studied by several authors in the integer order case.

For $u, v \in C([0,T]; X)$ and $x \in [0, \pi]$, defining the operators

$$\begin{aligned} p(u,v)(x) &:= \int_0^T P(u(s), v(s))(x) ds, \\ q(u,v)(x) &:= \int_0^T Q(u(s), v(s))(x) ds, \\ g(t,u,v)(x) &:= G(t,x,u(t,x),v(t,x)), \\ f(t,u,v)(x) &:= F(t,x,u(t,x),v(t,x)), \end{aligned}$$

allows us to write (26) abstractly as

$$\begin{cases} u''(t) = Au(t) + f\left(t, u(t), t^{\lambda_\alpha} D^\alpha u(t)\right) + I^\sigma g\left(t, u(t), t^{\lambda_\nu} D^\nu u(t)\right), \; t \in I \\ u(0) = u^0 + p\left(u, t^{\lambda_\beta} D^\beta u(t)\right), \\ u'(0) = u^1 + q\left(u, t^{\lambda_\gamma} D^\gamma u(t)\right). \end{cases}$$

Under appropriate conditions on F, G, P and Q which make (H2)-(H4) hold for the corresponding f, g, p and q, Theorem 2 ensures the existence of a mild solution to problem (26).

Some special cases of this problem may be found in many models of phenomena with hereditary properties.

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