A MEMORY TYPE BOUNDARY STABILIZATION OF A MIDLY DAMPED WAVE EQUATION

1) Mokhtar KIRANE and 2) Nasser-eddine TATAR

1) Université de Picarde Jules Verne,
Faculté de Mathématiques et d'Informatique, LAMFA UPRES A 6119
33, rue Saint Leu, 80039 Amiens Cedex 1, France
e-mail: Mokhtar.Kirane@u-picardie.fr

2) Université de Annaba
Institut de Mathématiques,
B.P 12, 23000 Annaba, Algeria

ABSTRACT: We consider the wave equation with a mild internal dissipation. It is proved that any small dissipation inside the domain is sufficient to uniformly stabilize the solution of this equation by means of a nonlinear feedback of memory type acting on a part of the boundary. This is established without any restriction on the space dimension and without geometrical conditions on the domain or its boundary.

Mathematics Subject Classification: 35L20, 35B40
Key words and Phrases: wave equations, internal dissipation, stabilization

1 INTRODUCTION

In this paper we are concerned with the uniform stability of the solution to the following mixed problem:

$$\begin{cases} u_{tt}(t,x) + \alpha u_t(t,x) = \triangle u(t,x) + g(t,x), \qquad t > 0,\, x \in \Omega, \\ \frac{\partial u}{\partial \nu}(t,x) + \int_0^t k(t-s,x) u_s(s,x) ds = h(t,x), \qquad t > 0, \qquad x \in \Gamma_0, \\ u(t,x) = 0, \qquad t > 0,\, x \in \Gamma_1, \\ u(0,x) = u_0(x),\, u_t(x) = u_1(x), \qquad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n (the n-dimensional Euclidean space, $n \geq 1$) with a boundary $\Gamma = \partial\Omega$ of class C^2; (Γ_0, Γ_1) is a partition of Γ such that $\text{int}(\Gamma_1) \neq \emptyset$; $\nu(x)$ denotes the outward normal vector to Γ at $x \in \Gamma$; $\frac{\partial}{\partial \nu}$ is the normal derivative on Γ; α is a positive number and g, h, u_0, u_1 are given functions; $\triangle$ is the Laplacian with respect to the spatial variable x and the subscript t denotes differentiation with respect to the variable t.

Problem (1.1) models, for instance, the evolution of sound in a compressible fluid with reflection of sound at the surface of the material. The boundary condition in (1.1) is general and covers a fairly large variety of different physical configurations. The physical meaning of this boundary condition as well as the following three particular cases

$$\frac{\partial p}{\partial \nu}(t,x) + \zeta(x) p_t(t,x) = 0, \qquad t > 0, \qquad x \in \Gamma, \tag{1.2}$$

$$\frac{\partial p}{\partial \nu}(t,x) + \beta(x) p_t(t,x) + \alpha(x) p(t,x) = 0, \qquad t > 0, \qquad x \in \Gamma, \tag{1.3}$$

$$\begin{array}{l} m(x)\delta_{tt}(t,x) + d(x)\delta_t(t,x) + K(x)\delta(t,x) = -p(t,x), \\ \frac{\partial p}{\partial \nu}(t,x) + \delta_{tt}(t,x) = 0, \qquad t > 0, \qquad x \in \Gamma, \end{array} \tag{1.4}$$

is discussed in [4]. See also references therein for questions of existence, uniqueness, regularity and asymptotic behavior. In [1] the exponential decay of the energy of problem (1.1) with the boundary condition (1.2) in the case $\zeta(x) \equiv C$ a positive constant, $g \equiv h \equiv 0$ and $\alpha < 0$ on the n-dimensional open unit cube was established. More delicate is the same problem with boundary condition (1.2), $g \equiv h \equiv 0$ without internal damping *i.e* $\alpha = 0$. This is discussed in Komornik and Zuazua [2] and Zuazua [6].

Inspired by the method developed in [2], we shall prove exponential decay for solutions of problem (1.1) $(h \equiv 0)$ using an appropriately chosen energy functional. In fact, we shall uniformly stabilize the solution of the wave equation by a nonlinear feedback of memory type acting on a part of the boundary provided the equation contains a mild damping (however small it is) in the interior of the domain.

Let R_+ denote the set of nonnegative real numbers and

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u\,|_{\Gamma_1} = 0\} \tag{1.5}$$

where $H^1(\Omega)$ is the usual Sobolev space.

By a real function $a(t,x) \in L^1_{loc}(R_+; L^\infty(\Gamma_0))$ of positive type we mean a function satisfying

$$\int_0^T \int_{\Gamma_0} v(t) \int_0^t a(t-s) v(s) ds\, d\sigma\, dt \geq 0 \tag{1.6}$$

for all $v \in C(R_+; H^1_{\Gamma_1}(\Omega))$ and for every $T > 0$. See [3] for more information on functions of positive type.

In [4], Propst and Prüss have reformulated problem (1.1) (with $\alpha = 0$) as an integral equation of variational type and then have used results and methods developed in the second author's monograph [5] to derive, among others the following theorem:

Theorem 1.1 *Suppose that Γ_0 and Γ_1 are closed in Γ. Let $u_0 \in H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega)$, $u_1 \in H^1_{\Gamma_1}(\Omega)$, $g \in W^{1,1}_{loc}(R_+; L^2(\Omega))$, $h \in W^{2,1}_{loc}(R_+; L^2(\Gamma_0))$ and $h \in C(R_+; H^{1/2}(\Gamma_0))$, $k \in BV_{loc}(R_+; C^1(\Gamma_0))$ of positive type, either $u_1 = 0$ on Γ_0 or k is locally absolutely continuous in t, uniformly with respect to $x \in \Gamma_0$ and k' (the derivative of k with respect to t) is in $BV_{loc}(R_+; L^\infty(\Gamma_0))$, then there is a unique solution $u \in C(R_+; H^2(\Omega)) \cap C^1(R_+; H^1_{\Gamma_1}(\Omega)) \cap C^2(R_+; L^2(\Omega))$ and $u(t,x)$ satisfies (1.1) for all $t \geq 0$ and almost all x.*

$W^{m,p}$ and C^m are the usual Sobolev space and the space of continuously differentiable functions up to order m respectively. BV is the space of functions of bounded variation.

2 Exponential decay

In this section we assume the existence of a regular strong solution to problem (1.1) in the sense of the preceding theorem with $h \equiv 0$.

Note that the Poincaré inequality holds in $H^1_{\Gamma_1}(\Omega)$ *i.e*

$$\exists \beta > 0, \;\; \|v\|_2^2 \leq \beta \|\nabla v\|_2^2, \quad \text{for all} \quad v \in H^1_{\Gamma_1}(\Omega). \tag{2.7}$$

Combined with the trace inequality the preceding inequality (2.7) yields

$$\exists \gamma > 0, \;\; \int_{\Gamma_0} v^2 d\sigma \leq \gamma \int_\Omega |\nabla v|^2 \, d\sigma, \quad \text{for all} \quad v \in H^1_{\Gamma_1}(\Omega). \tag{2.8}$$

We suppose that our boundary material is characterized by the function

$$k(t,x) = p(x) e^{-t}, \;\; t \geq 0, \qquad x \in \Gamma_0, \tag{2.9}$$

with $0 \leq p(x) \in C^1(\Gamma_0)$ and $\|p(x)\|_\infty = M$.
Let us introduce the energy functional

$$E(u;\, t) = \frac{1}{2} \int_\Omega \left(|u_t|^2 + |\nabla u|^2 \right) dx + \frac{1}{2} \int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right)^2 d\sigma. \tag{2.10}$$

Differentiating the energy functional (2.10) using (1.1) we obtain

$$\frac{d}{dt}E(u;\,t) = -\alpha \int_\Omega u_t^2 dx - \int_{\Gamma_0} p(x) \left(\int_0^t e^{-(t-s)} u_s(s) ds \right)^2 d\sigma + \int_\Omega u_t g dx. \tag{2.11}$$

Remark 2.1 *Note that if $g \equiv 0$, then it is readily seen that the energy is decreasing.*

Next, for $\varepsilon > 0$ we will define

$$E_\varepsilon(u;\,t) = E(u;\,t) + \varepsilon\varphi(u;\,t), \qquad t \geq 0, \tag{2.12}$$

where

$$\varphi(u;\,t) = \int_\Omega u_t u dx. \tag{2.13}$$

For the sake of brevity, we will write $E_\varepsilon(t)$ for $E_\varepsilon(u;\,t)$ and $\varphi(t)$ for $\varphi(u;\,t)$. Using the Poincaré inequality (2.7) we have

$$|\varphi(t)| \leq \frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\beta \int_\Omega |\nabla u|^2 dx \leq (1+\beta)E(t). \tag{2.14}$$

It then follows that

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon(1+\beta)E(t), \;\; t \geq 0. \tag{2.15}$$

We are now ready to prove our main theorem.

Theorem 2.2 *Assume that h and k are as above. Let $u_0 \in H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega)$, $u_1 \in H^1_{\Gamma_1}(\Omega)$ and $g \in W^{1,1}_{loc}(R_+; L^2(\Omega))$. If*

$$\int_0^t e^{\varepsilon\omega s} \left(\int_\Omega g^2 dx \right) ds$$

grows no faster than a polynomial as $t \to \infty$ for some ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{2\alpha}{5 + 4\alpha^2\beta}, \frac{2}{1+2M\gamma}, \frac{1}{1+\beta} \right\}$$

where β and γ are the constants in (2.7) and (2.8) and $1/2 < \omega < 1$, then there exists a positive constant C such that

$$E(t) \leq Ce^{-\varepsilon\omega t}, \qquad t \geq 0.$$

Proof: Differentiating the functional $E_\varepsilon(t)$ we find

$$\begin{aligned} E_\varepsilon^{'}(t) &= E^{'}(t)+\varepsilon\varphi^{'}(t) \\ &= -\alpha\int_\Omega u_t^2dx-\int_{\Gamma_0}p(x)\left(\int_0^t e^{-(t-s)}u_s(s)ds\right)^2 d\sigma \\ &\quad +\int_\Omega u_t g dx+\varepsilon\int_\Omega u_{tt}udx+\varepsilon\int_\Omega u_t^2dx. \end{aligned} \tag{2.16}$$

Using problem (1.1) we get

$$\begin{aligned} \int_\Omega u_{tt}udx &= -\alpha\int_\Omega u_t u dx+\int_\Omega(\triangle u)udx+\int_\Omega ugdx \\ &= -\alpha\int_\Omega u_t udx-\int_{\Gamma_0}p(x)u\int_0^t e^{-(t-s)}u_s(s)dsd\sigma \\ &\quad -\int_\Omega|\nabla u|^2dx+\int_\Omega ugdx. \end{aligned} \tag{2.17}$$

Making use of the Hölder inequality and the algebraic inequality

$$ab\le\lambda a^2+\frac{1}{4\lambda}b^2,\qquad a,\ b\in R,\ \lambda>0, \tag{2.18}$$

we have the following estimates

$$\int_\Omega u_t udx\le c_1\int_\Omega u_t^2dx+\frac{1}{4c_1}\int_\Omega u^2dx \tag{2.19}$$

$$\int_\Omega u_t gdx\le c_2\int_\Omega u_t^2dx+\frac{1}{4c_2}\int_\Omega g^2dx \tag{2.20}$$

$$\int_\Omega ugdx\le c_3\int_\Omega u^2dx+\frac{1}{4c_3}\int_\Omega g^2dx \tag{2.21}$$

$$\begin{aligned} &\int_{\Gamma_0}p(x)u\int_0^t e^{-(t-s)}u_s(s)dsd\sigma\le \\ &\qquad c_4M\int_{\Gamma_0}u^2dx+\tfrac{1}{4c_4}\int_{\Gamma_0}p(x)\left(\int_0^t e^{-(t-s)}u_s(s)ds\right)^2d\sigma \end{aligned} \tag{2.22}$$

Replacing the expression (2.17) into (2.16) and taking into account the estimates (2.19)-(2.22) we obtain

$$\begin{aligned} E_\varepsilon^{'}(t) &\le -2\varepsilon E(t)-(\alpha-2\varepsilon)\int_\Omega u_t^2dx+(\alpha c_1\varepsilon+c_2)\int_\Omega u_t^2dx \\ &\quad +\beta\left(\tfrac{\alpha\varepsilon}{4c_1}+c_3\varepsilon\right)\int_\Omega|\nabla u|^2dx+2Mc_4\gamma\varepsilon\int_\Omega|\nabla u|^2dx \\ &\quad -\left(1-\varepsilon-\tfrac{\varepsilon}{4c_4}\right)\int_{\Gamma_0}p(x)\left(\int_0^t e^{-(t-s)}u_s(s)ds\right)^2d\sigma \\ &\quad +\left(\tfrac{\varepsilon}{4c_3}+\tfrac{1}{4c_2}\right)\int_\Omega g^2dx. \end{aligned} \tag{2.23}$$

Note that we have used the inequalities (2.7) and (2.8) in (2.23). Let us choose $c_1=2\alpha\beta$, $c_2=\varepsilon$, $c_3=1/8\beta$ and $c_4=1/4M\gamma$, then (2.23) yields

$$\begin{aligned} E_\varepsilon^{'}(t) &\le -\varepsilon E(t)-\left\{\alpha-(\tfrac{5}{2}+2\alpha^2\beta)\varepsilon\right\}\int_\Omega u_t^2dx+\left(2\beta\varepsilon+\tfrac{1}{4\varepsilon}\right)\int_\Omega g^2dx \\ &\quad -\left\{1-(M\gamma+\tfrac{1}{2})\varepsilon\right\}\int_{\Gamma_0}p(x)^2\left(\int_0^t e^{-(t-s)}u_t(s)ds\right)^2d\sigma. \end{aligned} \tag{2.24}$$

Now we choose $\varepsilon > 0$ so that $\alpha - (\frac{5}{2} + 2\alpha^2\beta)\varepsilon \geq 0$ and $1 - (M\gamma + \frac{1}{2})\varepsilon \geq 0$, i.e.,

$$\varepsilon \leq \min\left\{\frac{2\alpha}{5 + 4\alpha^2\beta}, \frac{2}{1 + 2M\gamma}\right\}. \tag{2.25}$$

Hence,

$$E_\varepsilon^{'}(t) \leq -\varepsilon E(t) + K(\varepsilon, \beta)\int_\Omega g^2 dx. \tag{2.26}$$

It follows from (2.15) that

$$(1 - (1+\beta)\varepsilon)E(t) \leq E_\varepsilon(t) \leq (1 + (1+\beta)\varepsilon)E(t), \;\; t \geq 0. \tag{2.27}$$

If moreover $\varepsilon > 0$ satisfies $\varepsilon < 1/(1+\beta)$, let a be any real number such that $0 < a \leq 1 - (1+\beta)\varepsilon$, then

$$aE(t) \leq E_\varepsilon(t) \leq (2 - a)E(t), \;\; t \geq 0. \tag{2.28}$$

Using (2.28) in (2.16) we deduce

$$E_\varepsilon^{'}(t) \leq -\frac{\varepsilon}{2-a}E_\varepsilon(t) + K(\varepsilon, \beta)\int_\Omega g^2 dx. \tag{2.29}$$

Consequently,

$$E_\varepsilon(t) \leq \left\{E_\varepsilon(0) + K(\varepsilon, \beta)\int_0^t e^{\varepsilon\omega s}\left(\int_\Omega g^2 dx\right) ds\right\} e^{-\varepsilon\omega t}, \qquad t \geq 0, \tag{2.30}$$

where $\omega = 1/(2-a)$. Once again in view of (2.28) we infer from (2.30) that

$$E(t) \leq \left\{\frac{2-a}{a}E(0) + \frac{K(\varepsilon, \beta)}{a}\int_0^t e^{\varepsilon\omega s}\left(\int_\Omega g^2 dx\right) ds\right\} e^{-\varepsilon\omega t}, \;\; t \geq 0. \tag{2.31}$$

The proof is now complete.

Remark 2.2 *It is clear from the proof that α may depend on the spatial variable x.*

Remark 2.3 *If the mild damping is in the boundary instead of the equation, i.e,*

$$\begin{cases} u_{tt}(t,x) = \triangle u(t,x) + g(t,x), \;\; t > 0, \; x \in \Omega, \\ \frac{\partial u}{\partial \nu}(t,x) + \alpha u_t(t,x) + \int_0^t k(t-s,x)u_t(s,x)ds = h(t,x), \;\; t > 0, \; x \in \Gamma_0, \\ u(t,x) = 0, \;\; t > 0, \; x \in \Gamma_1, \\ u(0,x) = u_0(x), \; u_t(x) = u_1(x), \; x \in \Omega, \end{cases}$$

then considering the energy functional (2.10) we proceed as in [2] with $\alpha(x) = m(x).\nu(x)$ *where* $m(x) = x - x^0$, $x^0 \in R^n$, *and*

$$\Gamma_0 = \{x \in \Gamma : m(x).\nu(x) > 0\},$$
$$\Gamma_1 = \{x \in \Gamma : m(x).\nu(x) \leq 0\}.$$

The appropriate perturbed energy functional is

$$E_\varepsilon(u;\, t) = E(u;\, t) + \varepsilon \int_\Omega u_t \{(n-1)u + 2(m(x).\nabla u)\}\, dx, \ \ t \geq 0.$$

In this case we do not impose to the function h *(and* g*) to vanish identically, we are restricted however to the space dimension condition* $n \leq 3$ *when* $cl(\Gamma_0) \cap cl(\Gamma_1) \neq \emptyset$ *because of the limited validity of Grisvard's inequality (see [2] and [6]).*

References

[1] A. B. Aliev and A. Kh. Khanmamedov, *Energy estimates for solutions of the mixed problem for linear second-order hyperbolic equations*, Math. Notes, Vol. 59, No 4 (1996), 345-149.

[2] V. Komornik and E. Zuazua, *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures et appl., 69 (1990), 33-54.

[3] J. A. Nohel and D. F. Shea, *Frequency domain methods for Volterra equations*, Adv. Math., 22 (1976), 278-304.

[4] G. Propst and J. Prüss, *On wave equations with boundary dissipation of memory type*, J. Integral eq. Appl., Vol. 8, No 1 (1996), 99-123.

[5] J. Prüss, Evolutionary integral equations and applications, Birkhäuser Verlag, Basel, 1993.

[6] E. Zuazua, *Uniform stabilization of the wave equation by nonlinear boundary feedback*, SIAM J. Control Optim., Vol 28, No 2 (1990), 466-477.