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METHODS OF EXTENDING LOWER ORDER PROBLEMS TO HIGHER ORDER PROBLEMS IN THE CONTEXT OF SMALLEST EIGENVALUE COMPARISONS

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Abstract. The theory of u_0-positive operators with respect to a cone in a Banach space is applied to the linear differential equations $u^{(4)} + \lambda_1 p(x)u = 0$ and $u^{(4)} + \lambda_2 q(x)u = 0$, $0 \le x \le 1$, with each satisfying the boundary conditions $u(0) = u'(r) = u''(r) = u'''(1) = 0$, $0 < r < 1$. The existence of smallest positive eigenvalues is established, and a comparison theorem for smallest positive eigenvalues is obtained. These results are then extended to the nth order problem using two different methods. One method involves finding sign conditions for the Green's function for $-u^{(n)} = 0$ satisfying the higher order boundary conditions, and the other involves making a substitution that allows us to work with a variation of the fourth order problem.

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1. Introduction

In this paper, we will consider the eigenvalue problems

$$u^{(4)} + \lambda_1 p(x)u = 0, \tag{1.1}$$

$$u^{(4)} + \lambda_2 q(x)u = 0, \tag{1.2}$$

satisfying the boundary conditions

$$u(0) = u'(r) = u''(r) = u'''(1) = 0, \tag{1.3}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

The focus of this paper will be on comparing the smallest eigenvalues for these eigenvalue problems. First, using the theory of u_0-positive operators with respect to a cone in a Banach space, we establish the existence of smallest eigenvalues for (1.1),(1.3), and (1.2),(1.3), and then compare these smallest eigenvalues after assuming a relationship between $p(x)$ and $q(x)$. We then extend these results to the nth order case using two different methods. First, we establish the sign properties of the Green's function for the nth order problem, and by using these properties, we are able to again establish the existence of smallest eigenvalues and then derive the comparison results. We then use a substitution method so that we can work with fourth order eigenvalue problems that have the same eigenvalues as the nth order problem. Comparison results are then obtained.

The technique for the comparison of these eigenvalues involve the application of sign properties of the Green's function, followed by the application of u_0-positive operators with respect to a cone in a Banach space. These applications are presented in books by Krasnoselskii [19] and by Krein and Rutman [18].

Several authors have before applied these techniques in comparing eigenvalues for different boundary problems than the ones seen here. Previous work has been devoted to boundary value problems for ordinary differential equations involving conjugate, Lidstone, and right focal conditions. For example, Eloe and Henderson have studied smallest eigenvalue comparisons for a class of two-point boundary value problems [4] and for a class of multipoint boundary value problems [5]. Karna has also studied smallest eigenvalue comparisons for m-point boundary value problems [14] and three-point boundary value problems [15]. In addition, comparison results have been obtained for difference equations [9] and for boundary value problems on time scales [1, 3, 12, 13, 20]. For additional work on this field, see [2, 6, 7, 8, 10, 16, 22, 23].

2. Preliminary Definitions and Theorems

Definition 2.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and
(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.2. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 2.1. Krasnosel'skii [19] showed that every solid cone is reproducing.

Definition 2.3. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition 2.4. A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $0 \neq u_0 \in \mathcal{P}$ such that for each $0 \neq u \in \mathcal{P}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

The following three results are fundamental to our comparison results and are attributed to Krasnosel'skii [19]. The proof of Lemma 2.1 is provided, the proof of

Theorem 2.1 can be found in Krasnosel'skii's book [19], and the proof of Theorem 2.2 is provided by Keener and Travis [17] as an extension of Krasonel'skii's results.

Lemma 2.1. *Let $\mathcal{B}$ be a Banach space over the reals, and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P}\backslash\{0\} \to \mathcal{P}^\circ$, then M is u_0-positive with respect to $\mathcal{P}$.*

Proof. Choose any $u_0 \in \mathcal{P}\backslash\{0\}$, and let $u \in \mathcal{P}\backslash\{0\}$. So $Mu \in \Omega \subset \mathcal{P}^\circ$. Choose $k_1 > 0$ sufficiently small and k_2 sufficiently large so that $Mu - k_1u_0 \in \mathcal{P}^\circ$ and $u_0 - \frac{1}{k_2}Mu \in \mathcal{P}^\circ$. So $k_1u_0 \leq Mu$ with respect to $\mathcal{P}$ and $Mu \leq k_2u_0$ with respect to $\mathcal{P}$. Thus $k_1u_0 \leq Mu \leq k_2u_0$ with respect to $\mathcal{P}$ and so M is u_0-positive with respect to P. $\square$

Theorem 2.1. *Let $\mathcal{B}$ be a real Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.2. *Let $\mathcal{B}$ be a real Banach space and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If $M \leq N$, $Mu_1 \geq \lambda_1u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Nu_2 \leq \lambda_2u_2$ for some $u_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Futhermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

3. The Fourth Order Problem

In this section, we consider the fourth order eigenvalue problems

$$u^{(4)} + \lambda_1 p(x)u = 0, \tag{3.1}$$

$$u^{(4)} + \lambda_2 q(x)u = 0, \tag{3.2}$$

satisfying the boundary conditions

$$u(0) = u'(r) = u''(r) = u'''(1) = 0, \tag{3.3}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

We derive comparison results for these fourth order eigenvalue problems by applying the theorems previously mentioned. To do this, we will define integral operators whose kernel is the Green's function for $-u^{(4)} = 0$ satisfying (3.3).

This Green's function is given by

$$G(x,s) = \begin{cases} \frac{s^3}{6}, & s \le r, s \le x, \\ \frac{(x-r)^3+r^3}{6}, & s > r, s > x, \\ \frac{(x-s)^3+s^3}{6}, & s \le r, s > x, \\ \frac{r^3+(s-x)^3+(x-r)^3}{6}, & s > r, s \le x. \end{cases}$$

So $u(x)$ solves (3.1),(3.3) if and only if $u(x) = \lambda_1 \int_0^1 G(x,s)p(s)u(s)ds$, and $u(x)$ solves (3.2),(3.3) if and only if $u(x) = \lambda_2 \int_0^1 G(x,s)q(s)u(s)ds$. Note $G(x,s) \ge 0$ on $[0,1] \times [0,1]$, $G(x,s) > 0$ on $(0,1] \times (0,1]$, and $\frac{\partial}{\partial x} G(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 2.1 and 2.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^1[0,1] \mid u(0) = 0\}$$

with the norm

$$||u|| = \sup_{0 \le x \le 1} |u'(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(x) \ge 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \le x \le 1$,

$$\begin{aligned} |u(x)| = |u(x) - u(0)| &= \left| \int_0^x u'(s)ds \right| \\ &\le ||u||x \\ &\le ||u||, \end{aligned}$$

and so $\sup\limits_{0 \le x \le 1} |u(x)| \le ||u||$.

Lemma 3.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0,1] \text{ and } u'(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. Choose $u \in \Omega$ and define $B_\epsilon(u) = \{v \in B \mid ||u - v|| < \epsilon\}$ for $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $u'(0) - \epsilon_0 > 0$ and $u(x) - \epsilon_0 > 0$ for $0 < x \le 1$. So for $v \in B_{\epsilon_0}(u)$, $\sup\limits_{0 \le x \le 1} |v'(x) - u'(x)| < \epsilon_0$. So $v'(0) > u'(0) - \epsilon_0 > 0$. Also, $|v(x) - u(x)| \le ||v - u|| < \epsilon_0$, and so $v(x) > 0$ on $(0,1]$. So $v \in \Omega$ and hence $B_{\epsilon_0}(u) \subset \Omega \subset \mathcal{P}$ and $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(x) = \int_0^1 G(x,s)p(s)u(s)ds, \ 0 \le x \le 1,$$

and

$$Nu(x) = \int_0^1 G(x,s)q(s)u(s)ds, \ 0 \le x \le 1.$$

A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 3.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. We show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(x) \ge 0$. Then since $G(x,s) \ge 0$ on $[0,1] \times [0,1]$ and $p(x) \ge 0$ on $[0,1]$,

$$Mu(x) = \int_0^1 G(x,s)p(s)u(s)ds \ge 0,$$

for $0 \le x \le 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0,1]$ such that $u(x) > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $G(x,s) > 0$ on $(0,1] \times (0,1]$,

$$\begin{aligned} Mu(x) &= \int_0^1 G(x,s)p(s)u(s)ds \\ &\ge \int_\alpha^\beta G(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \le 1$. Also, since $\frac{\partial}{\partial x}G(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mu)'(0) &= \int_0^1 \frac{\partial}{\partial x}G(0,s)p(s)u(s)ds \\ &\ge \int_\alpha^\beta \frac{\partial}{\partial x}G(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $Mu \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 2.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. □

Remark 3.1. Notice that

$$\Lambda u = Mu = \int_0^1 G(x,s)p(s)u(s)ds,$$

if and only if

$$u(x) = \frac{1}{\Lambda} \int_0^1 G(x,s)p(s)u(s)ds,$$

if and only if

$$-u^{(4)}(x) = \frac{1}{\Lambda} p(x)u(x),\ 0 \le x \le 1,$$

with

$$u(0) = u'(r) = u''(r) = u'''(1) = 0.$$

So the eigenvalues of (3.1),(3.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (3.2),(3.3) are reciprocals of N, and conversely.

Theorem 3.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. □

Theorem 3.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \le q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.1 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \le q(x)$ on $[0,1]$. So for any $u \in \mathcal{P}$ and $x \in [0,1]$,

$$(Nu - Mu)(x) = \int_0^1 G(x,s)(q(s) - p(s))u(s)ds \ge 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \le N$ with respect to $\mathcal{P}$. Then by Theorem 2.2, $\Lambda_1 \le \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \le Nu_1$, implying $Nu_1 \ge (\Lambda_1 + \epsilon)u_1$. Since $N \le N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \epsilon \le \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

By Remark 3.1, the following theorem is an immediate consequence of Theorems 3.1 and 3.2.

Theorem 3.3. *Assume the hypotheses of Theorem 3.2. Then there exists smallest positive eigenvalues λ_1 and λ_2 of (3.1),(3.3) and (3.2),(3.3), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \leq x \leq 1$.*

4. Extending the Fourth Order Problem Using Sign Properties of the Green's Function

Let $n \in \mathbb{N}$, $n \geq 5$. In this section, we will consider the eigenvalue problems

$$u^{(n)} + \lambda_1 p(x)u = 0, \tag{4.1}$$

$$u^{(n)} + \lambda_2 q(x)u = 0, \tag{4.2}$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-4)}(0) = u^{(n-3)}(r) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0, \tag{4.3}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanish identically on any compact subinterval of $[0, 1]$.

Here we will use methods similar to the methods used previously to derive comparison theorems for these nth order eigenvalue problems. We will do this by finding the the sign properties of the Green's function, which we will call $G_n(x, s)$, for $-u^{(n)} = 0$ satisfying (4.3). This Green's function, as a function of x, is $C^{n-4}[0, 1]$, and $\frac{\partial^{(n-4)}}{\partial x^{(n-4)}} G_n(x, s) = G(x, s)$, where $G(x, s)$ is as defined earlier.

Now $u(x)$ solves (4.1),(4.3) if and only if $u(x) = \lambda_1 \int_0^1 G_n(x, s)p(s)u(s)ds$, and $u(x)$ solves (4.2),(4.3) if and only if $u(x) = \lambda_2 \int_0^1 G_n(x, s)q(s)u(s)ds$.

Since $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x, s) = G(x, s)$, then $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x, s) \geq 0$ on $[0, 1] \times [0, 1]$ and $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x, s) > 0$ on $(0, 1] \times (0, 1]$. Also, since $\frac{\partial^{n-3}}{\partial x^{n-3}} G_n(x, s) = \frac{\partial}{\partial x} G(x, s)$, then $\frac{\partial^{n-3}}{\partial x^{n-3}} G_n(x, s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 2.1 and 2.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^{(n-3)}[0, 1] \mid u(0) = u'(0) = \cdots = u^{(n-4)}(0) = 0\}$$

with the norm

$$||u|| = \sup_{0 \leq x \leq 1} |u^{(n-3)}(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u^{(n-4)}(x) \geq 0 \text{ on } [0, 1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \le x \le 1$,

$$\begin{aligned} |u^{(n-4)}(x)| = |u^{(n-4)}(x) - u^{(n-4)}(0)| &= \left| \int_0^x u^{(n-3)}(s)ds \right| \\ &\le ||u||x \\ &\le ||u||, \end{aligned}$$

and so $\sup_{0 \le x \le 1} |u^{(n-4)}(x)| \le ||u||$.

Lemma 4.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u^{(n-4)}(x) > 0 \text{ on } (0,1] \text{ and } u^{(n-3)}(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. Choose $u \in \Omega$ and define $B_\epsilon(u) = \{v \in B \mid ||u - v|| < \epsilon\}$ for $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $u^{(n-3)}(0) - \epsilon_0 > 0$ and $u^{(n-4)}(x) - \epsilon_0 > 0$ for $0 < x \le 1$. So for $v \in B_{\epsilon_0}(u)$, $\sup_{0 \le x \le 1} |v^{(n-3)}(x) - u^{(n-3)}(x)| < \epsilon_0$. So $v^{(n-3)}(0) > u^{(n-3)}(0) - \epsilon_0 > 0$. Also, $|v^{(n-4)}(x) - u^{(n-4)}(x)| \le ||v - u|| < \epsilon_0$, and so $v^{(n-4)}(x) > 0$ on $(0, 1]$. So $v \in \Omega$ and hence $B_{\epsilon_0}(u) \subset \Omega \subset \mathcal{P}$, and $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators M and N by

$$Mu(x) = \int_0^1 G_n(x,s)p(s)u(s)ds, \ 0 \le x \le 1,$$

and

$$Nu(x) = \int_0^1 G_n(x,s)q(s)u(s)ds, \ 0 \le x \le 1.$$

Note that since $\frac{\partial^{n-i}}{\partial x^{n-i}} G_n(x,s)|_{x=0} = 0$ for $i = 4, 5, \ldots, n$, then $M, N : \mathcal{B} \to \mathcal{B}$. A standard application of the Arzela-Ascoli theorem shows that M and N are compact.

Lemma 4.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P} \backslash \{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(x) \ge 0$. Then, since $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s) = G(x,s) \ge 0$ on $[0,1] \times [0,1]$ and $p(x) \ge 0$ on $[0,1]$,

$$Mu^{(n-4)}(x) = \int_0^1 \frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s)p(s)u(s)ds \ge 0,$$

for $0 \le x \le 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0, 1]$ such that $u(x) > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x, s) > 0$ on $(0, 1] \times (0, 1]$,

$$\begin{aligned} Mu^{(n-4)}(x) &= \int_0^1 \frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x, s) p(s) u(s) ds \\ &\geq \int_\alpha^\beta \frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x, s) p(s) u(s) ds \\ &> 0, \end{aligned}$$

for $0 < x \leq 1$. Also, since $\frac{\partial^{n-3}}{\partial x^{n-3}} G_n(x, s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mu)^{(n-3)}(0) &= \int_0^1 \frac{\partial^{n-3}}{\partial x^{n-3}} G_n(0, s) p(s) u(s) ds \\ &\geq \int_\alpha^\beta \frac{\partial^{n-3}}{\partial x^{n-3}} G_n(0, s) p(s) u(s) ds \\ &> 0, \end{aligned}$$

and so $Mu \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 2.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. □

Remark 4.1. Notice that

$$\Lambda u = Mu = \int_0^1 G_n(x, s) p(s) u(s) ds,$$

if and only if

$$u(x) = \frac{1}{\Lambda} \int_0^1 G_n(x, s) p(s) u(s) ds,$$

if and only if

$$-u^{(n)}(x) = \frac{1}{\Lambda} p(x) u(x), \ 0 \leq x \leq 1,$$

with

$$u(0) = u'(0) = \cdots = u^{(n-4)}(0) = u^{(n-3)}(r) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0.$$

So the eigenvalues of (4.1),(4.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (4.2),(4.3) are reciprocals of N, and conversely.

Theorem 4.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. □

Theorem 4.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \leq q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 4.1 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$ and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \leq q(x)$ on $[0,1]$. So for any $u \in \mathcal{P}$ and $x \in [0,1]$,

$$(Nu - Mu)^{(n-4)}(x) = \int_0^1 \frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s)(q(s) - p(s))u(s)ds \geq 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 2.2, $\Lambda_1 \leq \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

By Remark 4.1, the following theorem is an immediate consequence of Theorems 4.1 and 4.2.

Theorem 4.3. *Assume the hypotheses of Theorem 4.2. Then there exists smallest positive eigenvalues λ_1 and λ_2 of (4.1),(4.3) and (4.2),(4.3), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$ and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \leq x \leq 1$.*

5. Extending the Fourth Order Problem Using Substitution

Instead of using the sign properties of the Green's function for the nth order equation to derive the comparison theorems, we will instead make a substitution and work with a variation of the fourth order problem. This method has its benefits, since we do not need to find the sign properties of the Green's function of the nth order problem, and can instead work with the fourth order problem. The techniques used in this section have been used previously by Henderson and Parmjet [11] and by Maroun [21] to reduce the order of singular problems. However, they have not been used in the context of smallest eigenvalue comparisons.

Let $n \in \mathbb{N}$, $n \geq 5$. We consider the eigenvalue problems

$$u^{(n)} + \lambda_1 p(x) u = 0, \tag{5.1}$$

$$u^{(n)} + \lambda_2 q(x) u = 0, \tag{5.2}$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-4)}(0) = u^{(n-3)}(r) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0, \tag{5.3}$$

and the eigenvalue problems

$$v^{(4)} + \lambda_1 p(x) \frac{1}{(n-5)!} \int_0^x (x-s)^{n-5} v(s) ds = 0, \tag{5.4}$$

$$v^{(4)} + \lambda_2 q(x) \frac{1}{(n-5)!} \int_0^x (x-s)^{n-5} v(s) ds = 0, \tag{5.5}$$

satisfying the boundary condtions

$$v(0) = v'(r) = v''(r) = v'''(1) = 0, \tag{5.6}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0,1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0,1]$.

First we note that if $u(x)$ is a solution to (5.1),(5.3), then $u^{(n-4)}(x)$ solves (5.4),(5.6). Also, if $v(x)$ is a solution to (5.4),(5.6), then $\frac{1}{(n-5)!} \int_0^x (x-s)^{n-5} v(s) ds$ is a solution to (5.1),(5.3). Similarly, if $u(x)$ is a solution to (5.2),(5.3), then $u^{(n-4)}(x)$ solves (5.5),(5.6) and if $v(x)$ is a solution to (5.5),(5.6), then $\frac{1}{(n-5)!} \int_0^x (x-s)^{n-5} v(s) ds$ is a solution to (5.2),(5.3).

Now let λ be an eigenvalue of (5.1),(5.3) with the corresponding eigenvector $u(x)$. Then $u^{(n-4)}(x)$ is a solution to (5.4),(5.6) with the same eigenvalue λ. Also, if λ is an eigenvalue of (5.4),(5.6) with corresponding eigenvector $v(x)$, then $\frac{1}{(n-5)!} \int_0^x (x-s)^{n-5} v(s) ds$ is a solution to (5.1),(5.3) with the corresponding eigenvalue λ. So eigenvalues of (5.1),(5.3) are eigenvalues of (5.4),(5.6), and vice versa. Similarly, eigenvalues of (5.2),(5.3) are eigenvalues of (5.5),(5.6), and vice versa. So any comparison theorems for (5.4),(5.6), and (5.5),(5.6) will apply to (5.1),(5.3), and (5.2),(5.3).

For these reasons, we will derive comparison theorems for eigenvalue problems (5.4),(5.6), and (5.5),(5.6), and then use these theorems to derive the comparison theorems for (5.1),(5.3), and (5.2),(5.3).

Let $G(x,s)$ by the Green's function for $-v^{(4)} = 0$ satisfying (5.6), which was defined earlier. So $v(x)$ solves (5.4),(5.6) if and only if

$$v(x) = \lambda_1 \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,$$

and $v(x)$ solves (5.5),(5.6) if and only if

$$v(x) = \lambda_2 \int_0^1 G(x,s)q(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds.$$

Again, note $G(x,s) \geq 0$ on $[0,1]\times[0,1]$, $G(x,s) > 0$ on $(0,1]\times(0,1]$, and $\frac{\partial}{\partial x}G(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 2.1 and 2.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{v \in C^1[0,1] \mid v(0) = 0\}$$

with the norm

$$||v|| = \sup_{0\leq x\leq 1} |v'(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{v \in \mathcal{B} \mid v(x) \geq 0 \text{ on } [0,1]\}.$$

Notice that for $v \in \mathcal{B}$, $0 \leq x \leq 1$,

$$\begin{aligned} |v(x)| = |v(x) - v(0)| &= \left|\int_0^x v'(s)ds\right| \\ &\leq ||v||x \\ &\leq ||v||, \end{aligned}$$

and so $\sup_{0\leq x\leq 1} |v(x)| \leq ||v||$.

Lemma 5.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{v \in \mathcal{B} \mid v(x) > 0 \text{ on } (0,1] \text{ and } v'(0) > 0\}.$$

It was shown earlier that $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. □

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mv(x) = \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,\ 0 \leq x \leq 1,$$

and

$$Nv(x) = \int_0^1 G(x,s)q(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,\ 0 \le x \le 1.$$

A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 5.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. We again show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $v \in \mathcal{P}$. So $v(x) \ge 0$. Then since $G(x,s) \ge 0$ on $[0,1] \times [0,1]$, $p(x) \ge 0$ on $[0,1]$ and $\frac{1}{(n-5)!}\int_0^x (x-s)^{n-5}v(s)ds \ge 0$,

$$Mv(x) = \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \ge 0,$$

for $0 \le x \le 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $v \in \mathcal{P}\backslash\{0\}$. Since $(x-s)^{n-5} > 0$ for $0 \le s < x$, there exists a compact interval $[\alpha,\beta] \subset [0,1]$ such that $\frac{1}{(n-5)!}\int_0^x (x-s)^{n-5}v(s)ds > 0$ and $p(x) > 0$ for all $x \in [\alpha,\beta]$. Then, since $G(x,s) > 0$ on $(0,1] \times (0,1]$,

$$\begin{aligned} Mv(x) &= \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \\ &\ge \int_\alpha^\beta G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \\ &> 0, \end{aligned}$$

for $0 < x \le 1$. Also, since $\frac{\partial}{\partial x}G(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mv)'(0) &= \int_0^1 \frac{\partial}{\partial x}G(0,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \\ &\ge \int_\alpha^\beta \frac{\partial}{\partial x}G(0,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \\ &> 0, \end{aligned}$$

and so $Mv \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 2.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. □

Remark 5.1. Notice that

$$\Lambda v = Mv = \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,$$

if and only if

$$v(x) = \frac{1}{\Lambda}\int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,$$

if and only if

$$-v^{(4)}(x) = \frac{1}{\Lambda}p(x)\frac{1}{(n-5)!}\int_0^x (x-s)^{n-5}v(s)ds,\ 0 \le x \le 1,$$

with

$$v(0) = v'(r) = v''(r) = v'''(1) = 0.$$

So the eigenvalues of (5.4),(5.6) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (5.5),(5.6) are reciprocals of N, and conversely.

Theorem 5.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $v \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $v \neq 0$, $Mv \in \Omega \subset \mathcal{P}^\circ$ and $v = M\left(\frac{1}{\Lambda}v\right) \in \mathcal{P}^\circ$. □

Theorem 5.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \le q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 5.1 associated with M and N, respectively, with the essentially unique eigenvectors v_1 and $v_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \le q(x)$ on $[0,1]$. So for any $v \in \mathcal{P}$, $x \in [0,1]$,

$$(Nv - Mv)(x) = \int_0^1 G(x,s)(q(s)-p(s))\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \ge 0.$$

So $Nv - Mv \in \mathcal{P}$ for all $v \in \mathcal{P}$, or $M \le N$ with respect to $\mathcal{P}$. Then by Theorem 2.2, $\Lambda_1 \le \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha,\beta] \subset [0,1]$. Then $(N-M)v_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N-M)v_1 - \epsilon v_1 \in \mathcal{P}$. So $\Lambda_1 v_1 + \epsilon v_1 = Mv_1 + \epsilon v_1 \le Nv_1$, implying $Nv_1 \ge (\Lambda_1 + \epsilon)v_1$. Since $N \le N$ and $Nv_2 = \Lambda_2 v_2$, by Theorem 2.2, $\Lambda_1 + \epsilon \le \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

By Remark 5.1, the following theorem is an immediate consequence of Theorems 5.1 and 5.2.

Theorem 5.3. *Assume the hypotheses of Theorem 5.2. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (5.4),(5.6) *(and hence* (5.1),(5.3)*) and* (5.5),(5.6) *(and hence* (5.2),(5.3)*), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \leq x \leq 1$.*

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