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Multiple positive solutions to systems of nonlinear semipositone fractional differential equations with coupled boundary conditions*

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Abstract. In this paper, we consider four-point coupled boundary value problem for systems of the nonlinear semipositone fractional differential equation

$$\begin{cases} \mathbf{D}_{0+}^{\alpha}u + \lambda f(t,u,v) = 0, \quad 0 < t < 1, \lambda > 0, \\ \mathbf{D}_{0+}^{\alpha}v + \lambda g(t,u,v) = 0, \\ u^{(i)}(0) = v^{(i)}(0) = 0, \ \ 0 \le i \le n-2, \\ u(1) = av(\xi), v(1) = bu(\eta), \ \ \xi, \eta \in (0,1) \end{cases}$$

where λ is a parameter, a, b, ξ, η satisfy $\xi, \eta \in (0,1)$, $0 < ab\xi\eta < 1$, $\alpha \in (n-1, n]$ is a real number and $n \ge 3$, and $\mathbf{D}_{0+}^{\alpha}$ is the Riemann-Liouville's fractional derivative, and f, g are continuous and semipositone. We derive an interval on λ such that for any λ lying in this interval, the semipositone boundary value problem has multiple positive solutions.

Key words. Riemann-Liouville's fractional derivative; semipositone fractional differential equation; four-point coupled boundary value problem; positive solution; fixed-point theorem.

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1 Introduction

We consider the four-point coupled boundary value problem for nonlinear fractional differential equation involving the Riemann-Liouville's derivative

$$\begin{cases} \mathbf{D}_{0+}^{\alpha}u + \lambda f(t,u,v) = 0, \quad 0 < t < 1, \lambda > 0, \\ \mathbf{D}_{0+}^{\alpha}v + \lambda g(t,u,v) = 0, \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ u(1) = av(\xi), v(1) = bu(\eta), \quad \xi, \eta \in (0,1) \end{cases} \tag{1.1}$$

where λ is a parameter, a, b, ξ, η satisfy $\xi, \eta \in (0,1)$, $0 < ab\xi\eta < 1$, $\alpha \in (n-1, n]$ is a real number and $n \ge 2$, $\mathbf{D}_{0+}^{\alpha}$ is the Riemann-Liouville's fractional derivative, and f, g are sign-changing continuous functions.

Fractional differential equation's modeling capabilities in engineering, science, economics, and other fields, over the last few decades has resulted in the rapid development of the theory of fractional differential equations, see

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[1]-[7] for a good overview. To our knowledge there are only a few papers which deal with the boundary value problem for nonlinear fractional differential equations (see for example [8]-[20]). Coupled boundary conditions arise in the study of reaction-diffusion equations and Sturm-Liouvillie problems, see [21, 22] and have wide applications in various fields of sciences and engineering, for example the heat equation [23, 24, 25] and mathematical biology [26, 27].

In [23], the authors study the case of two equations

$$\begin{array}{l} u_t = \triangle u, \quad v_t = \triangle v, \quad x \in \Omega, \quad 0 < t < T, \\ \frac{\partial u}{\partial \eta} = v^p, \quad \frac{\partial v}{\partial \eta} = u^p, \quad X \in \partial\Omega, \quad 0 < t < T, \end{array}$$

and it was shown that if $pq \leq 1$, all nonnegative solutions are global, while if $pq > 1$, every nonnegative solution blows up in finite time.

In [26], the authors study the blow-up properties of the positive solutions to the system of heat equations with nonlinear boundary conditions

$$\begin{array}{l} u_{it} = \triangle u_i, i = l, \cdots, k, u_{k+l} := u_l, \quad x \in \Omega, \quad 0 < t < T, \\ \frac{\partial u_i}{\partial \eta} = u_{i+1}^{p_i}, \quad X \in \partial\Omega, \quad 0 < t < T, \\ u_i(x, 0) = u_{i,0}(x), \quad x \in \Omega, \end{array}$$

where $p_i > 0$, $i = 1, \cdots, k$. $\Omega \in R^N$ is a bounded domain with smooth boundary $\partial\Omega$, η is the unit outward normal vector, $u_{i,0}(x)$ are nonnegative nontrivial functions and satisfy appropriate compatibility conditions. The upper and lower bounds of the blow-up rate is derived.

In [28], Leung studied the reaction-diffusion system for prey-predator interaction

$$\begin{array}{l} u_t(t, x) = \sigma_1 \triangle u + u(a + f(u, v)), t \geq 0; x \in \Omega \subset R^n, \\ v_t(t, x) = \sigma_2 \triangle v + v(r + g(u; v)), t \geq 0; x \in \Omega \subset R^n, \end{array}$$

subject to the coupled boundary conditions

$$\frac{\partial u}{\partial \eta} = 0; \frac{\partial v}{\partial \eta} - p(u) - q(v) = 0 \text{ on } \partial\Omega,$$

where the functions $u(t, x), v(t, x)$ respectively represent the density of prey and predator at time $t \geq 0$ and at position $x = (x_1, \cdots, x_n)$. Similar coupled boundary conditions are also studied in [27] for a biochemical system.

The above mentioned work and wide applications of coupled boundary conditions motivate us to study equation (1.1). In this paper, we give sufficient conditions for the existence of positive solution of the semipositone boundary value problems (1.1) for a sufficiently small $\lambda > 0$ where f, g may change sign. Our analysis relies on a nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorems.

2 Preliminaries

For completeness, in this section, we first present some fundamental facts of the Riemann-Liouville's derivatives of fractional order which can been found in [3].

Definition 2.1 *[3]* The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0,$$

where $\alpha > 0$, is called Riemann-Liouville fractional integral of order α.

Definition 2.2 *[3]* For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dx})^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order s.

As examples, for $\mu > -1$, we have

$$\mathbf{D}_{0+}^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}$$

giving in particular $\mathbf{D}_{0+}^{\alpha} x^{\alpha-m},\ m = i, 2, 3, \cdots, N$, where N is the smallest integer greater than or equal to α.

Lemma 2.1 *Let $\alpha > 0$. Then the differential equation*

$$\mathbf{D}_{0+}^{\alpha} u(t) = 0$$

has solutions $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, , 2 \ldots, n$, where n is the smallest integer greater than or equal to α.

Lemma 2.2 *Let $\alpha > 0$. Then*

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, n is the smallest integer greater than or equal to α.

Lemma 2.3 *Let $x, y \in C[0,1]$ be given functions. Then the boundary-value problem*

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} u + x(t) = 0, \quad 0 < t < 1, \lambda > 0, \\ \mathbf{D}_{0+}^{\alpha} v + y(t) = 0, \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ u(1) = av(\xi), v(1) = bu(\eta), \quad \xi, \eta \in (0,1) \end{cases} \tag{2.1}$$

has an integral representation

$$\begin{cases} u(t) = \int_0^1 G_{\xi\eta}(t,s) x(s) ds + \int_0^1 K_{\xi\eta}(t,s) y(s) ds, \\ v(t) = \int_0^1 G_{\eta\xi}(t,s) y(s) ds + \int_0^1 K_{\eta\xi}(t,s) x(s) ds \end{cases} \tag{2.2}$$

where

$$G_{\xi\eta}(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, s \le \eta, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, s \ge \eta, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le 1, s \le \eta, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le 1, s \ge \eta \end{cases} \tag{2.3}$$

$$G_{\eta\xi}(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{ab\eta^{\alpha-1}t^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, s \le \xi, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, s \ge \xi, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{ab\eta^{\alpha-1}t^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le 1, s \le \xi, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \le t \le s \le 1, s \ge \xi \end{cases} \tag{2.4}$$

$$K_{\xi\eta}(t,s) = \begin{cases} \frac{a\xi^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{at^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, & s \le \xi, \\ \frac{a\xi^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, & s \ge \xi \end{cases} \tag{2.5}$$

$$K_{\eta\xi}(t,s) = \begin{cases} \frac{b\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{bt^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, & s \le \eta, \\ \frac{b\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, & s \ge \eta. \end{cases} \tag{2.6}$$

Proof. From Lemma 2.2 we can reduce (2.1) to an equivalent integral equation

$$\begin{aligned} u(t) &= c_{11} t^{\alpha-1} + c_{12} t^{\alpha-2} + \cdots + c_{1n} t^{\alpha-n} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds \\ v(t) &= c_{21} t^{\alpha-1} + c_{22} t^{\alpha-2} + \cdots + c_{2n} t^{\alpha-n} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds. \end{aligned} \tag{2.7}$$

From $u^{(j)}(0) = v^{(j)}(0) = 0, 0 \leq j \leq n-2$, we have $c_{in} = c_{i(n-1)} = \cdots = c_{i2} = 0, (i = 1, 2)$. Then

$$\begin{aligned} u(t) &= c_{11}t^{\alpha-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds \\ v(t) &= c_{21}t^{\alpha-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \end{aligned}$$

and from the condition $u(1) = av(\xi), v(1) = bu(\eta)$ we have

$$\begin{aligned} c_{11} - a\xi^{\alpha-1}c_{21} &= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds - a\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds, \\ c_{21} - b\eta^{\alpha-1}c_{11} &= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - b\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds. \end{aligned}$$

Solving for c_{11} and c_{21}, we have

$$\begin{aligned} c_{11} = &\frac{1}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds - \frac{a}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\ &+\frac{a\xi^{\alpha-1}}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - \frac{ab\xi^{\alpha-1}}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, \\ c_{21} = &\frac{1}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - \frac{b}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds \\ &+\frac{b\eta^{\alpha-1}}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds - \frac{ab\eta^{\alpha-1}}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds. \end{aligned}$$

Hence, we have

$$\begin{aligned} u(t) = &\frac{1}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds - \frac{ab\xi^{\alpha-1}}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^\eta \frac{t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds \\ &+\frac{a\xi^{\alpha-1}}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - \frac{a}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^\xi \frac{t^{\alpha-1}(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds, \\ v(t) = &\frac{1}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - \frac{ab\eta^{\alpha-1}}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^\xi \frac{t^{\alpha-1}(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\ &+\frac{b\eta^{\alpha-1}}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds - \frac{b}{1-ab\xi^{\alpha-1}\eta^{\alpha-1}}\int_0^\eta \frac{t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds. \end{aligned}$$

Thus

$$\begin{cases} u(t) = \int_0^1 G_{\xi\eta}(t,s)x(s)ds + \int_0^1 K_{\xi\eta}(t,s)y(s)ds, \\ v(t) = \int_0^1 G_{\eta\xi}(t,s)y(s)ds + \int_0^1 K_{\eta\xi}(t,s)x(s)ds. \end{cases}$$

Lemma 2.4 *The function $G_{\xi\eta}(t,s)$ and $K_{\xi\eta}(t,s)$ defined respectively by (2.3) and (2.5) have the following properties:*

(R1) $c_0t^{\alpha-1}(1-s)^{\alpha-1}s \leq G_{\xi\eta}(t,s) \leq C_0(1-s)^{\alpha-1}s$, $G_{\xi\eta}(t,s) \leq C_0t^{\alpha-1}$ *for* $t, s \in [0,1]$,

(R2) $c_0t^{\alpha-1}(1-s)^{\alpha-1}s \leq K_{\xi\eta}(t,s) \leq C_0t^{\alpha-1}(1-s)^{\alpha-1}s$ *for* $t, s \in [0,1]$,

where

$$\begin{aligned} &c_G = \frac{ab\xi^{\alpha-1}(1-\xi)\eta^{\alpha-1}(1-\eta)(1-ab\xi\eta)}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, \quad C_G = \frac{(\alpha-1)(1-ab\xi^{\alpha-1}\eta^{\alpha-1}+ab\xi^{\alpha-2}\eta^{\alpha-2})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, \quad C_G^* = \frac{1}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, \\ &c_K = \frac{\min\{a\xi^{\alpha-2}(1-\xi),\ b\eta^{\alpha-2}(1-\eta)\}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, \qquad C_K = \frac{(\alpha-1)(a\xi^{\alpha-2}+b\eta^{\alpha-2})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}, \\ &c_0 = \min\{c_G, c_K\}, \qquad C_0 = \min\{C_G, C_G^*, C_K\}. \end{aligned} \tag{2.8}$$

Proof. (R_1) For $(t,s) \in [0,1] \times [0,1]$, from (2.3), we discuss various cases.

Case 1. For $s \le t, s \le \eta$, we have

$$\begin{aligned}
G_{\xi\eta}(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{t^{\alpha-1}(1-s)^{\alpha-1}-ab\xi^{\alpha-1}t^{\alpha-1}(\eta-s)^{\alpha-1}-(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(t-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1})+ab\xi^{\alpha-1}t^{\alpha-1}(\eta^{\alpha-1}(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\ge \frac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta^{\alpha-1}(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{ab\xi^{\alpha-1}t^{\alpha-1}((\eta-\eta s)^{\alpha-1}-(\eta-s)^{\alpha-1})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{ab\xi^{\alpha-1}t^{\alpha-1}((\eta-\eta s)^{\alpha-1-[\alpha-1]}(\eta-\eta s)^{[\alpha-1]}-(\eta-s)^{\alpha-1-[\alpha-1]}(\eta-s)^{[\alpha-1]})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\ge \frac{ab\xi^{\alpha-1}t^{\alpha-1}((\eta-\eta s)^{\alpha-1-[\alpha-1]}(\eta-\eta s)^{[\alpha-1]}-(\eta-\eta s)^{\alpha-1-[\alpha-1]}(\eta-s)^{[\alpha-1]})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-\eta s)^{\alpha-1-[\alpha-1]}((\eta-\eta s)^{[\alpha-1]}-(\eta-s)^{[\alpha-1]})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\ge \frac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-\eta s)^{\alpha-1-[\alpha-1]}(\eta-\eta s)^{[\alpha-1]-1}((\eta-\eta s)-(\eta-s))}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-\eta s)^{\alpha-2}(s-\eta s)}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{ab\xi^{\alpha-1}\eta^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-2}(1-\eta)s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\ge \frac{ab\xi^{\alpha-1}\eta^{\alpha-2}(1-\eta)t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\ge c_G t^{\alpha-1}(1-s)^{\alpha-1}s,
\end{aligned}$$

$$\begin{aligned}
G_{\xi\eta}(t,s) &= \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1})+ab\xi^{\alpha-1}t^{\alpha-1}(\eta^{\alpha-1}(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(\alpha-1)\int_{t-s}^{t-ts} z^{\alpha-2}dz+ab\xi^{\alpha-1}t^{\alpha-1}(\alpha-1)\int_{\eta-s}^{\eta-\eta s} z^{\alpha-2}dz}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\le \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(\alpha-1)(t-ts)^{\alpha-2}(t-ts-t+s)+ab\xi^{\alpha-1}t^{\alpha-1}(\alpha-1)(\eta-\eta s)^{\alpha-2}(\eta-\eta s-\eta+s)}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}(1-t)s+ab\xi^{\alpha-1}(\alpha-1)\eta^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-2}(1-\eta)s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\le \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}(1-s)s+ab\xi^{\alpha-1}(\alpha-1)\eta^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-2}(1-s)s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\le \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(\alpha-1)(1-s)^{\alpha-1}s+ab\xi^{\alpha-1}(\alpha-1)\eta^{\alpha-2}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{(\alpha-1)((1-ab\xi^{\alpha-1}\eta^{\alpha-1})+ab\xi^{\alpha-1}\eta^{\alpha-2})(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{(\alpha-1)(1-ab\xi^{\alpha-1}\eta^{\alpha-1}+ab\xi^{\alpha-1}\eta^{\alpha-2})(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\le C_G(1-s)^{\alpha-1}s,
\end{aligned}$$

$$G_{\xi\eta}(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \le \frac{t^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \le C_G^* t^{\alpha-1}.$$

Case 2. For $s \le t, s \ge \eta$, from (2.3), we have

$$\begin{aligned}
G_{\xi\eta}(t,s) &= \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1})+ab\xi^{\alpha-1}\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\ge \frac{ab\xi^{\alpha-1}\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\ge \frac{ab\xi^{\alpha-1}\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\ge c_G t^{\alpha-1}(1-s)^{\alpha-1}s,
\end{aligned}$$

$$\begin{aligned}
G_{\xi\eta}(t,s) &= \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1})+ab\xi^{\alpha-1}\eta^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1})+ab\xi^{\alpha-1}\eta^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-1}\eta}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\le \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}(1-t)s+ab\xi^{\alpha-1}\eta^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\le \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}s+ab\xi^{\alpha-1}\eta^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\le \frac{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})(\alpha-1)(1-s)^{\alpha-1}s+ab\xi^{\alpha-1}\eta^{\alpha-2}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\le C_G(1-s)^{\alpha-1}s,
\end{aligned}$$

$$G_{\xi\eta}(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \le \frac{t^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \le C_G^* t^{\alpha-1}.$$

Case 3. For $t \leq s, s \leq \eta$, from (2.3), we have

$$\begin{aligned}
G_{\xi\eta}(t,s) &= \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \tfrac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}-ab\xi^{\alpha-1}t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \tfrac{(1-ab\xi^{\alpha-1})t^{\alpha-1}(1-s)^{\alpha-1}+ab\xi^{\alpha-1}t^{\alpha-1}((1-s)^{\alpha-1}-(\eta-s)^{\alpha-1})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\geq \tfrac{(1-ab\xi^{\alpha-1})t^{\alpha-1}(1-s)^{\alpha-1}+ab\xi^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-2}(1-\eta)}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\geq \tfrac{(1-ab\xi^{\alpha-1})t^{\alpha-1}(1-s)^{\alpha-1}+ab\xi^{\alpha-1}(1-\eta)t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\geq \tfrac{(1-ab\xi^{\alpha-1}+ab\xi^{\alpha-1}(1-\eta))t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\geq \tfrac{(1-ab\xi^{\alpha-1}\eta)t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\geq c_G t^{\alpha-1}(1-s)^{\alpha-1}s,
\end{aligned}$$

$$\begin{aligned}
G_{\xi\eta}(t,s) &= \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} - \tfrac{ab\xi^{\alpha-1}t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\leq \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\leq \tfrac{t^{\alpha-2}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\leq \tfrac{(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\leq C_G(1-s)^{\alpha-1}s,
\end{aligned}$$

$$G_{\xi\eta}(t,s) \leq \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \leq \tfrac{t^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \leq C_G^* t^{\alpha-1}.$$

Case 4. For $t \leq s, s \geq \eta$, from (2.3), we have

$$G_{\xi\eta}(t,s) = \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \geq \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \geq c_G t^{\alpha-1}(1-s)^{\alpha-1}s,$$

$$G_{\xi\eta}(t,s) = \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \leq \tfrac{t^{\alpha-2}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \leq \tfrac{(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \leq C_G(1-s)^{\alpha-1}s,$$

$$G_{\xi\eta}(t,s) = \tfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \leq \tfrac{t^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \leq C_G^* t^{\alpha-1}.$$

Then, $c_0 t^{\alpha-1}(1-s)^{\alpha-1}s \leq G_{\xi\eta}(t,s) \leq C_0(1-s)^{\alpha-1}s$, $G_{\xi\eta}(t,s) \leq C_0 t^{\alpha-1}$ for $t, s \in [0,1]$.

(R_2) For $(t,s) \in [0,1]\times[0,1]$, from (2.5), we also discuss various cases.

Case 1. For $s \leq \xi$, we have

$$\begin{aligned}
K_{\xi\eta}(t,s) &= \tfrac{a\xi^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}-at^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \tfrac{at^{\alpha-1}(\xi^{\alpha-1}(1-s)^{\alpha-1}-(\xi-s)^{\alpha-1})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\geq \tfrac{at^{\alpha-1}\xi^{\alpha-2}(1-s)^{\alpha-2}(1-\xi)s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\geq \tfrac{a\xi^{\alpha-2}(1-\xi)t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\geq c_K t^{\alpha-1}(1-s)^{\alpha-1}s,
\end{aligned}$$

$$\begin{aligned}
K_{\xi\eta}(t,s) &= \tfrac{a\xi^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}-at^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&= \tfrac{at^{\alpha-1}(\xi^{\alpha-1}(1-s)^{\alpha-1}-(\xi-s)^{\alpha-1})}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\leq \tfrac{at^{\alpha-1}(\alpha-1)\xi^{\alpha-2}(1-s)^{\alpha-2}(1-\xi)s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\leq \tfrac{at^{\alpha-1}(\alpha-1)\xi^{\alpha-2}(1-s)^{\alpha-2}(1-s)s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\leq \tfrac{(\alpha-1)a\xi^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \\
&\leq C_K t^{\alpha-1}(1-s)^{\alpha-1}s.
\end{aligned}$$

Case 2. For $s \geq \xi$, we have

$$K_{\xi\eta}(t,s) = \tfrac{a\xi^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \geq \tfrac{a\xi^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)} \geq c_K t^{\alpha-1}(1-s)^{\alpha-1}s,$$

$K_{\xi\eta}(t,s)=\frac{a\xi^{\alpha-1}t^{\alpha-1}(1-s)^{\alpha-1}}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}=\frac{a\xi^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-1}\xi}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}\leq\frac{a\xi^{\alpha-2}t^{\alpha-1}(1-s)^{\alpha-1}s}{(1-ab\xi^{\alpha-1}\eta^{\alpha-1})\Gamma(\alpha)}\leq C_K t^{\alpha-1}(1-s)^{\alpha-1}s.$

Thus, we have $c_0t^{\alpha-1}(1-s)^{\alpha-1}s\leq K_{\xi\eta}(t,s)\leq C_0t^{\alpha-1}(1-s)^{\alpha-1}s$ for $t,s\in[0,1]$.

Similarly we have the following lemma.

Lemma 2.5 *The function $G_{\eta\xi}(t,s)$ and $K_{\eta\xi}(t,s)$ defined respectively by* (2.4) *and* (2.6) *have the following properties:*

(R1) $c_0t^{\alpha-1}(1-s)^{\alpha-1}s\leq G_{\eta\xi}(t,s)\leq C_0(1-s)^{\alpha-1}s$, $G_{\xi\eta}(t,s)\leq C_0t^{\alpha-1}$ *for* $t,s\in[0,1]$,

(R2) $c_0t^{\alpha-1}(1-s)^{\alpha-1}s\leq K_{\eta\xi}(t,s)\leq C_0t^{\alpha-1}(1-s)^{\alpha-1}s$ *for* $t,s\in[0,1]$,

where c_0, C_0 *are as in Lemma 2.4*

Employing Lemma 2.3, the system (1.1) can be expressed as

$$\begin{cases} u(t)=\lambda(\int_0^1 G_{\xi\eta}(t,s)f(s,u(s),v(s))ds+\int_0^1 K_{\xi\eta}(t,s)g(s,u(s),v(s))ds),\\ v(t)=\lambda(\int_0^1 G_{\eta\xi}(t,s)g(s,u(s),v(s))ds+\int_0^1 K_{\eta\xi}(t,s)f(s,u(s),v(s))ds).\end{cases}\tag{2.9}$$

The following theorems (the first a nonlinear alternative of Leray-Schauder type and the second Krasnosel'skii's fixed-point theorem) will play a major role in Section 3.

Theorem 2.6 *[29] Let X be a Banach space with $\Omega\subset X$ closed and convex. Assume U is a relatively open subset of Ω with $0\in U$, and let $S:\overline{U}\to\Omega$ be a compact, continuous map. Then either*

1. *S has a fixed point in $\overline{U}$, or*
2. *there exists $u\in\partial U$ and $\nu\in(0,1)$, with $u=\nu Su$.*

Theorem 2.7 *[30] Let X be a Banach space, and let $P\subset X$ be a cone in X. Assume Ω_1,Ω_2 are bounded open subsets of X with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $S:P\to P$ be a completely continuous operator such that, either*

1. *$\|Sw\|\leq\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\geq\|w\|$, $w\in P\cap\partial\Omega_2$, or*
2. *$\|Sw\|\geq\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\leq\|w\|$ $w\in P\cap\partial\Omega_2$.*

Then S has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3 Main Results

We make the following assumption:

(H_1) $f(t,u,v),g(t,u,v)\in C([0,1]\times[0,+\infty)\times[0,+\infty),(-\infty,+\infty))$, moreover there exist function $e_i(t)\in L^1([0,1],(0,+\infty))$ $(i=1,2)$ such that $f(t,u,v)\geq -e_1(t)$ and $g(t,u,v)\geq -e_2(t)$, for any $t\in[0,1]$, $u,v\in[0,+\infty)$.

(H_1^*) $f(t,u,v),g(t,u,v)\in C((0,1)\times[0,+\infty),(-\infty,+\infty))$, f,g may be singular at $t=0,1$, moreover there exist functions $e_i(t)\in L^1((0,1),(0,+\infty))$ $(i=1,2)$ such that $f(t,u,v)\geq -e_1(t)$ and $g(t,u,v)\geq -e_2(t)$, for any $t\in(0,1)$, $u,v\in[0,+\infty)$.

(H_2) $f(t,0,0)>0, g(t,0,0)>0$ for $t\in[0,1]$.

(H_3) There exists $[\theta_1,\theta_2]\subset(0,1)$ such that $\lim\limits_{u\uparrow+\infty}\inf\min\limits_{t\in[\theta_1,\theta_2]}\frac{f(t,u,v)}{u}=+\infty$, $\lim\limits_{v\uparrow+\infty}\inf\min\limits_{t\in[\theta_1,\theta_2]}\frac{g(t,u,v)}{v}=+\infty$.

(H_3^*) There exists $[\theta_1,\theta_2]\subset(0,1)$ such that $\lim\limits_{v\uparrow+\infty}\inf\min\limits_{t\in[\theta_1,\theta_2]}\frac{f(t,u,v)}{v}=+\infty$, $\lim\limits_{u\uparrow+\infty}\inf\min\limits_{t\in[\theta_1,\theta_2]}\frac{g(t,u,v)}{u}=+\infty$.

(H_4) $\int_0^1(1-s)^{\alpha-1}se_i(s)ds<+\infty$, $\int_0^1(1-s)^{\alpha-1}sf(s,u,v)ds<+\infty$ and $\int_0^1(1-s)^{\alpha-1}sg(s,u,v)ds<+\infty$ for any $u,v\in[0,m]$, $m>0$ is any constant $(i=1,2)$.

We consider the boundary value problem

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} x + \lambda(f(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + e_1(t)) = 0, \quad 0 < t < 1, \lambda > 0, \\ \mathbf{D}_{0+}^{\alpha} y + \lambda(g(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + e_2(t)) = 0, \\ x^{(i)}(0) = y^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ x(1) = ay(\xi), y(1) = bx(\eta), \quad \xi, \eta \in (0,1) \end{cases} \tag{3.1}$$

where

$$z(t)^* = \begin{cases} z(t), \; z(t) \geq 0, \\ 0, \quad\; z(t) < 0, \end{cases}$$

and

$$\begin{cases} w_1(t) = \lambda \int_0^1 G_{\xi\eta}(t,s) e_1(s) ds + \lambda \int_0^1 K_{\xi\eta}(t,s) e_2(s) ds, \\ w_2(t) = \lambda \int_0^1 G_{\eta\xi}(t,s) e_2(s) ds + \lambda \int_0^1 K_{\eta\xi}(t,s) e_1(s) ds, \end{cases}$$

which is the solution of the coupled boundary value problem

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} w_1 + \lambda e_1(t) = 0, \quad 0 < t < 1, \lambda > 0, \\ \mathbf{D}_{0+}^{\alpha} w_2 + \lambda e_2(t) = 0, \\ w_1^{(i)}(0) = w_2^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ w_1(1) = aw_2(\xi), w_2(1) = bw_1(\eta), \quad \xi, \eta \in (0,1). \end{cases}$$

We will show there exists a solution (x, y) for the boundary value problem (3.1) with $x(t) \geq w_1(t)$ and $y(t) \geq w_2(t)$ for $t \in [0,1]$. If this is true, then $u(t) = x(t) - w_1(t)$ and $v(t) = y(t) - w_2(t)$ is a nonnegative solution (positive on $(0,1)$) of the boundary value problem (1.1). Since for any $t \in (0,1)$,

$$\begin{aligned} -\mathbf{D}_{0+}^{\alpha} x = -\mathbf{D}_{0+}^{\alpha} u + (-\mathbf{D}_{0+}^{\alpha} w_1) = \lambda[f(t,u,v) + e_1(t)], \\ -\mathbf{D}_{0+}^{\alpha} y = -\mathbf{D}_{0+}^{\alpha} v + (-\mathbf{D}_{0+}^{\alpha} w_2) = \lambda[g(t,u,v) + e_2(t)], \end{aligned}$$

we also have

$$-\mathbf{D}_{0+}^{\alpha} u = \lambda f(t,u,v) \text{ and } -\mathbf{D}_{0+}^{\alpha} v = \lambda g(t,u,v).$$

On the other hand, from the coupled value condition $x^{(i)}(0) = y^{(i)}(0) = 0, 0 \leq i \leq n-2$ and $x(1) = ay(\xi), y(1) = bx(\eta)$, we have

$$u^{(i)}(0) = v^{(i)}(0) = 0 \;\text{ for }\; 0 \leq i \leq n-2; \; u(1) = av(\xi), v(1) = bu(\eta) \;\text{ for }\; \xi, \eta \in (0,1).$$

As a result, we will concentrate our study on the boundary value problem (3.1).

Employing Lemma 2.3, we note that the system (3.1) is equivalent to

$$\begin{cases} x(t) = \lambda \int_0^1 G_{\xi\eta}(t,s)(f(s, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + e_1(s)) ds \\ \qquad\quad + \lambda \int_0^1 K_{\xi\eta}(t,s)(g(s, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + e_2(s)) ds, \\ y(t) = \lambda \int_0^1 G_{\eta\xi}(t,s)(g(s, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + e_2(s)) ds \\ \qquad\quad + \lambda \int_0^1 K_{\eta\xi}(t,s)(f(s, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + e_1(s)) ds. \end{cases} \tag{3.2}$$

We consider the Banach space $E = C[0,1]$ equipped with the standard norm $\|x\| = \max_{0 \leq t \leq 1} |x(t)|, x \in X$. We define a cone P of E by

$$P = \{x \in X | x(t) \geq \frac{c_0 t^{\alpha-1}}{C_0} \|x\|, \quad t \in [0,1], \alpha \in (n-1, n], n \geq 3\}.$$

For each $(x, y) \in E \times E$, we write $||(x,y)||_1 = ||x|| + ||y||$. Clearly, $(E \times E, ||\cdot||_1)$ is a Banach space and $P \times P$ is a cone of $E \times E$.

Define an integral operator $T : P \times P \to P \times P$ by

$$T(x,y) = (A(x,y), B(x,y)),$$

where the operators $A, B : P \times P \to P$ are defined by

$$\begin{cases} A(x,y)(t) = & \lambda \int_0^1 G_{\xi\eta}(t,s)(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_1(s))ds \\ & +\lambda \int_0^1 K_{\xi\eta}(t,s)(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_2(s))ds, \\ B(x,y)(t) = & \lambda \int_0^1 G_{\eta\xi}(t,s)(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_2(s))ds \\ & +\lambda \int_0^1 K_{\eta\xi}(t,s)(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_1(s))ds. \end{cases} \tag{3.3}$$

Clearly, if $(x, y) \in P \times P$ is a fixed point of T, then (x, y) is a solution of system (3.1).

Notice, from Lemma 2.4, we have $T(x, y)(t) \geq (0, 0)$ on $[0, 1]$ and for $(x, y) \in P \times P$

$$\begin{aligned} A(x,y)(t) &= \lambda \int_0^1 G_{\xi\eta}(t,s)(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_1(s))ds \\ &\quad + \lambda \int_0^1 K_{\xi\eta}(t,s)(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_2(s))ds, \\ &\leq \lambda \int_0^1 C_0(1-s)^{\alpha-1}s(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_1(s))ds \\ &\quad + \lambda \int_0^1 C_0(1-s)^{\alpha-1}s(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_2(s))ds, \end{aligned}$$

and then $\|A(x,y)\| \leq \lambda \int_0^1 C_0(1-s)^{\alpha-1}s(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_1(s))ds+\lambda \int_0^1 C_0(1-s)^{\alpha-1}s(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_2(s))ds$.

On the other hand, for $(x, y) \in P \times P$, $t \in [0, 1]$ we have

$$\begin{aligned} A(x,y)(t) &= \lambda \int_0^1 G_{\xi\eta}(t,s)(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_1(s))ds \\ &\quad + \lambda \int_0^1 K_{\xi\eta}(t,s)(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_2(s))ds, \\ &\geq \lambda \int_0^1 c_0 t^{\alpha-1}(1-s)^{\alpha-1}s(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_1(s))ds \\ &\quad + \lambda \int_0^1 c_0 t^{\alpha-1}(1-s)^{\alpha-1}s(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_2(s))ds, \\ &\geq \tfrac{c_0}{C_0}t^{\alpha-1}\lambda \int_0^1 C_0(1-s)^{\alpha-1}s(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_1(s))ds \\ &\quad + \tfrac{c_0}{C_0}t^{\alpha-1}\lambda \int_0^1 C_0(1-s)^{\alpha-1}s(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)+e_2(s))ds, \\ &\geq \tfrac{c_0}{C_0}t^{\alpha-1}\|A(x,y)\|. \end{aligned}$$

Consequently, $A(x, y) \in P$, i.e. $A(P \times P) \in P$. Similarly, we can show that $B(P \times P) \in P$. Hence, $T(P \times P) \subset P$. In addition, standard arguments in the literature guarantee that T is a completely continuous operator.

Theorem 3.1 *Suppose that (H_1) and (H_2) hold. Then there exists a constant $\overline{\lambda} > 0$ such that, for any $0 < \lambda \leq \overline{\lambda}$, the boundary value problem (1.1) has at least one positive solution.*

Proof. Fix $\delta \in (0, 1)$. From (H_2), let $0 < \varepsilon < 1$ be such that

$$f(t,u,v) \geq \delta f(t,0,0), \quad g(t,u,v) \geq \delta g(t,0,0), \quad \text{for} \quad 0 \leq t \leq 1, \quad 0 \leq z_1, z_2 \leq \varepsilon. \tag{3.4}$$

Let $\overline{f}(\varepsilon) = \max_{0\leq t\leq 1, 0\leq u,v\leq\varepsilon}\{f(t,u,v)+e_1(t)\}$, $\overline{g}(\varepsilon) = \max_{0\leq t\leq 1, 0\leq u,v\leq\varepsilon}\{g(t,u,v)+e_2(t)\}$ and $c = \int_0^1 C_0(1-s)^{\alpha-1}sds$. We have

$$\lim_{z\downarrow 0}\frac{\overline{f}(z)}{z} = +\infty, \quad \lim_{z\downarrow 0}\frac{\overline{g}(z)}{z} = +\infty.$$

Suppose

$$0 < \lambda < \frac{\varepsilon}{8c\overline{h}(\varepsilon)} := \overline{\lambda},$$

where $\overline{h}(\varepsilon) = \max\{\overline{f}(\varepsilon), \overline{g}(\varepsilon)\}$. Since

$$\lim_{z\downarrow 0}\frac{\overline{h}(z)}{z} = +\infty$$

and

$$\frac{\overline{h}(\varepsilon)}{\varepsilon} < \frac{1}{8c\lambda},$$

then exists a $R_0 \in (0, \varepsilon)$ such that

$$\frac{\overline{h}(R_0)}{R_0} = \frac{1}{8c\lambda}.$$

Let $U = \{(x,y) \in P \times P : \|(x,y)\|_1 < R_0\}$, $(x,y) \in \partial U$ and $\nu \in (0,1)$ be such that $(x,y) = \nu T(x,y)$, i.e. $x = \nu A(x,y), y = \nu B(x,y)$. We claim that $\|(x,y)\|_1 \neq R_0$. In fact, for $(x,y) \in \partial U$ and $\|(x,y)\|_1 = R_0$, we have

$$\begin{aligned} x(t) &= \nu A(x,y)(t) \\ &\leq \lambda \int_0^1 G_{\xi\eta}(t,s)(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*) + e_1(s))ds \\ &\quad + \lambda \int_0^1 K_{\xi\eta}(t,s)(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*) + e_2(s))ds, \\ &\leq \lambda \int_0^1 G_{\xi\eta}(t,s)\overline{f}(R_0)ds + \lambda \int_0^1 K_{\xi\eta}(t,s)\overline{g}(R_0)ds, \\ &\leq \lambda \int_0^1 C_0(1-s)^{\alpha-1}s\overline{h}(R_0)ds + \lambda \int_0^1 C_0(1-s)^{\alpha-1}s\overline{h}(R_0)ds, \\ &\leq 2\lambda \int_0^1 C_0(1-s)^{\alpha-1}sds\overline{h}(R_0), \\ &\leq 2\lambda c\overline{h}(R_0), \end{aligned} \tag{3.5}$$

and similarly, we also have

$$y(t) = \nu B(x,y)(t) \leq 2\lambda c\overline{h}(R_0). \tag{3.6}$$

It follows that

$$R_0 = \|(x,y)\|_1 \leq 4\lambda c\overline{h}(R_0),$$

that is

$$\frac{\overline{h}(R_0)}{R_0} \geq \frac{1}{4c\lambda} > \frac{1}{8c\lambda} = \frac{\overline{h}(R_0)}{R_0},$$

which implies that $\|(x,y)\|_1 \neq R_0$. By the nonlinear alternative of Leray-Schauder type, T has a fixed point $(x,y) \in \overline{U}$. Moreover, combining (3.4)-(3.6) and the fact that $R_0 < \varepsilon$, we obtain

$$\begin{aligned} x(t) &= \lambda \int_0^1 G_{\xi\eta}(t,s)(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*) + e_1(s))ds \\ &\quad + \lambda \int_0^1 K_{\xi\eta}(t,s)(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*) + e_2(s))ds, \\ &\geq \lambda \int_0^1 G_{\xi\eta}(t,s)(\delta f(s,0,0) + e_1(s))ds + \lambda \int_0^1 K_{\xi\eta}(t,s)(\delta g(s,0,0) + e_2(s))ds, \\ &> \lambda \int_0^1 G_{\xi\eta}(t,s)e_1(s)ds + \lambda \int_0^1 K_{\xi\eta}(t,s)e_2(s)ds, \\ &= w_1(t) \quad \text{for} \quad t \in (0,1), \end{aligned}$$

and similarly, we also have

$$y(t) > w_2(t) \quad \text{for} \quad t \in (0,1).$$

Then T has a positive fixed point (x,y) and $\|(x,y)\|_1 \leq R_0 < 1$. Namely, (x,y) is positive solution of the boundary value problem (3.1) with $x(t) > w_1(t)$ and $y(t) > w_2(t)$ for $t \in (0,1)$.

Let $u(t) = x(t) - w_1(t) \geq 0$ and $v(t) = y(t) - w_2(t) \geq 0$. Then (u,v) is a nonnegative solution (positive on $(0,1)$) of the boundary value problem (1.1).

Theorem 3.2 *Suppose that (H_1^*) and (H_3)-(H_4) hold. Then there exists a constant $\lambda^* > 0$ such that, for any $0 < \lambda \leq \lambda^*$, the boundary value problem (1.1) has at least one positive solution.*

Proof. Let $\Omega_1 = \{(x,y) \in E \times E : \|x\| < R_1, \|y\| < R_1\}$, where $R_1 = \max\{1, r\}$ and $r = \frac{C_0^2}{c_0} \int_0^1 (e_1(s) + e_2(s))ds)$. Choose

$$\lambda^* = \min\{1, \frac{R_1}{2}(R+1)^{-1}, \frac{R_1}{2r}\},$$

where $R = \int_0^1 C_0(1-s)^{\alpha-1}s[\max\limits_{0 \leq z_1,z_2 \leq R_1} f(s,z_1,z_2) + \max\limits_{0 \leq z_1,z_2 \leq R_1} g(s,z_1,z_2) + e_1(s) + e_2(s)]ds$ and $R \geq 0$.

Then, for any $(x,y) \in (P \times P) \cap \partial\Omega_1$, we have $\|x\| = R_1$ or $\|y\| = R_1$. Moreover $x(s) - w_1(s) \leq x(s) \leq \|x\| \leq R_1$, $y(s) - w_2(s) \leq y(s) \leq \|y\| \leq R_1$, and it follows that

$$\begin{aligned} \|A(x,y)(t)\| &\leq \lambda \int_0^1 C_0(1-s)^{\alpha-1}s(f(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*) + e_1(s))ds \\ &\quad + \lambda \int_0^1 C_0(1-s)^{\alpha-1}s(g(s,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*) + e_2(s))ds, \\ &\leq \lambda \int_0^1 C_0(1-s)^{\alpha-1}s(\max_{0 \leq z_1,z_2 \leq R_1} f(s,z_1,z_2) + e_1(s))ds \\ &\quad + \lambda \int_0^1 C_0(1-s)^{\alpha-1}s(\max_{0 \leq z_1,z_2 \leq R_1} g(s,z_1,z_2) + e_2(s))ds, \\ &\leq \lambda \int_0^1 C_0(1-s)^{\alpha-1}s[\max_{0 \leq z_1,z_2 \leq R_1} f(s,z_1,z_2) + \max_{0 \leq z_1,z_2 \leq R_1} g(s,z_1,z_2) + e_1(s) + e_2(s)]ds, \\ &\leq \lambda R, \\ &\leq \tfrac{R_1}{2}, \end{aligned}$$

and similarly, we also have

$$\|B(x,y)(t)\| \le \tfrac{R_1}{2}.$$

This implies

$$\|T(x,y)\|_1 = \|A(x,y)\| + \|B(x,y)\| \le R_1 \le \|(x,y)\|_1,\ (x,y) \in (P\times P)\cap \partial\Omega_1.$$

On the other hand, choose a constant $N > 1$ such that

$$\lambda N \frac{c_0^2}{2C_0}\gamma \int_{\theta_1}^{\theta_2} (1-s)^{\alpha-1} s^{\alpha} ds \ge 1,$$

where $\gamma = \min\limits_{\theta_1\le t\le\theta_2}\{t^{\alpha-1}\}$.

By assumptions (H_3) and (H_4), there exists a constant $B > R_1$ such that

$$\frac{f(t,z_1,z_2)}{z_1} > N, \quad \text{namely} \quad f(t,z_1,z_2) > Nz_1, \quad \text{for} \quad t\in[\theta_1,\theta_2],\ z_2>0, z_1>B$$

and

$$\frac{g(t,z_1,z_2)}{z_2} > N, \quad \text{namely} \quad g(t,z_1,z_2) > Nz_2, \quad \text{for} \quad t\in[\theta_1,\theta_2],\ z_1>0, z_2>B.$$

Choose $R_2 = \max\{R_1+1, 2\lambda r, \frac{2C_0(B+1)}{c_0\gamma}\}$, and let $\Omega_2 = \{(x,y)\in E\times E : \|x\| < R_2, \|y\| < R_2\}$. Then for any $(x,y)\in(P\times P)\cap\partial\Omega_2$, we have $\|x\| = R_2$ or $\|y\| = R_2$. If $\|x\| = R_2$, then

$$\begin{aligned}
x(t) - w_1(t) &= x(t) - (\lambda \textstyle\int_0^1 G_{\xi\eta}(t,s)e_1(s)ds + \lambda\int_0^1 K_{\xi\eta}(t,s)e_2(s)ds)\\
&\ge x(t) - (\lambda \textstyle\int_0^1 C_0 t^{\alpha-1} e_1(s)ds + \lambda\int_0^1 C_0 t^{\alpha-1} e_2(s)ds)\\
&= x(t) - (\lambda C_0 t^{\alpha-1} \textstyle\int_0^1 (e_1(s)+e_2(s))ds)\\
&= x(t) - (\lambda \tfrac{c_0}{C_0} t^{\alpha-1} \tfrac{C_0^2}{c_0} \textstyle\int_0^1 (e_1(s)+e_2(s))ds)\\
&= x(t) - \lambda \tfrac{c_0}{C_0} t^{\alpha-1} r\\
&\ge x(t) - \tfrac{x(t)}{\|x\|}\lambda r\\
&\ge x(t) - \tfrac{x(t)}{R_2}\lambda r\\
&\ge (1 - \tfrac{\lambda r}{R_2}) x(t)\\
&\ge \tfrac{1}{2}x(t) \ge 0,\ t\in[0,1],
\end{aligned}$$

and then

$$\begin{aligned}
\min_{\theta_1\le t\le\theta_2}\{[x(t)-w_1(t)]^*\} &= \min_{\theta_1\le t\le\theta_2}\{x(t)-w_1(t)\} \ge \min_{\theta_1\le t\le\theta_2}\{\tfrac{1}{2}x(t)\}\\
&\ge \min_{\theta_1\le t\le\theta_2}\{\tfrac{c_0}{2C_0} t^{\alpha-1}\|x\|\} = \tfrac{c_0}{2C_0} R_2 \min_{\theta_1\le t\le\theta_2}\{t^{\alpha-1}\} \ge B+1 > B.
\end{aligned}$$

Since $B > R_1 \ge m_0$, we have

$$f(t,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*) > N[x(t)-w_1(t)]^* \ge \frac{N}{2}x(t), \quad \text{for} \quad t\in[\theta_1,\theta_2].$$

It follows that

$$\begin{aligned}
A(x,y)(t) &= \lambda \textstyle\int_0^1 G_{\xi\eta}(t,s)(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + e_1(s))ds\\
&\quad + \lambda \textstyle\int_0^1 K_{\xi\eta}(t,s)(g(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + e_2(s))ds,\\
&\ge \lambda \textstyle\int_0^1 G_{\xi\eta}(t,s)(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + e_1(s))ds\\
&\ge \lambda \textstyle\int_{\theta_1}^{\theta_2} G_{\xi\eta}(t,s) f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)ds\\
&\ge \lambda \textstyle\int_{\theta_1}^{\theta_2} c_0 t^{\alpha}(1-s)^{\alpha-1} s \tfrac{N}{2} x(s)ds\\
&\ge \lambda t^{\alpha} \textstyle\int_{\theta_1}^{\theta_2} c_0 (1-s)^{\alpha-1} s \tfrac{N}{2}\tfrac{c_0}{C_0} s^{\alpha-1}\|x\|ds\\
&\ge \lambda t^{\alpha} \textstyle\int_{\theta_1}^{\theta_2} c_0 (1-s)^{\alpha-1} s N \tfrac{c_0}{2C_0} s^{\alpha-1} R_2 ds\\
&\ge \lambda N \tfrac{c_0^2}{2C_0}\gamma \textstyle\int_0^1 (1-s)^{\alpha-1} s^{\alpha} ds R_2\\
&\ge R_2, \quad t\in[\theta_1,\theta_2].
\end{aligned}$$

If $\|y\| = R_2$, we have

$$y(t) - w_2(t) = y(t) - (\lambda \int_0^1 G_{\eta\xi}(t,s)e_1(s)ds + \lambda \int_0^1 K_{\eta\xi}(t,s)e_2(s)ds) \geq \frac{1}{2}y(t) \geq 0, \ t \in [0,1],$$

and

$$\begin{aligned} \min_{\theta_1 \leq t \leq \theta_2} \{[y(t) - w_2(t)]^*\} &= \min_{\theta_1 \leq t \leq \theta_2} \{y(t) - w_2(t)\} \geq \min_{\theta_1 \leq t \leq \theta_2} \{\tfrac{1}{2}y(t)\} \\ &\geq \min_{\theta_1 \leq t \leq \theta_2} \{\tfrac{c_0}{2C_0} t^{\alpha-1}\|y\|\} = \tfrac{c_0}{2C_0} R_2 \min_{\theta_1 \leq t \leq \theta_2} \{t^{\alpha-1}\} \geq B + 1 > B. \end{aligned}$$

Then, for any $(x,y) \in (P \times P) \cap \partial\Omega_2$, we also have

$$g(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) > N[y(t) - w_2(t)]^* \geq \frac{N}{2} y(t), \quad \text{for} \quad t \in [\theta_1, \theta_2].$$

It follows that

$$\begin{aligned} A(x,y)(t) &= \lambda \int_0^1 G_{\xi\eta}(t,s)(f(s,[x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) + e_1(s))ds \\ &\quad + \lambda \int_0^1 K_{\xi\eta}(t,s)(g(s,[x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) + e_2(s))ds, \\ &\geq \lambda \int_{\theta_1}^{\theta_2} K_{\xi\eta}(t,s) g(s,[x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*)ds, \\ &\geq \lambda \int_{\theta_1}^{\theta_2} c_0 t^\alpha (1-s)^{\alpha-1} s g(s,[x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*)ds, \\ &\geq \lambda t^\alpha \int_{\theta_1}^{\theta_2} c_0 (1-s)^{\alpha-1} s \tfrac{N}{2} y(s) ds \\ &\geq \lambda t^\alpha \int_{\theta_1}^{\theta_2} c_0 (1-s)^{\alpha-1} s \tfrac{N}{2} \tfrac{c_0}{C_0} s^{\alpha-1}\|y\| ds \\ &\geq \lambda t^\alpha \int_{\theta_1}^{\theta_2} c_0 (1-s)^{\alpha-1} s^\alpha N \tfrac{c_0}{2C_0} R_2 ds \\ &\geq \lambda N \tfrac{c_0^2}{2C_0} \gamma \int_0^1 (1-s)^{\alpha-1} s^\alpha ds R_2 \\ &\geq R_2, \quad t \in [\theta_1, \theta_2]. \end{aligned}$$

Thus, for any $(x,y) \in (P \times P) \cap \partial\Omega_2$, we always have

$$A(x,y)(t) \geq R_2, \quad t \in [\theta_1, \theta_2].$$

Similarly, for any $(x,y) \in (P \times P) \cap \partial\Omega_2$, we also have

$$B(x,y)(t) \geq R_2, \quad t \in [\theta_1, \theta_2].$$

This implies

$$\|T(x,y)\|_1 \geq \|(x,y)\|_1, (x,y) \in (P \times P) \cap \partial\Omega_2.$$

Thus condition (2) of Krasnoesel'skii's fixed-point theorem is satisfied. As a result T has a fixed point (x,y) with $r \leq R_1 < \|x\| < R_2, \ r \leq R_1 < \|y\| < R_2$.

Also since $r < R_1 < \|x\|$ and $r < R_1 < \|y\|$, then

$$\begin{aligned} x(t) - w_1(t) &\geq \tfrac{c_0}{C_0} t^{\alpha-1}\|x\| - (\lambda \int_0^1 G_{\xi\eta}(t,s)e_1(s)ds + \lambda \int_0^1 K_{\xi\eta}(t,s)e_2(s)ds) \\ &\geq \tfrac{c_0}{C_0} t^{\alpha-1}\|x\| - \lambda \tfrac{c_0}{C_0} t^{\alpha-1} r \\ &\geq \tfrac{c_0}{C_0} t^{\alpha-1} r - \lambda \tfrac{c_0}{C_0} t^{\alpha-1} r \\ &\geq (1-\lambda) \tfrac{c_0}{C_0} t^{\alpha-1} r \\ &> 0, \ t \in (0,1), \end{aligned}$$

and

$$\begin{aligned} y(t) - w_2(t) &= y(t) - (\lambda \int_0^1 G_{\eta\xi}(t,s)e_2(s)ds + \lambda \int_0^1 K_{\eta\xi}(t,s)e_1(s)ds) \\ &\geq y(t) - (\lambda \int_0^1 C_0 t^{\alpha-1} e_2(s)ds + \lambda \int_0^1 C_0 t^{\alpha-1} e_1(s)ds) \\ &= y(t) - (\lambda C_0 t^{\alpha-1} \int_0^1 (e_1(s) + e_2(s))ds) \\ &= \tfrac{c_0}{C_0} t^{\alpha-1}\|y\| - \lambda \tfrac{c_0}{C_0} t^{\alpha-1} r \\ &\geq \tfrac{c_0}{C_0} t^{\alpha-1} r - \lambda \tfrac{c_0}{C_0} t^{\alpha-1} r \\ &\geq (1-\lambda) \tfrac{c_0}{C_0} t^{\alpha-1} r \\ &> 0, \ t \in (0,1). \end{aligned}$$

Thus, (x, y) is positive solution of the boundary value problem (3.1) with $x(t) > w_1(t)$ and $y(t) > w_2(t)$ for $t \in (0, 1)$.

Let $u(t) = x(t) - w_1(t) \geq 0$ and $v(t) = y(t) - w_2(t) \geq 0$. Then (u, v) is a nonnegative solution (positive on $(0, 1)$) of the boundary value problem (1.1).

Remark From the proof of Theorem 3.2, clearly condition (H_3) can be replaced by condition (H_3^*)

Theorem 3.3 *Suppose that (H_1^*), (H_3^*) and (H_4) hold. Then there exists a constant $\lambda^* > 0$ such that, for any $0 < \lambda \leq \lambda^*$, the boundary value problem (1.1) has at least one positive solution.*

Since condition (H_1) implies conditions (H_1^*) and (H_4), then from the proof of Theorem 3.1 and 3.2, we immediately have the following theorem:

Theorem 3.4 *Suppose that (H_1)-(H_3) hold. Then the boundary value problem (1.1) has at least two positive solutions for $\lambda > 0$ sufficiently small.*

In fact, let $0 < \lambda < \min\{\overline{\lambda}, \lambda^*\}$, then the boundary value problem (1.1) has at least two positive solutions.

Similarly we have

Theorem 3.5 *Suppose that (H_1)-(H_2) and (H_3^*) hold. Then the boundary value problem (1.1) has at least two positive solutions for $\lambda > 0$ sufficiently small.*

4 Example

To illustrate the usefulness of the results, we give some examples.

Example 4.1 Consider the boundary value problem

$$\begin{cases} -\mathbf{D}_{0+}^{\alpha} u = \lambda(u^c + \frac{1}{(t-t^2)^{\frac{1}{2}}} \cos(\pi v)), \quad t \in (0,1), \lambda > 0, \\ -\mathbf{D}_{0+}^{\alpha} v = \lambda(v^d + \frac{1}{(t-t^2)^{\frac{1}{2}}} \sin(2\pi u)), \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ u(1) = av(\xi), v(1) = bu(\eta), \quad \xi, \eta \in (0,1) \end{cases} \tag{4.1}$$

where $c, d > 1$. Then, if $\lambda > 0$ is sufficiently small, (4.1) has a positive solution (u, v) with $u > 0, v > 0$ for $t \in (0, 1)$.

To see this we will apply Theorem 3.2 with

$$f(t,u,v) = u^c + \frac{1}{(t-t^2)^{\frac{1}{2}}} \cos(\pi v), \quad g(t,u,v) = v^d + \frac{1}{(t-t^2)^{\frac{1}{2}}} \sin(2\pi u),$$
$$e_i(t) = e(t) = \frac{2}{(t-t^2)^{\frac{1}{2}}} \quad (i = 1, 2).$$

Clearly, for $t \in (0, 1)$,

$$f(t,u,v) + e(t) \geq u^c + \frac{1}{(t-t^2)^{\frac{1}{2}}} > 0, \quad g(t,u,v) + e(t) \geq v^d + \frac{1}{(t-t^2)^{\frac{1}{2}}} > 0, \text{ for } t \in (0,1);$$
$$\lim_{u\uparrow+\infty} \inf \frac{f(t,u,v)}{u} = +\infty, \quad \lim_{v\uparrow+\infty} \inf \frac{g(t,u,v)}{v} = +\infty, \text{ for } \forall\ t \in [\theta_1, \theta_2] \subset (0,1),$$

for $u, v \geq 0$. Thus (H_1^*) and (H_3)-(H_4) hold. Let $r = \frac{2C_0^2}{c_0} \int_0^1 \frac{2}{(s-s^2)^{\frac{1}{2}}} ds = \frac{2\pi C_0^2}{c_0}$ and let $R_1 = 1 + r$.

We have

$$R = \int_0^1 C_0(1-s)^{\alpha-1} s[\max_{0\leq z_1,z_2\leq R_1} f(s,z_1,z_2) + \max_{0\leq z_1,z_2\leq R_1} g(s,z_1,z_2) + \frac{4}{(s-s^2)^{\frac{1}{2}}}]ds \leq C_0(R_1^c + R_1^d + \pi).$$

Let

$$\lambda^* = \min\{1, R_1(R+1)^{-1}, \frac{R_1}{2r}\}.$$

Now, if $\lambda < \lambda^*$, Theorem 3.2 guarantees that (4.1) has a positive solution (u, v) with $\|u\| \geq 2\pi$ and $\|v\| \geq 2\pi$.

Example 4.2 Consider the boundary value problem

$$\begin{cases} -\mathbf{D}_{0+}^{\alpha} u = \lambda((u-a)(u-b) + \cos(\frac{\pi}{2a}v)), & t \in (0,1), \lambda > 0, \\ -\mathbf{D}_{0+}^{\alpha} v = \lambda((v-c)(v-d) + \sin(\frac{\pi}{c}u)), \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ u(1) = av(\xi), v(1) = bu(\eta), \quad \xi, \eta \in (0,1) \end{cases} \tag{4.2}$$

where $b > a > 0$, $d > c > 0$. Then, if $\lambda > 0$ is sufficiently small, (4.2) has two solutions (u_1, v_1), (u_2, v_2) with $u_i(t) > 0, v_i(t) > 0$ for $t \in (0,1), i = 1, 2$.

To see this we will apply Theorem 3.4 with

$$f(t,u,v) = (u-a)(u-b) + \cos(\frac{\pi}{2a}v) \quad \text{and} \quad g(t,u,v) = (v-c)(v-d) + \sin(\frac{\pi}{c}u).$$

Clearly, there exists a constant $e_1(t) = e_2(t) = M_0 > 0$ such that

$$f(t,u,v) + M_0 > 0, \quad g(t,u,v) + M_0 > 0, \text{ for } \forall\ t \in (0,1).$$

Let $\delta = \frac{1}{16(ab+cd+1)} \min\{ab, cd\}$, $\varepsilon = \frac{1}{4}\min\{1, a, b\}$ and $c = \int_0^1 C_0(1-s)^{\alpha-1} s ds$. We have

$$f(t, z_1, z_2) \geq \delta f(t,0,0) = \delta(ab+1), \quad g(t,z_1,z_2) \geq \delta g(t,0,0) = \delta cd, \quad \text{for} \quad 0 \leq t \leq 1, \quad 0 \leq z_1, z_2 \leq \varepsilon.$$

Thus (H_1)-(H_2) hold. Since

$$\begin{aligned} \overline{f}(\varepsilon) &= \max_{0 \leq t \leq 1, 0 \leq u, v \leq \varepsilon} \{f(t,u,v) + e_1(t)\} \leq ab + cd + 1, \\ \overline{g}(\varepsilon) &= \max_{0 \leq t \leq 1, 0 \leq u, v \leq \varepsilon} \{g(t,u,v) + e_2(t)\} \leq ab + cd + 1, \\ \overline{h}(\varepsilon) &= \max\{\overline{f}(\varepsilon), \overline{g}(\varepsilon)\} \leq ab + cd + 1, \end{aligned}$$

we can choose

$$\overline{\lambda} = \frac{\varepsilon}{8c(ab+cd+1)}. \tag{4.3}$$

Now, if $\lambda < \overline{\lambda}$, Theorem 3.1 guarantees that (4.2) has a positive solution (u_1, v_1) with $\|u_1\| \leq \frac{1}{4}$.

On the other hand,

$$\lim_{u \uparrow +\infty} \inf \frac{f(t,u,v)}{u} = +\infty, \quad \lim_{v \uparrow +\infty} \inf \frac{g(t,u,v)}{v} = +\infty \text{ for } \forall\ t \in [\theta_1, \theta_2] \subset (0,1), \quad u, v \in (0, \infty).$$

Thus (H_1)-(H_4) also hold. Let $r = \frac{2C_0^2}{c_0}$ and $R_1 > 1 + r$. We have

$$R = \int_0^1 C_0(1-s)^{\alpha-1} s[\max_{0 \leq z_1, z_2 \leq R_1} f(s, z_1, z_2) + \max_{0 \leq z_1, z_2 \leq R_1} g(s, z_1, z_2) + 2M_0] ds$$

and

$$\lambda^* = \min\{1, \frac{R_1}{2}(R+1)^{-1}, \frac{R_1}{2r}\}.$$

Now, if $0 < \lambda < \lambda^*$, Theorem 3.2 guarantees that (4.2) has a positive solution (u_2, v_2) with $\|u_2\| \geq 1$.

Since all the conditions of Theorem 3.4 are satisfied , if $\lambda < \min\{\overline{\lambda}, \lambda^*\}$, Theorem 3.4 guarantees that (4.2) has two solutions u_i with $u_i(t) > 0$ for $t \in (0,1), i = 1, 2$.

Example 4.3 Consider the boundary value problem

$$\begin{cases} -\mathbf{D}_{0+}^{\alpha} u = \lambda(v^c + \cos(2\pi u)), & t \in (0,1), \lambda > 0, \\ -\mathbf{D}_{0+}^{\alpha} v = \lambda(u^d + \cos(2\pi v)), \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ u(1) = av(\xi), v(1) = bu(\eta), \quad \xi, \eta \in (0,1) \end{cases} \tag{4.4}$$

where $c, d > 1$. Then, if $\lambda > 0$ is sufficiently small, (4.4) has two solutions (u_1, v_1), (u_2, v_2) with $u_i(t) > 0, v_i(t) > 0$ for $t \in (0,1), i = 1, 2$.

To see this we will apply Theorem 3.5 with

$$f(t,u,v) = v^c + \cos(2\pi u), \quad g(t,u,v) = u^d + \cos(2\pi v), \quad e(t) = 2.$$

Clearly,

$$\begin{aligned} &f(t,u,v) + e(t) \geq v^\alpha + 1 > 0, \quad g(t,u,v) + e(t) \geq u^\alpha + 1 > 0 \ \text{ for } \ t \in (0,1), \\ &f(t,0,0) = 1 > 0, \quad g(t,0,0) = 1 > 0, \\ &\lim_{v \uparrow +\infty} \inf \tfrac{f(t,u,v)}{v} = +\infty, \quad \lim_{u \uparrow +\infty} \inf \tfrac{g(t,u,v)}{u} = +\infty \ \text{ for } \ \forall\, t \in [\theta_1, \theta_2] \subset (0,1). \end{aligned}$$

Thus (H_1)-(H_2) and (H_3^*) hold.

First, let $\delta = \frac{1}{2}$, $\varepsilon = \frac{1}{8}$ and $c = \int_0^1 C_0(1-s)^{\alpha-1} s ds$. We have

$$\begin{aligned} \overline{f}(\varepsilon) &= \max_{0 \leq t \leq 1, 0 \leq u,v \leq \varepsilon} \{f(t,u,v) + e_1(t)\} \leq 8^{-c} + 3, \\ \overline{g}(\varepsilon) &= \max_{0 \leq t \leq 1, 0 \leq u,v \leq \varepsilon} \{g(t,u,v) + e_2(t)\} \leq 8^{-d} + 3, \\ \overline{h}(\varepsilon) &= \max\{\overline{f}(\varepsilon), \overline{g}(\varepsilon)\}, \end{aligned}$$

then $\frac{\varepsilon}{8c\overline{h}(\varepsilon)} \geq \frac{1}{8c(1+3)} = \frac{1}{32c}$.

Let $\overline{\lambda} = \frac{1}{32c}$. Now, if $0 < \lambda < \overline{\lambda}$ then $0 < \lambda < \frac{\varepsilon}{8c\overline{h}(\varepsilon)}$, Theorem 3.1 guarantees that (4.4) has a positive solution (u_1, v_1) with $\|u_1\| \leq \frac{1}{8}$.

Next, from $r = \frac{4C_0^2}{c_0}$ and let $R_1 = 1 + r$. Then, we have

$$R = \int_0^1 C_0(1-s)^{\alpha-1} s[\max_{0 \leq z_1, z_2 \leq R_1} f(s, z_1, z_2) + \max_{0 \leq z_1, z_2 \leq R_1} g(s, z_1, z_2) + 4] ds.$$

Let $\lambda^* = \min\{1, \frac{R_1}{2}(R+1)^{-1}, \frac{R_1}{2r}\}$. Now, if $0 < \lambda < \lambda^*$ then Theorem 3.3 guarantees that (4.4) has a positive solution (u_2, v_2) with $\|u_2\| \geq 1$.

So, if $\lambda < \min\{\overline{\lambda}, \lambda^*\}$, Theorem 3.5 guarantees that (4.4) has two solutions (u_1, v_1) and (u_2, v_2) with $u_i, v_i > 0$ for $t \in (0,1), i = 1, 2$.

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