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Existence of solutions for discontinuous systems of Stieltjes differential equations

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Abstract. We introduce new transversality conditions in terms of derivatives with respect to a nondecreasing function $g : \mathbb{R} \longrightarrow \mathbb{R}$, and we use them to prove the existence of solutions for discontinuous systems of Stieltjes differential equations, i.e. differential equations where usual derivatives are replaced with derivatives with respect to the derivator g. The proof of our main existence result leans on some interesting new results on g-derivatives also proven in this paper.

Keywords: discontinuous differential equations, Carathéodory solutions, Stieltjes differential equations.

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1 Introduction

We are concerned with initial value problems for first order systems of Stieltjes differential equations of the form

$$x'_g = f(t,x), \quad t \in I = [t_0, t_0 + L), \; x(t_0) = x_0, \tag{1.1}$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing function, $t_0, L \in \mathbb{R}$, $L > 0$, $x_0 \in \mathbb{R}^n$ ($n \in \mathbb{N}$) and $f = (f_1, f_2, \dots, f_n) : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ need not be continuous.

As usual, x'_g stands for the derivative with respect to g which, roughly speaking, means

$$x'_g(t) = \lim_{s \to t} \frac{x(s) - x(t)}{g(s) - g(t)},$$

and it allows us to study difference and differential equations, with or without impulses, under the unified formulation of (1.1).

Existence of Carathéodory type solutions for (1.1) has recently been well studied for the case when $f(t,x)$ is continuous with respect to the x variable. See [5,6,11–14]. In this paper,

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we intend to improve on the previous references by allowing the nonlinear part $f(t,x)$ to present many discontinuities. To do so, we show that discontinuities do not matter as long as they can be covered by some subsets of $I \times \mathbb{R}^n$ satisfying a novel transversality condition with respect to the g-differential equation in (1.1).

Our new transversality condition, inspired by an idea by Bressan and Shen [2], later generalized in [10], leans on some new results on g-differentiation which are interesting in its own right. We manage to translate the ideas the authors brought in [10] from usual differential equations to Stieltjes differental equations, generalising their results even further. We achieve this by introducing some new interesting properties on the g-derivatives.

This paper is organized as follows. In Section 2 we include the basic preliminaries on g-differential equations so that this paper be self-contained; in Section 3 we prove some new results on g-differentiation, specially, on a chain rule for functions of several variables which is essential for our definition of local transversality conditions; in Section 4 we prove our main existence result for (1.1).

2 Preliminaries on Stieltjes derivatives and differential equations

In what follows, $g : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone, nondecreasing and continuous from the left everywhere. Let us consider the set of points around which g is constant, namely

$$C_g = \{t \in \mathbb{R} : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}, \tag{2.1}$$

and the set of discontinuity points of g, which is at most countable and shall be denoted by

$$D_g = \{t \in \mathbb{R} : g(t^+) - g(t) > 0\}, \tag{2.2}$$

where, as usual, $g(t^+) = \lim_{s \to t^+} g(s) \in \mathbb{R}$.

Following [11], we define the *derivative with respect to* g (or g*-derivative*) of a real-valued real function f at a point $t_0 \in \mathbb{R} \setminus C_g$ as follows, provided that the corresponding limit exists:

$$f'_g(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{g(t) - g(t_0)} \quad \text{if } t_0 \notin D_g, \text{ or} \tag{2.3}$$

$$f'_g(t_0) = \lim_{t \to t_0^+} \frac{f(t) - f(t_0)}{g(t) - g(t_0)} \quad \text{if } t_0 \in D_g, \tag{2.4}$$

where C_g and D_g are defined in (2.1) and (2.2) respectively. Notice that the g-derivative at a point $t_0 \in D_g$ exists if, and only if, $f(t_0^+)$ exists and (2.4) can be rewritten as

$$f'_g(t_0) = \frac{f(t_0^+) - f(t_0)}{g(t_0^+) - g(t_0)}. \tag{2.5}$$

We say that f is *g-differentiable at t_0* if $f'_g(t_0)$ exists, and we say that f is *g-differentiable in a set $A \subset \mathbb{R}$* when f is g-differentiable at every $t_0 \in A \setminus C_g$. There will be no need to define g-derivatives for the points in C_g because, according to [10, Proposition 2.5], we have $\mu_g(C_g) = 0$, where μ_g stands for the Lebesgue–Stieltjes measure induced by g. However, it is possible to give a reasonable definition of a g-derivative everywhere, see [13], but we shall not need it in this paper.

Roughly speaking, g-differentiable functions can be recovered by integrating their g-derivatives in the Stieltjes sense with respect to g. More precisely, we have the following Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral [10, Theorem 5.4].

Theorem 2.1 (Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral)**.** *Let $a, b \in \mathbb{R}$, $a < b$, and $F : [a, b] \longrightarrow \mathbb{R}$. The following conditions are equivalent.*

(1) The function F is absolutely continuous with respect to g *(or g-*absolutely continuous*) according to the following definition: to each $\varepsilon > 0$, there is some $\delta > 0$ such that, for any family $\{(a_n, b_n)\}_{n=1}^{m}$ of pairwise disjoint open subintervals of $[a, b]$, the inequality*

$$\sum_{n=1}^{m} (g(b_n) - g(a_n)) < \delta$$

implies

$$\sum_{n=1}^{m} |F(b_n) - F(a_n)| < \varepsilon.$$

(2) The function F fulfills the following properties:

(a) there exists $F'_g(t)$ for g-almost all $t \in [a, b)$ (i.e., for all t except on a set of μ_g measure zero);

(b) $F'_g \in \mathcal{L}^1_g([a, b))$, the set of Lebesgue–Stieltjes integrable functions with respect to μ_g;

(c) for each $t \in [a, b]$, we have

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, d\mu_g. \tag{2.6}$$

We remark that $\mu_g(\{t\}) = g(t^+) - g(t) > 0$ when $t \in D_g$ (see equation (6) in [10]). In particular, if $t \in [a, b) \cap D_g$, then, (2.6) implies that

$$F(t^+) - F(t) = \int_{\{t\}} F'_g(s)\, d\mu_g = F'_g(t)(g(t^+) - g(t)),$$

which may be nonzero. Therefore, g-absolutely continuous functions need not be continuous at discontinuity points of g.

However, g-absolutely continuous functions have some nice properties.

Proposition 2.2 ([10, Proposition 5.3])**.** *If F is g-absolutely continuous on $[a, b]$, then it has bounded variation and it is continuous from the left at every $x \in [a, b)$.*

Moreover, F is continuous in $[a, b] \setminus D_g$, where D_g is the set of discontinuity points of g, and if g is constant on some $(\alpha, \beta) \subset [a, b]$, then F is constant on (α, β) as well.

We say that a vector-valued function $F : [a, b] \longrightarrow \mathbb{R}^n$ is g-absolutely continuous on $[a, b]$ if each of its components is g-absolutely continuous. We denote by $\mathcal{AC}_g([a, b])$ the set of g-absolutely continuous functions on $[a, b]$ with values in $\mathbb{R}^n$.

Definition 2.3. A (Carathéodory) *solution* of (1.1) is a g-absolutely continuous function $x : [t_0, t_0 + L] \longrightarrow \mathbb{R}^n$ such that $x(t_0) = x_0$, and

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } (g\text{-a.a.})\ t \in [t_0, t_0 + L).$$

Next we introduce a useful splitting of g into its continuous and discrete parts. This is well known for bounded variation functions on bounded intervals, see, for instance, [1, Theorem 2.6.1], but we have to adjust it for functions defined on the whole real line.

Denote

$$\Delta g(t) = g(t^+) - g(t),$$

for all $t \in \mathbb{R}$ and define the function $g^B : \mathbb{R} \to \mathbb{R}$ as follows:

$$g^B(t) = \begin{cases} \sum_{s\in[0,t)} \Delta g(s), & t > 0, \\ -\sum_{s\in[t,0)} \Delta g(s), & t \leq 0. \end{cases} \tag{2.7}$$

We have that g^B is nondecreasing and left-continuous. We will say that g^B is the *discontinuous* or *jump part* of g. We say g is *totally discontinuous* when $g = g^B \neq 0$. We define the *continuous part* of g as follows:

$$g^C(t) := g(t) - g^B(t), \quad \forall t \in \mathbb{R}.$$

Thus, g^C is nondecreasing and continuous (in the usual sense).

By definition we have that $g = g^C + g^B$. In particular we have, over Borel sets, that $\mu_g = \mu_{g^C} + \mu_{g^B}$. It holds that

$$g^C(s) - g^C(t) \leq g(s) - g(t)$$
$$g^B(s) - g^B(t) \leq g(s) - g(t)$$

for all $t, s \in \mathbb{R}$, $t < s$.

Now we recall the notion of g-topology in $\mathbb{R}$, introduced for the first time in [6].

Definition 2.4. Given any $t \in \mathbb{R}$ and $\varepsilon > 0$, we define the g-ball of center t and radius ε as the set

$$B_g(t,\varepsilon) := \{s \in \mathbb{R} \mid |g(s) - g(t)| < \varepsilon\}.$$

Define the g-topology as

$$\tau_g := \{ U \subset \mathbb{R} \mid \forall t \in U \ \exists \varepsilon > 0 : B_g(t,\varepsilon) \subset U \}.$$

For any $\lambda > 0$ and $B_g(t,\varepsilon)$ denote $\lambda B_g(t,\varepsilon) = B_g(t,\lambda\varepsilon)$. Denote also $B_g[t,\varepsilon]$ as the closed ball

$$B_g[t,\varepsilon] = \{s \in \mathbb{R} \mid |g(s) - g(t)| \leq \varepsilon\},$$

which need not be a closed set in the usual topology (for instance, $B_g(1,1/2) = (0,\infty)$ if $g(t) = 0$ for $t \leq 0$ and $g(t) = 1$ for $t > 0$).

Finally, since C_g is open, we can express it as a countable union of disjoint open intervals

$$C_g = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

and we denote by N_g the set of the real endpoints of all the intervals (a_n, b_n). The following technical information will be used in Section 3.

Proposition 2.5. *Let $t \in \mathbb{R}$ be such that $t \notin D_g \cup N_g \cup C_g$. Then any set N is a usual neighborhood of t iff N is a neighborhood of t in the topology τ_g.*

Proof. First, note that $\mathbb{R} - (C_g \cup N_g) = \{t \in \mathbb{R} : g(s) \neq g(t) \ \forall s \in \mathbb{R}, s \neq t\}$. Since $t \notin D_g$, g is continuous at t. Hence,

$$\forall \epsilon > 0 \ \exists \delta > 0 : (t - \delta, t + \delta) \subset B_g(t,\epsilon).$$

Fix $\epsilon > 0$ now, take $\delta = \min\{|g(t) - g(t-\epsilon)|, |g(t+\epsilon) - g(t)|\} > 0$, thus

$$B_g(t,\delta) \subset (t - \epsilon, t + \epsilon).$$

Equivalently, the function Id is g-continuous at t. In fact, g-continuity at t is equivalent to usual continuity at t. Every local basis of t (usual or on τ_g) is a local basis on both topologies. The claim is then proved. □

Remark 2.6. Let f be a function defined on a neighborhood $(t-\epsilon, t+\epsilon)$ of $t \notin D_g \cup N_g \cup C_g$. If $f'_g(t)$ exists, then f is continuous at t.

To prove it, simply observe that there exists $\delta > 0$ such that

$$|f(t) - f(s)| \leq (|f'_g(t)| + 1)|g(t) - g(s)| \quad \forall s \in (t-\delta, t+\delta),$$

which implies that f is continuous at t because so g is.

3 Covering lemmas and fine results on g-derivatives

This section is devoted to proving some technical results on g-null sets and g-derivatives which will play a fundamental role in the proof of our main existence result for (1.1).

We start with the following lemma on existence of special subcollections of arbitrary collections of sets.

Lemma 3.1. *Let X be a nonempty set and let $\mathcal{B}$ be a collection of subsets of X containing at least one nonempty set. Then, there exists a subcollection of subsets $\mathcal{C} \subset \mathcal{B}$ satisfying the following conditions:*

(1) $A \cap B = \emptyset$, for all $A, B \in \mathcal{C}$, $A \neq B$.

(2) For any $B \in \mathcal{B}$, there exists some $C \in \mathcal{C}$ such that $B \cap C \neq \emptyset$.

Proof. Let Ω be the set of all disjoint subcollections of $\mathcal{B}$ equipped with the partial order $\subset$. Clearly $\Omega \neq \emptyset$ as $\{B\} \in \Omega$ for any $B \in \mathcal{B}$. Let Λ be a nonempty chain in Ω. If it were to happen that

$$\mathcal{D} = \bigcup_{\mathcal{A} \in \Lambda} \mathcal{A} \in \Omega,$$

then $\mathcal{D}$ would be an upper bound of Λ. Let us prove the above. Clearly, $\mathcal{D} \subset \mathcal{B}$ since $\mathcal{A} \subset \mathcal{B}$ for all $\mathcal{A} \in \Lambda$. Besides, for any $A, B \in \mathcal{D}$, $A \neq B$, there exists some $\mathcal{A} \in \Lambda$ such that $A, B \in \mathcal{A}$, therefore, $A \cap B = \emptyset$. Hence, $\mathcal{D} \in \Omega$. By Zorn's lemma there exists some maximal element $\mathcal{C} \in \Omega$. $\mathcal{C}$ satisfies (1) and (2). □

As we will prove now, we can recover the Vitali Covering Theorem using the balls defined with the derivator. Our proof is basically identical to [4, Theorem 1.24], since the classical proof holds for the Stieltjes case as well.

Theorem 3.2 (The Stieltjes–Vitali Covering Theorem)**.** *Let Ω be an arbitrary collection of balls such that*

$$R = \sup\{\, \mathrm{rad}(B_g) \ : \ B_g \in \Omega \,\} < \infty$$

where $\mathrm{rad}(B_g)$ denotes the radius of B_g. Then, there exists a countable disjoint subcollection $\Lambda \subset \Omega$ of balls such that

$$\bigcup_{B_g \in \Omega} B_g \subset \bigcup_{A_g \in \Lambda} 5A_g.$$

Besides, each $B_g \in \Omega$ intersects some $A_g \in \Lambda$ such that $B_g \subset 5A_g$.

Proof. Denote

$$F_j = \left\{ B_g \in \Omega \ : \ \mathrm{rad}(B_g) \in \left(\frac{R}{2^{j+1}}, \frac{R}{2^j} \right] \right\} \quad \text{for} \quad j = 0, 1, 2, \ldots.$$

We define Λ_n as follows:

(1) Let Λ_0 be any maximal disjoint subcollection of F_0, which exists by Lemma 3.1.

(2) Suppose $\Lambda_0, \dots, \Lambda_{n-1}$ are already defined, define Λ_n to be the maximal disjoint subcollection of balls in

$$\left\{ B_g \in F_n \ : \ B_g \cap A_g = \emptyset, \ \forall A_g \in \bigcup_{k=0}^{n-1} \Lambda_k \right\}$$

given by Lemma 3.1.

Define now

$$\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n.$$

By construction, Λ is a disjoint subcollection of balls in Ω, and hence, countable, since τ_g is second countable [3, Proposition 2.5]. Take any $B_g \in \Omega$, there exist some index $n \geq 0$ such that $B_g \in F_n$. There must exist then some $A_g \in \cup_{k=0}^{n} \Lambda_k$ such that $B_g \cap A_g \neq \emptyset$. Hence, $\operatorname{rad}(B_g) \leq 2 \operatorname{rad}(A_g)$. Let $x \in B_g \cap A_g$ and x_1, x_2 be the centers of B_g and A_g respectively. Then, for every, $y \in B_g$,

$$d_g(y, x_2) \leq d_g(y, x_1) + d_g(x_1, x) + d_g(x, x_2) \leq 2 \operatorname{rad}(B_g) + \operatorname{rad}(A_g) \leq 5 \operatorname{rad}(A_g).$$

Thus, $B_g \subset 5A_g$. □

Since g is nondecreasing, every ball B_g is connected and thus an interval on $\mathbb{R}$. Fix any $t \in \mathbb{R}$ and $\varepsilon > 0$ and consider the closed ball $B_g[t, \varepsilon]$. Assume $B_g[t, \varepsilon]$ to be bounded, take $a = \inf B_g[t, \varepsilon]$ and $b = \sup B_g[t, \varepsilon]$. Hence

$$(a, b) \subset B_g[t, \varepsilon]. \tag{3.1}$$

Recall that B_g must be non degenerate since g is left-continuous, so $a \neq b$. We have that,

$$g(s) \leq g(t) + \varepsilon$$

for all $s < b$, hence $b \in B_g$ by left-continuity. Now, a may or may not be on $B_g[t, \varepsilon]$ depending whether $\Delta g(a)$ equals zero or not. In any case

$$\mu_g(B_g[t, \varepsilon]) = \mu_g((a, b]) = g(b^+) - g(a^+) = \Delta g(b) + g(b) - g(a^+) \leq \Delta g(b) + 2\varepsilon.$$

Note that, since g^C is continuous, singletons have zero measure. So,

$$\mu_{g^C}(B_g[t, \varepsilon]) = \mu_{g^C}((a, b]) = \mu_{g^C}((a, b)) = g^C(b) - g^C(a^+) \leq g(b) - g(a^+) \leq 2\varepsilon. \tag{3.2}$$

Lemma 3.3. *Let $\Lambda = \{ B_g(t_k, \varepsilon_k) \ : \ k = 1, \dots, n \}$ be a finite disjoint collection of open balls. Assume there exists some $t \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$\bigcup_{B_g \in \Lambda} B_g \subset B_g(t, \varepsilon).$$

Then, $\varepsilon > \frac{1}{4} \sum_{k=1}^{n} \varepsilon_k$.

Proof. Let us look first at the case $n = 2$. Suppose then we have two balls of center t_1 and t_2 and radius ε_1 and ε_2 respectively. Since the balls are disjoint, we have that $t_2 \notin B_g(t_1, \varepsilon_1)$ and $t_1 \notin B_g(t_2, \varepsilon_2)$. Hence

$$\max\{\varepsilon_1, \varepsilon_2\} \leq |g(t_2) - g(t_1)|.$$

Go back to the general case and assume without loss of generality that the center of the balls $t_1, \ldots, t_n$ are ordered from left to right. We have that

$$\begin{aligned} g(t_n) - g(t_1) &= \sum_{k=1}^{n-1} g(t_{k+1}) - g(t_k) \geq \sum_{k=1}^{n-1} \max\{\varepsilon_{k+1}, \varepsilon_k\} \geq \sum_{k=1}^{n-1} \frac{\varepsilon_{k+1} + \varepsilon_k}{2} \\ &\geq \sum_{k=2}^{n-1} \varepsilon_k + \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_n \geq \frac{1}{2} \sum_{k=1}^{n} \varepsilon_k. \end{aligned}$$

Now $t_1, t_n \in B_g(t, \varepsilon)$, hence

$$\frac{1}{2} \sum_{k=1}^{n} \varepsilon_k \leq g(t_n) - g(t_1) < 2\varepsilon.$$

□

Remark 3.4. The bounds

$$\frac{\varepsilon_1 + \varepsilon_2}{2} \leq \max\{\varepsilon_1, \varepsilon_2\}, \quad \frac{1}{2} \sum_{k=1}^{n} \varepsilon_k \leq \sum_{k=2}^{n-1} \varepsilon_k + \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_n$$

made on Lemma 3.3 are just arbitrary and of course suboptimal. We wish for our bounds just to be comfortable to manage with series as this next result is the whole purpose of the above.

Corollary 3.5. *Let $\Lambda = \{ B_g(t_k, \varepsilon_k) \mid k \in \mathbb{N}\}$ be a countable disjoint collection of open balls. If*

$$\bigcup_{B_g \in \Lambda} B_g \subset B_g(t, \varepsilon)$$

for some $t \in \mathbb{R}$ and $\varepsilon > 0$, the series $\sum_{k=1}^{\infty} \varepsilon_k$ converges.

Theorem 3.6. *Let X be a nonempty subset of $\mathbb{R}$. Let Ω be a collection of closed balls whose union covers X and satisfies the following property:*

For all $t \in X$ and $\delta > 0$, there exists some $A_g \in \Omega$ such that $A_g \subset B_g(t, \delta)$.

Then there exists a countable disjoint subcollection $\Lambda \subset \Omega$ which covers X up to a g^C-null measure set. In other words, such that

$$\mu_{g^C}\left(X - \bigcup_{B_g \in \Lambda} B_g \right) = 0.$$

Proof. Without loss of generality assume all balls in Ω have radius less than 1. Denote as Λ the countable disjoint subcollection given by Theorem 3.2. We show now that

$$E = X - \bigcup_{B_g \in \Lambda} B_g$$

has null g^C-measure. Take $E_n = E \cap B_g(0, n)$ for $n \in \mathbb{N}$, is enough to show that $\mu_{g^C}(E_n) = 0$ for all $n \in \mathbb{N}$. Denote $\Lambda_n = \{ B_g^k \equiv B_g[t_k, \varepsilon_k] \mid k \in \mathbb{N}\}$ the subcollection of balls of Λ that meet $B_g(0, n)$. Let $t \in E_n$ and $m \in \mathbb{N}$, since $K = \bigcup_{k=0}^{m} B_g^k$ is closed in τ_g, there exists some $A_g \in \Omega$

containing t, contained in $B_g(0,n)$ and disjoint from K. By the property of Λ, A_g intersects some $B_g^i \in \Lambda_n$ for which $A_g \subset 5B_g^i$. This means, $i > m$ and therefore,

$$E_n \subset \bigcup_{k>m} 5B_g^k.$$

This gives the inequality

$$\mu_{g^C}^*(E_n) \leq \mu_{g^C}\left(\bigcup_{k>m} 5B_g^k\right) \leq \sum_{k>m} \mu_{g^C}(5B_g^k) \tag{3.3}$$

for all $m \in \mathbb{N}$. Now, the balls Λ_n are all contained in $B_g(0,n+2)$. From Corollary 3.5, $\sum_{k=1}^{\infty} \varepsilon_k < \infty$, where $\varepsilon_k = \mathrm{rad}(B_g^k)$. Hence, by equation (3.2)

$$\sum_{k=1}^{\infty} \mu_{g^C}(5B_g[t_k,\varepsilon_k]) \leq 10 \sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

From (3.3), $\mu_{g^C}^*(E_n) = 0$. □

Theorem 3.7. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be an arbitrary function and E the set of points of $[a,b]$ at which f has nonzero g-derivative (ie. the g-derivative exists and takes nonzero values). Then, for every null Lebesgue measure set Z, the set $f^{-1}(Z) \cap E$ has null g^C-measure.*

Proof. It suffices to prove the result for the set of points where the g-derivative is positive, which we shall denote again by E. The result for the set of points where the g-derivative is negative follows from the other one applied to the function $-f$ and the null set $-Z$.

Moreover, since $\mu_{g^C}(N_g \cup C_g \cup D_g \cup \{a,b\}) = 0$, we can redefine E to be

$$E = \{t \in (a,b) : f_g'(t) \text{ exists and is positive, } g \text{ is continuous at } t \text{ and } g(s) \neq g(t)\ \forall s \in \mathbb{R},\ s \neq t\}.$$

We can express

$$E = \bigcup_{n=1}^{\infty} \left\{ t \in E \,:\, f_g'(t) > \frac{1}{n} \right\},$$

so it suffices to prove that, for any $\delta > 0$, the set

$$E_\delta = \left\{ t \in E \,:\, f_g'(t) > \delta \right\}$$

has a g^C-null intersection with $f^{-1}(Z)$, for any Z as in the statement.

For a fixed $\delta > 0$ and $q, p \in \mathbb{Q} \cap (a,b)$, $q < p$, we define

$$A_{q,p} = \left\{ t \in E \cap (q,p) : \frac{f(t) - f(s)}{g(t) - g(s)} > \delta,\ \forall s \in (q,p),\ s \neq t \right\},$$

and we have

$$E_\delta \subset \bigcup_{\substack{q,p \in \mathbb{Q} \cap (a,b) \\ q<p}} A_{q,p}.$$

This reduces the proof to showing that for each fixed null Lebesgue measure set Z, for each fixed $\delta > 0$ and for each fixed pair $q, p \in \mathbb{Q} \cap (a,b)$, $q < p$, we have $\mu_{g^C}(f^{-1}(Z) \cap A_{q,p}) = 0$.

First, observe that for $t_1, t_2 \in A_{q,p}$, $t_1 < t_2$, we have

$$\frac{f(t_1) - f(t_2)}{g(t_1) - g(t_2)} > \delta \Rightarrow f(t_1) - f(t_2) < \delta(g(t_1) - g(t_2)) < 0 \Rightarrow f(t_1) < f(t_2),$$

so f is increasing on $A_{q,p}$.

Let $\epsilon > 0$ be fixed and let U be a usual open set of measure less than ϵ that contains Z.

The definition of $A_{q,p}$ guarantees that for any $t \in A_{q,p}$ we can use Proposition 2.5 to ensure that the g-balls $B_g[t,r]$, $r > 0$, are usual neighborhoods of t and, by Remark 2.6, that f is continuous at t. Moreover, $B_g[t,r]$ shrinks to $\{t\}$ as r tends to zero. Therefore, for each $t \in f^{-1}(Z) \cap A_{q,p}$ there exists $r(t) > 0$ such that the usual closure $\overline{B_g[t,r(t)]} \subset (q,p)$ and $f\left(\overline{B_g[t,r(t)]}\right) \subset U$.

Define Ω as the collection of all closed g-balls $B_{t,k} = B_g[t, r(t)/k]$, $t \in f^{-1}(Z) \cap A_{q,p}$, $k \in \mathbb{N}$. Plainly, $f^{-1}(Z) \cap A_{q,p} \subset \bigcup_{t,k} B_{t,k}$ and, moreover, for each $t \in f^{-1}(Z) \cap A_{q,p}$ and each $\rho > 0$ there exists $k \in \mathbb{N}$ such that $B_{t,k} \subset B_g(t,\rho)$. By Vitali's Theorem 3.6 there exists a countable and disjoint subfamily of Ω that covers $f^{-1}(Z) \cap A_{q,p}$ up to a null g^C-measure set. Denote this subcollection as $\{B_g[t_n, r_n] : n \in \mathbb{N}\}$ and put

$$[a_n, b_n] = \overline{B_g[t_n, r_n]}, \quad n \in \mathbb{N}.$$

Since $[a_n, b_n] \subset (q,p)$ and $t_n \in A_{q,p}$,

$$\begin{aligned} f(b_n) - f(a_n) &= (f(b_n) - f(t_n)) + (f(t_n) - f(a_n)) \\ &> \delta((g(b_n) - g(t_n)) + (g(t_n) - g(a_n))) = \delta(g(b_n) - g(a_n)), \end{aligned} \tag{3.4}$$

and, moreover, $[f(a_n), f(b_n)] \subset U$.

Now, take $m, k \in \mathbb{N}$ such that $m \neq k$. Since the intervals $[t_m, b_m]$ and $[t_k, b_k]$ are disjoint we can suppose that $b_m < t_k$. Therefore,

$$f(t_m) < f(b_m) < f(t_k) < f(b_k)$$

since t_m and t_k belong in $A_{q,p}$. Hence, $\{(f(t_n), f(b_n)),\ n \in \mathbb{N}\}$ forms a disjoint family of nondegenerate intervals such that the sum of their lengths is less than ϵ since they are all contained in U. A similar argument proves the same for the intervals $(f(a_n), f(t_n))$, $n \in \mathbb{N}$. We deduce from (3.4) that

$$\begin{aligned} \mu_{g^C}\left(\bigcup_{n=1}^{\infty} B_g[t_n, r_n]\right) &\leq \sum_{n=1}^{\infty} \mu_{g^C}([a_n, b_n]) \leq \sum_{n=1}^{\infty} g(b_n) - g(a_n) \\ &< \frac{1}{\delta} \sum_{n=1}^{\infty} f(b_n) - f(a_n) = \frac{1}{\delta} \sum_{n=1}^{\infty} f(b_n) - f(t_n) + f(t_n) - f(a_n) \\ &= \frac{1}{\delta} \sum_{n=1}^{\infty} f(b_n) - f(t_n) + \frac{1}{\delta} \sum_{n=1}^{\infty} f(t_n) - f(a_n) \leq 2\frac{\epsilon}{\delta}, \end{aligned}$$

which implies that $\mu_{g^C}\left(f^{-1}(Z) \cap A_{a,p}\right) < 2\varepsilon/\delta$ for any $\epsilon > 0$. □

4 Chain rule

Here we shall prove a new version of the chain rule in higher dimensions which is crucial in the transversality conditions to be introduced in our main existence result for (1.1).

Proposition 4.1. *Let t, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, be fixed. Let $h : (t-\epsilon, t+\epsilon) \longrightarrow \mathbb{R}^n$, $f : U \longrightarrow \mathbb{R}$ where U is a usual open set of $\mathbb{R}^n$ such that $h(t-\varepsilon, t+\varepsilon) \subset U$.*

If $t \notin D_g \cup C_g$, h is g-differentiable at t and f is differentiable at $h(t)$, then $f \circ h$ is g-differentiable at t and

$$(f \circ h)'_g(t) = \nabla f(h(t)) \cdot h'_g(t),$$

where $\cdot$ denotes the usual inner product.

Proof. Since $t \notin C_g$ we have $g(s) < g(t)$ for all $s < t$, or $g(s) > g(t)$ for all $s > t$, or both. Since $t \notin D_g$, the existence of $h'_g(t)$ means, in any of the previous three cases, that there exists

$$h'_g(t) = \lim_{\substack{s \to t \\ g(s) \neq g(t)}} \frac{h(s) - h(t)}{g(s) - g(t)}$$

and we want to prove that

$$\lim_{\substack{s \to t \\ g(s) \neq g(t)}} \frac{f(h(s)) - f(h(t))}{g(s) - g(t)} = \nabla f(h(t)) \cdot h'_g(t).$$

Since g is continuous at t, there exists

$$\lim_{\substack{s \to t \\ g(s) \neq g(t)}} h(s) - h(t) = \lim_{\substack{s \to t \\ g(s) \neq g(t)}} \frac{h(s) - h(t)}{g(s) - g(t)} (g(s) - g(t)) = 0. \tag{4.1}$$

Observe that, as long as $g(s) \neq g(t)$, we have

$$\left| \frac{f(h(s)) - f(h(t))}{g(s) - g(t)} - \nabla f(h(t)) \cdot h'_g(t) \right| \leq \underbrace{\left| \frac{f(h(s)) - f(h(t)) - \nabla f(h(t)) \cdot (h(s) - h(t))}{g(s) - g(t)} \right|}_{=A(s)}$$
$$+ \underbrace{\|\nabla f(h(t))\| \left\| \frac{h(s) - h(t)}{g(s) - g(t)} - h'_g(t) \right\|}_{=B(s)},$$

and $B(s) \to 0$ as $s \to t$ $(g(s) \neq g(t))$ by definition of $h'_g(t)$. On the other hand, if $h(s) = h(t)$, then $A(s) = 0$, and otherwise we have

$$A(s) = \frac{|f(h(s)) - f(h(t)) - \nabla f(h(t)) \cdot (h(s) - h(t))|}{\|h(s) - h(t)\|} \left\| \frac{h(s) - h(t)}{g(s) - g(t)} \right\|,$$

which tends to zero as $s \to t$ $(g(s) \neq g(t))$ by definition of $\nabla f(h(t))$ and (4.1). □

Corollary 4.2. *Let t, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ be fixed. Let $x : (t-\epsilon, t+\epsilon) \longrightarrow \mathbb{R}^n$, $\tau : (t-\epsilon, t+\epsilon) \times U \longrightarrow \mathbb{R}$ where U is a usual open set of $\mathbb{R}^n$ such that $x(t-\varepsilon, t+\varepsilon) \subset U$.*

If $t \notin D_g \cup C_g$, $g'(t) > 0$, x is g-differentiable at t and τ is differentiable at $(t, x(t))$, then $\tau(\cdot, x(\cdot))$ is g-differentiable at t and

$$\frac{d}{dg(t)} \tau(t, x(t)) = \nabla \tau(t, x(t)) \cdot (1/g'(t), x'_g(t)).$$

Proof. Apply Proposition 4.1 with $h(t) = (t, x(t))$ and $f(y) = \tau(y)$ for $y = (t, x) \in (t - \varepsilon, t+\varepsilon) \times \mathbb{R}^n$. The condition $g'(t) > 0$ implies that h is g-differentiable at t and $h'_g(t) = (1/g'(t), x'_g(t))$. □

5 Main result

We are already in a position to prove our new existence result of Carathéodory solutions of (1.1). Our arguments lean on the chain rule, established in Section 4, along with a transversality condition of the g-differential equation with respect to the sets where discontinuities are allowed for $f(t,x)$.

A Krasovskij solution of (1.1) is defined as a g-absolutely continuous function $x : I \longrightarrow \mathbb{R}^n$ such that $x(t_0) = x_0$ and

$$x'_g(t) \in \bigcap_{\varepsilon>0} \overline{\text{co}}\, f\left(t, B_\varepsilon(x(t))\right) \quad \text{for } g\text{-a.a. } t \in [t_0, t_0+L), \tag{5.1}$$

where $\overline{\text{co}}$ means closed convex hull and $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|y-x\| < \varepsilon\}$. Observe that, in the scalar case ($n=1$), we have $B_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$.

We are now in a position to prove a result on the existence of Carathéodory solutions for (1.1).

Theorem 5.1. *Assume that for all $x \in \mathbb{R}^n$ the mapping $f(\cdot,x)$ is g-measurable, for all $t \in [t_0, t_0+L) \cap D_g$ the mapping $f(t,\cdot)$ is continuous, and there exists $M \in L^1_g([t_0,t_0+L))$ such that for g-almost all $t \in [t_0,t_0+L)$ and all $x \in \mathbb{R}^n$ we have $\|f(t,x)\| \le M(t)$.*

Moreover, assume that there exist null Lebesgue measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\tau_k : [a_k,b_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $[a_k,b_k] \subset I$, such that for g-a.a. $t \in [t_0,t_0+L) \setminus D_g$ the following conditions hold:

(a) $g'(t) > 0$;

(b) There exists a set $N(t) \subset \mathbb{R}^n$ such that $f(t,\cdot)$ is continuous in $\mathbb{R}^n \setminus N(t)$;

(c) For each $x \in N(t)$ there exists $k \in \mathcal{C}$ such that $t \in [a_k,b_k]$, $\tau_k(t,x) \in A_k$, and

$$\nabla \tau_k(t,x) \cdot (1/g'(t), z) \neq 0 \quad \text{for all } z \in \bigcap_{\varepsilon>0} \overline{\text{co}}\, f\left(t, B_\varepsilon(x)\right). \tag{5.2}$$

Then, problem (1.1) has at least one Carathéodory solution.

Proof. The assumptions imply that the multivalued mapping

$$(t,x) \longmapsto \mathcal{F}(t,x) = \bigcap_{\varepsilon>0} \overline{\text{co}}\, f\left(t, B_\varepsilon(x)\right)$$

is compact and convex-valued, and it is upper semicontinuous. Hence, Theorem 3.9 in [8] ensures that problem (1.1) has at least one Krasovskij solution $x : I \longrightarrow \mathbb{R}^n$. Let us prove that x is a Carathéodory solution of (1.1).

Let $E \subset I$ be a g-null measure set such that, first, conditions (a), (b) and (c) hold for all $t \in I \setminus E$, and, second,

$$x'_g(t) \in \mathcal{F}(t, x(t)) \quad \text{for all } t \in I \setminus E.$$

Observe that for each $t \in I \setminus E$ such that $t \in D_g$ or $x(t) \notin N(t)$ the assumptions ensure that $f(t,\cdot)$ is continuous at $x(t)$, and therefore

$$x'_g(t) \in \mathcal{F}(t, x(t)) = \{f(t, x(t))\}.$$

Hence, it suffices to prove that the set $J = \{t \in I \setminus E,\, t \notin D_g : x(t) \in N(t)\}$ is g-null.

We deduce from condition (c) that

$$J \subset \bigcup_{k \in \mathcal{C}} \{t \in [a_k, b_k] \setminus E,\, t \notin D_g \,:\, \tau_k(t, x(t)) \in A_k\},$$

so the proof is reduced to showing that each $J_k = \{t \in [a_k, b_k] \setminus E,\, t \notin D_g \,:\, \tau_k(t, x(t)) \in A_k\}$ is a null g-measure set. For an arbitrarily fixed $k \in \mathcal{C}$, we define $\varphi(t) = \tau_k(t, x(t))$ for all $t \in [a_k, b_k]$, so that $J_k \subset \varphi^{-1}(A_k) \setminus D_g$ and it suffices to prove that $\varphi^{-1}(A_k) \setminus D_g$ is g-null. Since $m(A_k) = 0$, Theorem 3.7 guarantees the existence of a set $B \subset \varphi^{-1}(A_k)$, with null g^c-measure, such that for every $t \in \varphi^{-1}(A_k) \setminus B$ we have $\varphi'_g(t) = 0$, i.e. $\varphi'_g(t) = 0$ for g^c-a.a. $t \in \varphi^{-1}(A_k)$ or, equivalently, for g-a.a. $t \in \varphi^{-1}(A_k) \setminus D_g$ we have

$$\frac{d}{dg(t)} \tau_k(t, x(t)) = 0. \tag{5.3}$$

Let us prove that $\varphi^{-1}(A_k) \setminus D_g \subset E$, thus showing that $\varphi^{-1}(A_k) \setminus D_g$ is g-null. Reasoning by contradiction, we assume that there is some $t \in \varphi^{-1}(A_k) \setminus D_g$ such that $t \notin E$, and then we can use the chain rule in (5.3) to deduce that

$$\nabla \tau_k(t, x(t)) \cdot (1/g'(t), x'_g(t)) = 0,$$

a contradiction with condition (5.2). □

Checking the transversality condition (5.2) might be a difficult task in practical situations. Our next corollary gives a more friendly sufficient condition for it. Basically, it tells us that (5.2) is satisfied provided that $\partial \tau_k / \partial t$ is far from zero and $g'(t)$ is small enough. This is an interesting condition from the viewpoint of Stieltjes differential equations because it leans precisely on the slopes of the derivator.

Corollary 5.2. *Assume that for all $x \in \mathbb{R}^n$ the mapping $f(\cdot, x)$ is g-measurable, for all $t \in [t_0, t_0 + L) \cap D_g$ the mapping $f(t, \cdot)$ is continuous, and there exists $M \in L^1_g([t_0, t_0 + L))$ such that for g-almost all $t \in [t_0, t_0 + L)$ and all $x \in \mathbb{R}^n$ we have $\|f(t, x)\| \le M(t)$.*

Moreover, assume that there exist null Lebesgue measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\tau_k : [a_k, b_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $[a_k, b_k] \subset I$, such that for g-a.a. $t \in [t_0, t_0 + L) \setminus D_g$ the following conditions hold:

(a) $g'(t) > 0$;

(b) There exists a set $N(t) \subset \mathbb{R}^n$ such that $f(t, \cdot)$ is continuous in $\mathbb{R}^n \setminus N(t)$;

(c) For each $x \in N(t)$ there exists $k \in \mathcal{C}$ such that $t \in [a_k, b_k]$, $\tau_k(t, x) \in A_k$, and*

$$\left| \frac{\partial \tau_k}{\partial t}(t, x) \right| > g'(t)\, M(t) \sum_{j=1}^{n} \left| \frac{\partial \tau_k}{\partial x_j}(t, x) \right|. \tag{5.4}$$

Then, problem (1.1) has at least one Carathéodory solution.

Proof. We only have to check that condition $(c*)$ implies condition (c) in Theorem 5.1. To do it, we take x as in (c^*) and we observe that $z = (z_1, z_2, \ldots, z_n) \in \bigcap_{\varepsilon > 0} \overline{\text{co}}\, f(t, B_\varepsilon(x))$ implies that $|z_j| \leq M(t)$ for $j = 1, 2, \ldots, n$, so we have

$$\begin{aligned}
\left|\nabla\tau_k(t,x)\cdot(1/g'(t),z)\right| &= \left|\frac{1}{g'(t)}\frac{\partial\tau_k}{\partial t}(t,x)+\sum_{j=1}^{n} z_j\frac{\partial\tau_k}{\partial x_j}(t,x)\right| \\
&= \left|\frac{1}{g'(t)}\frac{\partial\tau_k}{\partial t}(t,x)-\sum_{j=1}^{n}(-z_j)\frac{\partial\tau_k}{\partial x_j}(t,x)\right| \\
&\geq \frac{1}{g'(t)}\left|\frac{\partial\tau_k}{\partial t}(t,x)\right|-\left|\sum_{j=1}^{n}(-z_j)\frac{\partial\tau_k}{\partial x_j}(t,x)\right| \\
&\geq \frac{1}{g'(t)}\left|\frac{\partial\tau_k}{\partial t}(t,x)\right|-M(t)\sum_{j=1}^{n}\left|\frac{\partial\tau_k}{\partial x_j}(t,x)\right|
\end{aligned}$$

and the last term is positive thanks to condition (5.4). □

Observe that condition (5.4) is fulfilled when $\partial\tau_k/\partial t$ is positive and either g' is small enough or $\partial\tau_k/\partial t$ is large enough. Corollary 5.2 is new even in case $g(t) = t$, i.e. for ordinary differential equations, where condition (5.4) reduces to

$$\left|\frac{\partial\tau_k}{\partial t}(t,x)\right| > M(t)\sum_{j=1}^{n}\left|\frac{\partial\tau_k}{\partial x_j}(t,x)\right|.$$

Finally, we get an interesting consequence for the scalar case in connection with bounded variation (not necessarily continuous) nonlinear parts of the g-differential equations.

Corollary 5.3. *The scalar problem*

$$x'_g = f(t,x), \quad t \in I = [t_0, t_0+L), \; x(t_0) = x_0, \tag{5.5}$$

has at least one Carathéodory solution provided that there exists $M > 0$ such that $|f(t,x)| \leq M$ for all $(t,x) \in I \times \mathbb{R}$, $f(t,\cdot)$ is continuous whenever $t \in [t_0, t_0+L) \cap D_g$, and for g-a.a. $t \in [t_0, t_0+L) \setminus D_g$ we have $g'(t) > 0$ and we can express $f(t,x) = F(\tau(t,x))$ for some functions F and τ satisfying the following assumptions:

(i) $F : \mathbb{R} \longrightarrow \mathbb{R}$ *has bounded variation on any compact interval;*

(ii) $\tau : I \times \mathbb{R} \longrightarrow \mathbb{R}$ *is differentiable and*

$$\left|\frac{\partial\tau}{\partial t}(t,x)\right| > M g'(t)\left|\frac{\partial\tau}{\partial x}(t,x)\right|, \quad (t,x) \in ([t_0, t_0+L) \setminus D_g) \times \mathbb{R}. \tag{5.6}$$

Proof. The result follows from Corollary 5.2. First, observe that for any $x \in \mathbb{R}$ the composition $f(\cdot,x) = F(\tau(\cdot,x))$ is Borel measurable, hence g-measurable in the Lebesgue–Stieltjes sense. On the other hand, for any $t \in I \setminus D_g$ the function $f(t,\cdot) = F(\tau(t,\cdot))$ is continuous everywhere except at those $x \in \mathbb{R}$ such that $\tau(t,x)$ is a discontinuity point of F. Let A be a countable set such that F is continuous in $\mathbb{R} \setminus A$; then $f(t,\cdot)$ is continuous in $\mathbb{R} \setminus N(t)$, where

$$N(t) = \{x \in \mathbb{R} \, : \, \tau(t,x) \in A\} = \tau(t,\cdot)^{-1}(A).$$

Obviously, for each $x \in N(t)$ we have $\tau(t,x) \in A$, and condition (5.6) implies condition (5.4). □

Remark 5.4. In the conditions of Corollary 5.3, solutions $x(t)$ of (5.5) satisfy $|x(t) - x_0| \leq M\,(g(t) - g(t_0)) \leq M\,(g(t_0+L) - g(t_0))$ for all $t \in [t_0, t_0+L)$. Hence, condition (5.6) is really needed only for all $(t,x) \in ([t_0, t_0+L) \setminus D_g) \times \mathbb{R}$ such that $|x - x_0| \leq M\,(g(t) - g(t_0))$.

5.1 An example

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be nondecreasing, left-continuous everywhere and assume that $g'(t) > 0$ for g-a.a. $t \in [t_0, t_0 + L) \setminus D_g$.

We consider the following particular case of problem (5.5):

$$x'_g = F(\alpha\, t + \beta\, x)\chi_{I\setminus D_g}(t) + F(t)\chi_{I\cap D_g}(t), \quad t \in I = [t_0, t_0 + L),\ x(t_0) = x_0, \tag{5.7}$$

where $\alpha, \beta \in \mathbb{R}$, χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$, and $F : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$F(y) = \sum_{r_n<y} 2^{-n}, \quad y \in \mathbb{R},$$

where $r_n = r(n)$, $n \in \mathbb{N}$, and $r : \mathbb{N} \longrightarrow \mathbb{Q}$ is a bijection.

Observe that F is increasing, $0 < F(y) < 1$ for all $y \in \mathbb{R}$, and F is discontinuous at every rational number.

It is easy to check that Corollary 5.3 applies if

$$|\alpha| > g'(t)|\beta| \quad \text{for } g\text{-a.a. } t \in I \setminus D_g,$$

so in that case (5.7) has at least one solution.

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