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Multiple solutions for a parametric Steklov problem involving the $p(x)$-Laplacian operator

Aboubacar Abdou ✉ [1], **Mohamed Karimou Gazibo** [1] and **Aboubacar Marcos** [2]

[1]Université Abdou Moumouni, École Normale Supérieure, Département de Mathématiques, Niamey, Niger
[2]Université d'Abomey Calavi, Institut de Mathématiques et de Sciences Physiques, Porto-Novo, Bénin

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Abstract. In this paper, we study the existence and multiplicity of weak solutions for a Steklov problem involving $p(x)$-Laplacian operator in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. The boundary equation is perturbed with some weight functions belonging to approriate generalized Lebesgue spaces and two real parameters. Our arguments are based on variational method, using "Mountain Pass Theorem", "Fountain Theorem" and "Dual Fountain Theorem" combined with the critical points theory, we prove several existence results.

Keywords: Steklov problem, $p(x)$-Laplacian operator, generalized Lebesgue–Sobolev spaces, variational method, Mountain Pass Theorem, Fountain Theorem, Dual Fountain Theorem.

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1 Introduction

We are interested in the existence and multiplicity of weak solutions for the following quasilinear elliptic problem with Steklov boundary condition involving variable exponents

$$\begin{cases} \Delta_{p(x)}u = a(x)|u|^{p(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = \lambda V(x)|u|^{q(x)-2}u + \mu W(x)|u|^{r(x)-2}u \quad \text{on } \partial\Omega, \end{cases} \tag{$S_{\lambda,\mu}$}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$, λ, μ are real parameters, $p, q, r : \overline{\Omega} \to (1, +\infty)$ are continuous functions, $a \in L^\infty(\Omega)$ is a positive function, V, W are positive measurable weight functions belonging to some generalized Lebesgue space and $\Delta_{p(x)}u$ is the $p(x)$-Laplacian operator defined by $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ which is a natural extension of the p-Laplacian operator obtained in the constant case $p(x) \equiv p$, if $p \equiv 2$ we have the classical Laplacian operator $\Delta u = \operatorname{div}(\nabla u)$.

✉Corresponding author. Email: abdou.aboubacar@ymail.com

In recent years, nonlinear boundary value problems with variable exponents have received an increasing attention. This is partly due to their frequent appearance in a lot of models in mathematical physics such as the modeling of electrorheological fluids, elasticity problems, flow in porous medium and image processing, see [17, 42, 44, 51]. The study of variable exponent problems is more difficult than the constant one, since the $p(x)$-Laplacian operator possesses more complicated structures than the p-Laplacian operator, for example it is inhomogeneous. Recently, boundary value problems involving a such operator have been a subject of many researchers, we refer to [1, 15, 24, 29, 33, 34, 36, 39], in which the authors have extended some well known results from constant to variable exponents case.

Problems of Steklov type appear in the modeling of an elastic membrane problem whose mass is concentrated on the boundary [45]. In the past decades, many works have extended the results of [45] from Laplacian operator to p-Laplacian operator [7–10, 22, 38, 40, 47] and more general $p(x)$-Laplacian operator [2–6, 11, 12, 14, 19, 43].

In [11], problem ($S_{\lambda,\mu}$) is considered with $a(x) \equiv 1$, $V(x) \equiv 1$ and $\mu = 0$. For any $\lambda > 0$, if $q^+ < p^-$, the authors proved the existence of a sequence of nontrivial weak solutions, and if $p^+ < q^- \leq q^+ < p^\partial(x)$, the existence of nontrivial weak solution. In [14], the authors considered problem ($S_{\lambda,\mu}$) with $\mu = 0$ and $V(x)$ belonging to some generalized Lebesgue space. Under condition $1 \leq q^+ < p^-$, the authors proved the existence of at least one non trivial weak solution with negative energy for all $\lambda > 0$, and non trivial weak solution for all $\lambda \in (0, \lambda^*)$, where λ^* is a positive constant. It should be noticed that in [24, 36], problem like ($S_{\lambda,\mu}$) was considered but with Dirichlet boundary condition. Under appropriate conditions on the variable exponents, the authors proved several existence and multiplicity results by changing the role of parameters λ, μ with respect to their signs.

Inspired by the above mentioned works [11, 14, 24, 36], we study the Steklov problem ($S_{\lambda,\mu}$) with some weight functions and two parameters. The appropriate functional spaces and their basic properties used in this work are guaranteed by the paper of S. G. Deng [19], namely, the compact embedding from the generalized Sobolev space to the generalized Lebesgue space with weight. Under relaxed conditions on the variable exponents and real parameters λ, μ that the ones use in [11, 14], we obtain new criteria on existence and multiplicity results of ($S_{\lambda,\mu}$).

We make the following basic hypotheses on the data of ($S_{\lambda,\mu}$) :

$$\begin{gathered} s,t : \overline{\Omega} \to (1,+\infty) \quad \text{are variable exponents such that :} \\ s(x), t(x) > \frac{N-1}{p(x)-1}, \quad \text{for a.e. } x \in \partial\Omega, \\ 1 \leq q(x) < \frac{s(x)-1}{s(x)} p^\partial(x), \qquad 1 \leq r(x) < \frac{t(x)-1}{t(x)} p^\partial(x), \quad \text{for a.e. } x \in \partial\Omega, \end{gathered} \tag{$\mathbf{H_1}$}$$

where $p^\partial(x)$ is the critical Sobolev exponent on $\partial\Omega$;

$$a \in L^\infty(\Omega), \quad \text{with } \operatorname*{ess\,inf}_{x\in\Omega} a(x) = a^- > 0; \tag{$\mathbf{H_2}$}$$

$$V \in L^{s(x)}(\partial\Omega), \qquad W \in L^{t(x)}(\partial\Omega), \quad \text{with } V(x), W(x) > 0 \quad \text{for a.e. } x \in \partial\Omega. \tag{$\mathbf{H_3}$}$$

Our main results in this paper are given by the following theorems.

Theorem 1.1. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled. Denote by* p^+, q^+, r^+ *(resp.* p^-, q^-, r^-*) the maximum (resp. the minimum) of functions* $p(x)$, $q(x)$, $r(x)$ *on* $\overline{\Omega}$, *respectively.*

(i) *If* $\max\{q^+, r^+\} < p^-$, *then for any* $\lambda, \mu \in \mathbb{R}$, *problem* ($S_{\lambda,\mu}$) *has a nonnegative solution.*

(ii) *If* $\min\{q^-, r^-\} > p^+$, *then for any* $\lambda, \mu \geq 0$, *problem* ($S_{\lambda,\mu}$) *has a nontrivial nonnegative solution.*

Theorem 1.2. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled. Denote by* Ψ *the energy fuctional associated to* ($S_{\lambda,\mu}$) *and* p^+, q^+, r^+ *(resp.* p^-, q^-, r^-*) the maximum (resp. the minimum) of functions* $p(x)$, $q(x)$, $r(x)$ *on* $\overline{\Omega}$, *respectively.*

(i) *If* $1 < p^- \leq p^+ < r^- \leq q^-$, *then for any* $\lambda, \mu \geq 0$, *problem* ($S_{\lambda,\mu}$) *has a sequence of solutions* $(\pm u_n)$ *such that* $\Psi(\pm u_n) \to \infty$ *as* $n \to \infty$.

(ii) *If* $1 < q^- \leq q^+ < p^- \leq p^+ < r^-$, *then for any* $\lambda \in \mathbb{R}$ *and* $\mu > 0$, *problem* ($S_{\lambda,\mu}$) *has a sequence of solutions* $(\pm u_n)$ *such that* $\Psi(\pm u_n) \to \infty$ *as* $n \to \infty$.

(iii) *If* $1 < q^- \leq q^+ = p^- \leq p^+ < r^-$, *then for any* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \lambda_0 := \frac{\frac{1}{p^+} - \frac{1}{r^-}}{(\frac{1}{q^-} - \frac{1}{h^-})C^{q^+}}$ *and* $\mu > 0$, *problem* ($S_{\lambda,\mu}$) *has a sequence of solutions* $(\pm u_n)$ *such that* $\Psi(\pm u_n) \to \infty$ *as* $n \to \infty$.

(iv) *If* $1 < q^- \leq q^+ < p^- \leq p^+ < r^-$, *then for any* $\lambda > 0$ *and* $\mu \in \mathbb{R}$, *problem* ($S_{\lambda,\mu}$) *has a sequence of solutions* $(\pm v_n)$ *such that* $\Psi(\pm v_n) < 0$, *for all* n, *and* $\Psi(\pm v_n) \to 0$ *as* $n \to \infty$.

(v) *If* $1 < q^- \leq q^+ < \min\{p^-, r^-\}$ *and* $p^+ < r^+$, *then for any* $\lambda > 0$ *and* $\mu \leq 0$, *problem* ($S_{\lambda,\mu}$) *has a sequence of solutions* $(\pm v_n)$ *such that* $\Psi(\pm v_n) < 0$, *for all* n, *and* $\Psi(\pm v_n) \to 0$ *as* $n \to \infty$.

To prove these results, our approach is variational combined with the critical point theory. Theorem 1.1 is established by using the global minimizer method for the case (i) and Mountain Pass Theorem of Ambrosetti–Rabinowitz for the case (ii). For the proof of Theorem 1.2 we need to use the fountain theorem established by T. Bartsch in 1993 and the dual of fountain theorem established by T. Bartsch and M. Willem in 1995, which are both simple and powerful tools in searching solutions for large energy and small energy of several boundary value problems.

The remainder of this paper is organized in the following way. In Section 2, we give some preliminaries on variable exponent Lebesgue and Sobolev spaces. In particular, we give a weighted variable exponent Sobolev trace compact embedding theorem which will be useful in this work. In Section 3, we begin by the preliminary results of ($S_{\lambda,\mu}$), state some abstract abstract critical point theorems and establish the technical lemmas which allow us to use the abstract theorems in order to guarantee the existence of solutions. Section 4 is devoted to the proof of our main results : Theorem 1.1 and Theorem 1.2. Finally, in the Appendix we give some examples of variable exponents illustrating Theorem 1.1 and Theorem 1.2.

2 Preliminaries on generalized Lebesgue–Sobolev spaces

We need to recall here some basic properties of variable exponent Lebesgue and Sobolev spaces which will be used later. We refer to [19, 21, 24, 29, 30, 37] for more details on the properties of these space.

Let Ω be a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Let us denote by

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for any } x \in \overline{\Omega}\};$$

for $p \in C_+(\overline{\Omega})$,

$$p^- = \min_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^+ = \max_{x\in\overline{\Omega}} p(x).$$

Let $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} \text{ measurable function } : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the so-called Luxemburg norm given by

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Remark 2.1. The spaces $\left(L^{p(x)}(\Omega), \|\cdot\|_{p(x)}\right)$ resembles to the classical Lebesgue spaces in many case, namely, they are separable Banach spaces, the Hölder inequality holds, the inclusions between Lebesgue spaces are also valid, that is, if $0 < \text{meas}(\Omega) < \infty$ and $p(x) < q(x)$ a.e. in Ω, then there exists a continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

endowed with the norm

$$\|u\|_{1,p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} + \left| \frac{u(x)}{\lambda} \right|^{p(x)} \right) dx \le 1 \right\},$$

that is $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)} + \|u\|_{p(x)}$.

Proposition 2.2 ([30]). *Both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \overline{\Omega}$.

Proposition 2.3 ([30]). *The Hölder inequality holds, namely*

$$\int_\Omega |u(x)||v(x)| dx \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \le 2\|u\|_{p(x)} \|v\|_{p'(x)},$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

In what follows, for brevity we write $X = W^{1,p(x)}$ and denote X^* its dual space.

Let $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} a(x) > 0$. For any $u \in X$, let us define

$$\|u\|_a = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} + a(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} \right) dx \le 1 \right\}.$$

Then, $\|\cdot\|_a$ is a norm on X equivalent to the norm $\|\cdot\|_{1,p(x)}$ (see [2, 19]). In what follows, we will use $\|\cdot\|_a$ instead of $\|\cdot\|_{1,p(x)}$ on X.

The modular is an important tool in studying problems involving generalized Lebesgue–Sobolev spaces, it is a mapping ρ_a defined on X by

$$\rho_a(u) = \int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \right) dx.$$

The following proposition connect $\|u\|_a$ with $\rho_a(u)$, it will be very useful in this work.

Proposition 2.4 ([20,30]). *Then, for all $u, u_n \in X$ $(n = 1, 2, \dots)$, we have*

(i) $\|u\|_a < 1(= 1; > 1) \Leftrightarrow \rho_a(u) < 1(= 1; > 1)$;

(ii) $\|u\|_a \leq 1 \Rightarrow \|u\|_a^{p^+} \leq \rho_a(u) \leq \|u\|_a^{p^-}$;

(iii) $\|u\|_a \geq 1 \Rightarrow \|u\|_a^{p^-} \leq \rho_a(u) \leq \|u\|_a^{p^+}$;

(iv) $\|u_n\|_a \to 0 \Leftrightarrow \rho_a(u_n) \to 0$ *(as $n \to \infty$).*

Consider the real valued fuctional defined on X by

$$\phi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)} \right) dx.$$

The following properties of ϕ are very interesting in the study of variational problems.

Proposition 2.5 ([25,26]). *Then, ϕ is convex, sequentially weakly lower semi-continuous and continuously Fréchet differentiable such that*

$$\langle \phi'(u), v \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|u|^{p(x)-2} uv \right) dx, \quad \text{for all } u, v \in X.$$

Moreover, The derivative function ϕ' satisfies

(i) $\phi' : X \to X^*$ *is continuous, bounded and strictly monotone operator;*

(ii) $\phi' : X \to X^*$ *is a mapping of type (S^+), that is, if* $u_n \xrightarrow{weakly} u$ *in X and* $\limsup_{n\to\infty} \langle \phi'(u_n) - \phi'(u), u_n - u \rangle \leq 0$, *then* $u_n \xrightarrow{strongly} u$ *in X;*

(iii) $\phi' : X \to X^*$ *is a homeomorphism.*

Let $b : \partial\Omega \to \mathbb{R}$ be a measurable function. The weighted variable exponent Lebesgue space is defined by

$$L^{p(x)}_{b(x)}(\partial\Omega) = \left\{ u \colon \partial\Omega \to \mathbb{R} \text{ measurable function} : \int_{\partial\Omega} |b(x)||u(x)|^{p(x)} d\sigma < \infty \right\},$$

endowed with the norm

$$\|u\|_{(p(x),b(x))} = \|u\|_{L^{p(x)}_{b(x)}(\partial\Omega)} = \inf \left\{ \tau > 0 : \int_{\partial\Omega} |b(x)| \left| \frac{u(x)}{\tau} \right|^{p(x)} d\sigma \leq 1 \right\},$$

where $d\sigma$ is the surface measure on the boundary.

Remark 2.6. The space $L^{p(x)}_{b(x)}(\partial\Omega)$ is a Banach space. In particular, when $b(x) \equiv 1$ on $\partial\Omega$ or more general $b(x) \in L^\infty(\partial\Omega)$, $L^{p(x)}_{b(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

For $u \in L^{p(x)}_{b(x)}(\partial\Omega)$, consider the real valued functional

$$\rho(u) = \int_{\partial\Omega} |b(x)||u(x)|^{p(x)} d\sigma.$$

The following proposition connect $\|u\|_{(p(x),b(x))}$ with $\rho(u)$.

Proposition 2.7 ([19]). *Then, for all $u, u_n \in L^{p(x)}_{b(x)}(\partial\Omega)$ $(n = 1, 2, \dots)$, we have*

(i) $\|u\|_{(p(x),b(x))} \leq 1 \Rightarrow \|u\|^{p^+}_{(p(x),b(x))} \leq \rho(u) \leq \|u\|^{p^-}_{(p(x),b(x))}$;

(ii) $\|u\|_{(p(x),b(x))} \geq 1 \Rightarrow \|u\|^{p^-}_{(p(x),b(x))} \leq \rho(u) \leq \|u\|^{p^+}_{(p(x),b(x))}$;

(iii) $\|u_n\|_{(p(x),b(x))} \to 0 \Leftrightarrow \rho(u_n) \to 0 (\text{as } n \to \infty)$.

Consequently,

$$\|u\|^{p^-}_{(p(x),b(x))} - 1 \leq \rho(u) \leq \|u\|^{p^+}_{(p(x),b(x))} + 1, \quad \text{for all } u \in L^{p(x)}_{b(x)}(\partial\Omega).$$

Proposition 2.8 ([13]). *The map $u \in L^{p(x)}(\partial\Omega) \longmapsto |u|^{q(x)-2}u \in L^{p'(x)}(\partial\Omega)$ is continuous.*

For $p \in C_+(\overline{\Omega})$ and $r \in C_+(\partial\Omega)$, the Sobolev critical exponents are defined by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases} \quad \text{and} \quad p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Let $p^\partial_{r(x)}(x)$ defined by

$$p^\partial_{r(x)}(x) = \frac{r(x)-1}{r(x)} p^\partial(x).$$

Proposition 2.9 ([19, 28]). *Assume that the boundary of Ω possesses the cone property and $p \in C_+(\overline{\Omega})$. Suppose that $b \in L^{r(x)}(\partial\Omega)$, $r \in C_+(\partial\Omega)$ with $r(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. Then, we have the following properties.*

(i) *If $q \in C_+(\overline{\Omega})$ and $1 \leq q(x) < p^*(x)$ for any $x \in \overline{\Omega}$. Then, the embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ is compact and continuous.*

(ii) *If $q \in C_+(\partial\Omega)$ and $1 \leq q(x) < p^\partial_{r(x)}(x)$ for any $x \in \partial\Omega$. Then, the embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}_{b(x)}(\partial\Omega)$ is compact and continuous. In particular, there is a compact and continuous embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\partial\Omega)$, where $1 \leq q(x) < p^\partial(x)$, for any $x \in \partial\Omega$.*

3 Preliminary results, abstract theorems and technical lemmas

We investigate here the preliminary results, state some well known abstract critical point theorems and establish the technical lemmas which allow us to prove our main results. Here and henceforth, we denote by X the generalized Sobolev space $W^{1,p(x)}(\Omega)$ equipped with the norm $\|\cdot\|_a$ and X^* its dual space.

3.1 Preliminary results

Definition 3.1. We say that $u \in X$ is a weak solution of the Steklov problem ($S_{\lambda,\mu}$) if and only if for all $v \in X$

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_\Omega a(x)|u|^{p(x)-2} uv dx = \lambda \int_{\partial\Omega} V(x)|u|^{q(x)-2} uv d\sigma + \mu \int_{\partial\Omega} W(x)|u|^{r(x)-2} uv d\sigma,$$

where $d\sigma$ is the surface measure on the boundary of Ω.

Denote by $\Psi : X \to \mathbb{R}$ the Euler–Lagrange functional (energy functional) associated with the problem ($S_{\lambda,\mu}$), it is defined by

$$\Psi(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{a(x)}{p(x)} |u|^{p(x)} dx - \lambda \int_{\partial\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} d\sigma \\ - \mu \int_{\partial\Omega} \frac{W(x)}{r(x)} |u|^{r(x)} d\sigma.$$

For $u \in X$, we denote by $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and we define the functional

$$\Psi_+(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{a(x)}{p(x)} |u|^{p(x)} dx - \lambda \int_{\partial\Omega} \frac{V(x)}{q(x)} (u^+)^{q(x)} d\sigma \\ - \mu \int_{\partial\Omega} \frac{W(x)}{r(x)} (u^+)^{r(x)} d\sigma.$$

Introduce the functionals $\phi, \psi : X \to \mathbb{R}$ defined by

$$\phi(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{a(x)}{p(x)} |u|^{p(x)} dx$$
$$\psi(u) = \lambda \int_{\partial\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} d\sigma + \mu \int_{\partial\Omega} \frac{W(x)}{r(x)} |u|^{r(x)} d\sigma.$$

From assumptions (**H$_1$**)–(**H$_2$**)–(**H$_3$**) and Proposition 2.9, we deduce the following compact and continuous embeddings

$$X \hookrightarrow L^{q(x)}_{V(x)}(\partial\Omega) \quad \text{and} \quad X \hookrightarrow L^{r(x)}_{W(x)}(\partial\Omega),$$

that is, there exists a positive constant C such that

$$|u|_{(q(x),V(x))} \le C\|u\|_a \quad \text{and} \quad |u|_{(r(x),W(x))} \le C\|u\|_a, \quad \text{for all } u \in X. \tag{3.1}$$

Without loss of generality, we can suppose that $C > 1$.

Remark 3.2. If $u \in X$, then we have

$$V(x)^{\frac{1}{q(x)}} u \in L^{q(x)}(\partial\Omega) \quad \text{and} \quad V(x)^{\frac{1}{q'(x)}} u^{q(x)-1} \in L^{q'(x)}(\partial\Omega).$$

Moreover, $\|V(x)^{\frac{1}{q(x)}} u\|_{q(x),\partial} = \|u\|_{(q(x),V(x))}$.

Proposition 3.3. *The functionals $\Psi, \Psi_+ \in C^1(X, \mathbb{R})$ (i.e., Ψ and Ψ_+ are continuously Fréchet differentiable) and $u \in X$ is a critical point of Ψ (respectively Ψ_+) if and only if u is a solution (respectively a nonnegative solution) of the Steklov problem ($S_{\lambda,\mu}$).*

Proof. From Proposition 2.5, it is clear that the functional $\phi \in C^1(X, \mathbb{R})$. Then, it is enough to show that $\psi \in C^1(X, \mathbb{R})$. Consider the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \int_{\partial\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} d\sigma.$$

We prove that J is Fréchet differentiable, i.e., at any $u \in X$ there exists $J'(u) \in X^*$ such that

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = \langle J'(u), v\rangle \quad \text{for all } v \in X \quad \text{and} \quad J' : X \to X^* \quad \text{is continuous.}$$

For $\epsilon > 0$ small enough, let $\gamma : [-\epsilon, \epsilon] \to \mathbb{R}$ defined by $\gamma(t) = J(u + tv)$, then we have

$$\begin{aligned}\frac{d\gamma(t)}{dt} = \frac{dJ(u+tv)}{dt} &= \frac{d}{dt}\int_{\partial\Omega}\frac{V(x)}{q(x)}|u+tv|^{q(x)}d\sigma\\ &= \int_{\partial\Omega}\frac{\partial}{\partial t}\left(\frac{V(x)}{q(x)}|u+tv|^{q(x)}\right)d\sigma\\ &= \int_{\partial\Omega}V(x)|u+tv|^{q(x)-2}(u+tv)vd\sigma.\end{aligned}$$

Indeed, the differentiation under the integral is allowed for t close to zero, since for $|t| < 1$ we have

$$|V(x)|u+tv|^{q(x)-2}(u+tv)v| \leq V(x)(|u|+|v|)^{q(x)-1}|v| \in L^1(\partial\Omega),$$

because of Remark 3.2. It follows that

$$\lim_{t\to 0}\frac{J(u+tv)-J(u)}{t} = \lim_{t\to 0}\frac{\gamma(t)-\gamma(0)}{t} = \frac{d\gamma(t)}{dt}|t=0 = \int_{\partial\Omega}V(x)|u|^{q(x)-2}uvd\sigma.$$

For $u \in X$, let $J'(u) : X \to \mathbb{R}$ be an operator defined by

$$\langle J'(u), v\rangle = \int_{\partial\Omega}V(x)|u|^{q(x)-2}uvd\sigma, \quad \text{for all } v \in X.$$

Obviously, $J'(u)$ is a linear operator for each given $u \in X$. Using Remark 3.2, Hölder's inequality, Proposition 2.7, and inequality (3.1), the following inequalities hold true

$$\begin{aligned}|\langle J'(u), v\rangle| &\leq \int_{\partial\Omega}V(x)|u|^{q(x)-1}|v|d\sigma\\ &= \int_{\partial\Omega}V(x)^{\frac{1}{q'(x)}}|u|^{q(x)-1}V(x)^{\frac{1}{q(x)}}|v|d\sigma\\ &\leq 2\left|V(x)^{\frac{1}{q'(x)}}|u|^{q(x)-1}\right|_{q'(x),\partial}\left|V(x)^{\frac{1}{q(x)}}|v|\right|_{q(x),\partial}\\ &\leq 2\left(1+\int_{\partial\Omega}V(x)|u|^{q(x)}d\sigma\right)^{\frac{1}{(q')^-}}|v|_{(q(x),V(x))}\\ &\leq 2\left(2+|u|^{q^+}_{(q(x),V(x))}\right)^{\frac{1}{(q')^-}}|v|_{(q(x),V(x))}\\ &\leq 2C\left(2+|u|^{q^+}_{(q(x),V(x))}\right)^{\frac{1}{(q')^-}}\|v\|_a\\ &\leq M\|v\|_a,\end{aligned}$$

where $M = 2C\big(2+|u|^{q^+}_{(q(x),V(x))}\big)^{\frac{1}{(q')^-}}$ is a positive constant. So, $J'(u)$ is a linear and bounded operator, hence continuous, i.e., $J'(u) \in X^*$. The functional J is then Gâteaux differentiable at any $u \in X$ and its Gâteaux derivative is $J'(u)$. Next, we will prove that for $(u_n)_{n\in\mathbb{N}} \subset X$ and $u \in X$, we have

$$u_n \longrightarrow u \quad \text{as } n \longrightarrow \infty \text{ in } X \quad \Longrightarrow \quad J'(u_n) \longrightarrow J'(u) \quad \text{as } n \longrightarrow \infty \text{ in } X^*.$$

Using Remark 3.2, Hölder's inequality, Proposition 2.7, and inequality (3.1), the following inequalities hold true for all $v \in X$

$$\begin{aligned} |\langle J'(u_n) - J'(u), v\rangle| &\le \int_{\partial\Omega} V(x) \left| |u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u \right| |v| d\sigma \\ &\le 2 \left| V(x)^{\frac{1}{q'(x)}} |u_n|^{q(x)-2}u_n - V(x)^{\frac{1}{q'(x)}} |u|^{q(x)-2}u \right|_{q'(x),\partial} \left| V(x)^{\frac{1}{q(x)}} |v| \right|_{q(x),\partial} \\ &\le 2C \left| V(x)^{\frac{1}{q'(x)}} |u_n|^{q(x)-2}u_n - V(x)^{\frac{1}{q'(x)}} |u|^{q(x)-2}u \right|_{q'(x),\partial} \|v\|_a. \end{aligned}$$

So, we have

$$\|J'(u_n) - J'(u)\|_{X^*} \le 2C \left| V(x)^{\frac{1}{q'(x)}} |u_n|^{q(x)-2}u_n - V(x)^{\frac{1}{q'(x)}} |u|^{q(x)-2}u \right|_{q'(x),\partial}. \tag{3.2}$$

Since the mapping $u \mapsto |u|^{q(x)-2}u$ from $L^{q(x)}(\partial\Omega)$ into $L^{q'(x)}(\partial\Omega)$ is continuous, using the compact and continuous embedding $X \hookrightarrow L^{q(x)}_{V(x)}(\partial\Omega)$, we deduce that

$$V(x)^{\frac{1}{q(x)}} u_n \longrightarrow V(x)^{\frac{1}{q(x)}} u \quad \text{as } n \longrightarrow \infty \text{ in } L^{q(x)}(\partial\Omega),$$

and then

$$V(x)^{\frac{1}{q'(x)}} |u_n|^{q(x)-2}u_n \longrightarrow V(x)^{\frac{1}{q'(x)}} |u|^{q(x)-2}u \quad \text{as } n \longrightarrow \infty \text{ in } L^{q'(x)}(\partial\Omega). \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$\|J'(u_n) - J'(u)\|_{X^*} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

that is, the mapping $J' : X \to X^*$ is continuous. Then $J \in C^1(X, \mathbb{R})$, i.e., J is Fréchet differentiable. By similar computation the functional $\int_{\partial\Omega} \frac{W(x)}{r(x)} |u|^{r(x)} d\sigma$ is of class $C^1(X, \mathbb{R})$. Then $\Psi \in C^1(X, \mathbb{R})$, and for $u, v \in X$, we have

$$\begin{aligned} \langle \Psi'(u), v\rangle = &\int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|u|^{p(x)-2}uv \right) dx \\ &- \lambda \int_{\partial\Omega} V(x)|u|^{q(x)-2}uv d\sigma - \mu \int_{\partial\Omega} W(x)|u|^{r(x)-2}uv d\sigma. \end{aligned}$$

Similarly, one can show that $\Psi_+ \in C^1(X, \mathbb{R})$, and for $u, v \in X$, we have

$$\begin{aligned} \langle \Psi'_+(u), v\rangle = &\int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|u|^{p(x)-2}uv \right) dx \\ &- \lambda \int_{\partial\Omega} V(x)(u^+)^{q(x)-1} v d\sigma - \mu \int_{\partial\Omega} W(x)(u^+)^{r(x)-1} v d\sigma. \end{aligned}$$

It is easy to see that $u \in X$ is a weak solution (rep. nonnegative weak solution) of ($S_{\lambda,\mu}$) if and only if u is a critical point of Ψ (resp. Ψ_+). Indeed, let $u \in X$ be a critical point of Ψ_+, then

$$\begin{aligned} \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla u^- + a(x)|u|^{p(x)-2}uu^- \right) dx - \lambda \int_{\partial\Omega} V(x)(u^+)^{q(x)-1} u^- d\sigma& \\ - \mu \int_{\partial\Omega} W(x)(u^+)^{r(x)-1} u^- d\sigma = 0&, \end{aligned}$$

which gives $\int_\Omega \left(|\nabla u^-|^{p(x)} + a(x)|u^-|^{p(x)} \right) dx = 0$. From Proposition 2.4 (iv), it follows that $\|u^-\|_a = 0$ and then $u^- = 0$ in X, i.e., u is nonnegative function. Since u is nonnegative, we have $u^+ = u$, then u is also a critical point of Ψ. □

Proposition 3.4. *Ψ and Ψ_+ are sequentially weakly lower semicontinuous functionals.*

Proof. By Proposition 2.5, the functional ϕ is sequentially weakly lower semicontinuous. We have only to show that ψ is also sequentially weakly lower semicontinuous. Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a sequence weakly convergent to some u in X. Let us prove that, up to a subsequence, $(\psi(u_n))_n$ converge strongly to $\psi(u)$ in $\mathbb{R}$. By virtue of the compact embedding $X \hookrightarrow L^{q(x)}_{V(x)}(\partial\Omega)$, we have

$$|u_n - u|_{(q(x),V(x))} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \quad \Longleftrightarrow \quad \int_{\partial\Omega} V(x)|u_n - u|^{q(x)} d\sigma \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

According to the reciprocal of Lebesque's dominated convergence theorem, there is a subsequence of $(u_n)_n$ still denoted by $(u_n)_n$ such that

$$u_n(x) \longrightarrow u(x) \quad \text{as } n \longrightarrow \infty \text{ for a.e. } x \in \partial\Omega, \qquad V(x)|u_n - u|^{q(x)} \leq g,$$

with $g \in L^1(\partial\Omega)$. Then, we deduce that

$$\frac{V(x)}{q(x)}|u_n|^{q(x)} \longrightarrow \frac{V(x)}{q(x)}|u|^{q(x)} \quad \text{as } n \longrightarrow \infty \text{ for a.e. } x \in \partial\Omega.$$

Using the inequality $(a+b)^q \leq 2^{q-1}(a^q + b^q)$, for $a, b \geq 0$ and $q \geq 1$, we have

$$\begin{aligned}\frac{V(x)}{q(x)}|u_n|^{q(x)} &\leq V(x)|u_n|^{q(x)} \leq V(x)\left(|u| + |u_n - u|\right)^{q(x)} \\ &\leq 2^{q(x)-1}(V(x)|u|^{q(x)} + V(x)|u_n - u|^{q(x)}) \\ &\leq 2^{q^+-1}(V(x)|u|^{q(x)} + g(x)).\end{aligned}$$

By Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{\partial\Omega} \frac{V(x)}{q(x)}|u_n|^{q(x)} d\sigma = \int_{\partial\Omega} \frac{V(x)}{q(x)}|u|^{q(x)} d\sigma. \tag{3.4}$$

Similarly, we also have

$$\lim_{n\to\infty} \int_{\partial\Omega} \frac{W(x)}{r(x)}|u_n|^{r(x)} d\sigma = \int_{\partial\Omega} \frac{W(x)}{r(x)}|u|^{r(x)} d\sigma. \tag{3.5}$$

Then, $\lim_{n\to\infty} \psi(u_n) = \psi(u)$. Since $\liminf_{n\to\infty} \phi(u_n) \geq \phi(u)$, we have

$$\liminf_{n\to\infty} \Psi(u_n) \geq \Psi(u).$$

By similar steps one can show that the functional Ψ_+ is also sequentially weakly lower semicontinuous. □

Proposition 3.5. *The derivative functions Ψ' and Ψ'_+ are of (S^+) type operators.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a sequence weakly convergent to some u in X such that

$$\limsup_{n\to\infty} \langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle \leq 0.$$

As in the proof of Lemma 3.3, we obtain the following inequalities

$$\begin{aligned} |\langle J'(u_n) - J'(u), u_n - u\rangle| &\leq \int_{\partial\Omega} V(x) \left| |u_n(x)|^{q(x)-2}u_n(x) - |u(x)|^{q(x)-2}u(x) \right| |u_n - u| d\sigma \\ &\leq 2 \left| V(x)^{\frac{1}{q'(x)}} |u_n|^{q(x)-2}u_n - V^{\frac{1}{q'(x)}} |u|^{q(x)-2}u \right|_{q'(x),\partial} |u_n - u|_{(q(x),V(x))}. \end{aligned}$$

From the compact embedding $X \hookrightarrow L^{q(x)}_{V(x)}(\partial\Omega)$ and the continuity of the mapping $u \mapsto |u|^{q(x)-2}u$ from $L^{q(x)}(\partial\Omega)$ into $L^{q'(x)}(\partial\Omega)$, we obtain

$$\langle J'(u_n) - J'(u), u_n - u\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Similarly for the functional $u \mapsto \int_{\partial\Omega} \frac{W(x)}{r(x)} |u|^{r(x)} d\sigma$. Then, $\lim_{n\to\infty}\langle \psi'(u_n) - \psi'(u), u_n - u\rangle = 0$. We have

$$\limsup_{n\to\infty} \langle \phi'(u_n) - \phi'(u), u_n - u\rangle = \limsup_{n\to\infty} \langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle \leq 0.$$

Combine the last inequality and the (S^+) property of the operator ϕ' (see Proposition 2.5, it follows that $(u_n)_n$ converge strongly to u in X. The operator Ψ' is then of (S^+) type. Similarly, one can show that the operator Ψ'_+ is also of (S^+) type. □

We need to recall some classical well known results of the theory of Banach spaces which will be useful in the sequel. Let X be a reflexive and separable Banach space, there exist $(e_n)_{n\in\mathbb{N}^*} \subseteq X$ and $(f_n)_{n\in\mathbb{N}^*} \subseteq X^*$ such that $\langle f_n, e_m\rangle = \delta_{nm}$ for all $n, m \in \mathbb{N}^*$, $X = \overline{\text{Span}\{e_n \mid n \in \mathbb{N}^*\}}$, $X^* = \overline{\text{Span}\{f_n \mid n \in \mathbb{N}^*\}}^{w^*}$ (see [49], Section 17), where X^* denote the dual space of X. For $k \in \mathbb{N}^*$, consider the subsets of X :

$$X_k = \text{Span}\{e_k\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{k}^{\infty} X_j}.$$

Proposition 3.6 ([27]). *Assume that $\psi : X \to \mathbb{R}$ is weakly-strongly continuous and $\psi(0) = 0$. Let $r > 0$ be given and set*

$$\beta_k = \beta_k(r) = \sup\{|\psi(u)| \mid u \in Z_k, \|u\|_a \leq r\}.$$

Then $\beta_k \to 0$ as $k \to \infty$.

Definition 3.7. Let X be a reflexive Banach space and $\Phi : X \to \mathbb{R}$ be a functional such that $\Phi \in C^1(X, \mathbb{R})$.

(i) The C^1-functional Φ is said to satisfy the Palais–Smale condition (in short (PS) condition) if any sequence $(u_n)_{n\in\mathbb{N}} \subset X$ for which $(\Phi(u_n))_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

(ii) The C^1-functional Φ is said satisfy the $(PS)_c$ condition, where $c \in \mathbb{R}$, if any sequence $(u_n)_{n\in\mathbb{N}} \subset X$ for which $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

(iii) Let Y_n, $n \in \mathbb{N}^*$, as defined above. The C^1-functional Φ is said to satisfy the $(PS)^*_c$ condition, where $c \in \mathbb{R}$, if any sequence $(u_n)_{n\in\mathbb{N}^*} \subset X$ for which $u_n \in Y_n$ for any $n \in \mathbb{N}^*$, $\Phi(u_n) \to c$ and $(\Phi|_{Y_n})'(u_n) \to 0$ as $n \to \infty$, has a subsequence convergent to a critical point of Φ.

Remark 3.8. It is easy to remark that if Φ satisfies the (PS) condition, then Φ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.

3.2 Abstract theorems

Theorem 3.9 ([46]). *Let X be a reflexive Banach space and $\Phi : X \to \mathbb{R}$ be a functional such that*

(i) Φ is sequentially weakly lower semicontinuous, i.e., for any sequence $(u_n)_n$ and u in X

$$u_n \xrightarrow{\text{weakly}} u \quad \text{as } n \longrightarrow \infty \quad \Longrightarrow \quad \Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n);$$

(ii) Φ is coercive, i.e.,

$$\Phi(u) \longrightarrow \infty \quad \text{as } \|u\|_X \longrightarrow \infty.$$

Then Φ is bounded below and its minimum is achieved at a point $u_0 \in X$.

Theorem 3.10 (Mountain Pass Theorem, [48]). *Let X be a real Banach space and $\Phi \in C^1(X, \mathbb{R})$ be a functional satisfying the Palais-Smale (PS) condition. Suppose $\Phi(0) = 0$ and*

(i) there are constants $\alpha, \rho > 0$ such that $\Phi|_{\|x\|=\alpha} \geq \rho$;

(ii) there is an element $e \in X$ such that $\Phi(e) \leq 0$.

Then Φ possesses a critical value $c \geq \rho$. Moreover, c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} \Phi(g(t)) \text{ where } \Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}.$$

Theorem 3.11 (Fountain Theorem, [48]). *Let X be a reflexive and separable Banach space, $\Phi \in C^1(X, \mathbb{R})$ an even functional and the subspaces X_k, Y_k, Z_k as defined above. If for each $k \in \mathbb{N}^*$ there exist $\rho_k > r_k > 0$ such that*

(F1) $\inf_{x\in Z_k,\ \|x\|=r_k} \Phi(x) \to \infty$ as $k \to \infty$;

(F2) $\max_{x\in Y_k,\ \|x\|=\rho_k} \Phi(x) \leq 0$;

(F3) Φ satisfies the $(PS)_c$ condition for every $c > 0$.

Then, Φ has a sequence of critical values tending to $+\infty$.

Theorem 3.12 (Dual Fountain Theorem, [48]). *Let X be a reflexive and separable Banach space, $\Phi \in C^1(X, \mathbb{R})$ an even functional and the subspaces X_k, Y_k, Z_k as defined above. Assume there is a $k_0 \in \mathbb{N}^*$ such that, for each $k \in \mathbb{N}^*$, $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that*

(D1) $\inf_{x\in Z_k,\ \|x\|=\rho_k} \Phi(x) \geq 0$;

(D2) $b_k := \max_{x\in Y_k,\ \|x\|=r_k} \Phi(x) < 0$;

(D3) $d_k := \inf_{x\in Z_k,\ \|x\|\leq\rho_k} \Phi(x) \to 0$ as $k \to \infty$;

(D4) Φ satisfies the $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0)$.

Then, Φ has a sequence of negative critical values converging to 0.

3.3 Technical lemmas

Lemma 3.13. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled. If* $\min\{q^-,r^-\} > p^+$ *and* $\lambda,\mu \geq 0$, *then there exists* $\varphi \in X\setminus\{0\}$ *such that* $\varphi \geq 0$ *and* $\Psi_+(t\varphi) \to -\infty$ *as* $t \to +\infty$.

Proof. Suppose that $\theta = \min\{q^-,r^-\} > p^+$ and $\lambda,\mu \geq 0$. Choose $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \not\equiv 0$. For any $t > 1$, we have

$$\begin{aligned}
\Psi_+(t\varphi) = &\int_\Omega \frac{t^{p(x)}}{p(x)}\left(|\nabla\varphi(x)|^{p(x)} + a(x)|\varphi(x)|^{p(x)}\right)dx \\
&- \lambda\int_{\partial\Omega}\frac{t^{q(x)}}{q(x)}V(x)\varphi(x)^{q(x)}d\sigma - \mu\int_{\partial\Omega}\frac{t^{r(x)}}{r(x)}W(x)\varphi(x)^{r(x)}d\sigma \\
\leq &\frac{t^{p^+}}{p^-}\int_\Omega\left(|\nabla\varphi(x)|^{p(x)} + a(x)|\varphi(x)|^{p(x)}\right)dx \\
&- \frac{\lambda t^{q^-}}{q^+}\int_{\partial\Omega}W(x)\varphi(x)^{q(x)}d\sigma - \frac{\mu t^{r^-}}{r^+}\int_{\partial\Omega}W(x)\varphi(x)^{r(x)}d\sigma \\
\leq &\frac{t^{p^+}}{p^-}\int_\Omega\left(|\nabla\varphi(x)|^{p(x)} + a(x)|\varphi(x)|^{p(x)}\right)dx \\
&- t^\theta\left(\frac{\lambda}{q^+}\int_{\partial\Omega}V(x)\varphi(x)^{q(x)}d\sigma + \frac{\mu}{r^+}\int_{\partial\Omega}W(x)\varphi(x)^{r(x)}d\sigma\right).
\end{aligned}$$

Since $\theta > p^+$, it follows that $\Psi_+(t\varphi) \to -\infty$ as $t \to +\infty$. ☐

Lemma 3.14. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled.*

(i) If $\min\{q^-,r^-\} > p^+$ *and* $\lambda,\mu \geq 0$, *then there exist* $\tau,\rho > 0$ *such that* $\Psi_+(u) \geq \tau$ *if* $\|u\|_a = \rho$.

(ii) If $p^+ < r^-$, *then for each given* $\mu \in \mathbb{R}$, *there exists* $\lambda^0 = \lambda^0(\mu) > 0$ *such that, for any* $\lambda < \lambda^0$, *there exist* $\tau,\rho > 0$ *such that* $\Psi_+(u) \geq \tau$ *if* $\|u\|_a = \rho$.

(iii) If $q^+ < p^-$, *then for each given* $\lambda \in \mathbb{R}$, *there exists* $\mu^0 = \mu^0(\lambda) > 0$ *such that, for any* $\mu < \mu^0$, *there exist* $\overline{\tau},\overline{\rho} > 0$ *such that* $\Psi_+(u) \geq \overline{\tau}$ *if* $\|u\|_a = \overline{\rho}$.

Proof. We have seen, from (3.1), that there exists a positive constant $C > 1$ such that

$$|u|_{(q(x),V(x))} \leq C\|u\|_a \text{ and } |u|_{(r(x),W(x))} \leq C\|u\|_a, \text{ for all } u \in X.$$

(i) Suppose that $\min\{q^-,r^-\} > p^+$ and $\lambda,\mu \geq 0$. Fix a positive constant ρ such that

$$\rho < \frac{1}{C} < 1 \text{ and } \rho \leq \min\left\{\left(\frac{q^-}{4\lambda p^+ C^{q^-}}\right)^{\frac{1}{q^- - p^+}}, \left(\frac{r^-}{4\mu p^+ C^{r^-}}\right)^{\frac{1}{r^- - p^+}}\right\}.$$

Then, using Proposition 2.4, Proposition 2.7 and inequalities (3.1), for all $u \in X$ with $\|u\|_a =$

$\rho < 1$, we have

$$\begin{aligned}
\Psi_+(u) \geq & \frac{1}{p^+}\int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx \\
& - \frac{\lambda}{q^-}\int_{\partial\Omega} V(x)|u(x)|^{q(x)} d\sigma - \frac{\mu}{r^-}\int_{\partial\Omega} W(x)|u(x)|^{r(x)} d\sigma \\
\geq & \frac{1}{p^+}\rho_a(u) - \frac{\lambda}{q^-}|u|^{q^-}_{(q(x),V(x))} - \frac{\mu}{r^-}|u|^{r^-}_{(r(x),W(x))}\} \\
\geq & \frac{1}{p^+}\|u\|_a^{p^+} - \frac{\lambda C^{q^-}}{q^-}\|u\|_a^{q^-} - \frac{\mu C^{r^-}}{r^-}\|u\|_a^{r^-} \\
= & \frac{1}{p^+}\rho^{p^+} - \frac{\lambda C^{q^-}}{q^-}\rho^{q^-} - \frac{\mu C^{r^-}}{r^-}\rho^{r^-} \\
\geq & \frac{1}{p^+}\rho^{p^+} - \frac{1}{4p^+}\rho^{p^+} - \frac{1}{4p^+}\rho^{p^+} \\
= & \frac{1}{2p^+}\rho^{p^+}.
\end{aligned}$$

Then, there exist $\rho > 0$ and $\tau = \frac{1}{2p^+}\rho^{p^+} > 0$ such that $\Psi_+(u) \geq \tau$ for all $u \in X$ with $\|u\|_a = \rho$.

(ii) Suppose that $p^+ < h^-$. For each given $\mu \in \mathbb{R}$, fix ρ such that

$$0 < \rho < \frac{1}{C} \quad \text{and} \quad \frac{|\mu|C^{r^-}}{r^-}\rho^{r^-} \leq \frac{1}{2p^+}\rho^{p^+}.$$

Then, using Proposition 2.4, Proposition 2.7 and inequalities (3.1), for all $u \in X$ with $\|u\|_a = \rho < 1$, we have

$$\begin{aligned}
\Psi_+(u) \geq & \frac{1}{p^+}\int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx \\
& - \frac{\max\{\lambda, 0\}}{q^-}\int_{\partial\Omega} V(x)|u(x)|^{q(x)} d\sigma - \frac{|\mu|}{r^-}\int_{\partial\Omega} W(x)|u(x)|^{r(x)} d\sigma \\
\geq & \frac{1}{p^+}\rho_a(u) - \frac{\max\{\lambda, 0\}}{q^-}|u|^{q^-}_{(q(x),V(x))} - \frac{|\mu|}{r^-}|u|^{r^-}_{(r(x),W(x))} \\
\geq & \frac{1}{p^+}\|u\|_a^{p^+} - \frac{\max\{\lambda, 0\}C^{q^-}}{q^-}\|u\|_a^{q^-} - \frac{|\mu|C^{r^-}}{r^-}\|u\|_a^{r^-} \\
\geq & \frac{1}{2p^+}\rho^{p^+} - \frac{\max\{\lambda, 0\}C^{q^-}}{q^-}\rho^{q^-}.
\end{aligned}$$

We can choose

$$\lambda^0 = \frac{q^-\rho^{p^+-q^-}}{4p^+C^{q^-}} \quad \text{and} \quad \tau = \frac{1}{4p^+}\rho^{p^+}.$$

Then, for any $\lambda < \lambda^0$, $\Psi_+(u) \geq \tau > 0$ for all $u \in X$ with $\|u\|_a = \rho$.

(iii) From Proposition 2.7 and inequalities (3.1), for all $u \in X$, we have

$$\begin{aligned}
&\int_{\partial\Omega} V(x)|u(x)|^{q(x)} d\sigma \leq \max\{C^{q^-}\|u\|_a^{q^-}, C^{q^+}\|u\|_a^{q^+}\}, \\
&\int_{\partial\Omega} W(x)|u(x)|^{r(x)} d\sigma \leq \max\{C^{r^-}\|u\|_a^{r^-}, C^{r^+}\|u\|_a^{r^+}\}.
\end{aligned} \tag{3.6}$$

Suppose that $q^+ < p^-$. For each given $\lambda \in \mathbb{R}$, setting $\overline{\rho} \geq 1$ and $\frac{|\lambda| C^{q^+}}{q^-}\overline{\rho}^{q^+} \leq \frac{1}{2p^+}\overline{\rho}^{p^-}$. Then, using Proposition 2.4 and inequalities (3.6), for all $u \in X$ with $\|u\|_a = \overline{\rho} \geq 1$, we have

$$\begin{aligned} \Psi_+(u) &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx \\ &\quad - \frac{|\lambda|}{q^-}\int_{\partial\Omega} V(x)|u(x)|^{q(x)} d\sigma - \frac{\max\{\mu, 0\}}{h^-}\int_{\partial\Omega} W(x)|u(x)|^{r(x)} d\sigma \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{|\lambda| C^{q^+}}{q^-}\|u\|_a^{q^+} - \frac{\max\{\mu,0\} C^{r^+}}{r^-}\|u\|_a^{r^+} \\ &\geq \frac{1}{2p^+}\overline{\rho}^{p^-} - \frac{\max\{\mu,0\} C^{r^+}}{r^-}\overline{\rho}^{r^+}. \end{aligned} \tag{3.7}$$

We can choose

$$\mu^0 = \frac{r^- \overline{\rho}^{p^- - r^+}}{4p^+ C^{r^+}} \quad \text{and} \quad \overline{\tau} = \frac{1}{4p^+}\overline{\rho}^{p^-}.$$

Then, for any $\mu < \mu^0$, $\Psi_+(u) \geq \overline{\tau} > 0$ for all $u \in X$ with $\|u\|_a = \overline{\rho}$. □

Lemma 3.15. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled. Then, the functional* Ψ_+ *satisfies* (PS) *condition if one of the following assertions is satisfied :*

(i) $\min\{q^-, r^-\} > p^+$ *and* $\lambda, \mu \geq 0$;

(ii) $q^+ < p^- \leq p^+ < r^-$ *and* $\lambda \in \mathbb{R}$, $\mu \geq 0$;

(iii) $q^+ = p^- \leq p^+ < r^-$ *and* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \lambda_0 := \frac{\frac{1}{p^+} - \frac{1}{r^-}}{(\frac{1}{q^-} - \frac{1}{r^-})C^{q^+}}$, $\mu \geq 0$;

(iv) $p^+ < r^-$, $q^+ \leq r^-$ *and* $\lambda \leq 0$, $\mu \geq 0$.

Proof. In the first time, we show that every (PS) sequence of Ψ_+ is bounded in X under condition (i)–(iv). Let $(u_n)_{n\in\mathbb{N}}$ be a (PS) sequence of Ψ_+, that is

$$|\Psi_+(u_n)| < M, \quad \text{for all } n \in \mathbb{N}, \text{ with } M > 0 \quad \text{and} \quad \Psi'_+(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty.$$

Arguing by contradiction, let us suppose that

$$\|u_n\|_a \to +\infty \quad \text{as} \quad n \to \infty.$$

Case (i), assume that $\theta = \min\{q^-, r^-\} > p^+$ and $\lambda, \mu \geq 0$. If $\|u_n\|_a > 1$ for n large enough, using Proposition 2.4, we have

$$\begin{aligned} M + \|u_n\|_a &\geq \Psi_+(u_n) - \frac{1}{\theta}\langle \Psi'_+(u_n), u_n\rangle \\ &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u_n(x)|^{p(x)} + a(x)|u_n(x)|^{p(x)}\right) dx \\ &\quad - \frac{\lambda}{q^-}\int_{\partial\Omega} V(x)(u_n^+)^{q(x)} d\sigma - \frac{\mu}{r^-}\int_{\partial\Omega} W(x)(u_n^+)^{r(x)} d\sigma \\ &\quad - \frac{1}{\theta}\int_\Omega \left(|\nabla u_n(x)|^{p(x)} + a(x)|u_n(x)|^{p(x)}\right) dx \\ &\quad + \frac{\lambda}{\theta}\int_{\partial\Omega} V(x)(u_n^+)^{q(x)} d\sigma + \frac{\mu}{\theta}\int_{\partial\Omega} W(x)(u_n^+)^{r(x)} d\sigma \end{aligned}$$

$$\geq \left(\frac{1}{p^+}-\frac{1}{\theta}\right)\int_\Omega \left(|\nabla u_n(x)|^{p(x)}+a(x)|u_n(x)|^{p(x)}\right)dx$$
$$\geq \left(\frac{1}{p^+}-\frac{1}{\theta}\right)\|u_n\|_a^{p^-}.$$

From the last inequality, for $n \in \mathbb{N}$ large, we obtain a contradiction since $p^- > 1$. So, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X.
Cases (ii) and (iii), assume that $q^+, p^+ \leq r^-$ and $\lambda \in \mathbb{R}, \mu \geq 0$. If $\|u_n\|_a > 1$ for any $n \in \mathbb{N}$, then

$$\begin{aligned} M+\|u_n\|_a &\geq \Psi_+(u_n)-\frac{1}{r^-}\langle \Psi_+'(u_n),u_n\rangle \\ &\geq \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_\Omega \left(|\nabla u_n(x)|^{p(x)}+a(x)|u_n(x)|^{p(x)}\right)dx \\ &\quad -\lambda\int_{\partial\Omega}\frac{1}{q(x)}V(x)(u_n^+)^{q(x)}d\sigma+\frac{\lambda}{r^-}\int_{\partial\Omega}V(x)(u_n^+)^{q(x)}d\sigma \\ &\geq \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_\Omega \left(|\nabla u_n(x)|^{p(x)}+a(x)|u_n(x)|^{p(x)}\right)dx \\ &\quad -|\lambda|\left(\frac{1}{q^-}-\frac{1}{r^-}\right)\int_{\partial\Omega}V(x)(u_n^+)^{q(x)}d\sigma. \end{aligned} \tag{3.8}$$

Taking into account inequalities (3.6), inequality (3.8) and Proposition 2.4 imply that

$$M+\|u_n\|_a \geq \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\|u_n\|_a^{p^-}-\left(\frac{1}{q^-}-\frac{1}{r^-}\right)|\lambda|C^{q^+}\|u_n\|_a^{q^+}.$$

This last inequality yields a contradiction if $n \in \mathbb{N}$ is large, since $p^- > 1$, $q^+ \leq p^-$ and $|\lambda| < \lambda_0$. So, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X in both cases (ii) and (iii).
Case (iv), by a similar computation as above, we have

$$M+\|u_n\|_a \geq \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\|u_n\|_a^{p^-},$$

which gives a contradiction if $n \in \mathbb{N}$ is large, since $p^- > 1$.
Thus, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X.

Next, we show that there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ strongly convergent in X. Because of the reflexiveness of X, since $(u_n)_{n\in\mathbb{N}}$ is bounded in X, we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ weakly convergent to some $u \in X$. We have

$$\begin{aligned} |\langle \Psi_+'(u_{n_k})-\Psi_+'(u),u_{n_k}-u\rangle| &\leq |\langle \Psi_+'(u_{n_k}),u_{n_k}-u\rangle|+|\langle \Psi_+'(u),u_{n_k}-u\rangle| \\ &\leq |\langle \Psi_+'(u_{n_k}),u_{n_k}\rangle|+|\langle \Psi_+'(u_{n_k}),u\rangle| \\ &\quad +|\langle \Psi_+'(u),u_{n_k}\rangle-\langle \Psi_+'(u),u\rangle| \\ &\leq \|\Psi_+'(u_{n_k})\|_{X^*}\|u_{n_k}\|_a+\|\Psi_+'(u_{n_k})\|_{X^*}\|u\|_a \\ &\quad +|\langle \Psi_+'(u),u_{n_k}\rangle-\langle \Psi_+'(u),u\rangle|. \end{aligned} \tag{3.9}$$

Since $\Psi_+'(u_{n_k}) \longrightarrow 0$ in X^* as $k \longrightarrow \infty$ and $(u_{n_k})_{k\in\mathbb{N}}$ is bounded in X, inequality (3.9) and the weak convergence $u_{n_k} \xrightarrow{\text{weakly}} u$ as $k \to \infty$ yield

$$\lim_{k\to\infty}\langle \Psi_+'(u_{n_k})-\Psi_+'(u),u_{n_k}-u\rangle = 0. \tag{3.10}$$

Combine (3.10) and the (S^+) property of Ψ_+', we obtain that the subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converges strongly to u in X. □

Remark 3.16. All statements of Lemmas 3.13–3.15 remain valid if we replace Ψ_+ by Ψ.

4 Proof of main theorems

4.1 Proof of Theorem 1.1

(i) Assume that $\max\{q^+, r^+\} < p^-$ and $\lambda, \mu \in \mathbb{R}$. From Lemma 3.4 the C^1-functional Ψ_+ is sequentially weakly lower semicontinuous, we only have to show that Ψ_+ is coercive to apply Theorem 3.9. Let $u \in X$ be such that $\|u\|_a > 1$, from Proposition 2.4 and inequalities (3.6), we have

$$\begin{aligned}\Psi_+(u) &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx \\ &\quad - \frac{|\lambda|}{q^-}\int_{\partial\Omega} V(x)|u(x)|^{q(x)} d\sigma - \frac{|\mu|}{r^-}\int_{\partial\Omega} W(x)|u(x)|^{r(x)} d\sigma \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{|\lambda| C^{q^+}}{q^-}\|u\|_a^{q^+} - \frac{|\mu| C^{r^+}}{r^-}\|u\|_a^{r^+},\end{aligned}$$

Since $\max\{q^+, r^+\} < p^-$, we have $\Psi_+(u) \to \infty$ as $\|u\|_a \to \infty$, then Ψ_+ is coercive. Applying Theorem 3.9, the functional Ψ_+ has a minimum point u_0 in X which is a nonnegative solution of ($S_{\lambda,\mu}$), as a critical point of Ψ_+.

(ii) Assume that $\min\{q^-, r^-\} > p^+$ and $\lambda, \mu \geq 0$. From Lemma 3.14 (i), there exist $\tau, \rho > 0$ such that $\Psi_+(u) \geq \tau$ if $\|u\|_a = \rho$. From Lemma 3.13, there exists $\varphi \in X \setminus \{0\}$ such that $\varphi \geq 0$ and $\Psi_+(t\varphi) \to -\infty$ as $t \to +\infty$. Therefore, for $t > 1$ large enough, there is $e = t\varphi \in X$ such that $\|e\|_a > \rho$ and $\Psi_+(e) < 0$. Moreover, from Lemma 3.15 (i), Ψ_+ satisfies the (PS) condition. Then, the functional Ψ_+ satisfies all requirements of the mountain geometry. Consequently, according to the Mountain Pass Theorem (see Theorem 3.10), Ψ_+ admits a critical value $c \geq \tau$ which is characterized by

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} \Psi_+(g(t)), \quad \text{where } \Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}.$$

We deduce the existence of a critical point u of Ψ_+ associated with the critical value c satisfying $\Psi_+(u) \geq \tau$. Hence, the Steklov problem ($S_{\lambda,\mu}$) has a nontrivial nonnegative solution.

4.2 Proof of Theorem 1.2

(i) Assume $1 < p^- \leq p^+ < r^- \leq q^-$ and $\lambda, \mu \geq 0$. We will prove that the functional Ψ satisfies the conditions of Theorem 3.11. Obviously, from Lemma 3.3 and Lemma 3.15, $\Psi \in C^1(X, \mathbb{R})$ is an even functional and satisfies (PS) condition. Then, Ψ satisfies also $(PS)_c$ condition for every $c > 0$. Next, we will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that conditions (F1) and (F2) of Theorem 3.11 hold. For $r > 0$ and $k \in \mathbb{N}^*$, let

$$\beta_k(r) = \sup\{|\psi(u)| \mid u \in Z_k, \|u\|_a \leq r\},$$

with $\psi(u) = \lambda \int_{\partial\Omega} \frac{V(x)}{q(x)}|u|^{q(x)} d\sigma + \mu \int_{\partial\Omega} \frac{W(x)}{r(x)}|u|^{r(x)} d\sigma$. The functional ψ is weakly-strongly continuous (see Lemma 3.4). Since $\beta_k(r) \to 0$ as $k \to \infty$ (see Proposition 3.6), it follows that, for each $n \in \mathbb{N}^*$, there exists $k_0(n) \in \mathbb{N}^*$ such that $\beta_k(n) \leq 1$ if $k \geq k_0(n)$. Without loss of generality, we may assume that $k_0(n) < k_0(n+1)$ for each $n \in \mathbb{N}^*$. Let us define the sequence r_k, $k \in \mathbb{N}^*$, by

$$r_k = \begin{cases} 1 & \text{if } 1 \leq k < k_0(1), \\ n & \text{if } k_0(n) \leq k < k_0(n+1). \end{cases}$$

It is easy to see that $r_k \to \infty$ as $k \to \infty$. For $k \in \mathbb{N}^*$ large enough, let $u \in Z_k$ with $\|u\|_a = r_k$, we have

$$\Psi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} dx + a(x)|u|^{p(x)} \right) dx - \psi(u) \geq \frac{1}{p^+}(r_k)^{p^-} - 1,$$

because $\psi(u) \leq \beta_k(n) \leq 1$. Therefore,

$$\inf_{u \in Z_k,\ \|u\|_a = r_k} \Psi(u) \to \infty \quad \text{as } k \to \infty.$$

So, the condition (F1) of Theorem 3.11 is satisfied.

Since Y_k is a finite dimensional space, the norms $\|\cdot\|_a$, $|\cdot|_{(q(x),V(x))}$ and $|\cdot|_{(r(x),W(x))}$ are equivalent. Then, there exist positive constants δ_1, δ_2 such that $|u|_{(q(x),V(x))} \geq \delta_1 \|u\|_a$ and $|u|_{(r(x),W(x))} \geq \delta_2 \|u\|_a$, for all $u \in Y_k$. For $k \in \mathbb{N}^*$, let $u \in Y_k$ with $\|u\|_a > 1$ large enough such that $|u|_{(q(x),V(x))} > 1$ and $|u|_{(r(x),W(x))} > 1$, from Proposition 2.7, we deduce that

$$\begin{aligned}
\Psi(u) &\leq \frac{1}{p^-}\|u\|_a^{p^+} - \frac{\lambda}{q^+}\int_{\partial\Omega} V(x)|u|^{q(x)} d\sigma - \frac{\mu}{r^+}\int_{\partial\Omega} W(x)|u|^{r(x)} d\sigma \\
&\leq \frac{1}{p^-}\|u\|_a^{p^+} - \frac{\lambda}{q^+}|u|^{q^-}_{(q(x),V(x))} - \frac{\mu}{r^+}|u|^{r^-}_{(r(x),W(x))} \\
&\leq \frac{1}{p^-}\|u\|_a^{p^+} - \frac{\lambda}{q^+}(|u|^{q^-}_{(q(x),V(x))} - 1) - \frac{\mu}{r^+}(|u|^{r^-}_{(r(x),W(x))} - 1 \\
&\leq \frac{1}{p^-}\|u\|_a^{p^+} - \frac{\lambda}{q^+}(\delta_1^{q^-}\|u\|_a^{q^-} - 1) - \frac{\mu}{r^+}(\delta_2^{r^-}\|u\|_a^{r^-} - 1) \\
&\leq \frac{1}{p^-}\|u\|_a^{p^+} - (\frac{\lambda\delta_1^{q^-}}{q^+} + \frac{\mu\delta_2^{r^-}}{r^+})\|u\|_a^{\theta} + C(\lambda, \mu, q^+, r^+),
\end{aligned} \tag{4.1}$$

where $\theta = \min\{q^-, r^-\}$ and $C(\lambda, \mu, q^+, r^+) > 0$ a constant. From the last inequality, since $\theta > p^+$, we obtain that $\Psi(u) \to -\infty$ as $\|u\|_a \to \infty$. Thus for each $k \in \mathbb{N}^*$, there exists a sufficiently large $\rho_k > r_k$ such that $\Psi(u) < 0$ for all $u \in Y_k$ with $\|u\|_a = \rho_k$. Therefore,

$$\max_{u \in Y_k,\ \|u\|_a = \rho_k} \Psi(u) \leq 0.$$

So, the condition (F2) of Theorem 3.11 is satisfied.

According to the Fountain Theorem 3.11, the functional Ψ has a sequence of critical values $(c_n)_n$ tending to $+\infty$, which correspond to a sequence $(u_n)_n \subset X$ of critical points of Ψ such that $\Psi(u_n) \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

(ii)–(iii) Assume $q^+ \leq p^- \leq p^+ < h^-$, $\lambda \in \mathbb{R}$ and $\mu > 0$. We will prove that the functional Ψ satisfies the conditions of Theorem 3.11. As above, there exists $r_k \to \infty$ as $k \to \infty$ such that for $k \in \mathbb{N}^*$ large enough, $u \in Z_k$ with $\|u\|_a = r_k$, we have

$$\Psi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} dx + a(x)|u|^{p(x)} \right) dx - \psi(u) \geq \frac{1}{p^+}(r_k)^{p^-} - 1.$$

Therefore,

$$\inf_{u \in Z_k,\ \|u\|_a = r_k} \Psi(u) \to \infty \quad \text{as } k \to \infty.$$

So, the condition (F1) of Theorem 3.11 is satisfied.

For $k \in \mathbb{N}^*$, let $u \in Y_k$ with $\|u\|_a > 1$, from Proposition 2.7, we deduce that

$$\begin{aligned}\Psi(u) &\leq \frac{1}{p^-}\|u\|_a^{p^+} + \frac{|\lambda|}{q^-}\left(|u|^{q^+}_{(q(x),V(x))} + 1\right) - \frac{\mu}{r^+}\left(|u|^{r^-}_{(r(x),W(x))} - 1\right) \\ &\leq \frac{1}{p^-}\|u\|_a^{p^+} + \frac{|\lambda|\delta_1^{q^+}}{q^-}\|u\|_a^{q^+} - \frac{\mu\delta_2^{r^-}}{r^+}\|u\|_a^{r^-} + C(\lambda,\mu,q^-,r^+),\end{aligned} \tag{4.2}$$

where $C(\lambda,\mu,q^-,r^+) > 0$ is a constant and δ_1, δ_2 as in the proof of (i). From the last inequality, since $q^+ \leq p^+ < r^-$, we obtain that $\Psi(u) \to -\infty$ as $\|u\|_a \to \infty$. Thus for each $k \in \mathbb{N}^*$, there exists a sufficiently large $\rho_k > r_k$ such that $\Psi(u) < 0$ for all $u \in Y_k$ with $\|u\|_a = \rho_k$. Therefore,

$$\max_{u \in Y_k,\ \|u\|_a = \rho_k} \Psi(u) \leq 0.$$

So, the condition (F2) of Theorem 3.11 is satisfied.

By Theorem 3.11, there is a sequence $(u_n)_n \subset X$ of critical points of Ψ such that $\Psi(u_n) \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

(iv)–(v) Assume $q^+ < \min\{p^-, r^-\}$, $\lambda > 0$ and $\mu \in \mathbb{R}$. We will prove that the functional Ψ satisfies the conditions of Theorem 3.12. We have already seen that $\Psi \in C^1(X,\mathbb{R})$ is an even functional. For $r > 0$ and $k \in \mathbb{N}^*$, let

$$\beta_k(r) = \sup\{|\psi(u)| \mid u \in Z_k, \|u\|_a \leq r\},$$

with $\psi(u) = \lambda \int_{\partial\Omega} \frac{V(x)}{q(x)}|u|^{q(x)}d\sigma + \mu \int_{\partial\Omega} \frac{W(x)}{r(x)}|u|^{r(x)}d\sigma$. The functional ψ is weakly-strongly continuous (see Lemma 3.4). Since $\beta_k(r) \to 0$ as $k \to \infty$ (see Proposition 3.6), then there exists $k_0 \in \mathbb{N}^*$ such that $\beta_k(1) \leq \frac{1}{2p^+}$ if $k \geq k_0$. Setting $\rho_k = 1$, then for $k \geq k_0$ and $u \in Z_k$ with $\|u\|_a = \rho_k = 1$, we have

$$\Psi(u) \geq \frac{1}{p^+}\rho_a(u) - \psi(u) \geq \frac{1}{p^+} - \frac{1}{2p^+} = \frac{1}{2p^+} > 0.$$

Therefore,

$$\inf_{u \in Z_k,\ \|u\|_a = \rho_k} \Psi(u) \geq 0.$$

So, the condition (D1) of Theorem 3.12 is satisfied.

Let r be a positive small real number such that $0 < r < \frac{1}{C}$, where $C > 1$ appears in (3.1). As above, since Y_k is a finite dimensional space, there are positive constants δ_1, δ_2 such that for all $u \in Y_k$ with $\|u\|_a = r < 1$, we have

$$\Psi(u) \leq \frac{1}{p^-}\|u\|_a^{p^-} - \frac{\lambda\delta_1^{q^+}}{q^+}\|u\|_a^{q^+} + \frac{|\mu|\delta_2^{r^-}}{r^-}\|u\|_a^{r^-}.$$

Since $q^+ < \min\{p^-, r^-\}$, we can choose $r_k \in (0, \rho_k)$ small enough such that $\Psi(u) < 0$ for all $u \in Y_k$ with $\|u\|_a = r_k$. Therefore,

$$\max_{u \in Y_k,\ \|u\|_a = r_k} \Psi(u) < 0.$$

So, the condition (D2) of Theorem 3.12 is satisfied.

From the definition of Y_k and Z_k, it follows that $Y_k \cap Z_k \neq \emptyset$ for each $k \in \mathbb{N}^*$. Let $u_0 \in Y_k \cap Z_k$ with $u_0 \not\equiv 0$, and set $u_k = \frac{r_k}{\|u_0\|_a}u_0$ for each $k \geq k_0$, then $\|u_k\|_a = r_k$. For $k \geq k_0$, since $0 < r_k < \rho_k$, we have

$$d_k := \inf_{u \in Z_k,\ \|u\|_a \leq \rho_k} \Psi(u) \leq \Psi(u_k) \leq b_k := \max_{u \in Y_k,\ \|u\|_a = r_k} \Psi(u) < 0.$$

For $u \in Z_k$ with $\|u\|_a \leq \rho_k = 1$, we have

$$\Psi(u) = \phi(u) - \psi(u) \geq -\psi(u) \geq -\beta_k(1),$$

and then

$$-\beta_k(1) \leq d_k := \inf_{u \in Z_k,\ \|u\|_a \leq \rho_k} \Psi(u) < 0.$$

Since $\beta_k(1) \to 0$ as $k \to \infty$, we deduce from the last inequality that

$$d_k := \inf_{u \in Z_k,\ \|u\|_a \leq \rho_k} \Psi(u) \to 0 \text{ as } k \to \infty.$$

So, the condition (D3) of Theorem 3.12 is satisfied.

Next, we will prove that Ψ satisfy condition $(PS)_c^*$ for every $c \in \mathbb{R}$. Let $(u_n)_{n\in\mathbb{N}^*} \subset X$ and $c \in \mathbb{R}$ such that

$$u_n \in Y_n \quad \text{for each } n \in \mathbb{N}^*, \qquad \Psi(u_n) \longrightarrow c \quad \text{and} \quad (\Psi|_{Y_n})'(u_n) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Let us show that the sequence (u_n) is bounded in X. Arguing by contradiction, assume $\|u_n\|_a \to \infty$ as $n \to \infty$. For $\lambda > 0, \mu \geq 0$ and $n \in \mathbb{N}^*$ large enough such that $\|u_n\|_a > 1$, we have

$$\begin{aligned} c + 1 + \|u_n\|_a \geq& \Psi(u_n) - \frac{1}{r^-}\langle (\Psi|_{Y_n})'(u_n), u_n\rangle \\ =& \Psi(u_n) - \frac{1}{r^-}\langle \Psi'(u_n), u_n\rangle \\ \geq& \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\|u_n\|_a^{p^-} - \lambda\left(\frac{1}{q^-} - \frac{1}{r^-}\right)C^{q^+}\|u_n\|_a^{q^+}. \end{aligned}$$

Since $p^- > 1$, we obtain a contradiction for a large $n \in \mathbb{N}^*$. For $\lambda > 0, \mu \leq 0$ and $n \in \mathbb{N}^*$ large enough such that $\|u_n\|_a > 1$, we have

$$\begin{aligned} c + 1 + \|u_n\|_a &\geq \Psi(u_n) - \frac{1}{r^+}\langle (\Psi|_{Y_n})'(u_n), u_n\rangle \\ &= \Psi(u_n) - \frac{1}{r^+}\langle \Psi'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{r^+}\right)\|u_n\|_a^{p^-} - \lambda\left(\frac{1}{q^-} - \frac{1}{r^+}\right)C^{q^+}\|u_n\|_a^{q^+}. \end{aligned}$$

Since $p^- > 1$, we obtain a contradiction for a large $n \in \mathbb{N}^*$. Then, (u_n) is bounded in X. As X is a reflexive Banach space, we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}^*}$ of $(u_n)_{n\in\mathbb{N}^*}$ weakly convergent to some $u \in X$. We can write X as

$$X = \overline{\bigcup_{n\in\mathbb{N}^*} Y_n}.$$

By choosing $(v_n)_{n\in\mathbb{N}^*}$ such that

$$v_n \in Y_n \quad \text{for each } n \in \mathbb{N}^* \quad \text{and} \quad \lim_{n\to\infty} v_n = u \quad \text{in } X,$$

we obtain

$$\begin{aligned} \lim_{k\to\infty}\langle \Psi'(u_{n_k}), u_{n_k} - u\rangle &= \lim_{k\to\infty}\langle \Psi'(u_{n_k}), u_{n_k} - v_{n_k}\rangle + \lim_{k\to\infty}\langle \Psi'(u_{n_k}), v_{n_k} - u\rangle \\ &= \lim_{k\to\infty}\langle (\Psi|_{Y_{n_k}})'(u_{n_k}), u_{n_k} - v_{n_k}\rangle = 0, \end{aligned}$$

because of $(\Psi|_{Y_{n_k}})'(u_{n_k}) \to 0$ as $k \to \infty$, $u_{n_k} - v_{n_k} \rightharpoonup 0$ in Y_{n_k} and $v_{n_k} \to u \in X$. Then, we deduce that

$$\lim_{k\to\infty} \langle \Psi'(u_{n_k}) - \Psi'(u), u_{n_k} - u \rangle = 0.$$

By the (S^+) property of the functional Ψ, we get the strong convergence of $(u_{n_k})_{k\in\mathbb{N}^*}$ to u in X. It remains to prove that u is a critical point of Ψ. Choosing an arbitrary $w_n \in Y_n$, for any $n_k \geq n$, we have

$$\begin{aligned} \langle \Psi'(u), w_n \rangle &= \langle \Psi'(u) - \Psi'(u_{n_k}), w_n \rangle + \langle \Psi'(u_{n_k}), w_n \rangle \\ &= \langle \Psi'(u) - \Psi'(u_{n_k}), w_n \rangle + \langle (\Psi|_{Y_{n_k}})'(u_{n_k}), w_n \rangle. \end{aligned} \tag{4.3}$$

Since $\Psi \in C^1(X, \mathbb{R})$ and $\lim_{k\to\infty} u_{n_k} = u$ in X, then $\lim_{k\to\infty} \Psi'(u_{n_k}) = \Psi'(u)$. In expression (4.3), letting $k \to \infty$ we deduce that $\langle \Psi'(u), w_n \rangle = 0$, for all $w_n \in Y_n$, it follows that $\Psi'(u) = 0$. Consequently, Ψ satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$.

According to the Dual Fountain Theorem 3.12, the functional Ψ has a sequence of negative critical values $(c_n)_n$ converging to 0, which correspond to a sequence $(u_n)_n \subset X$ of critical points of Ψ such that $\Psi(u_n) \leq 0$, for each $n \in \mathbb{N}$ and $\Psi(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$.

Appendix

Some examples of variable exponents illustrating (i)–(ii) of Theorem 1.1

Example 1 : Suppose that $N = 1$ and $\Omega = [a, b]$ the closed set of $\mathbb{R}$.

(i) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 4 + \frac{1}{1 + (x - c)^2}, \text{ with } c \in [a, b];\ q(x) = 1 + \sin(x);\ r(x) = 2 + \cos(x).$$

We have

$$\max\{q^+, r^+\} = \max\{2, 3\} = 3 < 4 < p^-.$$

(ii) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 1 + \frac{1}{1 + (x - c)^2}, \text{ with } c \in [a, b];\ q(x) = 4 + \sin(x);\ r(x) = 5 + \cos(x).$$

We have

$$\min\{q^-, r^-\} = \min\{3, 4\} = 3 > p^+ = 2.$$

Example 2 : Suppose that $N = 2$ and $\Omega = [0, 1] \times [0, 1]$ the closed set of $\mathbb{R}^2$.

(i) Consider the continuous functions $p, q, r : \Omega \longrightarrow (1, +\infty)$ defined by :

$$\begin{aligned} p(x, y) &= 5 + \frac{1}{1 + x^2 + y^2}; \\ q(x, y) &= 2 + \sin(\pi x)\cos(\pi x); \\ r(x, y) &= 3 + \cos(\pi x)\sin(\pi x). \end{aligned}$$

We have

$$\max\{q^+, r^+\} = \max\{3, 4\} = 4 < p^- = \frac{16}{3}.$$

(ii) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x, y) = 2 + \frac{1}{1 + x^2 + y^2};$$
$$q(x, y) = 5 + \sin(\pi x)\cos(\pi x);$$
$$r(x, y) = 6 + \cos(\pi x)\sin(\pi x).$$

We have

$$\min\{q^-, r^-\} = \max\{4, 5\} = 4 > p^+ = 3.$$

Some examples of variable exponents illustrating (i)–(v) of Theorem 1.2

Example 1 : Suppose that $N = 1$ and $\Omega = [a, b]$ the closed set of $\mathbb{R}$.

(i) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 1 + \frac{1}{1 + (x - c)^2}, \quad \text{with } c \in [a, b]; \qquad q(x) = 5 + \sin(x); \qquad r(x) = 4 + \cos(x).$$

We have

$$p^+ = 2 < r^- = 3 < q^- = 4.$$

(ii) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 4 + \sin(x); \qquad q(x) = 1 + \frac{1}{1 + (x - c)^2}, \quad \text{with } c \in [a, b]; \qquad r(x) = 7 + \cos(x).$$

We have

$$q^+ = 2 < p^- = 3 < p^+ = 5 < r^- = 6.$$

(iii) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 5 + \sin(x); \qquad q(x) = 3 + \frac{1}{1 + (x - c)^2}, \quad \text{with } c \in [a, b]; \qquad r(x) = 6 + \cos(x).$$

We have

$$q^+ = p^- = 4 < r^- = 5.$$

(iv) The same example of (ii).

(v) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 4 + \sin(x); \qquad q(x) = 1 + \frac{1}{1 + (x - c)^2}, \quad \text{with } c \in [a, b]; \qquad r(x) = 5 + \cos(x).$$

We have

$$q^+ = 2 < \min\{p^-, r^-\} = 3 \quad \text{and} \quad p^+ = 5 < r^+ = 6.$$

Example 2 : Suppose that $N = 2$ and $\Omega = [0, 1] \times [0, 1]$ the closed set of $\mathbb{R}^2$.

(i) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$
\begin{aligned}
p(x,y) &= 2 + \frac{1}{1 + x^2 + y^2}; \\
(x,y) &= 6 + \sin(\pi x)\cos(\pi y); \\
r(x,y) &= 5 + \cos(\pi x)\sin(\pi y).
\end{aligned}
$$

We have

$$p^+ = 3 < r^- = 4 < q^- = 5.$$

(ii) Consider the continuous functions $p, q, r : \Omega \longrightarrow (1, +\infty)$ defined by :

$$
\begin{aligned}
p(x,y) &= 5 + \sin(\pi x)\cos(\pi y); \\
q(x,y) &= 2 + \frac{1}{1 + x^2 + y^2}; \\
r(x,y) &= 8 + \cos(\pi x)\sin(\pi x).
\end{aligned}
$$

We have

$$q^+ = 3 < p^- = 4 < p^+ = 6 < r^- = 7.$$

(iii) Consider the continuous functions $p, q, r : \Omega \longrightarrow (1, +\infty)$ defined by :

$$
\begin{aligned}
p(x,y) &= 5 + \sin(\pi x)\cos(\pi y); \\
q(x,y) &= 3 + \frac{1}{1 + x^2 + y^2}; \\
r(x,y) &= 8 + \cos(\pi x)\sin(\pi x).
\end{aligned}
$$

We have

$$q^+ = 4 = p^- < p^+ = 6 < r^- = 7.$$

(iv) The same example of (ii).

(v) Consider the continuous functions $p, q, r : \Omega \longrightarrow (1, +\infty)$ defined by :

$$
\begin{aligned}
p(x,y) &= 5 + \sin(\pi x)\cos(\pi y); \\
q(x,y) &= 2 + \frac{1}{1 + x^2 + y^2}; \\
r(x,y) &= 6 + \cos(\pi x)\sin(\pi x).
\end{aligned}
$$

We have

$$q^+ = 3 < \min\{p^-, r^-\} = 4 \quad \text{and} \quad p^+ = 6 < r^+ = 7.$$

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