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A critical elliptic equation with a logarithmic type perturbation

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Abstract. In this note, we consider a critical elliptic equation perturbed by a logarithmic type subcritical term in $\mathbb{R}^4$, and investigate how the logarithmic term affects the existence of weak solutions to such a problem. Since the logarithmic term does not satisfy the standard monotonicity condition, essential difficulty arises when one looks for weak solutions to this problem in the variational framework. After some delicate estimates on the logarithmic term we can control the mountain pass level of the corresponding functional so that it satisfies the local compactness condition. Then a positive weak solution follows with the application of the Mountain Pass Lemma and the Brézis–Lieb Lemma. Our result implies that the logarithmic term plays a positive role for the problem to admit positive solutions.

Keywords: critical, logarithmic type perturbation, Brézis–Lieb Lemma.

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1 Introduction and the main results

In this note, we confine ourselves to the following critical elliptic problem with a logarithmic type perturbation

$$\begin{cases} -\Delta u = \lambda |u|^{q-2} u \ln u^2 + u^3, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded smooth domain with boundary $\partial\Omega$, $q \in (2,4)$ and $\lambda > 0$ is a parameter. Without loss of generality, we may assume that $0 \in \Omega$.

It is clear that problem (1.1) is a special form of the following elliptic boundary value problem

$$\begin{cases} (-\Delta)^m u = \mu u + f(x,u) + |u|^{p-2} u, & x \in \Omega, \\ D^\alpha u = 0, \quad \text{for } |\alpha| \le m-1, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N (N > 2m)$ is a bounded smooth domain, $\mu \in \mathbb{R}$, $p = \frac{2N}{N-2m}$ and $f(x,u)$ is a nonlinear term. One of the main features of the equation in (1.2) is that it involves a critical

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term in the sense that the embedding $H_0^m(\Omega) \hookrightarrow L^p(\Omega)$ is not compact, which brings essential difficulty in proving the existence of weak solutions to (1.2). Therefore, the study of elliptic problems with critical exponents is not only interesting, but also challenging. However, it was mainly after the pioneering work of Brézis and Nirenberg [2] that such problems were extensively investigated, and remarkable progress has been made on the existence, nonexistence and multiplicity of weak solutions to such problems.

On the other hand, partial differential equations with logarithmic nonlinearities have also been attracting more and more attention in recent years due to their wide applications in many applied sciences. Interested readers may refer to [3,5,8–11] and the references therein for the study of different kinds of equations with logarithmic nonlinearity. For example, when $m = 1$, $f(x,u) = \mu u \ln u^2$ $(\mu \neq 0)$, Deng et al. [3] considered the existence and nonexistence of positive solutions to problem (1.2). With the help of some delicate estimates on the logarithmic term, they showed, among many other interesting results, that problem (1.2) admits a positive mountain pass type solution when $N \geq 4$, $\lambda \in \mathbb{R}$ and $\mu > 0$, which is also a ground state solution. Later, the main results in [3] were extended to second order elliptic system by Hajaiej et al. [4] and to critical bi-harmonic equation by Li et al. [5].

It is worth pointing out that the exponent of the logarithmic term in all the above mentioned literature is $q = 2$, and it seems that there is no result for $q > 2$. Motivated mainly by [3,5,11] we will consider problem (1.1) and investigate what role the logarithmic term plays for the problem to admit weak solutions. Since the logarithmic term does not satisfy the standard monotonicity condition, additional difficulty arises when we look for weak solutions to problem (1.1) by using variational methods. A key step in this process is to control the mountain pass level of the corresponding energy functional from above by a small constant such that it satisfies the (PS) condition locally. This is done with the help of some very delicate estimates on the logarithmic function of the truncated Talenti's functions. Then by combining the Mountain Pass Lemma with the Brézis–Lieb Lemma, we can show for $q \in (2,4)$ that problem (1.1) admits a positive mountain pass type solution for all $\lambda > 0$, which, compared with the case $\lambda = 0$ (see Remark 1.2 below), means that the logarithmic type subcritical perturbation plays a positive role for problem (1.1) to admit positive solutions.

The notations used in this paper are almost standard. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$ norm for $1 \leq p \leq \infty$, and equip the Sobolev space $H_0^1(\Omega)$ with the norm $\|u\| := \|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2$. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ and the dual pair between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is written as $\langle \cdot, \cdot \rangle$. For each Banach space B, we use $\to$ and $\rightharpoonup$ to denote the strong and weak convergence in it, respectively. We also use $C, C_1, C_2, \ldots$ to denote (possibly different) positive constants. The positive constant S denotes the best embedding constant from $H_0^1(\Omega)$ to $L^4(\Omega)$, i.e.,

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|u\|^2}{\|u\|_4^2}. \tag{1.3}$$

Finally, $B_r(x_0)$ is a ball of radius r centered at x_0 and $B_r(0)$ is simply written as B_r when no confusion arises. The symbol $O(t)$ means $|\frac{O(t)}{t}| \leq C$ as $t \to 0$ and $o_n(1)$ is an infinitesimal as $n \to \infty$.

The energy functional associated with problem (1.1) and its Fréchet derivative are denoted respectively by $I(u)$ and $I'(u)$, i.e.,

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{2\lambda}{q^2}\|u\|_q^q - \frac{\lambda}{q}\int_\Omega |u|^q \ln u^2 \mathrm{d}x - \frac{1}{4}\|u\|_4^4, \qquad u \in H_0^1(\Omega), \tag{1.4}$$

and

$$\langle I'(u), \phi \rangle = \int_\Omega \nabla u \nabla \phi \mathrm{d}x - \lambda \int_\Omega |u|^{q-2} u \phi \ln u^2 \mathrm{d}x - \int_\Omega u^3 \phi \mathrm{d}x, \qquad u, \phi \in H_0^1(\Omega). \tag{1.5}$$

It is obviously that $I(u)$ is a C^1 functional on $H_0^1(\Omega)$, and each critical point of $I(u)$ is also a weak solution to problem (1.1). Since $I(u) = I(|u|)$, we may assume that $u \geq 0$ in the sequel. The main result of this paper is the following theorem.

Theorem 1.1. *Assume that $q \in (2,4)$. Then problem (1.1) admits a positive mountain pass type solution u for all $\lambda > 0$.*

Remark 1.2. In [6], Pohožaev showed that problem (1.1) with $\lambda = 0$ admits no positive solution if Ω is star-shaped. However, Theorem 1.1 shows that this situation can be reversed when the logarithmic term is introduced. This means that the logarithmic term plays a positive role for problem (1.1) to admit positive solutions.

Remark 1.3. The solution obtained in Theorem 1.1 is actually a ground state solution to problem (1.1). To see this, set

$$\mathcal{N} = \left\{ u \in H_0^1(\Omega) \setminus \{0\} : \langle I'(u), u \rangle = 0 \right\},$$

$c_1 = \inf_{u \in \mathcal{N}} I(u)$ and $c_2 = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \max_{t \geq 0} I(tu)$. Then it is easily to verify that $c_1 = c_2$. Let c_0 be the mountain pass level defined in Lemma 2.7. Since $q \in (2,4)$, one sees that $I(tu) < 0$ for $u \in H_0^1(\Omega) \setminus \{0\}$ and t large enough, which implies that $c_0 \leq c_2$. Noticing that the Nehari's manifold $\mathcal{N}$ separates $H_0^1(\Omega)$ into two components. The component containing the origin also contains a small ball around the origin. Moreover $I(u) \geq 0$ for all u in this component, by the assumption $q \in (2,4)$. Thus every $\gamma \in \Gamma$ has to cross $\mathcal{N}$ and $c_1 \leq c_0$. Therefore, $c_0 = c_1$ and the solution u obtained in Theorem 1.1 is a ground state solution.

Remark 1.4. When $q = 2$ and $N = 3, 4$, Deng et al. [3] also considered the case that $\lambda < 0$, and obtained a positive solution by using the Mountain Pass Lemma without compactness condition. However, it seems quite difficult to derive the same conclusion for $q \in (2,4)$. This can be seen from (2.23) and (2.24). Indeed, from (2.23) and (2.24) we know that compared with $\int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 \mathrm{d}x$, $\|u_\varepsilon\|_q^q$ is an infinitesimal of higher order as $\varepsilon \to 0$ and consequently we can not control the mountain pass level as was done in [3]. So the treatment for the case $q \in (2,4)$ must be different from those in [3] and new ideas and techniques may be needed.

2 Proof of the main result

We begin this section with some definitions, lemmas and basic inequalities, which will be used in the proof of the main result.

Definition 2.1. ($(PS)_c$ condition) Assume that B is a real Banach space, $I : B \to \mathbb{R}$ is a C^1 functional and $c \in \mathbb{R}$. We say that I satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subset B$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } B' \text{ (the dual space of } B) \quad \text{as } n \to \infty$$

has a convergent subsequence.

Lemma 2.2 (Mountain Pass Lemma [5,7])**.** *Assume that* $(B, \|\cdot\|_B)$ *is a real Banach space,* $I : B \to \mathbb{R}$ *is a* C^1 *functional and there exist* $\alpha > 0$ *and* $\rho > 0$ *such that* I *satisfies the following mountain pass geometry*

(i) $I(u) \geq \alpha > 0$ *if* $\|u\|_B = \rho$*;*

(ii) There exists a $\overline{u} \in B$ *such that* $\|\overline{u}\|_B > \rho$ *and* $I(\overline{u}) < 0$*.*

Then there exist a sequence $\{u_n\} \subset B$ *such that* $I(u_n) \to c_0$ *and* $I'(u_n) \to 0$ *in* B' *as* $n \to \infty$*, where*

$$c_0 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \alpha, \qquad \Gamma = \{\gamma \in C([0,1], B) : \gamma(0) = 0, \gamma(1) = \overline{u}\}.$$

Furthermore, c_0 *(which is usually called the mountain level) is a critical value of* I *if* I *satisfies the* $(PS)_{c_0}$ *condition.*

Lemma 2.3 (Brézis–Lieb Lemma [1])**.** *Let* $p \in (1, \infty)$*. Suppose that* $\{u_n\}$ *is a bounded sequence in* $L^p(\Omega)$ *and* $u_n \to u$ *a.e. in* Ω*. Then*

$$\lim_{n\to\infty} (\|u_n\|_p^p - \|u_n - u\|_p^p) = \|u\|_p^p.$$

Lemma 2.4. *For any* $\delta > 0$*, there exists a positive constant* C_δ *such that*

$$|\ln t| \leq C_\delta(t^\delta + t^{-\delta}), \qquad \forall\, t > 0, \tag{2.1}$$

$$\ln t \leq \frac{t^\delta}{e\delta}, \qquad \forall\, t \geq 1. \tag{2.2}$$

In order to apply the Mountain Pass Lemma, we then verify the mountain pass geometry for I around 0 when $q \in (2, 4)$ and $\lambda > 0$.

Lemma 2.5. *Assume that* $q \in (2, 4)$ *and* $\lambda > 0$*. Then* $I(u)$ *satisfies the mountain pass geometry around* 0*.*

Proof. By recalling (2.2) with $\delta \in (0, 4 - q)$ and making use of the Sobolev embedding inequality, one has

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{2\lambda}{q^2}\|u\|_q^q - \frac{\lambda}{q}\int_\Omega |u|^q \ln u^2 \mathrm{d}x - \frac{1}{4}\|u\|_4^4 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\Omega_1} |u|^q \ln u^2 \mathrm{d}x - \frac{\lambda}{q}\int_{\Omega_2} |u|^q \ln u^2 \mathrm{d}x - \frac{1}{4}\|u\|_4^4 \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\|^{q+\delta} - \frac{1}{4}S^{-2}\|u\|^4 \\ &= \|u\|^2 \left(\frac{1}{2} - C\|u\|^{q+\delta-2} - \frac{1}{4}S^{-2}\|u\|^2\right), \end{aligned}$$

where $\Omega_1 = \{x \in \Omega : |u(x)| < 1\}$, $\Omega_2 = \{x \in \Omega : |u(x)| \geq 1\}$. Hence, there exist positive constants α and ρ such that

$$I(u) \geq \alpha \quad \text{for all } \|u\| = \rho.$$

On the other hand, for any $v \in H_0^1(\Omega) \setminus \{0\}$, one has $\lim_{t\to\infty} I(tv) = -\infty$ since $q < 4$, which ensures that there exists a $t(v) > 0$ such that $\|t(v)v\| > \rho$ and $I(t(v)v) < 0$. Thus, I satisfies the mountain pass geometry around 0. The proof is complete. □

Next, we shall verify that I satisfies the $(PS)_c$ condition when $c < \frac{1}{4}S^2$. This condition will play a key role in showing the existence of weak solutions to problem (1.1).

Lemma 2.6. *Assume that $q \in (2,4)$ and $\lambda > 0$. Then $I(u)$ satisfies the $(PS)_c$ condition when $c < \frac{1}{4}S^2$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS) sequence for $I(u)$ at the level c with $c < \frac{1}{4}S^2$, i.e., $I(u_n) \to c$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$. First, we claim that $\{u_n\}$ is bounded. Indeed, from the definition of the (PS) sequence, one obtains, for n suitably large, that

$$\begin{aligned} c + 1 + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2 + \frac{2\lambda}{q^2}\|u_n\|_q^q + \left(\frac{1}{q} - \frac{1}{4}\right)\|u_n\|_4^4 \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Consequently, there is a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that, as $n \to \infty$,

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_0^1(\Omega), \\ u_n \to u & \text{in } L^r(\Omega)\ (1 \leq r < 4), \\ u_n \rightharpoonup u & \text{in } L^4(\Omega), \\ |u_n|^2 u_n \rightharpoonup |u|^2 u & \text{in } L^{\frac{4}{3}}(\Omega), \\ u_n \to u & \text{a.e. in } \Omega. \end{cases} \tag{2.3}$$

Next, we show that

$$\lim_{n\to\infty} \int_\Omega |u_n|^q \ln u_n^2 \mathrm{d}x = \int_\Omega |u|^q \ln u^2 \mathrm{d}x. \tag{2.4}$$

Indeed, since $u_n \to u$ a.e. in Ω as $n \to \infty$, we get

$$|u_n|^q \ln u_n^2 \to |u|^q \ln u^2 \quad \text{a.e. in } \Omega. \tag{2.5}$$

Moreover, by recalling (2.2) with $\delta < 4 - q$ and (2.3), one has

$$\left||u_n|^q \ln u_n^2\right| \leq C_\delta(|u_n|^{q-\delta} + |u_n|^{q+\delta}) \to C_\delta(|u|^{q-\delta} + |u|^{q+\delta}) \quad \text{in } L^1(\Omega). \tag{2.6}$$

Then (2.4) follows from (2.5), (2.6) and Lebesgue's dominated convergence theorem. Similarly,

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q-2} u_n \phi \ln u_n^2 \mathrm{d}x = \int_\Omega |u|^{q-2} u \phi \ln u^2 \mathrm{d}x, \qquad \forall\, \phi \in H_0^1(\Omega). \tag{2.7}$$

To prove that $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$, set $w_n = u_n - u$. Then $\{w_n\}$ is also a bounded sequence in $H_0^1(\Omega)$, and therefore there exists a subsequence of $\{w_n\}$ (which we still denote by $\{w_n\}$) such that

$$\lim_{n\to\infty} \|w_n\|^2 = l \geq 0. \tag{2.8}$$

We claim that $l = 0$. Indeed, in view of the weak convergence $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, we have

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o_n(1), \qquad n \to \infty. \tag{2.9}$$

Moreover, (2.3) implies that we can apply the Brézis–Lieb Lemma to obtain

$$\|u_n\|_4^4 = \|w_n\|_4^4 + \|u\|_4^4 + o_n(1), \qquad n \to \infty. \tag{2.10}$$

With (2.3), (2.4), (2.7), (2.9), (2.10) at hand and recalling the boundedness of $\{u_n\}$ and the assumption that $I'(u_n) \to 0$ in $H^{-1}(\Omega)$, we have

$$o_n(1) = \langle I'(u_n), u_n \rangle = \langle I'(u), u \rangle + \|w_n\|^2 - \|w_n\|_4^4 + o_n(1), \qquad n \to \infty, \tag{2.11}$$

and

$$o_n(1) = \langle I'(u_n), \phi \rangle = \langle I'(u), \phi \rangle + o_n(1), \qquad n \to \infty, \ \forall\, \phi \in H_0^1(\Omega). \tag{2.12}$$

It follows from (2.12) that u is a weak solution to problem (1.1).

Choosing $\phi = u$ in (2.12), one obtains

$$\langle I'(u), u \rangle = 0, \tag{2.13}$$

which, together with (2.11), implies that

$$\|w_n\|^2 - \|w_n\|_4^4 = o_n(1), \qquad n \to \infty. \tag{2.14}$$

In addition, by the Sobolev embedding one has

$$\|w_n\|_4^4 \le S^{-2}\|w_n\|^4, \qquad \forall\, n \in \mathbb{N}. \tag{2.15}$$

Letting $n \to \infty$ on both sides of (2.15) and recalling (2.8) and (2.14), one arrives at

$$l \le S^{-2} l^2. \tag{2.16}$$

If $l > 0$, then

$$l \ge S^2. \tag{2.17}$$

On one hand, by virtue of (2.13), we have

$$I(u) = I(u) - \frac{1}{q}\langle I'(u), u \rangle = \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 + \frac{2\lambda}{q^2}\|u\|_q^q + \left(\frac{1}{q} - \frac{1}{4}\right)\|u\|_4^4 \ge 0,$$

since $q \in (2, 4)$ and $\lambda > 0$.

On the other hand, in view of (2.3), (2.4), (2.9), (2.10) and the fact that $I(u_n) = c + o_n(1)$ as $n \to \infty$, we deduce that

$$o_n(1) + c = I(u_n) = I(u) + \frac{1}{2}\|w_n\|^2 - \frac{1}{4}\|w_n\|_4^4 + o_n(1), \qquad n \to \infty,$$

which yields

$$I(u) = c - \frac{1}{2}\|w_n\|^2 + \frac{1}{4}\|w_n\|_4^4 + o_n(1), \qquad n \to \infty.$$

From this and recalling (2.8), (2.14) and (2.17) one has

$$I(u) = c - \left(\frac{1}{2} - \frac{1}{4}\right) l \le c - \frac{1}{4} S^2 < 0,$$

which is a contradiction. Therefore, we have $\lim_{n\to\infty} \|w_n\|^2 = l = 0$, i.e., $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$. The proof is complete. □

Lemma 2.7. *Assume that $q \in (2,4)$ and $\lambda > 0$. If there exists a $v \in H_0^1(\Omega) \setminus \{0\}$ such that*

$$\sup_{t \geq 0} I(tv) < \frac{1}{4} S^2, \tag{2.18}$$

then problem (1.1) possesses a nontrivial weak solution.

Proof. From Lemma 2.5, we know that I satisfies the mountain pass geometry around 0. Consequently, in view of Lemma 2.2, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $I(u_n) \to c_0$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$, where

$$0 < \alpha \leq c_0 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \quad \Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = t(v)v\}, \tag{2.19}$$

and $t(v) > 0$ is the constant determined in the proof of Lemma 2.5. According to (2.18), we have

$$c_0 \leq \max_{t \in [0,1]} I(tt(v)v) \leq \sup_{t \geq 0} I(tv) < \frac{1}{4} S^2. \tag{2.20}$$

Then by (2.20) and Lemma 2.6, we know that there exists a convergent subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), such that $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$, which implies that $I(u) = c_0$ and $I'(u) = 0$, i.e., u is a nontrivial mountain pass type solution to problem (1.1). The proof is complete. □

From Lemma 2.7 it is easily seen that a nontrivial nonnegative mountain pass type solution to problem (1.1) follows once we can find a $v \in H_0^1(\Omega) \setminus \{0\}$ such that (2.18) holds. This is done with the help of some delicate estimates on the truncated Talenti's functions, inspired mainly by Brézis and Nirenberg [2], Deng et al. [3] and Li et al. [5].

For any $\varepsilon > 0$, define

$$U_\varepsilon(x) = \frac{\sqrt{8}\varepsilon}{\varepsilon^2 + |x|^2}, \qquad x \in \mathbb{R}^4. \tag{2.21}$$

It is well known that $U_\varepsilon(x)$ is a solution to the critical problem

$$-\Delta u = u^3, \qquad x \in \mathbb{R}^4,$$

which is also a minimizer for S.

Lemma 2.8. *Let $\varphi \in C_0^\infty(\Omega)$ satisfy $0 \leq \varphi(x) \leq 1$ in Ω and*

$$\varphi(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| > 2R, \end{cases}$$

where $R > 0$ is a constant such that $B_{2R}(0) \subset \Omega$. Set $u_\varepsilon(x) = \varphi(x) U_\varepsilon(x)$. Then, as $\varepsilon \to 0$,

$$\begin{aligned} \|u_\varepsilon\|^2 &= S^2 + O(\varepsilon^2), \\ \|u_\varepsilon\|_4^4 &= S^2 + O(\varepsilon^4), \end{aligned} \tag{2.22}$$

$$\|u_\varepsilon\|_q^q = O(\varepsilon^{4-q}), \qquad 2 < q < 4, \tag{2.23}$$

and

$$\int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 dx = C\varepsilon^{4-q} \ln\left(\frac{1}{\varepsilon}\right) + O(\varepsilon^{4-q}). \tag{2.24}$$

Proof. (2.22) is well known and can be referred, for example, to [2] and [3]. We only prove (2.23) and (2.24).

By the definition of u_ε, one has

$$\begin{aligned}\|u_\varepsilon\|_q^q &= C\int_{B_{2R}} \frac{\varphi^q(x)\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x\\ &= C\int_{B_R} \frac{\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x + C\int_{B_{2R}\setminus B_R} \frac{\varphi^q(x)\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x\\ &= I_1 + O(\varepsilon^q).\end{aligned}$$

By direct computation, we obtain

$$\begin{aligned}I_1 &= C\int_{B_{R/\varepsilon}} \frac{\varepsilon^q\cdot\varepsilon^4}{\varepsilon^{2q}(1+|y|^2)^q}\mathrm{d}y = C\varepsilon^{4-q}\int_0^{R/\varepsilon}\frac{r^3}{(1+r^2)^q}\mathrm{d}r\\ &= C\varepsilon^{4-q}\left(\int_0^\infty \frac{r^3}{(1+r^2)^q}\mathrm{d}r - \int_{R/\varepsilon}^\infty \frac{r^3}{(1+r^2)^q}\mathrm{d}r\right)\\ &= C\varepsilon^{4-q} - O(\varepsilon^q),\end{aligned}$$

where we have used the fact that

$$\left|\int_{R/\varepsilon}^\infty \frac{r^3}{(1+r^2)^q}\mathrm{d}r\right| \le \int_{R/\varepsilon}^\infty r^{3-2q}\mathrm{d}r = O(\varepsilon^{2q-4}).$$

Therefore,

$$\|u_\varepsilon\|_q^q = C\varepsilon^{4-q} + O(\varepsilon^q) = O(\varepsilon^{4-q}).$$

Next we prove (2.24). Set

$$\begin{aligned}J &= \int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 \mathrm{d}x\\ &= \int_\Omega \varphi^q(x)U_\varepsilon^q(x)\ln\varphi^2(x)\mathrm{d}x + \int_\Omega \varphi^q(x)U_\varepsilon^q(x)\ln U_\varepsilon^2(x)\mathrm{d}x\\ &= \int_\Omega \varphi^q(x)U_\varepsilon^q(x)\ln\varphi^2(x)\mathrm{d}x + \int_{B_R} U_\varepsilon^q(x)\ln U_\varepsilon^2(x)\mathrm{d}x + \int_{B_{2R}\setminus B_R} \varphi^q(x)U_\varepsilon^q(x)\ln U_\varepsilon^2(x)\mathrm{d}x\\ &= J_1 + J_2 + J_3,\end{aligned} \tag{2.25}$$

where

$$|J_1| = \left|\int_{B_{2R}\setminus B_R} \varphi^q(x)U_\varepsilon^q(x)\ln\varphi^2(x)\mathrm{d}x\right| \le C\int_{B_{2R}\setminus B_R} U_\varepsilon^q(x)\mathrm{d}x = O(\varepsilon^q). \tag{2.26}$$

By applying (2.1) with $\delta_1 \in (0, 2q-4)$ to J_3, one has

$$|J_3| \le C_{\delta_1}\int_{B_{2R}\setminus B_R}\left(U_\varepsilon^{q-\delta_1}(x) + U_\varepsilon^{q+\delta_1}(x)\right)\mathrm{d}x = O(\varepsilon^{q-\delta_1}). \tag{2.27}$$

To estimate J_2, we rewrite it as follows:

$$
\begin{aligned}
J_2 &= \int_{B_R} U_\varepsilon^q(x)\ln U_\varepsilon^2(x)\mathrm{d}x \\
&= C\int_{B_R} \frac{\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\ln\left(\frac{C\varepsilon^2}{(\varepsilon^2+|x|^2)^2}\right)\mathrm{d}x \\
&= C\varepsilon^{4-q}\int_{B_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{\varepsilon^2(1+|y|^2)^2}\right)\mathrm{d}y \\
&= C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\int_{B_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\mathrm{d}y + C\varepsilon^{4-q}\int_{B_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y \\
&= C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\left[\int_{\mathbb{R}^4} \frac{1}{(1+|y|^2)^q}\mathrm{d}y - \int_{B^c_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\mathrm{d}y\right] \\
&\quad + C\varepsilon^{4-q}\int_{B_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y \\
&= C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) - C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\int_{B^c_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\mathrm{d}y \\
&\quad + C\varepsilon^{4-q}\int_{B_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y,
\end{aligned}
\tag{2.28}
$$

where

$$
\left|\int_{B^c_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\mathrm{d}y\right| \le C\int_{R/\varepsilon}^{\infty} \frac{r^3}{(1+r^2)^q}\mathrm{d}r = O(\varepsilon^{2q-4}). \tag{2.29}
$$

The last term in (2.28) can be estimated as follows:

$$
\begin{aligned}
&\left|\int_{B_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y\right| \\
&\quad = \left|\int_{\mathbb{R}^4} \frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y - \int_{B^c_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y\right| \\
&\quad = \left|C - \int_{B^c_{R/\varepsilon}} \frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y\right| \\
&\quad \le C + C_{\delta_2}\int_{B^c_{R/\varepsilon}}\left(\frac{1}{(1+|y|^2)^{q-\delta_2}} + \frac{1}{(1+|y|^2)^{q+\delta_2}}\right)\mathrm{d}y \\
&\quad = C + O(\varepsilon^{2q-4-2\delta_2}),
\end{aligned}
\tag{2.30}
$$

where we have used (2.1) with $\delta_2 \in (0, q-2)$. Substituting (2.26)–(2.30) into (2.25) and noticing the choice of δ_1 and δ_2 one sees that

$$
\begin{aligned}
J &= O(\varepsilon^q) + C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) + O\left(\varepsilon^q\ln\left(\frac{1}{\varepsilon}\right)\right) + O(\varepsilon^{4-q}) + O(\varepsilon^{q-2\delta_2}) + O(\varepsilon^{q-\delta_1}) \\
&= C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) + O(\varepsilon^{4-q}).
\end{aligned}
$$

Therefore, (2.24) is valid. The proof is complete. □

On the basis of the estimates in Lemma 2.8, we can find a $v \in H_0^1(\Omega)\setminus\{0\}$ such that (2.18) holds.

Lemma 2.9. *Assume that $q \in (2,4)$ and $\lambda > 0$. Then for $\varepsilon > 0$ suitably small, we have*

$$\sup_{t\geq 0} I(tu_\varepsilon) < \frac{1}{4}S^2. \tag{2.31}$$

Proof. Let u_ε be given in Lemma 2.8 and set $\psi_{u_\varepsilon}(t) = I(tu_\varepsilon)$. Then $\lim_{t\to 0}\psi_{u_\varepsilon}(t) = 0$, $\psi_{u_\varepsilon}(t) > 0$ for suitably small $t > 0$ and $\lim_{t\to+\infty}\psi_{u_\varepsilon}(t) = -\infty$ uniformly for $\varepsilon \in (0,\varepsilon_1)$, where $\varepsilon_1 > 0$ is a suitably small but fixed number. Consequently, for each $\varepsilon \in (0,\varepsilon_1)$, there exists a $t_\varepsilon \in (0,+\infty)$ such that

$$\sup_{t\geq 0} I(tu_\varepsilon) = \sup_{t\geq 0}\psi_{u_\varepsilon}(t) = \psi_{u_\varepsilon}(t_\varepsilon),$$

and

$$\psi'_{u_\varepsilon}(t_\varepsilon) = t_\varepsilon\left(\|u_\varepsilon\|^2 - \lambda t_\varepsilon^{q-2}\int_\Omega |u_\varepsilon|^q \ln(t_\varepsilon u_\varepsilon)^2 dx - t_\varepsilon^2\|u_\varepsilon\|_4^4\right) = 0,$$

which implies that

$$\|u_\varepsilon\|^2 = t_\varepsilon^2\|u_\varepsilon\|_4^4 + \lambda t_\varepsilon^{q-2}\int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 dx + \lambda t_\varepsilon^{q-2}\ln t_\varepsilon^2\|u_\varepsilon\|_q^q. \tag{2.32}$$

Combining (2.22)–(2.24) with (2.32) we see that there exist $0 < T_0 < T^0$ such that, for $\varepsilon > 0$ uniformly small,

$$T_0 \leq t_\varepsilon \leq T^0. \tag{2.33}$$

By the boundedness of t_ε and (2.22)–(2.24) one gets

$$\begin{aligned}
\sup_{t\geq 0} I(tu_\varepsilon) = I(t_\varepsilon u_\varepsilon) &= \frac{1}{2}t_\varepsilon^2\|u_\varepsilon\|^2 - \frac{1}{4}t_\varepsilon^4\|u_\varepsilon\|_2^4 + \frac{2\lambda}{q^2}t_\varepsilon^q\|u_\varepsilon\|_q^q - \frac{\lambda}{q}\int_\Omega |t_\varepsilon u_\varepsilon|^q \ln(t_\varepsilon u_\varepsilon)^2 dx \\
&= \frac{1}{2}t_\varepsilon^2\|u_\varepsilon\|^2 - \frac{1}{4}t_\varepsilon^4\|u_\varepsilon\|_2^4 + \frac{\lambda}{q}t_\varepsilon^q\left(\frac{2}{q} - \ln t_\varepsilon^2\right)\|u_\varepsilon\|_q^q - \frac{\lambda}{q}t_\varepsilon^q\int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 dx \\
&= \left(\frac{1}{2}t_\varepsilon^2 - \frac{1}{4}t_\varepsilon^4\right)(S^2 + O(\varepsilon^2)) + O(\varepsilon^{4-q}) - \frac{\lambda}{q}t_\varepsilon^q\left(C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) + O(\varepsilon^{4-q})\right) \\
&\leq \frac{1}{4}S^2 + O(\varepsilon^2) + O(\varepsilon^{4-q}) - C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right).
\end{aligned} \tag{2.34}$$

Since $2 < q < 4$, we see that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)} = 0, \qquad \lim_{\varepsilon\to 0}\frac{\varepsilon^{4-q}}{\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)} = 0. \tag{2.35}$$

Consequently, it follows from (2.34) and (2.35) that $\sup_{t\geq 0} I(tu_\varepsilon) < \frac{1}{4}S^2$ for sufficiently small $\varepsilon > 0$. The proof is complete. □

With the above lemmas at hand, we are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Assume that $q \in (2,4)$ and $\lambda > 0$. By virtue of Lemmas 2.5, 2.7 and 2.9 we know that problem (1.1) admits a nontrivial mountain pass type solution u. The nonnegativity of u follows from the nonnegativity of the (PS) sequence $\{u_n\}$ which we have assumed without loss of generality to be valid since the energy functional $I(u)$ satisfies $I(u) = I(|u|)$ for any $u \in H_0^1(\Omega)$. Finally, by applying the standard arguments used in the proof of Theorem 1.2 in [3] we can show that the solution is positive in Ω. The proof is complete. □

Remark 2.10. Although we only consider problem (1.1) when the space dimension $N = 4$, the same result can be obtained for general $N \geq 5$ with little modification when $q \in (2, \frac{2N}{N-2})$ and $\lambda > 0$. Interested readers may check it themselves.

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