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Center manifolds for random dynamical systems with generalized trichotomies

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Abstract. In this paper, we consider small perturbations of linear random dynamical systems evolving on a Banach space and admitting a general form of trichotomy. We prove the existence of invariant center manifolds in both continuous and discrete-time. Furthermore, we provide several illustrative examples.

Keywords: invariant manifolds, random dynamical systems, trichotomies.

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1 Introduction

The theory of center manifolds plays a crucial role in stability and bifurcation theory, as it often enables the reduction of the dimension of the state space (see [19,29,31–33]). The origins of this theory date back to the 1960s, with the works of Pliss [49] and Kelley [34,35]. Subsequently, various results on this subject were developed by several authors. In the context of autonomous differential equations, we recommend the surveys by Vanderbauwhede [54] (see also Vanderbauwhede and Gils [56]) for the finite-dimensional case and by Vanderbauwhede and Iooss [55] in the infinite-dimensional case. For the nonautonomous case we recommend the survey by Aulbach and Wanner [3]. We also recommend [23,24] and [22,25,26,44,53] for, respectively, finite and infinite dimension.

The concept of trichotomy is an essential tool for obtaining center manifolds. The (uniform) exponential trichotomies were introduced, independently, by Sacker and Sell [51], Aulbach [2] and Elaydi and Hájek [28]. This notion was motivated by the idea of (uniform) exponential dichotomy that started in the thirties with Perron [47,48].

Several generalizations of exponential trichotomies have since emerged. Fenner and Pinto [42] introduced the (h, k)-trichotomies that use non exponential growth rates and Barreira and Valls [4,5] introduced nonuniform exponential trichotomies that take into account the initial time. Later, Barreira and Valls [6,7] introduced the ρ-nonuniform exponential trichotomies that are nonuniform and non exponential, but do not include the (h, k)-trichotomies.

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In [12, 15], a general type of trichotomies was introduced, for linear differential equations and linear difference equations, respectively. This new framework contains as special cases the notions of trichotomies mentioned above and also contains additional new cases (the case of dichotomies was done in [13, 14]).

Invariant manifold theory has also been extended to dynamical systems with randomness. In this work, we focus on random dynamical systems (RDS), which can be generated, for instance, by random or stochastic differential equations. In this context, various studies have addressed center, stable, unstable, and inertial invariant manifolds, both locally and globally, across a range of spaces that goes from finite to infinite dimension, including Hilbert spaces and separable Banach spaces. Arnold's monograph [1] provides a detailed exposition on the Multiplicative Ergodic Theorem and invariant manifold theory for finite-dimensional RDS. Smooth systems are discussed in [41]. For results on infinite-dimensional RDS, we refer to [8–11, 18, 27, 37, 40, 43, 45, 46, 50, 52] and the references therein.

Center manifolds for RDS have also garnered attention, either in finite or infinite dimensions. In the finite-dimensional context, Wanner [57] discusses invariant manifolds, including center manifolds, in terms of linearization in $\mathbb{R}^n$. Boxler [17] proved the existence of center manifolds for discrete random maps (random diffeomorphisms). Existence, smooth conjugacy theorems, and Takens-type theorems based on Lyapunov exponents were established by Li and Lu in [38] and by Guo and Shen in [30], in the presence of zero Lyapunov exponents. On the other hand, infinite-dimensional RDS hold significant interest not only due to their inherent mathematical richness but also for their applications in understanding stochastic and partial differential equations. Under the assumption of an exponential trichotomy, Chen, Roberts, and Duan [20] established the existence and smoothness of center manifolds for a class of stochastic evolution equations with linear multiplicative noise. In [21], Chen, Roberts and Duan established the existence of center manifolds for both discrete and continuous-time infinite-dimensional RDS, assuming an exponential trichotomy, by employing the Lyapunov–Perron method. Moreover, they provided examples illustrating the application of these results to stochastic evolution equations through their conversion into infinite-dimensional RDS. In a similar vein, Kuehn and Neamţu [36] addressed the issue of center manifolds for rough partial differential equations, which also translates into center manifolds within the RDS framework. Li, Zeng and Huan [39] established the existence and smoothness of center-unstable invariant manifolds and center-stable foliations for a class of stochastic PDE with non-dense domain, by converting them into infinite-dimensional RDS.

Exponential trichotomies have played an important role in invariant manifold theory for infinite-dimensional dynamical systems and non-autonomous systems, whether in deterministic or random scenarios, as discussed. In this work, we extend the results on the existence of center manifolds for infinite-dimensional RDS by assuming a generalized trichotomy. This type of general assumption was considered in [16] for dichotomies, and in this work, it is extended to include a central direction. This generalization allows various types of non exponential behaviours along the three subspaces of the invariant splitting. In our context, each subspace is governed by a very general type of rate for controlling the growth of the evolution operator, described in terms of a cocycle. In specific cases, these subspaces correspond to the traditional central, stable, and unstable subspaces. However, our assumptions are sufficiently general to accommodate behaviours beyond exponential-type, such as those observed in (non)uniformly (pseudo-)hyperbolic settings.

This paper is organized as follows. Section 2 introduces the setup and provides preliminaries on RDS and generalized random trichotomies, as well as a description of auxiliary spaces of functions which are essential for handling nonlinear RDS components and deriving the center manifold as the graph of a suitable regular function. Section 3 presents the main result for continuous-time RDS (Theorem 3.1), while Section 4 focuses on the discrete-time counterpart (Theorem 4.1). In Section 5, continuous-time examples are discussed, including tempered exponential trichotomies and a general framework called ψ-trichotomies, which extend beyond exponential bounds. Corresponding discrete-time examples are provided in Section 6.

2 Generalized trichotomies for RDS

2.1 Random Dynamical Systems

Consider *time* $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$, and set $\mathbb{T}^- = \mathbb{T} \cap]-\infty, 0]$ and $\mathbb{T}^+ = \mathbb{T} \cap [0, +\infty[$. A *measure-preserving dynamical system* is a quadruplet $\Sigma \equiv (\Omega, \mathcal{F}, \mathbb{P}, \theta)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space and

$\theta \colon \mathbb{T} \times \Omega \to \Omega$ is measurable;

$\theta^t(\cdot) = \theta(t, \cdot) \colon \Omega \to \Omega$ preserves $\mathbb{P}$ for all $t \in \mathbb{T}$;

$\theta^0 = \mathrm{Id}_\Omega$;

$\theta^{t+s} = \theta^t \circ \theta^s$ for all $t, s \in \mathbb{T}$.

A (Bochner) measurable *random dynamical system*, henceforth abbreviated as RDS, on a Banach space X over a measure-preserving dynamical system Σ with time $\mathbb{T}$ is a map

$$\Phi : \mathbb{T} \times \Omega \times X \to X$$

such that

i) $\Phi(\cdot, \cdot, x)$ is (Bochner) measurable for all $x \in X$;

ii) $\Phi^t_\omega(\cdot) = \Phi(t, \omega, \cdot) \colon X \to X$ satisfies

  a) $\Phi^0_\omega = \mathrm{Id}_X$ for all $\omega \in \Omega$;

  b) $\Phi^{t+s}_\omega = \Phi^t_{\theta^s\omega} \circ \Phi^s_\omega$, for all $\omega \in \Omega$ and all $s, t \in \mathbb{T}$.

When Φ^t_ω is a bounded linear operator for all $(t, \omega) \in \mathbb{T} \times \Omega$, the RDS Φ is called *linear*.

We may restrict the *driving system* Σ to a θ^t-invariant subset $\Omega' \subset \Omega$ with $\mathbb{P}$-full measure, obtaining a (Bochner) measurable RDS $\Phi|_{\mathbb{T}\times\Omega'\times X}$ over $\Sigma' \equiv (\Omega', \mathcal{F}', \mathbb{P}|_{\mathcal{F}'}, \theta|_{\Omega'})$, where $\mathcal{F}' = \{B \cap \Omega' \colon B \in \mathcal{F}\}$. In view of this, without any loss of generality, throughout this work, requiring a property to hold for all $\omega \in \Omega'$, for a θ^t-invariant subset $\Omega' \subset \Omega$ with $\mathbb{P}$-full measure, can be replaced by simply requiring it for all $\omega \in \Omega$ by restricting, if necessary, the RDS Φ to $\Phi|_{\mathbb{T}\times\Omega'\times X}$ over Σ'.

2.2 Generalized trichotomies

For every $i \in \{c, s, u\}$, consider a map $P^i \colon \Omega \times X \to X$, and set $P^i_\omega(\cdot) = P(\omega, \cdot) \colon X \to X$. Let $\mathcal{P} = (P^c, P^s, P^u)$. A (Bochner) measurable linear RDS Φ over Σ admits a (Bochner) measurable *$\mathcal{P}$-invariant splitting* if *i*) $P^i(\cdot, x)$ is (Bochner) measurable, for all $x \in X$ and every $i \in \{c, s, u\}$;

ii) P^i_ω is a bounded linear projection, for all $\omega \in \Omega$ and every $i \in \{c, s, u\}$;

iii) $P^c_\omega + P^s_\omega + P^u_\omega = \mathrm{Id}$, for all $\omega \in \Omega$;

iv) $P^c_\omega P^s_\omega = 0$, for all $\omega \in \Omega$;

v) $P^i_{\theta^t\omega}\Phi^t_\omega = \Phi^t_\omega P^i_\omega$, for all $(t, \omega) \in \mathbb{T} \times \Omega$ and every $i \in \{c, s, u\}$;

Notice that for all $\omega \in \Omega$ and $i, j \in \{c, s, u\}$, with $i \neq j$, we have $P^i_\omega P^j_\omega = 0$. To shorten the writing during future computations, for $t \in \mathbb{T}$, $\omega \in \Omega$, and $i \in \{c, s, u\}$ we will adopt the notation

$$\Phi^{i,t}_\omega = \Phi^t_\omega P^i_\omega.$$

We define the linear subspaces $E^i_\omega = P^i_\omega(X)$ for each $i \in \{c, s, u\}$. As usual, we identify $E^c_\omega \times E^s_\omega \times E^u_\omega$ and $E^c_\omega \oplus E^s_\omega \oplus E^u_\omega$. Given the maps

$$\begin{aligned} \alpha^c &: \mathbb{T} \times \Omega \to (0, +\infty), \\ \alpha^s &: \mathbb{T}^+ \times \Omega \to (0, +\infty), \\ \alpha^u &: \mathbb{T}^- \times \Omega \to (0, +\infty), \end{aligned}$$

we define $\alpha = (\alpha^c, \alpha^s, \alpha^u)$. Denote $\alpha^i(t, \omega)$ by $\alpha^i_{t,\omega}$. We say that a (Bochner) measurable linear RDS Φ over Σ exhibits a *generalized trichotomy with bounds* α (or simply an α*-trichotomy*) if it admits a (Bochner) measurable $\mathcal{P}$-invariant splitting satisfying

(T1) $\|\Phi^{c,t}_\omega\| \leqslant \alpha^c_{t,\omega}$ for all $(t, \omega) \in \mathbb{T} \times \Omega$,

(T2) $\|\Phi^{s,t}_\omega\| \leqslant \alpha^s_{t,\omega}$ for all $(t, \omega) \in \mathbb{T}^+ \times \Omega$,

(T3) $\|\Phi^{u,t}_\omega\| \leqslant \alpha^u_{t,\omega}$ for all $(t, \omega) \in \mathbb{T}^- \times \Omega$,

where the operators in (T1)-(T3) are considered as operators from X into X. In what follows, we always consider the operators defined in the whole Banach space X.

In Section 5 and Section 6, we present several examples of generalized trichotomies with both exponential and non-exponential bounds α.

In the remainder of this article, Φ will always denote a measurable (when $\mathbb{T} = \mathbb{Z}$) or Bochner measurable (when $\mathbb{T} = \mathbb{R}$) linear RDS on a Banach space X over a measure-preserving dynamical system $\Sigma \equiv (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ exhibiting a trichotomy with bounds $\alpha = (\alpha^c, \alpha^s, \alpha^u)$.

2.3 Auxiliary spaces

Let $\mathscr{F}$ denote the space of maps $f\colon \Omega \times X \to X$ such that $f(\cdot, x)$ is measurable for every $x \in X$, and for which, setting $f_\omega(\cdot) = f(\omega, \cdot)$, for every $\omega \in \Omega$ we have

$$f_\omega(0) = 0 \tag{2.1}$$

and

$$\mathrm{Lip}(f_\omega) = \sup\left\{\frac{\|f_\omega(x) - f_\omega(y)\|}{\|x - y\|} : x, y \in X,\ x \neq y\right\} < +\infty. \tag{2.2}$$

Conditions (2.2) and (2.1) ensure that for all $\omega \in \Omega$ and $x, y \in X$

$$\|f_\omega(x) - f_\omega(y)\| \leqslant \mathrm{Lip}(f_\omega)\|x - y\|, \tag{2.3}$$

and

$$\|f_\omega(x)\| \leqslant \mathrm{Lip}(f_\omega)\|x\|. \tag{2.4}$$

Let $\mathscr{F}^{(B)}$ represent the collection of functions $f \in \mathscr{F}$ for which $f(\cdot, x)$ is Bochner measurable for each $x \in X$. Additionally, define $\mathscr{F}_\alpha^{(B)}$ as the subset of $\mathscr{F}^{(B)}$ consisting of functions f such that, for every $\omega \in \Omega$, the maps

$$\begin{aligned}
&[a,b] \ni r \mapsto \alpha^c_{t-r,\theta^r\omega} \,\mathrm{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega},\\
&[c,0] \ni r \mapsto \alpha^s_{-r,\theta^r\omega} \,\mathrm{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega},\\
&[0,d] \ni r \mapsto \alpha^u_{-r,\theta^r\omega} \,\mathrm{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega}
\end{aligned}$$

are measurable for every $a < b$, $c < 0$, $d > 0$ and $t \in \mathbb{R}$.

We define the set

$$\mathcal{C} = \{(t, \omega, \xi) \in \mathbb{T} \times \Omega \times X \colon \xi \in E^c_\omega\}.$$

For a given $M > 0$, let $\mathfrak{C}_M$ (resp. $\mathfrak{C}_M^{(B)}$) denote the space of all functions $h\colon \mathcal{C} \to X$ such that, for each $(t, \omega) \in \mathbb{T} \times \Omega$, the map $h_{t,\omega}(\cdot) = h(t, \omega, \cdot)$ satisfies

$$h(\cdot, \cdot, P^c_\omega x) \text{ is measurable (resp. Bochner measurable) for all } x \in X; \tag{2.5}$$

$$h_{t,\omega}(0) = 0 \text{ for all } (t, \omega) \in \mathbb{T} \times \Omega; \tag{2.6}$$

$$h_{0,\omega} = \mathrm{Id}_{E^c_\omega} \text{ for all } \omega \in \Omega; \tag{2.7}$$

$$h_{t,\omega}(E^c_\omega) \subseteq E^c_{\theta^t\omega} \text{ for all } (t, \omega) \in \mathbb{T} \times \Omega; \tag{2.8}$$

$$\|h_{t,\omega}(\xi) - h_{t,\omega}(\xi')\| \leqslant M\alpha^c_{t,\omega}\|\xi - \xi'\| \text{ for all } (t, \omega, \xi), (t, \omega, \xi') \in \mathcal{C}. \tag{2.9}$$

From (2.9) and (2.6), it follows that

$$\|h_{t,\omega}(\xi)\| \leqslant M\alpha^c_{t,\omega}\|\xi\| \text{ for all } (t, \omega, \xi) \in \mathcal{C}. \tag{2.10}$$

Defining

$$d_1(h, g) = \sup\left\{ \frac{\|h_{t,\omega}(\xi) - g_{t,\omega}(\xi)\|}{\alpha^c_{t,\omega}\|\xi\|} \colon (t, \omega, \xi) \in \mathcal{C},\ \xi \neq 0 \right\} \tag{2.11}$$

we have that $(\mathfrak{C}_M, d_1)$ and $(\mathfrak{C}_M^{(B)}, d_1)$ are complete metric spaces.

We now consider the set

$$\mathcal{D} = \{(\omega, \xi) \in \Omega \times X \colon \xi \in E^c_\omega\}.$$

For a given $N > 0$, let $\mathfrak{D}_N$ (resp. $\mathfrak{D}_N^{(B)}$) denote the space of all functions $\varphi\colon \mathcal{D} \to X$ such that, for each $\omega \in \Omega$, the map $\varphi_\omega(\cdot) = \varphi(\omega, \cdot)$ satisfies

$$\varphi(\cdot, P^c_\omega x) \text{ is measurable (resp. Bochner measurable) for all } x \in X; \tag{2.12}$$

$$\varphi_\omega(0) = 0 \text{ for all } \omega \in \Omega; \tag{2.13}$$

$$\varphi_\omega(E^c_\omega) \subseteq E^s_\omega \oplus E^u_\omega \text{ for all } \omega \in \Omega; \tag{2.14}$$

$$\|\varphi_\omega(\xi) - \varphi_\omega(\xi')\| \leqslant N\|\xi - \xi'\| \text{ for all } (\omega, \xi), (\omega, \xi') \in \mathcal{D}. \tag{2.15}$$

By (2.15) and (2.13), taking $\xi' = 0$, we get

$$\|\varphi_\omega(\xi)\| \leqslant N\|\xi\| \text{ for all } (\omega,\xi) \in \mathcal{D}. \tag{2.16}$$

For future use, we set the notation $\varphi^s_\omega = P^s_\omega \varphi_\omega$ and $\varphi^u_\omega = P^u_\omega \varphi_\omega$. Given $\varphi \in \mathfrak{D}_N$ and $\omega \in \Omega$, we denote the *graph* of φ_ω by

$$\Gamma_{\varphi,\omega} = \{(\xi, \varphi_\omega(\xi)) : \xi \in E^c_\omega\} \subseteq X.$$

Defining now

$$d_2(\varphi,\psi) = \sup\left\{\frac{\|\varphi_\omega(\xi) - \psi_\omega(\xi)\|}{\|\xi\|} : (\omega,\xi) \in \mathcal{D}, \xi \neq 0\right\} \tag{2.17}$$

it follows that $(\mathfrak{D}_N, d_2)$ and $(\mathfrak{D}^{(B)}_N, d_2)$ are complete metric spaces.

To conclude this section, let $\mathfrak{U}_{M,N} = \mathfrak{C}_M \times \mathfrak{D}_N$ and $\mathfrak{U}^{(B)}_{M,N} = \mathfrak{C}^{(B)}_M \times \mathfrak{D}^{(B)}_N$. Setting

$$d((h,\varphi),(g,\psi)) = d_1(h,g) + d_2(\varphi,\psi),$$

we also have that $(\mathfrak{U}_{M,N}, d)$ and $(\mathfrak{U}^{(B)}_{M,N}, d)$ are complete metric spaces.

3 Invariant manifolds in continuous-time RDS

Throughout this section, we focus on the continuous-time case by considering $\mathbb{T} = \mathbb{R}$. Given a Bochner measurable linear RDS Φ and a map $f \in \mathscr{F}^{(B)}_\alpha$, we define

$$\sigma = \sup_{(t,\omega)\in\mathbb{R}\times\Omega} \frac{1}{\alpha^c_{t,\omega}} \left| \int_0^t \alpha^c_{t-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega}\, dr \right| \tag{3.1}$$

and

$$\tau = \sup_{\omega\in\Omega} \int_{-\infty}^0 \alpha^s_{-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega}\, dr + \int_0^{+\infty} \alpha^u_{-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega}\, dr. \tag{3.2}$$

If for every $(\omega,x) \in \Omega \times X$ there is a unique solution $\Psi(\cdot,\omega,x)$ of the equation

$$u(t) = \Phi^t_\omega x + \int_0^t \Phi^{t-r}_{\theta^r\omega} f_{\theta^r\omega}(u(r))\, dr \tag{3.3}$$

then $\Psi\colon \mathbb{R} \times \Omega \times X \to X$ is a Bochner measurable RDS on X over Σ. In particular, $\Psi(\cdot,\cdot,x)$ is Bochner measurable for all $x \in X$, and

$$\Psi^t_\omega x = \Phi^t_\omega x + \int_0^t \Phi^{t-r}_{\theta^r\omega} f_{\theta^r\omega}(\Psi^r_\omega x)\, dr. \tag{3.4}$$

Theorem 3.1. *Let Φ be a Bochner measurable linear RDS exhibiting an α-trichotomy, and let $f \in \mathscr{F}^{(B)}_\alpha$. Suppose that Ψ is a Bochner measurable RDS such that $\Psi(\cdot,\omega,x)$ is the unique solution of (3.3) for all $(\omega,x) \in \Omega \times X$. If*

$$\lim_{t\to-\infty} \alpha^s_{-t,\theta^t\omega}\alpha^c_{t,\omega} = \lim_{t\to+\infty} \alpha^u_{-t,\theta^t\omega}\alpha^c_{t,\omega} = 0 \tag{3.5}$$

for all $\omega \in \Omega$, and

$$\sigma + \tau < 1/2, \tag{3.6}$$

then there are $N \in]0,1[$ *and a unique* $\varphi \in \mathfrak{D}_N^{(B)}$ *such that*

$$\Psi_\omega^t(\Gamma_{\varphi,\omega}) \subseteq \Gamma_{\varphi,\theta^t\omega} \tag{3.7}$$

for all $(t,\omega) \in \mathbb{R} \times \Omega$. *Moreover, for all* $(t,\omega,\xi), (t,\omega,\xi') \in \mathcal{C}$ *we have*

$$\|\Psi_\omega^t(\xi, \varphi_\omega(\xi)) - \Psi_\omega^t(\xi', \varphi_\omega(\xi'))\| \leqslant (N/\tau)\alpha_{t,\omega}^c \|\xi - \xi'\|. \tag{3.8}$$

The remaining part of this section is devoted to proving Theorem 3.1.

From [15, Lemma 5.1], we may find constants $M \in]1,2[$ and $N \in]0,1[$ such that

$$\sigma = \frac{M-1}{M(1+N)} \quad \text{and} \quad \tau = \frac{N}{M(1+N)}. \tag{3.9}$$

Lemma 3.2. *Consider* $(h,\varphi) \in \mathfrak{U}_{M,N}^{(B)}$.

a) For every $x \in X$ *the maps*

$$\begin{aligned}
&(t,r,\omega) \mapsto \Phi_{\theta^r\omega}^{c,t-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x))),\\
&(r,\omega) \mapsto \Phi_{\theta^r\omega}^{s,-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x))),\\
&(r,\omega) \mapsto \Phi_{\theta^r\omega}^{u,-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x)))
\end{aligned}$$

are Bochner measurable on $\mathbb{R} \times \mathbb{R} \times \Omega$, $\mathbb{R}^- \times \Omega$ *and* $\mathbb{R}^+ \times \Omega$, *respectively.*

b) For every $(t,\omega,x) \in \mathbb{R} \times \Omega \times X$ *the map*

$$r \mapsto \Phi_{\theta^r\omega}^{c,t-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x)))$$

is Bochner integrable in every closed interval with bounds 0 *and* t.

c) For every $(\omega,x) \in \Omega \times X$ *and* $t > 0$, *the maps*

$$\begin{aligned}
&r \mapsto \Phi_{\theta^r\omega}^{s,-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x))),\\
&r \mapsto \Phi_{\theta^r\omega}^{u,-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x)))
\end{aligned}$$

are Bochner integrable in $[-t,0]$ *and* $[0,t]$, *respectively.*

The proof follows similarly as [16, Lemma 3.6]. Given $\omega \in \Omega$ and $x_\omega = (x_\omega^c, x_\omega^s, x_\omega^u) \in E_\omega^c \times E_\omega^s \times E_\omega^u$, it follows from (3.4) that the trajectory $x_{\theta^t\omega} = \Psi_\omega^t x_\omega = (x_{\theta^t\omega}^c, x_{\theta^t\omega}^s, x_{\theta^t\omega}^u)$ satisfies, for all $i \in \{c,s,u\}$ and all $t \in \mathbb{R}$,

$$x_{\theta^t\omega}^i = \Phi_\omega^{i,t} x_\omega + \int_0^t \Phi_{\theta^s\omega}^{i,t-s} f_{\theta^s\omega}(x_{\theta^s\omega}^c, x_{\theta^s\omega}^s, x_{\theta^s\omega}^u)\, ds. \tag{3.10}$$

Taking into account the invariance required in (3.7), for any given $x_\omega \in \Gamma_{\varphi,\omega}$ and $t \in \mathbb{R}$ we must have $x_{\theta^t\omega} \in \Gamma_{\varphi,\theta^t\omega}$. Thus, in this situation, the equations given by (3.10) can be written as

$$\begin{aligned}
x_{\theta^t\omega}^c &= \Phi_\omega^{c,t} x_\omega + \int_0^t \Phi_{\theta^s\omega}^{c,t-s} f_{\theta^s\omega}(x_{\theta^s\omega}^c, \varphi_{\theta^s\omega}(x_{\theta^s\omega}^c))\, ds,\\
\varphi_{\theta^t\omega}^s(x_{\theta^t\omega}) &= \Phi_\omega^t \varphi_\omega^s(x_\omega^c) + \int_0^t \Phi_{\theta^s\omega}^{s,t-s} f_{\theta^s\omega}(x_{\theta^s\omega}^c, \varphi_{\theta^s\omega}(x_{\theta^s\omega}^c))\, ds,\\
\varphi_{\theta^t\omega}^u(x_{\theta^t\omega}) &= \Phi_\omega^t \varphi_\omega^u(x_\omega^c) + \int_0^t \Phi_{\theta^s\omega}^{u,t-s} f_{\theta^s\omega}(x_{\theta^s\omega}^c, \varphi_{\theta^s\omega}(x_{\theta^s\omega}^c))\, ds.
\end{aligned}$$

Lemma 3.3. *Consider* $(h,\varphi) \in \mathfrak{U}^{(B)}_{M,N}$ *such that, for all* $(t,\omega,\xi) \in \mathcal{C}$,

$$h_{t,\omega}(x) = \Phi^t_\omega \xi + \int_0^t \Phi^{c,t-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr. \tag{3.11}$$

The following properties a) and b) are equivalent:

a) For each $j \in \{s,u\}$ *and all* $(t,\omega,\xi) \in \mathcal{C}$,

$$\varphi^j_{\theta^t\omega}(h_{t,\omega}(\xi)) = \Phi^t_\omega \varphi^j_\omega(\xi) + \int_0^t \Phi^{j,t-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr \tag{3.12}$$

b) For all $(\omega,\xi) \in \mathcal{D}$

$$\varphi^s_\omega(\xi) = \int_{-\infty}^0 \Phi^{s,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr \tag{3.13}$$

and

$$\varphi^u_\omega(\xi) = -\int_0^{+\infty} \Phi^{u,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr. \tag{3.14}$$

Proof. From (2.4), (2.16) and (2.10) we have

$$\begin{aligned}\|f_{\theta^r\omega}(h_{r,\omega}(\xi),\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\| &\leqslant \operatorname{Lip}(f_{\theta^r\omega})(\|h_{r,\omega}(\xi)\| + \|\varphi_{\theta^r\omega}(h_{r,\omega}(\xi))\|)\\ &\leqslant M(1+N)\operatorname{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega}\|\xi\|\end{aligned}$$

for every $(\omega,\xi) \in \mathcal{D}$. Thus, by (T2),

$$\int_{-\infty}^0 \left\|\Phi^{s,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\right\| dr \leqslant M(1+N)\tau\|\xi\|,$$

and by (T3) we obtain

$$\int_0^{+\infty} \left\|\Phi^{u,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\right\| dr \leqslant M(1+N)\tau\|\xi\|.$$

Hence the integrals are convergent.

Suppose that (3.12) holds for $j = s$ and all $(t,\omega,\xi) \in \mathcal{C}$. By applying $\Phi^{-t}_{\theta^t\omega}$ to both sides, it is equivalent to

$$\varphi^s_\omega(\xi) = \Phi^{s,-t}_{\theta^t\omega}\varphi^s_{\theta^t\omega}(h_{t,\omega}(\xi)) - \int_0^t \Phi^{s,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr. \tag{3.15}$$

Using (T2), (2.16) and (2.10), for $t \leqslant 0$ we have

$$\left\|\Phi^{s,-t}_{\theta^t\omega}\varphi^s_{\theta^t\omega}(h_{t,\omega}(\xi))\right\| \leqslant MN\alpha^s_{-t,\theta^t\omega}\,\alpha^c_{t,\omega}\|\xi\|,$$

which converges to zero as $t \to -\infty$ by (3.5). Thus, by taking $t \to -\infty$ in equation (3.15) we obtain (3.13). Similarly, equation (3.12) with $j = u$ can be written as

$$\varphi^u_\omega(\xi) = \Phi^{u,-t}_{\theta^t\omega}\varphi_{\theta^t\omega}(h_{t,\omega}(\xi)) - \int_0^t \Phi^{u,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr. \tag{3.16}$$

Using (T3), (2.16) and (2.10), for $t \geqslant 0$ we have

$$\left\|\Phi^{u,-t}_{\theta^t\omega}\varphi_{\theta^t\omega}(h_{t,\omega}(\xi))\right\| \leqslant MN\alpha^u_{-t,\theta^t\omega}\alpha^c_{t,\omega}\,\|\xi\|\,,$$

which, by (3.5), converges to zero as $t \to +\infty$. Thus, we obtain (3.14) by taking $t \to +\infty$ in equation (3.16).

For the converse, assume now that (3.13) and (3.14) hold for all $(\omega, \xi) \in \mathcal{D}$. For all $t \in \mathbb{R}$, we have

$$\begin{aligned}\Phi^t_\omega \varphi^s_\omega(\xi) = & \int_t^0 \Phi^{s,t-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\, , \varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr \\ & + \int_{-\infty}^0 \Phi^{s,-r}_{\theta^{t+r}\omega} f_{\theta^{t+r}\omega}(h_{t+r,\omega}(\xi)\, , \varphi_{\theta^{t+r}\omega}(h_{t+r,\omega}(\xi)))\, dr\end{aligned}$$

and

$$\begin{aligned}\Phi^t_\omega \varphi^u_\omega(\xi) = & -\int_0^t \Phi^{u,t-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\, , \varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr \\ & - \int_{-\infty}^0 \Phi^{u,-r}_{\theta^{t+r}\omega} f_{\theta^{t+r}\omega}(h_{t+r,\omega}(\xi)\, , \varphi_{\theta^{t+r}\omega}(h_{t+r,\omega}(\xi)))\, dr.\end{aligned}$$

Since $h_{t+s,\omega}(\xi) = h_{s,\theta^t\omega}(h_{t,\omega}(\xi))$ due to the uniqueness of the solution of (3.3), we get the identity (3.12) for $j = s$ and $j = u$. □

Consider the operator C, which assigns each pair $(h, \varphi) \in \mathfrak{U}^{(B)}_{M,N}$ to the map $C(h, \varphi)\colon \mathcal{C} \to X$ given by

$$[C(h,\varphi)]\,(t,\omega,\xi) = \Phi^t_\omega \xi + \int_0^t \Phi^{c,t-r}_{\theta^r\omega}\varphi(r,\omega)\, dr.$$

Lemma 3.4. $C\big(\mathfrak{U}^{(B)}_{M,N}\big) \subseteq \mathfrak{C}^{(B)}_M$.

Proof. Fix a pair $(h, \varphi) \in \mathfrak{U}^{(B)}_{M,N}$. It is straightforward to check that $C(h, \varphi)$ satisfies conditions (2.5) to (2.8). Define

$$\gamma_{\theta^r\omega}(\xi,\xi') = \|f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi))) - f_{\theta^r\omega}(h_{r,\omega}(\xi'), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi')))\|.$$

From (2.3), (2.15) and (2.9) we have

$$\gamma_{\theta^r\omega}(\xi,\xi') \leqslant \operatorname{Lip}(f_{\theta^r\omega})M(1+N)\|\xi-\xi'\|\alpha^c_{r,\omega}. \tag{3.17}$$

Following the previous notation, $C(h, \varphi)_{t,\omega}(\xi)$ stands for $[C(h, \varphi)](t, \omega, \xi)$. By (T1), (3.1), (3.17) and (3.9), we have

$$\begin{aligned}\|C(h,\varphi)_{t,\omega}(\xi) - C(h,\varphi)_{t,\omega}(\xi')\| &\leqslant \|\Phi^{c,t}_\omega\|\|\xi-\xi'\| + \int_0^t \|\Phi^{c,t-r}_{\theta^r\omega}\|\gamma_{\theta^r\omega}(\xi,\xi')\, dr \\ &\leqslant (1+\sigma M(1+N))\,\alpha^c_{t,\omega}\|\xi-\xi'\| \\ &= M\alpha^c_{t,\omega}\|\xi-\xi'\|.\end{aligned}$$

Hence $C(h, \varphi)$ also satisfies (2.9). □

Consider now the operator D, which assigns each pair $(h, \varphi) \in \mathfrak{U}^{(B)}_{M,N}$ the map

$$D(h,\varphi)\colon \mathcal{D} \to X$$

given by

$$[D(h,\varphi)]\,(\omega,\xi) = [D^s(h,\varphi)]\,(\omega,\xi) + [D^u(h,\varphi)]\,(\omega,\xi)$$

where
$$[D^s(h,\varphi)](\omega,\xi) = \int_{-\infty}^{0} \Phi^{s,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr$$
and
$$[D^u(h,\varphi)](\omega,\xi) = -\int_{0}^{+\infty} \Phi^{u,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr.$$

Lemma 3.5. $D(\mathfrak{U}^{(B)}_{M,N}) \subseteq \mathfrak{D}^{(B)}_N$.

Proof. Fix $(h,\varphi) \in \mathfrak{U}^{(B)}_{M,N}$. It is immediate to check that $[D(h,\varphi)](\omega,\xi)$ satisfies conditions (2.12) to (2.14). Again, $D(h,\varphi)_\omega(\xi)$ stands for $[D(h,\varphi)](\omega,\xi)$. From (T2), (T3), (3.17), (3.2) and (3.9) we have

$$\begin{aligned}\|D(h,\varphi)_\omega(\xi) - D(h,\varphi)_\omega(\xi')\| &\leqslant \int_{-\infty}^{0} \|\Phi^{s,-r}_{\theta^r\omega}\|\, \gamma_{\theta^r\omega}(\xi,\xi')\, dr + \int_0^{+\infty} \|\Phi^{u,-r}_{\theta^r\omega}\|\, \gamma_{\theta^r\omega}(\xi,\xi')\, dr\\ &\leqslant \tau M(1+N)\,\|\xi-\xi'\|\\ &= N\,\|\xi-\xi'\|.\end{aligned}$$

Hence (2.15) also holds for $D(h,\varphi)$. □

Consider now $U\colon \mathfrak{U}^{(B)}_{M,N} \to \mathfrak{U}^{(B)}_{M,N}$ given by
$$U(h,\varphi) = (C(h,\varphi), D(h,\varphi)).$$

Lemma 3.6. *The operator U is a contraction in $(\mathfrak{U}^{(B)}_{M,N}, d)$.*

Proof. Consider $(h,\varphi), (g,\psi) \in \mathfrak{U}^{(B)}_{M,N}$. Define
$$\hat{\gamma}_{\theta^r\omega}(\xi) = \|f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi))) - f_{\theta^r\omega}(g_{r,\omega}(\xi), \psi_{\theta^r\omega}(g_{r,\omega}(\xi)))\|.$$
By (2.3), (2.15), (2.11), (2.17) and (2.10), for all $(r,\omega) \in \mathbb{R}^+_0 \times \Omega$ and all $\xi \in E_\omega$,
$$\hat{\gamma}_{\theta^r\omega}(\xi) \leqslant \mathrm{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega}((1+N)d_1(h,g) + Md_2(\varphi,\psi))\,\|\xi\|. \tag{3.18}$$
Hence, in one hand, from (T1), (3.18) and (3.1), we have
$$\begin{aligned}\|C(h,\varphi)_{t,\omega}(\xi) - C(g,\psi)_{t,\omega}(\xi)\| &\leqslant \int_0^t \left\|\Phi^{c,t-r}_{\theta^r\omega}\right\| \hat{\gamma}_{\theta^r\omega}(\xi)\, dr\\ &\leqslant \sigma\alpha^c_{t,\omega}((1+N)d_1(h,g) + Md_2(\varphi,\psi))\,\|\xi\|,\end{aligned}$$
which implies
$$d_1(C(h,\varphi), C(g,\psi)) \leqslant \sigma((1+N)d_1(h,g) + Md_2(\varphi,\psi)).$$
On the other hand, from (T2), (T3), (3.18) and (3.2) we get
$$\begin{aligned}\|D(h,\varphi)_\omega(\xi) - D(g,\psi)_\omega(\xi)\| &\leqslant \int_{-\infty}^{0} \|\Phi^{s,-r}_{\theta^r\omega}\|\, \hat{\gamma}_{\theta^r\omega}(\xi)\, dr + \int_0^{+\infty} \|\Phi^{u,-r}_{\theta^r\omega}\|\, \hat{\gamma}_{\theta^r\omega}(\xi)\, dr\\ &\leqslant \tau((1+N)d_1(h,g) + Md_2(\varphi,\psi))\,\|\xi\|,\end{aligned}$$
which implies
$$d_2(D(h,\varphi), D(g,\psi)) \leqslant \tau((1+N)d_1(h,g) + Md_2(\varphi,\psi)).$$

In overall we get

$$\begin{aligned} d(U(h,\varphi),U(g,\psi)) &\leqslant (\sigma+\tau)((1+N)d_1(h,g)+Md_2(\varphi,\psi)) \\ &\leqslant \frac{1}{2}\max\{1+N,M\}\, d((h,\varphi),(g,\psi)) \end{aligned}$$

and because $N<1$ and $M<2$, U is a contraction. □

Proof of Theorem 3.1. Since U is a contraction, by the Banach Fixed Point Theorem, U has a unique fixed point (h,φ), that satisfies (3.11), (3.13) and (3.14). By Lemma 3.3, the pair (h,φ) also satisfies conditions (3.12). Therefore, for given initial condition $x_\omega = (\xi, \varphi^s_\omega(\xi), \varphi^u_\omega(\xi)) \in E^c_\omega \times E^s_\omega \times E^u_\omega$, the trajectory $x_{\theta^t\omega} = (h_{t,\omega}(\xi), \varphi_{\theta^t\omega}(h_{t,\omega}(\xi)))$ is the solution of (3.3). The graphs $\Gamma_{\varphi,\omega}$ are the required invariant manifolds of Ψ. To obtain (3.8), it follows from (2.15), (2.9) and (3.9) that, for each $(t,\omega,\xi),(t,\omega,\xi') \in \mathcal{C}$

$$\begin{aligned} &\|\Psi^t_\omega(\xi,\varphi^s_\omega(\xi),\varphi^u_\omega(\xi)) - \Psi^t_\omega(\xi',\varphi^s_\omega(\xi'),\varphi^u_\omega(\xi'))\| \\ &\quad= \|(h_{t,\omega}(\xi),\varphi_{\theta^t\omega}(h_{t,\omega}(\xi))) - (h_{t,\omega}(\xi'),\varphi_{\theta^t\omega}(h_{t,\omega}(\xi')))\,\| \\ &\quad\leqslant M(1+N)\alpha^c_{t,\omega}\|\xi-\bar{\xi}\| \\ &\quad\leqslant \frac{N}{\tau}\alpha^c_{t,\omega}\|\xi-\bar{\xi}\|. \end{aligned}$$

□

4 Invariant manifolds in discrete-time RDS

Throughout this section we consider $\mathbb{T}=\mathbb{Z}$. Given a measurable linear RDS Φ and a map $f\in\mathscr{F}$, we define

$$\sigma^-_\omega = \sup_{n\in\mathbb{N}} \frac{1}{\alpha^c_{-n,\omega}} \sum_{k=-n}^{-1} \alpha^c_{-n-k-1,\theta^{k+1}\omega} \operatorname{Lip}(f_{\theta^k\omega})\alpha^c_{k,\omega},$$

$$\sigma^+_\omega = \sup_{n\in\mathbb{N}} \frac{1}{\alpha^c_{n,\omega}} \sum_{k=0}^{n-1} \alpha^c_{n-k-1,\theta^{k+1}\omega} \operatorname{Lip}(f_{\theta^k\omega})\alpha^c_{k,\omega}$$

and

$$\sigma = \sup_{\omega\in\Omega} \max\{\sigma^-_\omega,\sigma^+_\omega\}.$$

Moreover, writing

$$\tau^-_\omega = \sum_{k=-\infty}^{-1} \alpha^s_{-k-1,\theta^{k+1}\omega} \operatorname{Lip}(f_{\theta^k\omega})\alpha^c_{k,\omega},$$

$$\tau^+_\omega = \sum_{k=0}^{+\infty} \alpha^u_{-k-1,\theta^{k+1}\omega} \operatorname{Lip}(f_{\theta^k\omega})\alpha^c_{k,\omega},$$

we also define

$$\tau = \sup_{\omega\in\Omega}(\tau^-_\omega+\tau^+_\omega).$$

Consider the measurable RDS $\Psi\colon \mathbb{Z}\times\Omega\times X\to X$ given by

$$\Psi^n_\omega(x) = \begin{cases} \Phi^n_\omega x + \sum_{k=0}^{n-1} \Phi^{n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(\Psi^k_\omega(x)) & \text{if } n\geqslant 1, \\ x & \text{if } n=0, \\ \Phi^n_\omega x - \sum_{k=n}^{-1} \Phi^{n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(\Psi^k_\omega(x)) & \text{if } n\leqslant -1 \end{cases} \tag{4.1}$$

which encapsulates the solutions of the random nonlinear difference equation

$$x_{n+1} = \Phi^1_{\theta^n\omega} x_n + f_{\theta^n\omega}(x_n).$$

Theorem 4.1. *Let Φ be a measurable linear RDS exhibiting an α-trichotomy and let $f \in \mathscr{F}$. If*

$$\lim_{n\to-\infty} \alpha^s_{-n,\theta^n\omega}\alpha^c_{n,\omega} = \lim_{n\to+\infty} \alpha^u_{-n,\theta^n\omega}\alpha^c_{n,\omega} = 0$$

for all $\omega \in \Omega$, and

$$\sigma + \tau < 1/2,$$

then there are $N \in \,]0,1[$ and a unique $\varphi \in \mathfrak{D}_N$ such that for the RDS Ψ given by (4.1) we have

$$\Psi^n_\omega(\Gamma_{\varphi,\omega}) \subseteq \Gamma_{\varphi,\theta^n\omega} \tag{4.2}$$

for all $(n,\omega) \in \mathbb{Z}\times\Omega$. Moreover, for every $(n,\omega,\xi),(n,\omega,\xi') \in \mathcal{C}$ we have

$$\|\Psi^n_\omega(\xi,\varphi_\omega(\xi)) - \Psi^n_\omega(\xi',\varphi_\omega(\xi'))\| \leqslant (N/\tau)\alpha^c_{n,\omega}\,\|\xi-\xi'\|.$$

The proof of Theorem 4.1 is analogous to the proof of Theorem 3.1. Therefore, in the remainder of this section, we provide a guide to the necessary adaptations. Fix M and N as in (3.9). Given $\omega \in \Omega$ and

$$x_\omega = (x^c_\omega, x^s_\omega, x^u_\omega) \in E^c_\omega \times E^s_\omega \times E^u_\omega,$$

the trajectory

$$x_{\theta^n\omega} = \Psi^n_\omega x_\omega = (x^c_{\theta^n\omega}, x^s_{\theta^n\omega}, x^u_{\theta^n\omega}) \in E^c_\omega \times E^s_\omega \times E^u_\omega$$

satisfies the following equations for each $i \in \{c,s,u\}$:

$$x^i_{\theta^n\omega} = \begin{cases} \Phi^{i,n}_\omega x^i_\omega + \displaystyle\sum_{k=0}^{n-1} \Phi^{i,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, x^s_{\theta^k\omega}, x^u_{\theta^k\omega}) & \text{if } n \geqslant 1,\\ \Phi^{i,n}_\omega x^i_\omega - \displaystyle\sum_{k=n}^{-1} \Phi^{i,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, x^s_{\theta^k\omega}, x^u_{\theta^k\omega}) & \text{if } n \leqslant -1. \end{cases} \tag{4.3}$$

In view of the invariance required in (4.2), if $x_\omega \in \Gamma_{\varphi,\omega}$ then $x_{\theta^n\omega}$ must be in $\Gamma_{\varphi,\theta^n\omega}$ for every $n \in \mathbb{Z}$, and thus, in this situation, the equations from (4.3) can be written as

$$x^c_{\theta^n\omega} = \begin{cases} \Phi^{c,n}_\omega x^c_\omega + \displaystyle\sum_{k=0}^{n-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, \varphi_{\theta^k\omega}(x^c_{\theta^k\omega})) & \text{if } n \geqslant 1,\\ \Phi^{c,n}_\omega x^c_\omega - \displaystyle\sum_{k=n}^{-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, \varphi_{\theta^k\omega}(x^c_{\theta^k\omega})) & \text{if } n \leqslant -1 \end{cases} \tag{4.4}$$

and, for $j \in \{s,u\}$,

$$\varphi^j_{\theta^n\omega}(x_{\theta^n\omega}) = \begin{cases} \Phi^{j,n}_\omega \varphi_\omega(x_\omega) + \displaystyle\sum_{k=0}^{n-1} \Phi^{j,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, \varphi_{\theta^k\omega}(x^c_{\theta^k\omega})) & \text{if } n \geqslant 1,\\ \Phi^{j,n}_\omega \varphi_\omega(x_\omega) - \displaystyle\sum_{k=n}^{-1} \Phi^{j,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, \varphi_{\theta^k\omega}(x^c_{\theta^k\omega})) & \text{if } n \leqslant -1. \end{cases} \tag{4.5}$$

Let us prove prove that equations (4.4) and (4.5) have solutions. First, we rewrite them, by a discrete version of Lemma 3.3.

Lemma 4.2. *Consider $(h,\varphi) \in \mathfrak{U}_{M,N}$ such that, for all $(n,\omega) \in \mathbb{Z} \times \Omega$ and all $\xi \in E^c_\omega$*

$$h_{n,\omega}(\xi) = \begin{cases} \Phi^{c,n}_\omega \xi + \sum_{k=0}^{n-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \geqslant 1, \\ \Phi^{c,n}_\omega \xi - \sum_{k=n}^{-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \leqslant -1. \end{cases} \tag{4.6}$$

Then the following conditions a) and b) are equivalent:

a) For each $j \in \{u,s\}$ and all $(n,\omega,\xi) \in \mathcal{C}$

$$\varphi^j_{\theta^n\omega}(h_{n,\omega}(\xi)) = \begin{cases} \Phi^{j,n}_\omega \varphi_\omega(\xi) + \sum_{k=0}^{n-1} \Phi^{j,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \geqslant 1, \\ \Phi^{j,n}_\omega \varphi_\omega(\xi) - \sum_{k=n}^{-1} \Phi^{j,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \leqslant -1. \end{cases} \tag{4.7}$$

b) For all $(\omega,\xi) \in \mathcal{D}$

$$\varphi^s_\omega(\xi) = \sum_{k=-\infty}^{-1} \Phi^{s,-(k+1)}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) \tag{4.8}$$

and

$$\varphi^u_\omega(\xi) = -\sum_{k=0}^{+\infty} \Phi^{u,-(k+1)}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))). \tag{4.9}$$

Consider here the operator C, which assigns each pair $(h,\varphi) \in \mathfrak{U}^{(B)}_{M,N}$ to the map

$$C(h,\varphi)\colon \mathcal{C} \to X$$

given by

$$[C(h,\varphi)]\,(n,\omega,\xi) = \begin{cases} \Phi^{c,n}_\omega \xi + \sum_{k=0}^{n-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \geqslant 1, \\ \Phi^{c,n}_\omega \xi - \sum_{k=n}^{-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \leqslant -1, \end{cases}$$

and D be the operator that assigns to each pair $(h,\varphi) \in \mathfrak{U}_{M,N}$ the map $D(h,\varphi)\colon \mathcal{D} \to X$ defined by

$$[D(h,\varphi)]\,(\omega,\xi) = [D^s(h,\varphi)]\,(\omega,\xi) + [D^u(h,\varphi)]\,(\omega,\xi),$$

where

$$[D^s(h,\varphi)]\,(\omega,\xi) = \sum_{k=-\infty}^{-1} \Phi^{s,-(k+1)}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi)))$$

and

$$[D^u(h,\varphi)]\,(\omega,\xi) = -\sum_{k=0}^{+\infty} \Phi^{u,-(k+1)}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))).$$

To finalize, define $U\colon \mathfrak{U}_{M,N} \to \mathfrak{U}_{M,N}$ by

$$U(h, \varphi) = (C(h, \varphi), D(h, \varphi))\,.$$

The operator U is a contraction in $(\mathfrak{U}_{M,N}, d)$. By the Banach Fixed Point Theorem, U as a unique fixed point (h, φ), which satisfies conditions (4.6), (4.8) and (4.9). By Lemma 4.2 the pair (h, φ) also satisfy the conditions in (4.7). Hence, by (4.4) and (4.5), we get that $(h_{n,\omega}(\xi), \varphi_{\theta^n\omega}(h_{n,\omega}(\xi)))$ is the orbit by Ψ of the initial condition

$$(\xi, \varphi^s_\omega(\xi), \varphi^u_\omega(\xi)) \in E^c_\omega \times E^s_\omega \times E^u_\omega.$$

The graphs $\Gamma_{\varphi,\omega}$ are the required invariant manifolds of Ψ. Furthermore, for all $\omega \in \Omega$, all $n \in \mathbb{Z}$ and all $\xi, \xi' \in E^c_\omega$ it follows from (2.15), (2.9) and (3.9) that

$$\|\Psi^n_\omega(\xi, \varphi_\omega(\xi)) - \Psi^n_\omega(\xi', \varphi_\omega(\xi'))\| \leqslant \frac{N}{\tau}\alpha^c_{n,\omega}\|\xi - \bar{\xi}\|,$$

which finishes the proof of Theorem 4.1.

5 Continuous-time examples

For this section assume $\mathbb{T} = \mathbb{R}$. Throughout this entire section we consider a constant $\delta \in]0, 1/6[$ and a random variable $G\colon \Omega \to]0, +\infty[$ satisfying

$$\int_{-\infty}^{+\infty} G(\theta^r\omega)\, dr \leqslant 1 \quad \text{for all } \omega \in \Omega.$$

In all the following examples we may consider different growth rates along the *central directions* E^c_ω, depending if we are looking to the *future* ($t \to +\infty$) or to the *past* ($t \to -\infty$).

5.1 Tempered exponential trichotomies

Let

$$\lambda^{\bar{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}$$

be θ-invariant random variables, i.e. satisfying $\lambda^\ell(\theta^t\omega) = \lambda^\ell(\omega)$ for all $\omega \in \Omega$, $t \in \mathbb{R}$ and $\ell \in \{\bar{c}, \underline{c}, s, u\}$. A Bochner measurable linear RDS Φ exhibits an *exponential trichotomy* if it exhibits a generalized trichotomy with bounds

$$\alpha^c_{t,\omega} = \begin{cases} K(\omega)\, e^{\lambda^{\bar{c}}(\omega)t}, & t \geqslant 0, \\ K(\omega)\, e^{\lambda^{\underline{c}}(\omega)t}, & t \leqslant 0, \end{cases}$$
$$\alpha^s_{t,\omega} = K(\omega)\, e^{\lambda^s(\omega)t}, \quad t \geqslant 0,$$
$$\alpha^u_{t,\omega} = K(\omega)\, e^{\lambda^u(\omega)t}, \quad t \leqslant 0$$

for some random variable $K : \Omega \to [1, +\infty[$. If the random variable K is *tempered*, *i.e.*, if

$$\Lambda_{K,\gamma,\omega} := \sup_{t\in\mathbb{T}} \left[e^{-\gamma|t|}\, K(\theta^t w)\right] < +\infty \tag{5.1}$$

for all $\gamma > 0$ and all $\omega \in \Omega$, we say that Φ exhibits an *tempered exponential trichotomy*.

Corollary 5.1. *Let Φ be a Bochner measurable linear RDS exhibiting a tempered exponential trichotomy such that*

$$\lambda^{\underline{c}}(\omega) > \lambda^s(\omega) \quad \text{and} \quad \lambda^{\overline{c}}(\omega) < \lambda^u(\omega)$$

for all $\omega \in \Omega$, and let $f \in \mathscr{F}_\alpha^{(B)}$. Assume that Ψ is a Bochner measurable RDS such that (3.3) has unique solution $\Psi(\cdot, \omega, x)$ for every $(\omega, x) \in \Omega \times X$. Consider a θ-invariant random variable $\gamma(\omega) > 0$ satisfying

$$a(\omega) := \lambda^{\underline{c}}(\omega) - \lambda^s(\omega) - \gamma(\omega) > 0 \quad \text{and} \quad b(\omega) := \lambda^u(\omega) - \lambda^{\overline{c}}(\omega) - \gamma(\omega) > 0$$

for all $\omega \in \Omega$. If

$$\mathrm{Lip}(f_\omega) \leqslant \frac{\delta}{K(\omega)} \min\left\{ G(\omega), \frac{a(\omega)}{\Lambda_{K,\gamma(\omega),\omega}}, \frac{b(\omega)}{\Lambda_{K,\gamma(\omega),\omega}} \right\}$$

for all $\omega \in \Omega$, then the same conclusions of Theorem 3.1 hold.

Proof. Since K is a tempered random variable, we have

$$\lim_{t \to -\infty} \alpha^s_{-t,\theta^t\omega} \alpha^c_{t,\omega} = \lim_{t \to -\infty} K(\omega) K(\theta^t \omega)\, \mathrm{e}^{(\lambda^{\underline{c}}(\omega) - \lambda^s(\omega))t} \leqslant \lim_{t \to -\infty} K(\omega) \Lambda_{K,a(\omega),\omega}\, \mathrm{e}^{\gamma(\omega)t} = 0$$

and

$$\lim_{t \to +\infty} \alpha^u_{-t,\theta^t\omega} \alpha^c_{t,\omega} = \lim_{t \to +\infty} K(\omega) K(\theta^t \omega)\, \mathrm{e}^{(\lambda^{\overline{c}}(\omega) - \lambda^u(\omega))t} \leqslant \lim_{t \to +\infty} K(\omega) \Lambda_{K,b(\omega),\omega}\, \mathrm{e}^{\gamma(\omega)t} = 0$$

for all $\omega \in \Omega$. Therefore condition (3.5) holds. Let us check (3.6). Indeed, for every $t \geqslant 0$ and every $\omega \in \Omega$ we have

$$\begin{aligned} \frac{1}{\alpha^c_{t,\omega}} \int_0^t \alpha^c_{t-r,\theta^r\omega} \mathrm{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega}\, dr &= \int_0^t K(\theta^r\omega) \mathrm{Lip}(f_{\theta^r\omega})\, dr \\ &\leqslant \delta \int_{-\infty}^{+\infty} G(\theta^r\omega)\, dr \\ &\leqslant \delta, \end{aligned}$$

and, similarly, for every $t \leqslant 0$ and every $\omega \in \Omega$ we have

$$\frac{1}{\alpha^c_{t,\omega}} \int_t^0 \alpha^c_{t-r,\theta^r\omega} \mathrm{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega}\, dr \leqslant \delta.$$

Thus, $\sigma \leqslant \delta$. Moreover, since $K(\omega) \leqslant \mathrm{e}^{\gamma(\omega)|r|} \Lambda_{K,\gamma(\omega),\theta^r\omega}$ for every $\omega \in \Omega$ and $r \in \mathbb{R}$, we have

$$\begin{aligned} \int_{-\infty}^0 \alpha^s_{-r,\theta^r\omega} \mathrm{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega}\, dr &= \int_{-\infty}^0 K(\omega) K(\theta^r\omega)\, \mathrm{e}^{(\lambda^{\underline{c}}(\omega) - \lambda^s(\omega))r} \mathrm{Lip}(f_{\theta^r\omega})\, dr \\ &\leqslant \delta \int_{-\infty}^0 a(\omega)\, \mathrm{e}^{a(\omega)r}\, dr \\ &\leqslant \delta. \end{aligned}$$

and

$$\begin{aligned} \int_0^{+\infty} \alpha^u_{-r,\theta^r\omega} \mathrm{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega}\, dr &= \int_0^{+\infty} K(\omega) K(\theta^r\omega)\, \mathrm{e}^{(\lambda^{\overline{c}}(\omega) - \lambda^u(\omega))r} \mathrm{Lip}(f_{\theta^r\omega})\, dr \\ &\leqslant \delta \int_0^{+\infty} b(\omega)\, \mathrm{e}^{-b(\omega)r}\, dr \\ &\leqslant \delta. \end{aligned}$$

Henceforth, $\sigma + \tau \leqslant 3\delta < 1/2$. □

5.2 ψ-trichotomies

Consider measurable functions

$$\psi^{\bar{c}}, \psi^{\underline{c}}, \psi^{s}, \psi^{u} \colon \mathbb{R} \times \Omega \to]0, +\infty[$$

such that for $\ell \in \{\bar{c}, \underline{c}, s, u\}$ we have

$$\psi^{\ell}(t+s,\omega) = \psi^{\ell}(t,\theta^{s}\omega)\psi^{\ell}(s,\omega) \tag{5.2}$$

for all $t,s \in \mathbb{R}$ and all $\omega \in \Omega$. A *ψ-trichotomy* is a generalized trichotomy with bounds

$$\alpha^{c}_{t,\omega} = \begin{cases} K(\omega)\psi^{\bar{c}}(t,\omega), & t \geqslant 0, \\ K(\omega)\psi^{\underline{c}}(t,\omega), & t \leqslant 0, \end{cases}$$
$$\alpha^{s}_{t,\omega} = K(\omega)\psi^{s}(t,\omega), \quad t \geqslant 0,$$
$$\alpha^{u}_{t,\omega} = K(\omega)\psi^{u}(t,\omega), \quad t \leqslant 0$$

for a random variable $K \colon \Omega \to [1, +\infty[$.

For all $\ell \in \{\underline{c}, \bar{c}, u, s\}$ set

$$d_{\psi^{\ell}}(\omega) = \lim_{h \to 0} \frac{\psi^{\ell}(h,\omega) - 1}{h}. \tag{5.3}$$

Since $\psi^{\ell}(0,\omega) = 1$, from (5.2) we have

$$\frac{d}{dt}\psi^{\ell}(t,\omega) = d_{\psi^{\ell}}(\theta^{t}\omega)\psi^{\ell}(t,\omega)$$

whenever limits (5.3) exist. Moreover, in this situation we also have

$$\frac{d}{dt}\psi^{\ell}(-t,\theta^{t}\omega) = \frac{d}{dt}\frac{1}{\psi^{\ell}(t,\omega)} = -d_{\psi^{\ell}}(\theta^{t}\omega)\psi^{\ell}(-t,\theta^{t}\omega).$$

From now on we also assume that for all $\omega \in \Omega$ the following limit exists:

$$d_{K}(\omega) = \lim_{h \to 0} \frac{K(\theta^{h}\omega) - K(\omega)}{h}. \tag{5.4}$$

We notice that for all $t \in \mathbb{R}$, $\frac{d}{dt}K(\theta^{t}\omega) = d_{K}(\theta^{t}\omega)$.

Corollary 5.2. *Let Φ be a Bochner measurable linear RDS exhibiting a ψ-trichotomy such that the limits in* (5.3) *and* (5.4) *exist and satisfy*

$$d_{\psi^{\bar{c}}}(\omega) - d_{\psi^{u}}(\omega) < \frac{d_{K}(\omega)}{K(\omega)} < d_{\psi^{\underline{c}}}(\omega) - d_{\psi^{s}}(\omega)$$

for all $\omega \in \Omega$. Let $f \in \mathscr{F}^{(B)}_{\alpha}$ be such that

$$\operatorname{Lip}(f_{\omega}) \leqslant \frac{\delta}{K(\omega)} \min\left\{G(\omega), \frac{a(\omega)}{K(\omega)}, \frac{b(\omega)}{K(\omega)}\right\}$$

for all $\omega \in \Omega$, where

$$a(\omega) = \frac{d_{K}(\omega)}{K(\omega)} - d_{\psi^{\bar{c}}}(\omega) + d_{\psi^{u}}(\omega) \quad \text{and} \quad b(\omega) = -\frac{d_{K}(\omega)}{K(\omega)} + d_{\psi^{\underline{c}}}(\omega) - d_{\psi^{s}}(\omega).$$

Assume that Ψ is a Bochner measurable RDS such that (3.3) *has unique solution $\Psi(\cdot, \omega, x)$ for every $\omega \in \Omega$ and every $x \in X$. If, for all $\omega \in \Omega$,*

$$\lim_{t\to-\infty} K(\theta^t\omega)\psi^s(-t,\theta^t\omega)\psi^{\underline{c}}(t,\omega) = \lim_{t\to+\infty} K(\theta^t\omega)\psi^u(-t,\theta^t\omega)\psi^{\overline{c}}(t,\omega) = 0 \tag{5.5}$$

then the same conclusions of Theorem 3.1 hold.

Proof. Conditions in (5.5) are equivalent to those in (3.5), and, as in the proof of Corollary 5.1 we have $\sigma \leqslant \delta$. Moreover, since

$$\begin{aligned}\frac{d}{dt}\left(\frac{\psi^u(-t,\theta^t\omega)\psi^{\overline{c}}(t,\omega)}{K(\theta^t\omega)}\right) &= \frac{\left(-d_{\psi^u}(\theta^t\omega) + d_{\psi^{\overline{c}}}(t,\omega)\right)K(\theta^t\omega) - d_K(\theta^t\omega)}{[K(\theta^t\omega)]^2}\,\psi^u(-t,\theta^t\omega)\,\psi^{\overline{c}}(t,\omega)\\ &= -\frac{a(\theta^t\omega)}{K(\theta^t\omega)}\,\psi^u(-t,\theta^t\omega)\,\psi^{\overline{c}}(t,\omega),\end{aligned}$$

we have

$$\begin{aligned}\int_0^{+\infty}\alpha^u_{-r,\theta^r\omega}\operatorname{Lip}(f_{\theta^r\omega})\alpha^{\overline{c}}_{r,\omega}\,dr &= K(\omega)\int_0^{+\infty}K(\theta^r\omega)\psi^u(-r,\theta^r\omega)\operatorname{Lip}(f_{\theta^r\omega})\psi^{\overline{c}}(r,\omega)\,dr\\ &\leqslant \delta K(\omega)\int_0^{+\infty}\frac{a(\theta^r\omega)}{K(\theta^r\omega)}\,\psi^u(-r,\theta^r\omega)\,\psi^{\overline{c}}(r,\omega)\,dr\\ &= \delta - \delta K(\omega)\lim_{r\to+\infty}\frac{\psi^u(-r,\theta^r\omega)\psi^{\overline{c}}(r,\omega)}{K(\theta^r\omega)}\\ &= \delta.\end{aligned}$$

Similarly, since

$$\begin{aligned}\frac{d}{dt}\left(\frac{\psi^s(-t,\theta^t\omega)\psi^{\underline{c}}(t,\omega)}{K(\theta^t\omega)}\right) &= \frac{\left(-d_{\psi^s}(\theta^t\omega) + d_{\psi^{\underline{c}}}(t,\omega)\right)K(\theta^t\omega) - d_K(\theta^t\omega)}{[K(\theta^t\omega)]^2}\,\psi^s(-t,\theta^t\omega)\,\psi^{\underline{c}}(t,\omega)\\ &= \frac{b(\theta^t\omega)}{K(\theta^t\omega)}\,\psi^s(-t,\theta^t\omega)\,\psi^{\underline{c}}(t,\omega),\end{aligned}$$

we have

$$\begin{aligned}\int_{-\infty}^{0}\alpha^s_{-r,\theta^r\omega}\operatorname{Lip}(f_{\theta^r\omega})\alpha^{\underline{c}}_{r,\omega}\,dr &= K(\omega)\int_{-\infty}^{0}K(\theta^r\omega)\psi^s(-r,\theta^r\omega)\operatorname{Lip}(f_{\theta^r\omega})\psi^{\underline{c}}(r,\omega)\,dr\\ &\leqslant \delta K(\omega)\int_{-\infty}^{0}\frac{b(\theta^r\omega)}{K(\theta^r\omega)}\,\psi^s(-r,\theta^r\omega)\,\psi^{\underline{c}}(r,\omega)\,dr\\ &= \delta - \delta K(\omega)\lim_{r\to-\infty}\frac{\psi^s(-r,\theta^r\omega)\psi^{\underline{c}}(r,\omega)}{K(\theta^r\omega)}\\ &= \delta.\end{aligned}$$

Thus $\sigma + \tau \leqslant 3\delta < 1/2$. □

In the following we provide a particular example of a ψ-trichotomy in $\mathbb{R}^4$.

Example 5.3. Let $\psi^{\overline{c}}, \psi^{\underline{c}}, \psi^s, \psi^u \colon \mathbb{R}\times\Omega \to]0,+\infty[$ be measurable functions satisfying (5.2) and let $K\colon \Omega \to [1,+\infty[$ be a random variable. In $X = \mathbb{R}^4$, equipped with the maximum norm, consider the projections

$$\begin{aligned}P^{\overline{c}}_\omega(x_1,x_2,x_3,x_4) &= (0,0,x_3+(K(\omega)-1)x_4,0)\\ P^{\underline{c}}_\omega(x_1,x_2,x_3,x_4) &= ((1-K(\omega))x_2,x_2,0,0)\\ P^s_\omega(x_1,x_2,x_3,x_4) &= (x_1+(K(\omega)-1)x_2,0,0,0)\\ P^u_\omega(x_1,x_2,x_3,x_4) &= (0,0,(1-K(\omega))x_4,x_4)\end{aligned}$$

For all $\omega', \omega \in \Omega$,

$$\begin{aligned} P^{\bar{c}}_{\omega'} P^u_\omega &= (0, 0, (K(\omega') - K(\omega))x_4, 0), \\ P^s_{\omega'} P^{\underline{c}}_\omega &= ((K(\omega') - K(\omega))x_2, 0, 0, 0) \end{aligned}$$

and for all the remaining $i, j \in \{\bar{c}, \underline{c}, s, u\}$, with $i \neq j$,

$$P^i_{\omega'} P^j_\omega = 0.$$

Notice that for all $\omega, \omega' \in \Omega$

$$P^s_{\omega'} P^s_\omega = P^s_{\omega}, \quad P^u_{\omega'} P^u_\omega = P^u_{\omega'}, \quad P^{\bar{c}}_{\omega'} P^{\bar{c}}_\omega = P^{\bar{c}}_\omega \quad \text{and} \quad P^{\underline{c}}_{\omega'} P^{\underline{c}}_\omega = P^{\underline{c}}_{\omega'}.$$

Moreover,

$$\|P^{\bar{c}}_\omega\| = \|P^s_\omega\| = K(\omega)$$

and

$$\|P^{\underline{c}}_\omega\| = \|P^u_\omega\| = \max\{K(\omega) - 1, 1\} \leqslant K(\omega).$$

We define $\Phi \colon \mathbb{R} \times \Omega \times \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\Phi^t_\omega = \psi^{\bar{c}}(t,\omega)\, P^{\bar{c}}_\omega + \frac{K(\omega)}{K(\theta^t\omega)} \psi^{\underline{c}}(t,\omega)\, P^{\underline{c}}_{\theta^t\omega} + \psi^s(t,\omega) P^s_\omega + \frac{K(\omega)}{K(\theta^t\omega)} \psi^u(t,\omega)\, P^u_{\theta^t\omega}.$$

Let $P^c = P^{\bar{c}} + P^{\underline{c}}$ and $\mathcal{P} = (P^c, P^s, P^u)$. We have that Φ is a measurable linear RDS over Σ that admits a measurable $\mathcal{P}$-invariant splitting, and

$$\begin{aligned} \|\Phi^{c,t}_\omega\| &= \max\left\{\psi^{\bar{c}}(t,\omega)\|P^{\bar{c}}_\omega\|, \frac{1}{K(\theta^t\omega)}\psi^{\underline{c}}(t,\omega)\|P^{\underline{c}}_{\theta^t\omega}\|\right\} \leqslant K(\omega)\max\left\{\psi^{\bar{c}}(t,\omega), \psi^{\underline{c}}(t,\omega)\right\}, \\ \|\Phi^{s,t}_\omega\| &= \psi^s(t,\omega)\|P^s_\omega\| = K(\omega)\psi^s(t,\omega), \\ \|\Phi^{u,t}_\omega\| &= \frac{K(\omega)}{K(\theta^t\omega)}\psi^u(t,\omega)\|P^u_{\theta^t\omega}\| \leqslant K(\omega)\psi^u(t,\omega). \end{aligned}$$

Hence the linear RDS Φ exhibits a generalized trichotomy with bounds

$$\begin{aligned} \alpha^c_{t,\omega} &= K(\omega)\max\left\{\psi^{\bar{c}}(t,\omega), \psi^{\underline{c}}(t,\omega)\right\}, \\ \alpha^s_{t,\omega} &= K(\omega)\psi^s(t,\omega), \\ \alpha^u_{t,\omega} &= K(\omega)\psi^u(t,\omega). \end{aligned}$$

If we assume $\psi^{\bar{c}}(t,\omega) \geqslant \psi^{\underline{c}}(t,\omega)$ for all $t \geqslant 0$ then

$$\alpha^c_{t,\omega} = \begin{cases} K(\omega)\psi^{\bar{c}}(t,\omega) & \text{if } t \geqslant 0, \\ K(\omega)\psi^{\underline{c}}(t,\omega) & \text{if } t \leqslant 0, \end{cases}$$

and Φ exhibits a ψ-trichotomy.

Next, based on the previous example, we provide an example of a ψ-trichotomy on an infinite dimensional Banach space.

Example 5.4. Let $K_n\colon \Omega \to [1,+\infty[$ be a sequence of random variables such that

$$K(\omega) := \sup_{n\in\mathbb{N}} K_n(\omega) < +\infty \text{ for all } \omega \in \Omega.$$

In ℓ_∞, the space of bounded sequences equipped with the supremum norm, consider, for all $n \in \mathbb{N}$ and for all $\omega \in \Omega$, the projections

$$P^{\bar{c}}_{n,\omega}, P^{\underline{c}}_{n,\omega}, P^{s}_{n,\omega}, P^{u}_{n,\omega}\colon \ell_\infty \to \ell_\infty$$

defined by

$$\begin{aligned}
P^{\bar{c}}_{n,\omega}(x_1,x_2,x_3,x_4,x_5,\dots) &= (\underbrace{0,\dots,0}_{4n-4\text{ zeros}},0,0,x_{4n-1}+L_n(\omega)x_{4n},0,0,0,\dots),\\
P^{\underline{c}}_{n,\omega}(x_1,x_2,x_3,x_4,x_5,\dots) &= (\underbrace{0,\dots,0}_{4n-4\text{ zeros}},-L_n(\omega)x_{4n-2},x_{4n-2},0,0,0,\dots),\\
P^{s}_{n,\omega}(x_1,x_2,x_3,x_4,x_5,\dots) &= (\underbrace{0,\dots,0}_{4n-4\text{ zeros}},x_{4n-3}+L_n(\omega)x_{4n-2},0,0,0,\dots),\\
P^{u}_{n,\omega}(x_1,x_2,x_3,x_4,x_5,\dots) &= (\underbrace{0,\dots,0}_{4n-4\text{ zeros}},0,0,-L_n(\omega)x_{4n},x_{4n},0,0,0,\dots),
\end{aligned}$$

where $L_n(\omega) = K_n(\omega) - 1$. It follows that for all $\omega',\omega \in \Omega$, for all $i,j \in \{\bar{c},\underline{c},s,u\}$ and for all $n,m \in \mathbb{N}$, we have

$$\begin{aligned}
P^{i}_{n,\omega}P^{j}_{m,\omega'} &= 0 \ \text{ with } \ n \neq m,\\
P^{i}_{n,\omega'}P^{j}_{n,\omega} &= 0 \ \text{ with } \ i \neq j,\ (i,j) \neq (\bar{c},u) \ \text{ and } \ (i,j) \neq (s,\underline{c}),\\
P^{\bar{c}}_{n,\omega'}P^{u}_{n,\omega} &= (\underbrace{0,\dots,0}_{4n-4\text{ zeros}},0,0,(K_n(\omega')-K_n(\omega))x_{4n},0,0,0,\dots),\\
P^{s}_{n,\omega'}P^{\underline{c}}_{n,\omega} &= (\underbrace{0,\dots,0}_{4n-4\text{ zeros}},(K_n(\omega')-K_n(\omega))x_{4n-2},0,0,0,\dots).
\end{aligned}$$

Moreover, for all $\omega,\omega' \in \Omega$ and for all $n \in \mathbb{N}$,

$$P^{i}_{n,\omega'}P^{i}_{n,\omega} = P^{i}_{n,\omega} \ \text{ and } \ P^{j}_{n,\omega'}P^{j}_{n,\omega} = P^{j}_{n,\omega'} \text{ with } i \in \{\bar{c},s\} \ \text{ and } j \in \{\underline{c},u\}.$$

Let $\psi^{\bar{c}}_n, \psi^{\underline{c}}_n, \psi^{s}_n, \psi^{u}_n\colon \mathbb{R}\times\Omega \to]0,+\infty[$ be sequences of measurable functions satisfying (5.2) and such that

$$\psi^{i}(t,\omega) := \sup_{n\in\mathbb{N}} \psi^{i}_n(t,\omega) < +\infty$$

for all $\omega \in \Omega$, for all $t \in \mathbb{R}$ and for all $i \in \{\bar{c},\underline{c},s,u\}$. If $\Phi\colon \mathbb{R}\times\Omega\times\ell_\infty \to \ell_\infty$ is given by

$$\Phi^t_\omega = \sum_{n=1}^{+\infty}\left[\psi^{\bar{c}}_n(t,\omega)\,P^{\bar{c}}_{n,\omega} + \frac{K_n(\omega)}{K_n(\theta^t\omega)}\psi^{\underline{c}}_n(t,\omega)\,P^{\underline{c}}_{n,\theta^t\omega} + \psi^{s}_n(t,\omega)P^{s}_{n,\omega} + \frac{K_n(\omega)}{K_n(\theta^t\omega)}\psi^{u}_n(t,\omega)\,P^{u}_{n,\theta^t\omega}\right],$$

then Φ is a measurable linear RDS over Σ that admits a measurable (P^c,P^s,P^u)-invariant splitting, where, for all $\omega \in \Omega$,

$$P^{i}_{\omega} = \sum_{n=1}^{+\infty} P^{i}_{n,\omega}, \quad i \in \{\bar{c},\underline{c},s,u\}, \quad \text{and} \quad P^{c}_{\omega} = P^{\bar{c}}_{\omega} + P^{\underline{c}}_{\omega}.$$

Moreover, since

$$\|P^{\bar{c}}_{n,\omega}\| = \|P^s_{n,\omega}\| = K_n(\omega)$$

and

$$\|P^{\underline{c}}_{n,\omega}\| = \|P^u_{n,\omega}\| = \max\{K_n(\omega)-1, 1\},$$

it follows that

$$\left\|P^{\bar{c}}_{\omega}\right\| = \sup_{n\in\mathbb{N}} \left\|P^s_{n,\omega}\right\| = \sup_{n\in\mathbb{N}} K_n(\omega) = K(\omega)$$

and

$$\left\|P^{\underline{c}}_{\omega}\right\| = \left\|P^u_{n,\omega}\right\| = \sup_{n\in\mathbb{N}}(\max\{K_n(\omega)-1, 1\}) = \max\{K(\omega)-1,1\} \leqslant K(\omega).$$

Hence

$$\begin{aligned}
\|\Phi^{c,t}_{\omega}\| &= \sup_{n\in\mathbb{N}}\left[\max\left\{\psi^{\bar{c}}_n(t,\omega)\|P^{\bar{c}}_{n,\omega}\|, \frac{1}{K_n(\theta^t\omega)}\psi^{\underline{c}}_n(t,\omega)\|P^{\underline{c}}_{n,\theta^t\omega}\|\right\}\right]\\
&\leqslant \sup_{n\in\mathbb{N}}\left[K_n(\omega)\max\left\{\psi^{\bar{c}}_n(t,\omega),\psi^{\underline{c}}_n(t,\omega)\right\}\right]\\
&\leqslant K(\omega)\max\left\{\psi^{\bar{c}}(t,\omega),\psi^{\underline{c}}(t,\omega)\right\},\\
\|\Phi^{s,t}_{\omega}\| &= \sup_{n\in\mathbb{N}}\left[\psi^s_n(t,\omega)\|P^s_{n,\omega}\|\right] = \sup_{n\in\mathbb{N}}\left[\psi^s_n(t,\omega)K_n(\omega)\right] \leqslant K(\omega)\psi^s(t,\omega),\\
\|\Phi^{u,t}_{\omega}\| &= \sup_{n\in\mathbb{N}}\left[\frac{K_n(\omega)}{K_n(\theta^t\omega)}\psi^u_n(t,\omega)\|P^u_{n,\theta^t\omega}\|\right] \leqslant K(\omega)\psi^u(t,\omega).
\end{aligned}$$

This implies that the linear RDS Φ admits a generalized trichotomy with bounds

$$\begin{aligned}
\alpha^c_{t,\omega} &= K(\omega)\max\left\{\psi^{\bar{c}}(t,\omega),\psi^{\underline{c}}(t,\omega)\right\},\\
\alpha^s_{t,\omega} &= K(\omega)\psi^s(t,\omega),\\
\alpha^u_{t,\omega} &= K(\omega)\psi^u(t,\omega).
\end{aligned}$$

If, in addition, $\psi^{\bar{c}}, \psi^{\underline{c}}, \psi^s, \psi^u$ satisfy (5.2) and $\psi^{\bar{c}}(t,\omega) \geqslant \psi^{\underline{c}}(t,\omega)$, for all $t \geqslant 0$ and for all $\omega\in\Omega$, then

$$\alpha^c_{t,\omega} = \begin{cases} K(\omega)\psi^{\bar{c}}(t,\omega) & \text{if } t \geqslant 0,\\ K(\omega)\psi^{\underline{c}}(t,\omega) & \text{if } t \leqslant 0,\end{cases}$$

and Φ exhibits a ψ-trichotomy.

In the next sections we consider particular ψ-trichotomies.

5.2.1 Integral exponential trichotomy

Let

$$\lambda^{\bar{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}$$

be random variables such that for all $\omega\in\Omega$ and $\ell \in \{\bar{c}, \underline{c}, s, u\}$, the map $r \mapsto \lambda^{\ell}(\theta^r\omega)$ is integrable in every interval $[0,t]$ An *integral exponential trichotomy* is a ψ-trichotomy with

$$\psi^{\ell}(t,\omega) = \mathrm{e}^{\int_0^t \lambda^{\ell}(\theta^r\omega)\,dr}$$

for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$, i.e., is a generalized trichotomy with bounds

$$\alpha^c_{t,\omega} = \begin{cases} K(\omega)\, e^{\int_0^t \lambda^{\bar{c}}(\theta^r\omega)dr}, & t \geqslant 0, \\ K(\omega)\, e^{\int_0^t \lambda^{\underline{c}}(\theta^r\omega)dr}, & t \leqslant 0, \end{cases}$$
$$\alpha^s_{t,\omega} = K(\omega)\, e^{\int_0^t \lambda^{s}(\theta^r\omega)dr}, \quad t \geqslant 0,$$
$$\alpha^u_{t,\omega} = K(\omega)\, e^{\int_0^t \lambda^{u}(\theta^r\omega)dr}, \quad t \leqslant 0.$$

Notice that if

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \lambda^\ell(\theta^r\omega)\, dr = \lambda^\ell(\omega) \tag{5.6}$$

then $d_{\psi^\ell}(\omega) = \lambda^\ell(\omega)$ for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$. From Corollary 5.2 we get the following.

Corollary 5.5. *Let Φ be a Bochner measurable linear RDS exhibiting an integral exponential trichotomy such that (5.6) holds and the limit (5.4) exists and satisfyies*

$$\lambda^{\bar{c}}(\omega) - \lambda^u(\omega) < \frac{d_K(\omega)}{K(\omega)} < \lambda^{\underline{c}}(\omega) - \lambda^s(\omega)$$

for all $\omega \in \Omega$. Let $f \in \mathscr{F}^{(B)}_\alpha$ be such that

$$\mathrm{Lip}(f_\omega) \leqslant \frac{\delta}{K(\omega)} \min\left\{ G(\omega), \frac{a(\omega)}{K(\omega)}, \frac{b(\omega)}{K(\omega)} \right\}$$

for all $\omega \in \Omega$, where

$$a(\omega) = \frac{d_K(\omega)}{K(\omega)} - \lambda^{\bar{c}}(\omega) + \lambda^u(\omega) \quad \textit{and} \quad b(\omega) = -\frac{d_K(\omega)}{K(\omega)} + \lambda^{\underline{c}}(\omega) - \lambda^s(\omega).$$

Assume that Ψ is a Bochner measurable RDS such that (3.3) has a unique solution $\Psi(\cdot, \omega, x)$ for every $\omega \in \Omega$ and every $x \in X$. If for all $\omega \in \Omega$,

$$\lim_{t\to+\infty} K(\theta^t\omega)\, e^{\int_0^t \lambda^{\bar{c}}(\theta^r\omega) - \lambda^u(\theta^r\omega)\, dr} = \lim_{t\to-\infty} K(\theta^t\omega)\, e^{\int_0^t \lambda^{\underline{c}}(\theta^r\omega) - \lambda^s(\theta^r\omega)\, dr} = 0,$$

then the same conclusions of Theorem 3.1 hold.

5.2.2 Non exponential trichotomies

We provide now a particular type of ψ-trichotomies that can be easily handled to construct trichotomies beyhond the exponential bounds. Let

$$\lambda^{\bar{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}$$

be random variables such that for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$ the following limit exists for all ω:

$$d_{\lambda^\ell}(\omega) := \lim_{h\to 0} \frac{\lambda^\ell(\theta^h\omega) - \lambda^\ell(\omega)}{h}. \tag{5.7}$$

Consider a ψ-trichotomy with

$$\psi^\ell(t,\omega) = \frac{\lambda^\ell(\omega)}{\lambda^\ell(\theta^t\omega)}$$

for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$, i.e., is a generalized trichotomy with bounds

$$\begin{aligned} \alpha^c_{t,\omega} &= \begin{cases} K(\omega)\dfrac{\lambda^{\bar{c}}(\omega)}{\lambda^{\bar{c}}(\theta^t\omega)}, & t \geqslant 0, \\ K(\omega)\dfrac{\lambda^{\underline{c}}(\omega)}{\lambda^{\underline{c}}(\theta^t\omega)}, & t \leqslant 0, \end{cases} \\ \alpha^s_{t,\omega} &= K(\omega)\frac{\lambda^s(\omega)}{\lambda^s(\theta^t\omega)}, \quad t \geqslant 0 \\ \alpha^u_{t,\omega} &= K(\omega)\frac{\lambda^u(\omega)}{\lambda^u(\theta^t\omega)}, \quad t \leqslant 0. \end{aligned} \tag{5.8}$$

Notice that

$$d_{\psi^\ell}(\omega) = -\frac{d_{\lambda^\ell}(\omega)}{\lambda^\ell(\omega)}.$$

for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$. From Corollary 5.2 we get the following.

Corollary 5.6. *Let Φ be a Bochner measurable linear RDS exhibiting an α-trichotomy, with bounds (5.8) and such that (5.7) and (5.4) exist and satisfy*

$$\frac{d_{\lambda^u}(\omega)}{\lambda^u(\omega)} - \frac{d_{\lambda^{\bar{c}}}(\omega)}{\lambda^{\bar{c}}(\omega)} < \frac{d_K(\omega)}{K(\omega)} < \frac{d_{\lambda^s}(\omega)}{\lambda^s(\omega)} - \frac{d_{\lambda^{\underline{c}}}(\omega)}{\lambda^{\underline{c}}(\omega)}.$$

Let $f \in \mathscr{F}^{(B)}_\alpha$ be such that for all $\omega \in \Omega$ we have

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\omega)} \min\left\{G(\omega), \frac{a(\omega)}{K(\omega)}, \frac{b(\omega)}{K(\omega)}\right\},$$

where

$$a(\omega) = \frac{d_K(\omega)}{K(\omega)} + \frac{d_{\lambda^{\bar{c}}}(\omega)}{\lambda^{\bar{c}}(\omega)} - \frac{d_{\lambda^u}(\omega)}{\lambda^u(\omega)} \quad \text{and} \quad b(\omega) = -\frac{d_K(\omega)}{K(\omega)} - \frac{d_{\lambda^{\underline{c}}}(\omega)}{\lambda^{\underline{c}}(\omega)} + \frac{d_{\lambda^s}(\omega)}{\lambda^s(\omega)}.$$

Assume that Ψ is a Bochner measurable RDS such that (3.3) has a unique solution $\Psi(\cdot,\omega,x)$ for every $\omega \in \Omega$ and every $x \in X$. If for all $\omega \in \Omega$,

$$\lim_{t\to+\infty} K(\theta^t\omega)\frac{\lambda^s(\theta^t\omega)}{\lambda^{\underline{c}}(\theta^t\omega)} = \lim_{t\to+\infty} K(\theta^t\omega)\frac{\lambda^u(\theta^t\omega)}{\lambda^{\bar{c}}(\theta^t\omega)} = 0$$

then the same conclusions of Theorem 3.1 hold.

Example 5.7 (Non exponential trichotomy)**.** Consider for the driving system the horizontal flow in $\mathbb{R}^2$ given by $\theta^t(x,y) = (x+t,y)$, which preserves the Lebesgue measure. Let $C, \xi^{\bar{c}}, \xi^{\underline{c}}, \xi^s, \xi^u$ and ε be some real constants with $C \geqslant 1$ and $\varepsilon \geqslant 0$, and set:

$$\begin{aligned} \lambda^\ell(x,y) &= (1+x^2)^{-(1+y^2)\xi_\ell}, \quad \ell \in \{\bar{c}, \underline{c}, s, u\}, \\ K(x,y) &= C(1+x^2)^{(1+y^2)\varepsilon}. \end{aligned}$$

In this case we obtain a polynomial type trichotomy. Let us assume $\lambda_{\overline{c}} \geqslant \lambda_{\underline{c}}$. Thus we have a trichotomy with

$$\alpha^c_{t,(x,y)} = \begin{cases} C\left(\dfrac{1+(x+t)^2}{1+x^2}\right)^{(1+y^2)\xi^{\overline{c}}} (1+x^2)^{(1+y^2)\varepsilon}, & t \geqslant 0, \\ C\left(\dfrac{1+(x+t)^2}{1+x^2}\right)^{(1+y^2)\xi^{\underline{c}}} (1+x^2)^{(1+y^2)\varepsilon}, & t \leqslant 0, \end{cases}$$

$$\alpha^s_{t,(x,y)} = C\left(\frac{1+(x+t)^2}{1+x^2}\right)^{(1+y^2)\xi^s} (1+x^2)^{(1+y^2)\varepsilon}, \quad t \geqslant 0,$$

$$\alpha^u_{t,(x,y)} = C\left(\frac{1+(x+t)^2}{1+x^2}\right)^{(1+y^2)\xi^u} (1+x^2)^{(1+y^2)\varepsilon}, \quad t \leqslant 0.$$

Notice that $d_{\lambda^\ell}(x,y) = \frac{\partial}{\partial x}\lambda^\ell(x,y)$.

6 Discrete-time examples

In this section we assume $\mathbb{T} = \mathbb{Z}$ and provide some corollaries to Theorem 4.1. Let X be a Banach space and let $\Sigma \equiv (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a measure-preserving dynamical system. Throughout this subsection we consider a real number $\delta \in]0, 1/6[$ and a random variable $G\colon \Omega \to]0, +\infty[$ such that for all $\omega \in \Omega$ we have

$$\sum_{k=-\infty}^{+\infty} G(\theta^k \omega) \leqslant 1.$$

6.1 Tempered exponential trichotomies

Consider θ-invariant random variables

$$\lambda^{\overline{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}.$$

We say that a measurable linear RDS Φ on X over Σ exhibits an *exponential trichotomy* if it admits a generalized trichotomy with bounds

$$\alpha^c_{n,\omega} = \begin{cases} K(\omega)\, \mathrm{e}^{\lambda^{\overline{c}}(\omega)n}, & n \geqslant 0, \\ K(\omega)\, \mathrm{e}^{\lambda^{\underline{c}}(\omega)n}, & n \leqslant 0, \end{cases}$$
$$\alpha^s_{n,\omega} = K(\omega)\, \mathrm{e}^{\lambda^s(\omega)n}, \quad n \geqslant 0,$$
$$\alpha^u_{n,\omega} = K(\omega)\, \mathrm{e}^{\lambda^u(\omega)n}, \quad n \leqslant 0$$

for some random variable $K : \Omega \to [1, +\infty[$. If the random variable K is *tempered* we say that Φ exhibits an *tempered exponential trichotomy*. Notice that in the discrete-time case the condition (5.1) is equivalent to

$$\lim_{n \to \pm\infty} \frac{1}{|n|} \log K(\theta^n \omega) = 0 \quad \text{for all } \omega \in \Omega.$$

Corollary 6.1. *Let Φ be a measurable linear RDS exhibiting a tempered exponential trichotomy such that, for all $\omega \in \Omega$, satisfies*

$$\lambda^{\underline{c}}(\omega) > \lambda^s(\omega) \quad \textit{and} \quad \lambda^{\overline{c}}(\omega) < \lambda^u(\omega)$$

and let $f \in \mathscr{F}$. Consider a θ-invariant random variable $\gamma(\omega) > 0$ satisfying for all $\omega \in \Omega$

$$a(\omega) := \lambda^{\underline{c}}(\omega) - \lambda^{s}(\omega) - \gamma(\omega) > 0 \quad \text{and} \quad b(\omega) := \lambda^{u}(\omega) - \lambda^{\overline{c}}(\omega) - \gamma(\omega) > 0.$$

If

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\theta\omega)} \min\left\{ \mathrm{e}^{\min\{\lambda^{\underline{c}}(\omega),\lambda^{\overline{c}}(\omega)\}} G(\omega), \mathrm{e}^{\lambda^u(\omega)} \frac{\mathrm{e}^{a(\omega)}-1}{\Lambda_{K,\gamma(\omega),\omega}}, \mathrm{e}^{\lambda^s(\omega)} \frac{1-\mathrm{e}^{-b(\omega)}}{\Lambda_{K,\gamma(\omega),\omega}} \right\}$$

for all $\omega \in \Omega$ then the same conclusions of Theorem 3.1 hold.

6.2 ψ-trichotomies

Consider measurable functions

$$\psi^{\overline{c}}, \psi^{\underline{c}}, \psi^{s}, \psi^{u} \colon \mathbb{Z} \times \Omega \to]0, +\infty[$$

such that for $\ell \in \{\overline{c}, \underline{c}, s, u\}$ we have

$$\psi^{\ell}(t+s,\omega) = \psi^{\ell}(t,\theta^s\omega)\psi^{\ell}(s,\omega)$$

for all $t, s \in \mathbb{Z}$ and all $\omega \in \Omega$. A *ψ-trichotomy* is a generalized trichotomy with bounds

$$\begin{aligned} \alpha^c_{n,\omega} &= \begin{cases} K(\omega)\psi^{\overline{c}}(n,\omega), & t \geqslant 0, \\ K(\omega)\psi^{\underline{c}}(n,\omega), & t \leqslant 0, \end{cases} \\ \alpha^s_{t,\omega} &= K(\omega)\psi^s(n,\omega), \quad t \geqslant 0, \\ \alpha^u_{t,\omega} &= K(\omega)\psi^u(n,\omega), \quad t \leqslant 0 \end{aligned}$$

where $K\colon \Omega \to [1, +\infty[$ is a random variable. We notice that, as in the continuous-time case, we may consider different growth rates along the central directions E^c_ω, depending if we are looking to the *future* ($n \to +\infty$) or to the *past* ($n \to -\infty$).

Corollary 6.2. *Let Φ be a measurable linear RDS exhibiting a ψ-trichotomy such that*

$$\frac{\psi^{\overline{c}}(1,\omega)}{\psi^u(1,\omega)} < \frac{K(\theta\omega)}{K(\omega)} < \frac{\psi^{\underline{c}}(1,\omega)}{\psi^s(1,\omega)}. \tag{6.1}$$

Let $f \in \mathscr{F}$ be such that

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\theta\omega)} \min\left\{\psi^{\overline{c}}(1,\omega)G(\omega), \psi^{\underline{c}}(1,\omega)G(\omega), a(\omega), b(\omega)\right\},$$

where

$$a(\omega) = \frac{\psi^u(1,\omega)}{K(\omega)} - \frac{\psi^{\overline{c}}(1,\omega)}{K(\theta\omega)} \quad \text{and} \quad b(\omega) = \frac{\psi^{\underline{c}}(1,\omega)}{K(\theta\omega)} - \frac{\psi^s(1,\omega)}{K(\omega)}.$$

If

$$\lim_{n\to-\infty} K(\theta^n\omega)\psi^s(-n,\theta^n\omega)\psi^{\underline{c}}(n,\omega) = \lim_{n\to+\infty} K(\theta^n\omega)\psi^u(-n,\theta^n\omega)\psi^{\overline{c}}(n,\omega) = 0$$

for all $\omega \in \Omega$, then the same conclusion of Theorem 4.1 holds.

Proof. We will check that we are in conditions to apply Theorem 4.1. Notice that from (6.1) we conclude $a(\omega), b(\omega) > 0$. We have

$$\begin{aligned}\sigma_\omega^- &= \sup_{n\in\mathbb{N}} \frac{1}{\psi^{\underline{c}}(-n,\omega)} \sum_{k=-n}^{-1} K(\theta^{k+1}\omega)\psi^{\underline{c}}(-n-k-1,\theta^{k+1}\omega)\operatorname{Lip}(f_{\theta^k\omega})\psi^{\overline{c}}(k,\omega)\\ &\leqslant \delta \sum_{k=-\infty}^{+\infty} G(\theta^k\omega) \leqslant \delta,\end{aligned}$$

and, similarly, $\sigma_\omega^+ \leqslant \delta$. Thus $\sigma \leqslant \delta$. Moreover,

$$\begin{aligned}\tau_\omega^+ &= \sum_{k=0}^{+\infty} K(\theta^{k+1}\omega)\psi^u(-k-1,\theta^{k+1}\omega)\operatorname{Lip}(f_{\theta^k\omega})K(\omega)\psi^{\overline{c}}(k,\omega)\\ &\leqslant \delta K(\omega)\sum_{k=0}^{+\infty}\left[\frac{\psi^u(-k,\theta^k\omega)\psi^{\overline{c}}(k,\omega))}{K(\theta^k\omega)} - \frac{\psi^u(-(k+1),\theta^{k+1}\omega)\psi^{\overline{c}}(k+1,\omega))}{K(\theta^{k+1}\omega)}\right]\\ &\leqslant \delta K(\omega)\left(\frac{1}{K(\omega)} - \lim_{k\to+\infty}\frac{\psi^u(-k,\theta^k\omega)\psi^{\overline{c}}(k,\omega))}{K(\theta^k\omega)}\right)\\ &= \delta.\end{aligned}$$

Similarly we get $\tau_\omega^- \leqslant \delta$. Therefore $\sigma + \tau \leqslant 3\delta < 1/2$. □

In the following we consider particular ψ-trichotomies.

6.2.1 Summable exponential trichotomies

We start by considering the integral (or summable) exponential trichotomies, which are a generalization of the exponential trichotomies and can be seen as the discrete counterpart of those in Section 5.2.1. Let

$$\lambda^{\overline{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}$$

be random variables and set For all $\ell \in \{\overline{c}, \underline{c}, s, u\}$ we set

$$S^\ell(n,\omega) = \begin{cases}\lambda^\ell(\omega) + \cdots + \lambda^\ell(\theta^{n-1}\omega), & n \geqslant 1,\\ 0, & n = 0,\\ -\lambda^\ell(\theta^n\omega) - \cdots - \lambda^\ell(\theta^{-1}\omega), & n \leqslant -1.\end{cases}$$

A *summable exponential trichotomy* is a ψ-trichotomy with

$$\psi^\ell(t,\omega) = \mathrm{e}^{S^\ell(n,\omega)}$$

for all $\ell \in \{\overline{c}, \underline{c}, s, u\}$, i.e., is a generalized trichotomy with bounds

$$\begin{aligned}\alpha^c_{n,\omega} &= \begin{cases} K(\omega)\,\mathrm{e}^{S^{\overline{c}}(n,\omega)}, & n \geqslant 0,\\ K(\omega)\,\mathrm{e}^{S^{\underline{c}}(n,\omega)}, & n \leqslant 0,\end{cases}\\ \alpha^s_{n,\omega} &= K(\omega)\,\mathrm{e}^{S^s(n,\omega)}, \quad n \geqslant 0,\\ \alpha^u_{n,\omega} &= K(\omega)\,\mathrm{e}^{S^u(n,\omega)}, \quad n \leqslant 0\end{aligned}$$

for some tempered random variable $K : \Omega \to [1, +\infty[$.

Corollary 6.3. *Let Φ be a measurable linear RDS exhibiting a summable exponential trichotomy such that*

$$\frac{e^{\lambda^{\overline{c}}(\omega)}}{e^{\lambda^u(\omega)}} < \frac{K(\theta\omega)}{K(\omega)} < \frac{e^{\lambda^{\underline{c}}(\omega)}}{e^{\lambda^s(\omega)}}.$$

Let $f \in \mathscr{F}$ be such that

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\theta\omega)} \min\left\{e^{\lambda^{\overline{c}}(\omega)}\, G(\omega), e^{\lambda^{\underline{c}}(\omega)}\, G(\omega), a(\omega), b(\omega)\right\},$$

where

$$a(\omega) = \frac{e^{\lambda^u(\omega)}}{K(\omega)} - \frac{e^{\lambda^{\overline{c}}(\omega)}}{K(\theta\omega)} \quad \text{and} \quad b(\omega) = \frac{e^{\lambda^{\underline{c}}(\omega)}}{K(\theta\omega)} - \frac{e^{\lambda^s(\omega)}}{K(\omega)}.$$

If

$$\lim_{n\to-\infty} K(\theta^n\omega)\, e^{S^s(-n,\theta^n\omega)+S^{\underline{c}}(n,\omega)} = \lim_{n\to+\infty} K(\theta^n\omega)\, e^{S^u(-n,\theta^n\omega)+S^{\overline{c}}(n,\omega)} = 0$$

for all $\omega \in \Omega$, then the same conclusion of Theorem 4.1 holds.

6.2.2 Non exponential trichotomies

We provide a particular type of ψ-trichotomies that can be easily handled to construct trichotomies beyhond the exponential bounds in the discrete-time scenario. Consider a ψ-trichotomy with

$$\psi^\ell(n,\omega) = \frac{\lambda^\ell(\omega)}{\lambda^\ell(\theta^n\omega)}$$

for all $\ell \in \{\overline{c}, \underline{c}, s, u\}$, *i.e.*, is a generalized trichotomy with bounds

$$\begin{aligned}
\alpha^c_{n,\omega} &= \begin{cases} K(\omega)\dfrac{\lambda^{\overline{c}}(\omega)}{\lambda^{\overline{c}}(\theta^n\omega)}, & n \geqslant 0, \\[2ex] K(\omega)\dfrac{\lambda^{\underline{c}}(\omega)}{\lambda^{\underline{c}}(\theta^n\omega)}, & n \leqslant 0, \end{cases} \\
\alpha^s_{n,\omega} &= K(\omega)\frac{\lambda^s(\omega)}{\lambda^s(\theta^n\omega)}, \quad n \geqslant 0, \\
\alpha^u_{n,\omega} &= K(\omega)\frac{\lambda^u(\omega)}{\lambda^u(\theta^n\omega)}, \quad n \leqslant 0.
\end{aligned} \tag{6.2}$$

For future use let us define

$$a(\omega) = \frac{\lambda^u(\omega)}{\lambda^u(\theta\omega)K(\omega)} - \frac{\lambda^{\overline{c}}(\omega)}{\lambda^{\overline{c}}(\theta\omega)K(\theta\omega)} \quad \text{and} \quad b(\omega) = \frac{\lambda^{\underline{c}}(\omega)}{\lambda^{\underline{c}}(\theta\omega)K(\theta\omega)} - \frac{\lambda^s(\omega)}{\lambda^s(\theta\omega)K(\omega)}.$$

Corollary 6.4. *Let Φ be a measurable linear RDS exhibiting an α-trichotomy with bounds (6.2) and such that*

$$\frac{\lambda^{\overline{c}}(\omega)\lambda^u(\theta\omega)}{\lambda^{\overline{c}}(\theta\omega)\lambda^u(\omega)} < \frac{K(\theta\omega)}{K(\omega)} < \frac{\lambda^{\underline{c}}(\omega)\lambda^s(\theta\omega)}{\lambda^{\underline{c}}(\theta\omega)\lambda^s(\omega)}.$$

Let $f \in \mathscr{F}$ be such that

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\theta\omega)} \min\left\{\frac{\lambda^{\overline{c}}(\omega)}{\lambda^{\overline{c}}(\theta\omega)}G(\omega), \frac{\lambda^{\underline{c}}(\omega)}{\lambda^{\underline{c}}(\theta\omega)}G(\omega), a(\omega), b(\omega)\right\}.$$

If

$$\lim_{n\to-\infty} \frac{K(\theta^n\omega)\lambda^s(\theta^n\omega)}{\lambda^{\underline{c}}(\theta^n\omega)} = \lim_{n\to+\infty} \frac{K(\theta^n\omega)\lambda^u(\theta^n\omega)}{\lambda^{\overline{c}}(\theta^n\omega)} = 0$$

for all $\omega \in \Omega$, then the same conclusion of Theorem 4.1 holds.

We may consider Example 5.7 with $\mathbb{T} = \mathbb{Z}$ to get an apllication of this result in a non exponential trichotomy situation.

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