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New oscillation criteria for third order nonlinear functional differential equations

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Abstract. The authors consider the general third order functional differential equation

$$\left(a_2(\nu)\left[\left(a_1(\nu)\left(x'(\nu)\right)^{\alpha_1}\right)'\right]^{\alpha_2}\right)' + q(\nu)x^{\beta}(\tau(\nu)) = 0, \qquad \nu \geq \nu_0,$$

and obtain sufficient conditions for the oscillation of all solutions. It is important to note that α_i for $i = 1, 2$, and β are somewhat independent of each other. The results obtained are illustrated with examples.

Keywords: oscillation, nonoscillation, delay differential equation, comparison method.

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1 Introduction

The primary objective of this work is to study the oscillatory behavior of solutions of the nonlinear third order differential equation

$$\left(a_2(\nu)\left[\left(a_1(\nu)\left(x'(\nu)\right)^{\alpha_1}\right)'\right]^{\alpha_2}\right)' + q(\nu)x^{\beta}(\tau(\nu)) = 0, \qquad \nu \geq \nu_0, \tag{1.1}$$

where α_i, $i = 1, 2$, and β are quotients of odd positive integers. A *solution* x of (1.1) is a continuous function on $[T_x, \infty)$, $T_x \geq \nu_0$ that satisfies (1.1) on $[T_x, \infty)$. We consider only those solutions $x(\nu)$ of (1.1) that are continuable, i.e., they satisfy $\sup\{|x(\nu)| : \nu \geq T\} > 0$ for all $T > T_x \geq \nu_0$. Such a solution is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and to be *nonoscillatory* otherwise.

Throughout, we always assume that

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$(\mathcal{A}_1)$ $a_i(\nu)$, $q(\nu) \in C([\nu_0,\infty),\mathbb{R}_+)$ for $i = 1,2$, with $q(\nu) \not\equiv 0$ and

$$\int_{\nu_0}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)ds = \infty = \int_{\nu_0}^{\infty} a_2^{-\frac{1}{\alpha_2}}(s)ds; \tag{1.2}$$

$(\mathcal{A}_2)$ $\tau \in C^1([\nu_0,\infty),\mathbb{R})$ with $\tau(\nu) \le \nu$, $\tau'(\nu) \ge 0$, and $\lim_{\nu\to\infty} \tau(\nu) = \infty$.

As equation (1.1) is regarded as a useful instrument for simulating processes in various fields of applied mathematics, physics, and chemistry (see the monographs [6,22,24]), it is important to analyze the qualitative properties of equation (1.1). For several years now, there has been a growing interest in the asymptotic behavior of solutions of various forms of linear and nonlinear third order differential equations and their applications; see, e.g., [1–5,7–16,18,21] and the references therein.

In particular, Baculíková and Džurina [4] considered the third-order nonlinear delay differential equation of the form

$$\left(a_1(\nu)\left[x''(\nu)\right]^{\alpha_1}\right)' + q(\nu)x^{\beta}(\tau(\nu)) = 0. \tag{1.3}$$

They used a comparison theorem with appropriate lower-order equations to derive sufficient condition for the asymptotic and oscillatory behaviour of Eq. (1.3). This work allows us to note the following:

(1) Eq. (1.3) is a particular case of Eq. (1.1);

(2) There is no general rule to choose the function $\xi(\nu)$ that plays a very important role in deriving the oscillation of Eq. (1.1).

Chatzarakis et al. [9] considered the third-order linear differential equation of the form

$$\left(a_2(\nu)\left[\left(a_1(\nu)\left(x'(\nu)\right)\right)'\right]\right)' + q(\nu)x(\tau(\nu)) = 0, \tag{1.4}$$

and using the integral technique, comparison method, and Gronwall inequality, they improved the results reported in [4] by relaxing the above mentioned second observation. Inspired by the papers referenced here, we wish to the study of the general equation (1.1) and derive some easily verifiable sufficient conditions for the oscillation of all it solutions.

2 Basic lemmas

In view of (1.2), we introduce the following notation:

$$A(\nu,\nu_0) = \int_{\nu_0}^{\nu} a_2^{-\frac{1}{\alpha_2}}(s)ds \quad \text{and} \quad A^*(\nu,\nu_0) = \int_{\nu_0}^{\nu} \left(\frac{A(s,\nu_0)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} ds.$$

Setting $G_1(x(\nu)) = (x'(\nu))^{\alpha_1}$ and $G_2(x(\nu)) = \left[(a_1(\nu)G_1(x(\nu)))'\right]^{\alpha_2}$, we can write equation (1.1) as the equivalent equation

$$(a_2(\nu)G_2(x(\nu)))' + q(\nu)x^{\beta}(\tau(\nu)) = 0 \quad \text{for } \nu \ge \nu_0. \tag{2.1}$$

To obtain our main results, we will utilize the following lemmas, the first of which is well known.

Lemma 2.1. *Let* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold. If* x *is an eventually positive solution of* (1.1) *for* $\nu \geq \nu_0$*, then there exists* $\nu_1 > \nu_0$ *such that either*

$$\text{(I)} \quad G_1(x(\nu)) \geq 0 \text{ and } G_2(x(\nu)) \geq 0, \quad \text{or} \quad \text{(II)} \quad G_1(x(\nu)) \leq 0 \text{ and } G_2(x(\nu)) \geq 0$$

for $\nu \geq \nu_1$.

Lemma 2.2. *Let* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold. If* x *is a positive solution of* (1.1) *such that Case I of Lemma* 2.1 *holds for* $\nu \geq \nu_1$*, then*

$$x(\nu) \geq A^*(\nu, \nu_1)\left((a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}}\right) \tag{2.2}$$

for $\nu \geq \nu_2 > \nu_1$.

Proof. Let x be a nonoscillatory solution of Eq. (1.1) such that $x(\nu) > 0$, $x(\tau(\nu)) > 0$, and which satisfies Case I of Lemma 2.1 for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. Then,

$$a_1(\nu)G_1(x(\nu)) \geq \int_{\nu_1}^{\nu} (a_1(s)G_1(x(s)))'\,ds = \int_{\nu_1}^{\nu} \frac{a_2^{\frac{1}{\alpha_2}}(s)G_2^{\frac{1}{\alpha_2}}(x(s))}{a_2^{\frac{1}{\alpha_2}}(s)}ds,$$

that is,

$$a_1(\nu)(x'(\nu))^{\alpha_1} \geq A(\nu, \nu_1)a_2^{\frac{1}{\alpha_2}}(\nu)G_2^{\frac{1}{\alpha_2}}(x(\nu)),$$

so

$$x'(\nu) \geq \left(\frac{A(\nu, \nu_1)}{a_1(\nu)}\right)^{\frac{1}{\alpha_1}} (a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}}. \tag{2.3}$$

Integrating from ν_1 to ν gives

$$x(\nu) \geq (a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}} \int_{\nu_1}^{\nu} \left(\frac{A(s, \nu_1)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} ds = A^*(\nu, \nu_1)\left((a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}}\right),$$

which completes the proof. □

For convenience, we let

$$B(\nu, s) = \left(\frac{A(\nu, s)}{a_1(s)}\right)^{\frac{1}{\alpha_1}}$$

and

$$\widehat{A}^*(\nu, \tau(\nu)) = \int_{\tau(\nu)}^{\nu} B(\nu, s)ds.$$

Lemma 2.3. *Let* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold. If* x *is a positive solution of* (1.1) *such that Case II of Lemma* 2.1 *holds for* $\nu \geq \nu_1$*, then*

$$x(\tau(\nu)) \geq \widehat{A}^*(\nu, \tau(\nu))\Big(a_2(\nu)G_2(x(\nu))\Big)^{\frac{1}{\alpha_1\alpha_2}} \tag{2.4}$$

for $\nu \geq \nu_2 > \nu_1$.

Proof. Let x be a nonoscillatory solution of Eq. (1.1) such that $x(\nu) > 0$, $x(\tau(\nu)) > 0$, and Case II of Lemma 2.1 is satisfied for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. For $\nu \geq s > \nu_1$, we have

$$a_1(\nu)G_1(x(\nu)) - a_1(s)G_1(x(s)) = \int_{s}^{\nu} (a_1(u)G_1(x(u)))'\,du = \int_{s}^{\nu} \frac{a_2^{\frac{1}{\alpha_2}}(u)G_2^{\frac{1}{\alpha_2}}(x(u))}{a_2^{\frac{1}{\alpha_2}}(s)}du.$$

That is,

$$-a_1(s)(x'(s))^{\alpha_1} \geq A(\nu,s)a_2^{\frac{1}{\alpha_2}}(\nu)G_2^{\frac{1}{\alpha_2}}(x(\nu)),$$

so

$$-x'(s) \geq \left(\frac{A(\nu,s)}{a_1(\nu)}\right)^{\frac{1}{\alpha_1}} (a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}} \geq B(\nu,s)\,(a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}}. \tag{2.5}$$

Integrating from $\tau(\nu)$ to ν, we obtain

$$-x(\nu)+x(\tau(\nu)) \geq \Big(a_2(\nu)G_2(x(\nu))\Big)^{\frac{1}{\alpha_1\alpha_2}} \int_{\tau(\nu)}^{\nu} B(\nu,s)ds,$$

or

$$x(\tau(\nu)) \geq \widehat{A}^*(\nu,\tau(\nu))\Big(a_2(\nu)G_2(x(\nu))\Big)^{\frac{1}{\alpha_1\alpha_2}}.$$

This proves the lemma. □

Remark 2.4. In view of Lemma 2.3, from (1.1) and (2.4), we see that

$$-\left(a_2(\nu)G_2(x(\nu))\right)' = q(\nu)x^{\beta}(\tau(\nu)) \geq q(\nu)\left(\widehat{A}^*(\nu,\tau(\nu))\right)^{\beta}\left(a_2(\nu)G_2(x(\nu))\right)^{\frac{\beta}{\alpha_1\alpha_2}}.$$

Integrating this inequality from $\tau(\nu)$ to ν, we have

$$\limsup_{\nu\to\infty}\int_{\tau(\nu)}^{\nu} q(u)\left(\widehat{A}^*(u,\tau(u))\right)^{\beta} du > 1$$

in the case where $\frac{\beta}{\alpha_1\alpha_2} = 1$.

We also have the following lemma.

Lemma 2.5. *In addition to the hypotheses of Lemma 2.3, assume that there exists a constant $\gamma > 1$ such that $\gamma\tau(\nu) \leq \nu$ for $\nu \geq \nu_2 > \nu_1$. Then*

$$x(\tau(\nu)) \geq \widehat{A}^*(\gamma\tau(\nu),\tau(\nu))\Big(a_2(\gamma\tau(\nu))G_2(x(\gamma\tau(\nu)))\Big)^{\frac{1}{\alpha_1\alpha_2}} \tag{2.6}$$

for $\nu \geq \nu_2 > \nu_1$.

Proof. If we integrate (2.5) from $\tau(\nu)$ to $\gamma\tau(\nu)$, we can obtain (2.6). □

3 Oscillation results

Our first oscillation result is as follows.

Theorem 3.1. *Let $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold and assume that there exists a constant $\gamma > 1$ such that $\gamma\tau(\nu) \leq \nu$ for $\nu \geq \nu_2 > \nu_1$. If the first-order delay equations*

$$Y'(\nu) + q(\nu)\left(A^*(\tau(\nu),\nu_1)\right)^{\beta}(Y(\tau(\nu)))^{\frac{\beta}{\alpha_1\alpha_2}} = 0 \tag{3.1}$$

and

$$Z'(\nu) + q(\nu)\left(\widehat{A}^*(\gamma\tau(\nu),\tau(\nu))\right)^{\beta}(Z(\gamma\tau(\nu)))^{\frac{\beta}{\alpha_1\alpha_2}} = 0 \tag{3.2}$$

are oscillatory, then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1) such that $x(\nu) > 0$ and $x(\tau(\nu)) > 0$ for $\nu \geq \nu_1 > \nu_0$. According to Lemma 2.1, we distinguish the following two cases.

Case I. Using (2.2) in (2.1), we obtain

$$\begin{aligned} -(a_2(\nu)G_2(x(\nu)))' &= q(\nu)x^\beta(\tau(\nu)) \\ &\geq q(\nu)\left(A^*(\tau(\nu),\nu_1)\right)^\beta\left(\left(a_2(\tau(\nu))G_2(x(\tau(\nu)))\right)^{\frac{1}{\alpha_1\alpha_2}}\right)^\beta. \end{aligned}$$

Setting $Y(\nu) = a_2(\nu)G_2(x(\nu))$, this becomes

$$Y'(\nu) + q(\nu)\left(A^*(\tau(\nu),\nu_1)\right)^\beta (Y(\tau(\nu)))^{\frac{\beta}{\alpha_1\alpha_2}} \leq 0.$$

By [3, Lemma 2.1(I)], the related differential equation (3.1) also has a positive solution, which is a contradiction.

Case II. Using (2.6) in Eq. (2.1), we obtain

$$\begin{aligned} -(a_2(\nu)G_2(x(\nu)))' &= q(\nu)x^\beta(\tau(\nu)) \\ &\geq q(\nu)\left(\widehat{A}^*(\gamma\tau(\nu),\tau(\nu))\left((a_2(\gamma\tau(\nu))G_2(x(\gamma\tau(\nu))))^{\frac{1}{\alpha_1\alpha_2}}\right)\right)^\beta. \end{aligned}$$

Setting $Z(\nu) = a_2(\nu)G_2(x(\nu))$, this becomes

$$Z'(\nu) + q(\nu)\left(\widehat{A}^*(\gamma\tau(\nu),\tau(\nu))\right)^\beta (Z(\gamma\tau(\nu)))^{\frac{\beta}{\alpha_1\alpha_2}} \leq 0.$$

Again by [3, Lemma 2.1(I)], the corresponding differential equation (3.2) must have a positive solution. This contradiction proves the theorem. □

Example 3.2. Consider the third-order delay equation

$$\left(\nu\left[\left(\frac{1}{\nu^2}\left(x'(\nu)\right)\right)'\right]^3\right)' + \frac{c}{\nu^2}x^{\frac{1}{3}}\left(\frac{\nu}{3}\right) = 0, \qquad \nu \geq 1, \tag{3.3}$$

where $c > 0$ is a constant, $\alpha_1 = 1$, $\alpha_2 = 3$, $a_1(\nu) = \frac{1}{\nu^2}$, $a_2(\nu) = \nu$, $q(\nu) = \frac{c}{\nu^2}$, $\beta = \frac{1}{3}$, and $\tau(\nu) = \frac{\nu}{3}$. Clearly, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and (1.2) hold. Using

$$A(\nu,1) = \int_1^\nu a_2^{-\frac{1}{\alpha_2}}(s)ds = \int_1^\nu s^{-\frac{1}{3}}ds = \frac{3\nu^{\frac{2}{3}} - 3}{2}$$

and

$$A^*(\tau(\nu),1) = \int_1^{\tau(\nu)}\left(\frac{A(s,1)}{a_1(s)}\right)^{\frac{1}{\alpha_1}}ds = \int_1^{\frac{\nu}{3}}\left(\frac{s^2\left(3s^{\frac{2}{3}} - 3\right)}{2}\right)ds = \frac{1}{2}\left(\frac{\nu^{\frac{11}{3}}}{33\cdot 3^{\frac{2}{3}}} - \frac{\nu^3}{27} + \frac{2}{11}\right),$$

it is not difficult to see that equation (3.1) becomes

$$Y'(\nu) + \frac{c}{2\nu^2}\left(\frac{\nu^{\frac{11}{3}}}{33\cdot 3^{\frac{2}{3}}} - \frac{\nu^3}{27} + \frac{2}{11}\right)^{\frac{1}{3}} Y^{\frac{1}{9}}\left(\frac{\nu}{3}\right) = 0. \tag{3.4}$$

Also, using $\gamma = 2$ and

$$B(\nu, s) = \left(\frac{A(\nu, s)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} = \frac{\int_s^\nu u^{-\frac{1}{3}} du}{\frac{1}{\nu^2}} = \frac{3\nu^2(\nu^{\frac{2}{3}} - s^{\frac{2}{3}})}{2},$$

we see that

$$\widehat{A}^*(\gamma\tau(\nu), \tau(\nu)) = \int_{\tau(\nu)}^{\gamma\tau(\nu)} B(\nu, s) ds = \int_{\frac{\nu}{3}}^{\frac{2\nu}{3}} \frac{3\nu^2(\nu^{\frac{2}{3}} - s^{\frac{2}{3}})}{2} ds = \frac{\nu^{\frac{11}{3}}}{2} - \frac{2^{\frac{5}{3}}\nu^{\frac{11}{3}} - \nu^{\frac{11}{3}}}{3^{\frac{5}{3}}},$$

and so equation (3.2) becomes

$$Z'(\nu) + \frac{c}{2\nu^2}\left(\frac{\nu^{\frac{11}{3}}}{2} - \frac{2^{\frac{5}{3}}\nu^{\frac{11}{3}} - \nu^{\frac{11}{3}}}{3^{\frac{5}{3}}}\right)^{\frac{1}{3}} Z^{\frac{1}{9}}\left(\frac{2\nu}{3}\right) = 0. \tag{3.5}$$

Clearly, [19, Theorem 5] guarantee that all solutions of Eqs. (3.4) and (3.5) are oscillatory. Thus, every solution of Eq. (3.3) oscillates.

Theorem 3.3. *Let $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. If the first-order delay equation (3.1) is oscillatory and*

$$\limsup_{\nu\to\infty} \int_{\tau(\nu)}^{\nu} q(u)\left(A^*(\tau(\nu), \tau(s))\right)^\beta ds > 1 \tag{3.6}$$

for $\beta = \alpha_1\alpha_2$, then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of Eq. (1.1) such that $x(\nu) > 0$ and $x(\tau(\nu)) > 0$ for $\nu \geq \nu_1 > \nu_0$. We again consider the two cases in Lemma 2.1.

Case I. Proceeding as in the proof of Theorem 3.1, we again obtain a contradiction.

Case II. Clearly, for $v \geq u > \nu_1$,

$$a_1(v)G_1(x(v)) - a_1(u)G_1(x(u)) = \int_u^v (a_1(s)G_1(x(s)))' ds = \int_u^v \frac{a_2^{\frac{1}{\alpha_2}}(s)G_2^{\frac{1}{\alpha_2}}(x(s))}{a_2^{\frac{1}{\alpha_2}}(s)} ds,$$

that is,

$$-a_1(u)G_1(x(u)) \geq a_2^{\frac{1}{\alpha_2}}(v)G_2^{\frac{1}{\alpha_2}}(x(v)) \int_u^v \frac{1}{a_2^{\frac{1}{\alpha_2}}(s)} ds,$$

and so

$$-a_1(u)(x'(u))^{\alpha_1} \geq a_2^{\frac{1}{\alpha_2}}(v)G_2^{\frac{1}{\alpha_2}}(x(v)) \int_u^v \frac{1}{a_2^{\frac{1}{\alpha_2}}(s)} ds,$$

Hence,

$$-x'(u) \geq (a_2(v)G_2(x(v)))^{\frac{1}{\alpha_1\alpha_2}} \left(\frac{1}{a_1(u)} \int_u^v \frac{1}{a_2^{\frac{1}{\alpha_2}}(s)} ds\right)^{\frac{1}{\alpha_1}},$$

and integrating from u to v gives

$$x(u) - x(v) \geq (a_2(v)G_2(x(v)))^{\frac{1}{\alpha_1\alpha_2}} \int_u^v \left(\frac{1}{a_1(y)} \int_y^v \frac{1}{a_2^{\frac{1}{\alpha_2}}(s)} ds\right)^{\frac{1}{\alpha_1}} dy,$$

or

$$x(u) \geq (a_2(v)G_2(x(v)))^{\frac{1}{\alpha_1\alpha_2}} A^*(v,u).$$

Now, for any $\nu \geq s > \nu_2$, for some $\nu_2 > \nu_1$, if we set $u = \tau(s)$ and $v = \tau(\nu)$ in the preceding inequality, gives

$$x(\tau(s)) \geq \Big(a_2(\tau(\nu))G_2(x(\tau(\nu)))\Big)^{\frac{1}{\alpha_1\alpha_2}} A^*(\tau(\nu),\tau(s)). \tag{3.7}$$

Integrating Eq. (1.1) from $\tau(\nu)$ to ν and then applying (3.7),

$$\begin{aligned} a_2(\tau(\nu))G_2(x(\tau(\nu))) &\geq \int_{\tau(\nu)}^{\nu} q(s)x^{\beta}(\tau(s))ds \\ &\geq \Big(a_2(\tau(\nu))G_2(x(\tau(\nu)))\Big)^{\frac{\beta}{\alpha_1\alpha_2}} \int_{\tau(\nu)}^{\nu} q(s)(A^*(\tau(\nu),\tau(s)))^{\beta}ds, \end{aligned}$$

which implies

$$\int_{\tau(\nu)}^{\nu} q(s)\left(A^*(\tau(\nu),\tau(s))\right)^{\beta} ds \leq 1,$$

and contradicts (3.6). □

Example 3.4. Consider the equation

$$\left(\frac{1}{\nu^2}\left[\left(\frac{1}{9\nu^2}\left(x'(\nu)\right)\right)'\right]^3\right)' + \frac{\delta}{\nu^7}x^3\left(\frac{\nu}{2}\right) = 0, \qquad \nu \geq 1, \tag{3.8}$$

where we have $\alpha_1 = 1$, $\alpha_2 = 3$, $a_1(\nu) = \frac{1}{9\nu^2}$, $a_2(\nu) = \frac{1}{\nu^2}$, $q(\nu) = \frac{\delta}{\nu^7}$ for $\delta > 0$, $\beta = 3$ and $\tau(\nu) = \frac{\nu}{2}$. Clearly, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and (1.2) hold. Using

$$A(\nu,1) = \int_1^{\nu} a_2^{-\frac{1}{\alpha_2}}(s)ds = \int_1^{\nu}\left(\frac{1}{s^2}\right)^{-\frac{1}{3}} ds = \frac{\left(3\nu^{\frac{5}{3}}-3\right)}{5}$$

and

$$\begin{aligned} A^*(\tau(\nu),1) = \int_1^{\tau(\nu)}\left(\frac{A(s,1)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} ds &= \int_1^{\frac{\nu}{2}} \frac{s^2\left(3s^{\frac{5}{3}}-3\right)}{5}ds \\ &= \frac{1}{5}\left(\frac{9\nu^{\frac{14}{3}}}{224\cdot 2^{\frac{2}{3}}} - \frac{\nu^3}{8} - \frac{5}{14}\right), \end{aligned}$$

it is not difficult to see that (3.1) becomes

$$Y'(\nu) + \frac{42}{125\cdot\nu^7}\left(\frac{9\nu^{\frac{14}{3}}}{7\cdot 2^{\frac{17}{3}}} - \frac{\nu^3}{8} - \frac{5}{14}\right)^3 Y\left(\frac{\nu}{2}\right) = 0. \tag{3.9}$$

Indeed, following [20, Theorem 2.1.1], Eq. (3.9) is oscillatory if

$$\lim_{\nu\to\infty}\int_{\frac{\nu}{2}}^{\nu} \frac{\delta}{125\cdot s^7}\left(\frac{9s^{\frac{14}{3}}}{7\cdot 2^{\frac{17}{3}}} - \frac{s^3}{8} - \frac{5}{14}\right)^3 ds > \frac{1}{e}.$$

And using

$$A(\nu,u)=\int_u^\nu a_2^{-\frac{1}{\alpha_2}}(s)ds=\int_u^\nu\left(\frac{1}{s^2}\right)^{\frac{-1}{3}}ds=\frac{3\nu^{\frac{5}{3}}-3u^{\frac{5}{3}}}{5}.$$

$$\begin{aligned}A^*(\tau(\nu),\tau(s))=\int_{\tau(s)}^{\tau(\nu)}\left(\frac{A(\nu,y)}{a_1(y)}\right)^{\frac{1}{\alpha_1}}dy&=\int_{\frac{s}{2}}^{\frac{\nu}{2}}\frac{27y^2\left(\nu^{\frac{5}{3}}-y^{\frac{5}{3}}\right)}{5}dy\\&=\frac{27}{25}\left(\frac{\nu^{\frac{5}{3}}\left(\nu^3-s^3\right)}{8}-\frac{3\nu^{\frac{14}{3}}-3s^{\frac{14}{3}}}{7\cdot 2^{\frac{17}{3}}}\right).\end{aligned}$$

Eq. (3.6) becomes

$$\begin{aligned}\int_{\tau(\nu)}^{\nu}q(s)\left(A^*(\tau(\nu),\tau(s))\right)^{\beta}ds&=\int_{\frac{\nu}{2}}^{\nu}\frac{\delta}{s^7}\left(\frac{27}{25}\left(\frac{\nu^{\frac{5}{3}}\left(\nu^3-s^3\right)}{8}-\frac{3\nu^{\frac{14}{3}}-3s^{\frac{14}{3}}}{7\cdot 2^{\frac{17}{3}}}\right)\right)^3ds\\&>1.\end{aligned}$$

By Theorem 3.3, every solution of (3.8) oscillates.

Theorem 3.5. *Let $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. If $\beta=\alpha_1\alpha_2$ and there is a nondecreasing function $\phi\in C^1([\nu_0,\infty),(0,\infty)$ such that (3.6) and*

$$\limsup_{\nu\to\infty}\int_{\nu_1}^{\nu}\left[\phi(s)q(s)-\frac{(\phi'(s))^2(\phi(s))^{\frac{1}{\alpha_1\alpha_2}-2}}{4\beta\tau'(s)}\left(\frac{A(\tau(s),\nu_1)}{a_1(s)}\right)^{\frac{-1}{\alpha_1}}\right]ds=\infty \tag{3.10}$$

hold, then equation (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of Eq. (1.1) such that $x(\nu)>0$ and $x(\tau(\nu))>0$ for $\nu\ge\nu_1>\nu_0$. We again consider cases.

Case I. Define

$$\mathcal{W}(\nu)=\phi(\nu)\frac{a_2(\nu)G_2(x(\nu))}{x^{\beta}(\tau(\nu))}.$$

Then $\mathcal{W}(\nu)>0$, and using Lemma 2.2, the decreasing nature of $a_2(\nu)G_2(x(\nu))$, and (2.3)

$$\begin{aligned}\mathcal{W}'(\nu)&=\frac{\phi(\nu)(a_2(\nu)G_2(x(\nu)))'}{x^{\beta}(\tau(\nu))}+\frac{a_2(\nu)G_2(x(\nu))\phi'(\nu)}{x^{\beta}(\tau(\nu))}-\beta\frac{\phi(\nu)(a_2(\nu)G_2(x(\nu)))x'(\tau(\nu))\tau'(\nu)}{x^{\beta+1}(\tau(\nu))}\\&\le-\phi(\nu)q(\nu)+\frac{\phi'(\nu)}{\phi(\nu)}\mathcal{W}(\nu)-\beta\tau'(\nu)\left(\frac{A(\tau(\nu),\nu_1)}{a_1(\nu)}\right)^{\frac{1}{\alpha_1}}\frac{\phi(\nu)(a_2(\nu)G_2(x(\nu)))^{1+\frac{1}{\alpha_1\alpha_2}}}{x^{\beta+1}(\tau(\nu))}\\&\le-\phi(\nu)q(\nu)+\frac{\phi'(\nu)}{\phi(\nu)}\mathcal{W}(\nu)-\frac{\beta\tau'(\nu)}{\phi^{\frac{1}{\alpha_1\alpha_2}}(\nu)}\left(\frac{A(\tau(\nu),\nu_1)}{a_1(\nu)}\right)^{\frac{1}{\alpha_1}}\mathcal{W}^2(\nu).\end{aligned}$$

If we complete the square on the right hand side, we find that

$$\mathcal{W}'(\nu)\le-\phi(\nu)q(\nu)+\frac{(\phi'(\nu))^2}{4\beta\tau'(\nu)}(\phi(\nu))^{\frac{1}{\alpha_1\alpha_2}-2}\left(\frac{A(\tau(\nu),\nu_1)}{a_1(\nu)}\right)^{\frac{-1}{\alpha_1}}.$$

Integrating the preceding inequality from ν_1 to ν, we see that (3.10) gives a contradiction to the fact that $W(\nu)\ge 0$.

Case II. Proceeding as in the proof of Theorem 3.3, leads to a contradiction in this case. □

Example 3.6. Consider the equation

$$\left(\frac{1}{\nu}\left[\left(\frac{1}{\nu}\left(x'(\nu)\right)^{\frac{1}{3}}\right)'\right]^3\right)' + \frac{\delta}{\nu^3}\, x\left(\frac{\nu}{3}\right) = 0, \qquad \nu \geq 1, \tag{3.11}$$

where we have $\alpha_1 = \frac{1}{3}$, $\alpha_2 = 3$, $a_1(\nu) = \frac{1}{\nu}$, $a_2(\nu) = \frac{1}{\nu}$, $q(\nu) = \frac{\delta}{\nu^3}$ for $\delta > 0$, $\beta = 1$ and $\tau(\nu) = \frac{\nu}{3}$. Clearly, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and (1.2) hold. Using $\phi(\nu) = \nu^4$ and $A(\tau(\nu), \nu_1) = \frac{3}{4}\left[\left(\frac{\nu}{3}\right)^{\frac{4}{3}} - 1\right]$ in Eq. (3.10), we have

$$\limsup_{\nu\to\infty} \int_1^\nu \left[\phi(s)q(s) - \frac{(\phi'(s))^2(\phi(s))^{\frac{1}{\alpha_1\alpha_2}-2}}{4\beta\tau'(s)}\left(\frac{A(\tau(s),1)}{a_1(s)}\right)^{\frac{-1}{\alpha_1}}\right] ds$$
$$= \limsup_{\nu\to\infty} \int_1^\nu \left[\delta s - \frac{3s^6}{s^4}\left(\frac{3s}{4}(s^{\frac{4}{3}} - 1)\right)^{-3}\right] ds = \infty.$$

It is not difficult to see that (3.6) holds, so by Theorem 3.5, every solution of (3.11) oscillates.

4 Concluding remark

Employing the methods of comparison, Riccati substitution, and the integral method, we introduced three novel conditions for the oscillation of a general third-order nonlinear delay differential equation. Interestingly, our results are applicable to linear, sublinear, and superlinear equations. Some illustrative examples are given to show the applicability of our results.

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