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Existence of solutions to Sturm–Liouville boundary value problems

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Abstract. We study the solvability of Sturm–Liouville boundary value problems for $x'' = f(t, x, x'), t \in (0, 1)$. The nonlinearity can be defined on a bounded set and is required to be continuous on its subset. The results obtained are based on combinations of well-known conditions with barrier strip type conditions.

Keywords: nonlinear boundary value problem, Sturm–Liouville boundary conditions, existence, barrier strips conditions.

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1 Introduction

This paper is devoted to the solvability of boundary value problems (BVPs) for the equation

$$x'' = f(t, x, x'), \qquad t \in (0, 1), \tag{1.1}$$

with Sturm–Liouville boundary conditions (BCs) either

$$-\alpha x(0) + \beta x'(0) = A, \qquad ax(1) + bx'(1) = B, \tag{1.2}$$

$$x'(0) = A, \qquad ax(1) + bx'(1) = B, \tag{1.3}$$

or

$$-\alpha x(0) + \beta x'(0) = A, \qquad x'(1) = B, \tag{1.4}$$

where $f : [0, 1] \times D_x \times D_p \to \mathbb{R}$, $D_x, D_p \subseteq \mathbb{R}$, $\alpha, \beta, a, b > 0$, and $A, B \in \mathbb{R}$.

This paper is motivated by A. Granas et al. [6]. The authors prove that BVP (1.1), (1.2) has a solution in $C^2[0, 1]$ assuming that the function $f(t, x, p)$ is continuous on $[0, 1] \times \mathbb{R}^2$ and there is a constant $M \geq 0$ such that

$$xf(t, x, 0) \geq M \quad \text{for } t \in [0, 1] \text{ and } |x| > M \tag{1.5}$$

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and the well-known Bernstein's growth condition holds, that is, there exist positive constants $G_i, i = 1, 2$, for which

$$|f(t, x, p)| \leq G_1 p^2 + G_2 \quad \text{for } (t, x) \in [0, 1] \times [-M_0, M_0], \tag{1.6}$$

where $M_0 = \max\{|A/\alpha|, |B/a|, M\}$. A similar result guarantees $C^2[0, 1]$-solutions to BVPs (1.1), (1.3) and (1.1), (1.4). In it, (1.5) is replaced by the assumption that $f(t, x, p)$ is differentiable with respect to x and there is a constant $K > 0$ such that

$$f_x(t, x, p) \geq K \quad \text{for } (t, x, p) \in [0, 1] \times \mathbb{R} \times \{C\}, \tag{1.7}$$

where $C = A$ or $C = B$. The conditions imposed guarantee a priori bounds. Moreover, each of the conditions (1.5) and (1.7) provide a priori bound for the solutions to the considered BVP, that is, for $|x(t)|$, and (1.6) provides the bound for $|x'(t)|$. The established a priori bounds are needed for applying the Topological transversality theorem.

One of the BVPs studied in C. Tisdell [10] is (1.1), (1.2). Here, the Leray–Schauder degree theory and a priori bounds are used. The well known condition

$$f(t, -R_1, 0) < 0 \quad \text{and} \quad f(t, R_2, 0) > 0, \qquad t \in [0, 1],$$

where R_1 and R_2 are some positive constants with $\min\{R_1, R_2\} > \max\{|A/\alpha|, |B/a|\}$, gives the bound for $|x(t)|$, and this for $|x'(t)|$ follows from the assumption that there exist non-negative constants α and K such that

$$|f(t, x, p)| \leq \alpha f(t, x, p) + K \quad \text{for all } t \in [0, 1], |x| \leq R, p \in \mathbb{R},$$

where $R = \max\{R_1, R_2\}$.

Other results on the solvability of BVPs for various equations with Sturm–Liouville boundary conditions can be found, for example, in M. Dobkevich and F. Sadyrbaev [3], A. M. A. El-Sayed et al. [4,5], T. Xue et al. [12], Y. Liu et al. [9], F. H. Wong et al. [11] and L. Zhang et al. [13].

The purpose of this paper is to give sufficient conditions for the existence of solutions in which growth restrictions on $f(t, x, p)$ are not imposed, that is, we do not use condition (1.6). It is replaced by sign conditions of barrier strips type.

The existence discussion is based on the basic existence theorem proved in R. P. Agarwal et al. [2], which is a variant of [6, Chapter V, Theorem 1.1]. Let us prepare its wording.

Consider the BVP

$$\begin{cases} x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)} = f(t, x, x', \ldots, x^{(n-1)}),\ t \in [0, 1], \\ V_i(x) = A_i,\ i = \overline{1, n}, \end{cases} \tag{1.8}$$

of which the considered boundary value problems (1.1), (1.2)–(1.4) are special cases. Here $s_k(t), k = \overline{0, n-1}$, are continuous on $[0, 1]$, $f : [0, 1] \times D_0 \times D_1 \times \cdots \times D_{n-1} \to \mathbb{R}$,

$$V_i(x) \equiv \sum_{j=0}^{n-1} [a_{ij} x^{(j)}(0) + b_{ij} x^{(j)}(1)], \qquad i = \overline{1, n},$$

with constants a_{ij} and b_{ij} for which $\sum_{j=0}^{n-1} (a_{ij}^2 + b_{ij}^2) > 0, i = \overline{1, n}$, and $A_i \in \mathbb{R}$.

For $\lambda \in [0, 1]$, consider also the family of BVPs

$$\begin{cases} x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)} = g(t, x, x', \ldots, x^{(n-1)}, \lambda),\ t \in [0, 1], \\ V_i(x) = A_i,\ i = \overline{1, n}, \end{cases} \tag{1.9}$$

where $g : [0,1] \times D_0 \times D_1 \times \cdots \times D_{n-1} \times [0,1] \to \mathbb{R}$, and $s_k(t), k = \overline{0, n-1}, V_i, A_i, i = \overline{1,n}$, are as above.

Let, as usual, $C[0,1]$ be the Banach space of continuous functions on $[0,1]$ with the norm $\|x\|_0 = \sup_{t\in[0,1]} |x(t)|$, and $C^n[0,1]$ be the Banach space of n-times continuously differentiable functions with $\|x\|_n = \max\{\|x\|_0, \dots, \|x^{(n)}\|_0\}$.

Let $\mathcal{B}$ denote the set of functions that satisfy the BCs $V_i(x) = A_i$, $i = \overline{1,n}$, and $\mathcal{B}_0$ be the set of functions satisfying $V_i(x) = 0, i = \overline{1,n}$. Finally, let $C^n_{\mathcal{B}}[0,1] = C^n[0,1] \cap \mathcal{B}$ and $C^n_{\mathcal{B}_0}[0,1] = C^n[0,1] \cap \mathcal{B}_0$.

We are now ready to formulate the basic existence theorem.

Theorem 1.1 ([2, Theorem 4])**.** *Assume that:*

(i) For $\lambda = 0$ problem (1.9) has a unique solution in $C^n[0,1]$.

(ii) Problems (1.8) and (1.9) are equivalent when $\lambda = 1$.

(iii) The map $\Lambda_h : C^n_{\mathcal{B}_0}[0,1] \to C[0,1]$, defined by

$$\Lambda_h x = x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)},$$

is one-to-one.

(iv) Each solution $x \in C^n[0,1]$ to family (1.9) satisfies the bounds

$$m_i \le x^{(i)}(t) \le M_i \quad \text{for } t \in [0,1],\ i = \overline{0,n},$$

where the constants $-\infty < m_i, M_i < \infty$, $i = \overline{0,n}$, are independent of λ and x.

(v) There is a sufficiently small $\delta > 0$ such that

$$[m_i - \delta, M_i + \delta] \subseteq D_i, \qquad i = \overline{0, n-1},$$

and the function $g(t, p_0, \dots, p_{n-1}, \lambda)$ is continuous on $[0,1] \times D \times [0,1]$, where

$$D = [m_0 - \delta, M_0 + \delta] \times [m_1 - \delta, M_1 + \delta] \times \cdots \times [m_{n-1} - \delta, M_{n-1} + \delta];$$

$m_i, M_i, i = \overline{0, n-1}$, are as in (iv).

Then, BVP (1.8) has at least one solution in $C^n[0,1]$.

To apply Theorem 1.1 for studying the considered BVPs, we use families of BVPs for

$$x'' = \lambda f(t, x, x') + (1-\lambda)(x - x'), \qquad t \in (0,1), \qquad (1.1)^-_\lambda$$

and

$$x'' = \lambda f(t, x, x') + (1-\lambda)(x + x'), \qquad t \in (0,1), \qquad (1.1)^+_\lambda$$

where $\lambda \in [0,1]$. They are adapted to the application of the barrier strips type conditions used here, namely conditions **(B_1)** and **(B_2)** below.

In order to achieve the a priori bounds of condition *(iv)* of Theorem 1.1, we impose three sets of conditions. The conditions of set **A**, these are (1.5) and (1.7), guarantee the a priori bounds for each eventual $C^2[0,1]$-solution $x(t)$ to the used families, those of the set **B** give the bounds for $x'(t)$, and **C** ensures the bounds for $x''(t)$.

Following are the hypotheses used in this article.

(A$_1$) There is a constant $M \geq 0$ such that $[-M, M] \subseteq D_x$ and

$$xf(t,x,0) \geq 0 \quad \text{for } t \in [0,1] \text{ and } x \in D_x \setminus [-M, M].$$

In the formulation of the next hypothesis, we use the functions

$$v(t) = At + \frac{B - A(a+b)}{a} \quad \text{and} \quad w(t) = Bt + \frac{B\beta - A}{\alpha}, \qquad t \in [0,1],$$

and more precisely the constants $m_v = \min_{[0,1]} v(t)$, $M_v = \max_{[0,1]} v(t)$, $m_w = \min_{[0,1]} w(t)$ and $M_w = \max_{[0,1]} w(t)$.

(A$_2$) $J \subseteq D_x$, here $J = [m_v, M_v]$ for (1.3) and $J = [m_w, M_w]$ for (1.4), and there is a constant $K > 0$ such that

$$f_x(t,x,C) \geq K \quad \text{for } (t,x) \in [0,1] \times D_x,$$

where $C = A$ for (1.3) and $C = B$ for (1.4).

For some constants M_0, G_m and G_M, they will be specified later for each problem, suppose that:

(B$_1$) There are constants $F_i, L_i, i = 1,2$, such that $[F_2, L_2] \subseteq D_p$,

$$F_2 < F_1 \leq \min\{-M_0, G_m\}, \qquad \max\{M_0, G_M\} \leq L_1 < L_2,$$

$$f(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0, M_0] \times [L_1, L_2], \tag{1.10}$$

$$f(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0, M_0] \times [F_2, F_1]. \tag{1.11}$$

(B$_2$) There are constants $F_i', L_i', i = 1,2$, such that $[F_2', L_2'] \subseteq D_p$,

$$F_2' < F_1' \leq \min\{-M_0, G_m\}, \qquad \max\{M_0, G_M\} \leq L_1' < L_2',$$

$$f(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0, M_0] \times [L_1', L_2'], \tag{1.12}$$

$$f(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0, M_0] \times [F_2', F_1'].$$

(C) There are constants $m_i \leq M_i, i = 0,1$, and a sufficiently small $\delta > 0$ such that

$$[m_0 - \delta, M_0 + \delta] \subseteq D_x, \qquad [m_1 - \delta, M_1 + \delta] \subseteq D_p,$$

and $f(t,x,p)$ is continuous on $[0,1] \times [m_0 - \delta, M_0 + \delta] \times [m_1 - \delta, M_1 + \delta]$.

The paper is organized as follows. In Section 2 we establish a priori bounds for $x(t)$ and $x'(t)$ for each solution $x \in C^2[0,1]$ to the families of BVPs for $(1.1)^-_\lambda$ with BCs (1.2) or (1.3) and for $(1.1)^+_\lambda$ with BCs (1.2) or (1.4). In Section 3 we use the obtained bounds to prove existence results for the considered BVPs. Examples illustrate the application of the obtained results in Section 4.

2 Auxiliary results

We need a well known maximum principle, see for example [6, Chapter II, Lemma 1.1], concerning equations of the form

$$x'' = h(t, x, x'), \qquad t \in [0,1]. \tag{2.1}$$

It is based on the following assumption.

(A) There is a constant $M \geq 0$ such that $[-M, M] \subseteq D_x$ and

$$xh(t, x, 0) > 0 \quad \text{for } t \in [0,1] \text{ and } x \in D_x \setminus [-M, M].$$

Lemma 2.1. *Let $x \in C^2[0,1]$ be a solution to equation (2.1) such that $|x(t)|$ does not achieve its maximum at $t = 0$ or $t = 1$. Assume further that* **(A)** *holds. Then $x(t)$ satisfies the bound*

$$|x(t)| \leq M \text{ for } t \in [0,1].$$

Proof. By the assumption of the lemma, $|x(t)|$ must achieve a positive maximum at a point $t_0 \in (0,1)$. Clearly, the function $y(t) = (x(t))^2$ also has a maximum at t_0. Thus,

$$y''(t_0) = 2x(t_0)x''(t_0) = 2x(t_0)h(t_0, x(t_0), 0) \leq 0.$$

Next, reasoning by contradiction, assume $|x(t_0)| > M$. Then from **(A)** it follows

$$x(t_0)h(t_0, x(t_0), 0) > 0$$

and the derived contradiction proves the lemma. □

The proofs of the following two lemmas follow the idea of proof of [6, Chapter II, Lemma 1.2].

Lemma 2.2. *Let* **(A)** *hold. Then each solution $x \in C^2[0,1]$ to (2.1), (1.3) with $A = B = 0$ satisfies the bound*

$$|x(t)| \leq M, \qquad t \in [0,1].$$

Proof. Suppose that $|x(0)|$ is the maximum value of $|x(t)|$. We claim that $|x(0)| > M$ is impossible. To verify this, by contradiction, assume it is true. Then,

$$x(0)x''(0) = x(0)h(0, x(0), 0) > 0.$$

Now, if $x(0) < 0$, then $x''(0) < 0$. Because of the continuity of $x''(t)$ on $[0,1]$, there is a neighborhood $N_0 \subseteq [0,1]$ of $t = 0$ where $x''(t) < 0$. This means that $x'(t)$ is strictly decreasing on N_0 and so $x'(t) < x'(0) = 0$ for $t \in N_0$. Consequently $x(t)$ is also strictly decreasing on N_0 and so $|x(0)|$ can not be the maximum of $|x(t)|$ on $[0,1]$, a contradiction. If $x(0) > 0$, then $x''(0) > 0$, from where conclude $x'(t) > x'(0) = 0$ for $t \in U_0$, where $U_0 \subseteq [0,1]$ is any neighborhood of $t = 0$. Thus, $x(t) > x(0) > 0$ for $t \in U_0$, which means that $|x(t)| > |x(0)|, t \in U_0$, again a contradiction. So, $|x(0)| \leq M$.

Let $|x(1)|$ be the maximum value of $|x(t)|, t \in [0,1]$. Then, $x(1)x'(1) \geq 0$ and

$$0 \leq x(1)bx'(1) = x(1)(-ax(1)) = -a(x(1))^2 \leq 0,$$

which is possible if $x(1) = 0$. So, we have $\max_{[0,1]} |x(t)| = 0$, which means $x(t) = 0$ for $t \in [0, 1]$ and the lemma is true.

Finally, if $|x(t)|$ achieve its maximum in $(0, 1)$, then

$$|x(t)| \le M, \qquad t \in [0,1],$$

by Lemma 2.1. □

Lemma 2.3. *Let* **(A)** *hold. Then each solution $x \in C^2[0,1]$ to (2.1), (1.4) with $A = B = 0$ satisfies the bound*

$$|x(t)| \le M, \qquad t \in [0,1].$$

Proof. If $|x(0)|$ is the maximum value of $|x(t)|$, we have $x(0)x'(0) \le 0$ from where obtain

$$0 \ge x(0)\beta x'(0) = x(0)(\alpha x(0)) = \alpha(x(0))^2 \ge 0.$$

This implies $\max_{[0,1]} |x(t)| = |x(0)| = 0$ which means $x(t) = 0$ on $[0, 1]$ and so the lemma is true.

If $|x(t)|$ achieves its maximum at $t = 1$, the assumption that $|x(1)| > M$ yields

$$x(1)x''(1) = x(1)h(1, x(1), 0) > 0.$$

Next, following the proof of Lemma 2.2, we derive contradictions in the cases $x(1) < 0$ and $x(1) > 0$ and conclude that $|x(1)| \le M$.

Finally, if $|x(t)|$ achieves its maximum somewhere at the interval $(0, 1)$, then the bound

$$|x(t)| \le M, \qquad t \in [0,1],$$

follows from Lemma 2.1. □

Following the proof of [6, Chapter II, Theorem 3.3], establish the assertion.

Lemma 2.4. *Let* **(A)** *hold for (1.1). Then each solution $x \in C^2[0,1]$ to $(1.1)^-_\lambda$, (1.2) or to $(1.1)^+_\lambda$, (1.2) satisfies the bound*

$$|x(t)| \le \max\{|A/\alpha|, |B/a|, M\}, \qquad t \in [0,1].$$

Proof. We will prove the assertion about family $(1.1)^-_\lambda$, (1.2), the proof for family $(1.1)^+_\lambda$, (1.2) is practically the same.

If $|x(0)|$ is the maximum value of $|x(t)|$ on $[0, 1]$ we have $x(0)x'(0) \le 0$, from where obtain

$$0 \ge x(0)\beta x'(0) = x(0)(A + \alpha x(0)) = \alpha(x(0))^2\Big[\frac{A}{\alpha x(0)} + 1\Big].$$

This yields consecutively

$$\frac{A}{\alpha x(0)} + 1 \le 0, \qquad \left|\frac{A}{\alpha x(0)}\right| \ge 1 \quad \text{and} \quad |x(0)| \le |A/\alpha|.$$

Likewise, if $|x(t)|$ achieves its maximum at $t = 1$, we obtain

$$|x(1)| \le |B/a|.$$

Now, let $|x(t)|$ achieve its maximum in an interior point of $(0,1)$. For the right-hand side of equation $(1.1)^-_\lambda$ we have

$$\lambda x f(t,x,0) + (1-\lambda)x^2 > 0 \quad \text{for each} \quad \lambda \in [0,1], t \in [0,1] \quad \text{and} \quad x \in D_x \setminus [-M,M],$$

that is, **(A)** is satisfied for $(1.1)^-_\lambda$. Thus,

$$|x(t)| \le M, \qquad t \in [0,1],$$

by Lemma 2.1. □

Now, let

$$L_- = \max\{\max |f(t,x,A)|, \max |x-A|\},$$

where the maximums are computed for $(t,x) \in [0,1] \times [m_v, M_v]$.

Lemma 2.5. *Let* **(A$_2$)** *hold for* $J = [m_v, M_v]$ *and* $C = A$. *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^-_\lambda$, (1.3) *satisfies the bound*

$$|x(t)| \le \frac{L_-}{\min\{1,K\}} + \max\{|m_v|, |M_v|\}, \qquad t \in [0,1].$$

Proof. Define the function $y(t) = x(t) - v(t), t \in [0,1]$. Since $v(t)$ satisfies BC (1.3), $y(t)$ is a $C^2[0,1]$-solution to the homogeneous BVP

$$y'' = h(t,y,y',\lambda), \qquad t \in (0,1),$$
$$y'(0) = 0, \qquad ay(1) + by'(1) = 0,$$

where $h(t,y,y',\lambda) = \lambda f(t, y+v, y'+v') + (1-\lambda)(y+v-y'-v')$, $\lambda[0,1]$. Besides,

$$\begin{aligned} yh(t,y,0,\lambda) &= y\Big(\lambda f(t,y+v,v') + (1-\lambda)(y+v-v')\Big) \\ &= y\Big(\lambda f(t,y+v,A) + (1-\lambda)(y+v-A) - \lambda f(t,v,A) \\ &\qquad - (1-\lambda)(v-A) + \lambda f(t,v,A) + (1-\lambda)(v-A)\Big) \\ &= y\Big(\lambda f_x(t,\theta y+v,A)y + (1-\lambda)y\Big) + y\Big(\lambda f(t,v,A) + (1-\lambda)(v-A)\Big) \end{aligned}$$

for any $\theta \in (0,1)$. Let us note, $f_x(t,\theta y + v, A), t \in [0,1]$, is well defined since for $t \in [0,1]$ $\theta y(t) + v(t) \in \big[\min\{m_v, \min_{[0,1]} x(t)\}, \max\{M_v, \max_{[0,1]} x(t)\}\big] \subseteq D_x$. Then

$$yh(t,y,0,\lambda) \ge y^2(\lambda K + (1-\lambda)) - |y|L_- \ge |y|(|y|\min\{1,K\} - L_-),$$

from where it follows

$$yh(t,y,0,\lambda) > 0 \quad \text{for } t \in [0,1] \text{ and } |y| > \frac{L_-}{\min\{1,k\}}$$

and for each $\lambda \in [0,1]$. Thus,

$$|y(t)| \le \frac{L_-}{\min\{1,k\}} \quad \text{for } t \in [0,1],$$

by Lemma 2.2. Keeping in mind that $x(t) = y(t) + v(t)$, we obtain the lemma. □

Now, let

$$L_+ = \max\{\max|f(t,x,B)|, \max|x+B|\},$$

where the maximums are computed over $(t,x) \in [0,1] \times [m_w, M_w]$.

Lemma 2.6. *Let* **(A$_2$)** *hold for $J = [m_w, M_w]$ and $C = B$. Then each solution $x \in C^2[0,1]$ to (1.1)$^+_\lambda$, (1.4) satisfies the bound*

$$|x(t)| \le \frac{L_+}{\min\{1,K\}} + \max\{|m_w|, |M_w|\}, \qquad t \in [0,1].$$

Proof. Following the proof of Lemma 2.5, introduce the function $y(t) = x(t) - w(t), t \in [0,1]$. Since $w(t)$ satisfies BC (1.4), $y(t)$ is a $C^2[0,1]$-solution to the homogeneous BVP

$$y'' = h(t,y,y',\lambda), \quad t \in (0,1),$$
$$-\alpha y(0) + \beta y'(0) = 0, \quad y'(0) = 0,$$

where $h(t,y,y',\lambda) = \lambda f(t,y+w,y'+w') + (1-\lambda)(y+w+y'+w')$. Besides,

$$\begin{aligned}
yh(t,y,0,\lambda) &= y\Big(\lambda f(t,y+w,w') + (1-\lambda)(y+w+w')\Big)\\
&= y\Big(\lambda f(t,y+w,B) + (1-\lambda)(y+w+B) - \lambda f(t,w,B)\\
&\qquad - (1-\lambda)(w+B) + \lambda f(t,w,B) + (1-\lambda)(w+B)\Big)\\
&= y\Big(\lambda f_x(t,\theta y+w,B)y + (1-\lambda)y\Big) + y\Big(\lambda f(t,w,B) + (1-\lambda)(w+B)\Big)
\end{aligned}$$

for any $\theta \in (0,1)$. Then

$$yh(t,y,0,\lambda) \ge y^2(\lambda K + (1-\lambda)) - |y|L_+ \ge |y|(|y|\min\{1,K\} - L_+),$$

from where it follows

$$yh(t,y,0,\lambda) > 0 \text{ for } t \in [0,1] \text{ and } |y| > \frac{L_+}{\min\{1,k\}}$$

and for each $\lambda \in [0,1]$. Thus,

$$|y(t)| \le \frac{L_+}{\min\{1,k\}} \quad \text{for } t \in [0,1],$$

by Lemma 2.3, which yields the bound for $|x(t)|$. □

The considerations used in the proofs of the following statements are standard for proofs of lemmas based on barrier strips conditions. These results provide a priori bounds for the $(n-1)$th derivative of the nth-order differential equations, see for example Lemma 2 of R. P. Agarwal and P. Kelevedjiev [1], Lemma 3.1 of P. Kelevedjiev and T. Todorov [7] and Theorem 3.1 of P. Kelevedjiev [8].

Lemma 2.7. *Let* **(B$_1$)** *hold for constants M_0, $G_m = (A - \alpha M_0)\beta^{-1}$ and $G_M = (A + \alpha M_0)\beta^{-1}$. Then each solution $x \in C^2[0,1]$ to (1.1)$^-_\lambda$, (1.2) with the property*

$$-M_0 \le x(t) \le M_0, \qquad t \in [0,1],$$

satisfies the bound

$$F_1 \le x'(t) \le L_1, \qquad t \in [0,1].$$

Proof. Reasoning by contradiction, assume that $x'(t) > L_1$ for some $t \in (0,1]$. Then, the continuity of $x'(t)$ on $[0,1]$ and

$$x'(0) = \frac{A + \alpha x(0)}{\beta} \leq \frac{A + \alpha M_0}{\beta} \leq L_1$$

imply that the set

$$S_- = \{t \in [0,1] : L_1 < x'(t) \leq L_2\}$$

is not empty. Also, that there is a $\gamma \in S_-$ such that

$$x''(\gamma) > 0.$$

Because $x(t)$ is a $C^2[0,1]$-solution to $(1.1)^-_\lambda$, we have in particular

$$x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) + (1-\lambda)(x(\gamma) - x'(\gamma)).$$

Now, from $(\gamma, x(\gamma), x'(\gamma)) \in S_- \times [-M_0, M_0] \times (L_1, L_2]$ and (1.10) we get

$$\lambda f(\gamma, x(\gamma), x'(\gamma)) \leq 0,$$

and $x'(\gamma) > L_1 \geq M_0 \geq x(\gamma)$ yields $x(\gamma) - x'(\gamma) < 0$. As a result

$$x''(\gamma) \leq 0,$$

a contradiction. Thus,

$$x'(t) \leq L_1 \text{ for } t \in [0,1].$$

Similarly, assuming that $x'(t) < F_1$ for some $t \in (0,1]$ and using that

$$x'(0) = \frac{A + \alpha x(0)}{\beta} \geq \frac{A - \alpha M_0}{\beta} \geq F_1,$$

we establish that

$$S_+ = \{t \in [0,1] : F_2 \leq x'(t) < F_1\}$$

is the empty set and so

$$x'(t) \geq F_1 \quad \text{for } t \in [0,1].$$

□

Lemma 2.8. *Let* **(B$_2$)** *hold for constants* M_0, $G_m = (B - aM_0)b^{-1}$ *and* $G_M = (B + aM_0)b^{-1}$. *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^+_\lambda$, (1.2) *with the property*

$$-M_0 \leq x(t) \leq M_0, \qquad t \in [0,1],$$

satisfies the bound

$$F_1' \leq x'(t) \leq L_1', \qquad t \in [0,1].$$

Proof. By contradiction, assume that $x'(t) > L_1'$ for some $t \in [0,1)$. This means that the set

$$S_+ = \{t \in [0,1] : L_1' < x'(t) \leq L_2'\}$$

is not empty because $x'(t)$ is continuous on $[0,1]$ and

$$x'(1) = \frac{B - ax(1)}{b} \leq \frac{B + aM_0}{b} \leq L_1'.$$

Also, there is a $\gamma \in S_+$ such that
$$x''(\gamma) < 0.$$
As $x(t)$ is a $C^2[0,1]$-solution to (1.1)$_\lambda^+$,
$$x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) + (1-\lambda)(x(\gamma) + x'(\gamma)).$$
Now, $(\gamma, x(\gamma), x'(\gamma)) \in S_+ \times [-M_0, M_0] \times (L_1', L_2']$ and (1.12) imply
$$\lambda f(\gamma, x(\gamma), x'(\gamma)) \geq 0.$$
Besides, $x(\gamma) + x'(\gamma) > 0$ because $x'(\gamma) > L_1' \geq M_0 \geq x(\gamma) \geq -M_0$. Thus,
$$x''(\gamma) \geq 0,$$
a contradiction. Consequently,
$$x'(t) \leq L_1' \quad \text{for } t \in [0,1].$$

Along similar lines, assuming on the contrary that the set
$$S_- = \{t \in [0,1] : F_2' \leq x'(t) < F_1'\}$$
is not empty and using that
$$x'(1) = \frac{B - ax(0)}{b} \geq \frac{B - aM_0}{b} \geq F_1',$$
we achieve a contradiction which implies that
$$F_1' \leq x'(t) \quad \text{for } t \in [0,1]. \qquad \square$$

Lemma 2.9. *Let* **(B$_1$)** *hold for constants* M_0, $G_m = \min\{-M_0, A\}$ *and* $G_M = \max\{M_0, A\}$. *Then each solution* $x \in C^2[0,1]$ *to* (1.1)$_\lambda^-$, (1.3) *with the property*
$$-M_0 \leq x(t) \leq M_0, \qquad t \in [0,1],$$
satisfies the bound
$$F_1 \leq x'(t) \leq L_1, \qquad t \in [0,1].$$

Proof. Let, on the contrary, $x'(t) < F_1$ for some $t \in (0,1]$. Then, the continuity of $x'(t)$ on $[0,1]$ and $x'(0) = A \geq F_1$ imply that the set
$$S_+ = \{t \in [0,1] : F_2 \leq x'(t) < F_1\}$$
is not empty and there is a $\gamma \in S_+$ such that
$$x''(\gamma) < 0.$$
On the other hand, since $x(t)$ is a $C^2[0,1]$-solution to (1.1)$_\lambda^-$, we have in particular
$$x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) + (1-\lambda)(x(\gamma) - x'(\gamma)).$$
But, $(\gamma, x(\gamma), x'(\gamma)) \in S_+ \times [-M_0, M_0] \times [F_2, F_1)$ and (1.11) imply
$$\lambda f(\gamma, x(\gamma), x'(\gamma)) \geq 0,$$

and $x(\gamma) - x'(\gamma) > 0$, because $x'(\gamma) < F_1 \le -M_0 \le x(\gamma)$. As a result

$$x''(\gamma) \ge 0$$

and a contradiction is achieved. Thus,

$$x'(t) \ge F_1 \quad \text{for } t \in [0,1].$$

Similar reasoning based on the use of (1.10) shows that

$$S_- = \{t \in [0,1] : L_1 < x'(t) \le L_2\}$$

is empty and so

$$x'(t) \le L_1 \quad \text{for } t \in [0,1]. \qquad \square$$

The proof of the next assertion is omitted, it follows the lines of proof of Lemmas 2.7, 2.8 and 2.9.

Lemma 2.10. *Let* **(B$_2$)** *hold for constants* M_0, $G_m = \min\{-M_0, B\}$ *and* $G_M = \max\{M_0, B\}$*. Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^+_\lambda$, (1.4) *with the property*

$$-M_0 \le x(t) \le M_0, \qquad t \in [0,1],$$

satisfies the bound

$$F_1' \le x'(t) \le L_1', \qquad t \in [0,1].$$

3 Existence results

BVP $(1.1)^-_0$, (1.2), arising from $(1.1)^-_\lambda$, (1.2) for $\lambda = 0$, has the form

$$\begin{aligned} x'' + x' - x &= 0, & t &\in (0,1), \\ -\alpha x(0) + \beta x'(0) &= A, & a x(1) + b x'(1) &= B. \end{aligned}$$

By standard reasoning, we obtain that the solutions of the differential equation are the functions $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$, and the system for C_1 and C_2 gives a unique solution to the BVP if

$$\begin{vmatrix} \beta r_1 - \alpha & \beta r_2 - \alpha \\ (a + b r_1) e^{r_1} & (a + b r_2) e^{r_2} \end{vmatrix} \ne 0, \tag{3.1}$$

where $r_1 = -\frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{-1+\sqrt{5}}{2}$ are the roots of the characteristic equation.

Theorem 3.1. *Let* **(A$_1$)** *hold,* **(B$_1$)** *hold for*

$$M_0 = \max\{|A/\alpha|, |B/a|, M\}, \qquad G_m = (A - \alpha M_0)\beta^{-1} \quad \textit{and} \quad G_M = (A + \alpha M_0)\beta^{-1},$$

(C) *hold for the same constant* M_0 *and for* $m_0 = -M_0, m_1 = F_1$, $M_1 = L_1$ *and* (3.1) *be satisfied. Then BVP* (1.1), (1.2) *has at least one solution in* $C^2[0,1]$.

Proof. We divide the proof into two steps.

First step. We will check the hypotheses of Theorem 1.1 for the family of BVPs $(1.1)_\lambda^-$, (1.2) and BVP (1.1), (1.2) assuming firstly that the inequality in **($\mathbf{A_1}$)** is strong, that is,

$$xf(t,x,0) > 0 \quad \text{for } t \in [0,1] \text{ and } x \in D_x \setminus [-M,M]. \tag{3.2}$$

Condition *(i)* is fulfilled because we know from above that (3.1) guarantees a unique $C^2[0,1]$-solution to BVP $(1.1)_0^-$, (1.2). Apparently *(ii)* also holds. To check *(iii)* we establish, by standard reasoning, that for an arbitrary $y(t) \in C[0,1]$ the BVP

$$\begin{gathered} x'' = y(t) \\ -\alpha x(0) + \beta x'(0) = 0, \qquad ax(1) + bx'(1) = 0, \end{gathered}$$

has a unique solution in $C^2[0,1]$, which means that the map $\Lambda_h : C^2_{B_0}[0,1] \to C[0,1]$, defined by $\Lambda_h x = x''$, is one-to-one. Besides, for each solution $x(t) \in C^2[0,1]$ to $(1.1)_\lambda^-$, (1.2) we have

$$m_0 \le x(t) \le M_0, \qquad t \in [0,1], \quad \text{by Lemma 2.4},$$

and

$$m_1 \le x'(t) \le M_1, \qquad t \in [0,1], \quad \text{by Lemma 2.7}.$$

In view of **(C)**, the function $f(t,x,p)$ is continuous on $[0,1] \times [m_0,M_0] \times [m_1,M_1]$. Thus, there are constants m_2 and M_2 such that

$$\begin{aligned} &m_2 \le \lambda f(t,x,p) + (1-\lambda)(x-p) \le M_2 \\ &\text{for} \quad \lambda \in [0,1] \quad \text{and} \quad (t,x,p) \in [0,1] \times [m_0,M_0] \times [m_1,M_1]. \end{aligned}$$

Since for $t \in [0,1]$ we have $(x(t),x'(t)) \in [m_0,M_0] \times [m_1,M_1]$, the equation $(1.1)_\lambda^-$ implies

$$m_2 \le x''(t) \le M_2 \quad \text{for } t \in [0,1].$$

Hence, *(iv)* also holds. Finally, *(v)* follows again from **(C)**. Therefore, we can apply Theorem 1.1 to conclude that assertion is true when we have (3.2).

Second step. Now, assuming that **($\mathbf{A_1}$)** is in the form given, consider the family of BVPs

$$\begin{cases} x'' = f_n(t,x,x'), \ t \in (0,1), \\ -\alpha x(0) + \beta x'(0) = A, \ ax(1) + bx'(1) = B, \end{cases} \tag{3.3}$$

where $f_n(t,x,x') = f(t,x,x') + (x-x') : n, \ n = 1,2,3,\ldots$. Clearly,

$$xf_n(t,x,0) = xf(t,x,0) + \frac{x^2}{n} > 0 \text{ for } t \in [0,1], x \in D_x \setminus [-M,M],$$

that is, (3.2) is satisfied for each $n = 1,2,3,\ldots$. Besides,

$$\begin{aligned} &\frac{x-p}{n} \le 0 \quad \text{for } x \in [-M_0,M_0] \text{ and } p \ge L_1 \ge M_0, \\ &\frac{x-p}{n} \ge 0 \quad \text{for } x \in [-M_0,M_0] \text{ and } p \le F_1 \le -M_0, \end{aligned}$$

and **($\mathbf{B_1}$)** imply

$$f_n(t,x,p) \le 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0,M_0] \times [L_1,L_2],$$

and

$$f_n(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0, M_0] \times [F_2, F_1].$$

Thus, family (3.3) satisfies **(B_1)** for all $n = 1,2,3,\dots$. Obviously, (3.3) satisfies **(C)** as well. Finally, to verify (3.1), we just have to consider that for each fixed $n = 1,2,3,\dots$, the λ-family corresponding to (3.3) is

$$x'' = \lambda\left(f(t,x,x') + \frac{x-x'}{n}\right) + (1-\lambda)(x-x'), \qquad \lambda \in [0,1],$$
$$-\alpha x(0) + \beta x'(0) = A, \qquad ax(1) + bx'(1) = B,$$

and to see that for $\lambda = 0$ we have again BVP $(1.1)_0^-$, (1.2). So, taking into consideration the proved in *First step*, we conclude that family (3.3) has a solution $x_n \in C^2[0,1]$ for each $n = 1,2,3,\dots$.

Further, just as in the proof of Lemma 2.4, we find that

$$|x_n(t)| \leq M_0 \quad \text{for } t \in [0,1],$$

and just as in the proof of Lemma 2.7 establish that

$$F_1 \leq x_n'(t) \leq L_1 \quad \text{for } t \in [0,1]$$

for each $n = 1,2,3,\dots$. Finally,

$$|f_n(t,x,p)| \leq |f(t,x,p)| + \frac{|x-p|}{n} \leq |f(t,x,p)| + |x-p|$$

together with the continuity of $f(t,x,p)$ on $[0,1] \times [-M_0, M_0] \times [F_1, L_1]$ implies that there is a constant M_2 independent of n such that

$$|x_n''(t)| \leq M_2 \quad \text{for } t \in [0,1].$$

The estimates for $|x_n(t)|$, $|x_n'(t)|$ and $|x_n''(t)|$ allow us to use the Arzelà–Ascoli theorem to extract a subsequence $\{x_{n_k}\}$, $k = 1,2,3,\dots$, of $\{x_n\}$ converging uniformly on $[0,1]$ to a function $x \in C^1[0,1]$. We will show that $x(t)$ is a $C^2[0,1]$-solution to BVP (1.1), (1.2). For this purpose introduce the sequence $\{y_{n_k}\}$, $k = 1,2,3,\dots$, where $y_{n_k}(t) = x_{n_k}(t) - r(t)$, $t \in [0,1]$, with

$$r(t) = \frac{\alpha B + aA}{\alpha a + \alpha b + a\beta} t - \frac{Aa + Ab - \beta B}{\alpha a + \alpha b + a\beta}.$$

Clearly, $\{y_{n_k}\}$ converges uniformly on $[0,1]$ to the function $y(t) = x(t) - r(t)$ and $y \in C^1[0,1]$. Besides, since $r(t)$ satisfies BCs (1.2), $y_{n_k}(t)$ is a solution to the BVP

$$y_{n_k}'' = f_{n_k}(t, y_{n_k} + r, y_{n_k}' + r'), \qquad t \in (0,1),$$
$$-\alpha y_{n_k}(0) + \beta y_{n_k}'(0) = 0, \qquad a y_{n_k}(1) + b y_{n_k}'(1) = 0,$$

and its integral form

$$y_{n_k}(t) = \int_0^1 G(t,s) f_{n_k}(s, y_{n_k}(s) + r(s), y_{n_k}'(s) + r'(s))ds, \tag{3.4}$$

for $k = 1,2,3,\dots$, where $G(t,s)$ is the Green function for the BVP

$$x'' = 0, \qquad t \in (0,1),$$

$$-\alpha x(0) + \beta x'(0) = 0, \qquad ax(1) + bx'(1) = 0,$$

and

$$f_{n_k}(t, y_{n_k} + r, y'_{n_k} + r') = f(t, y_{n_k} + r, y'_{n_k} + r') + \frac{y_{n_k} + r - y'_{n_k} - r'}{n_k}.$$

Since $|y_{n_k} + r - y'_{n_k} - r'| = |x_{n_k} - x'_{n_k}| \le M_0 + \max\{|F_1|, L_1\}$, letting $k \to \infty$ in (3.4), we obtain

$$y(t) = \int_0^1 G(t,s) f(s, y(s) + r(s), y'(s) + r'(s)) ds.$$

Thus,

$$x(t) = \int_0^1 G(t,s) f(s, x(s), x'(s)) ds + r(t),$$

which means that $x(t)$ is a $C^2[0,1]$-solution to BVP (1.1), (1.2). □

Now let us look at BVP $(1.1)_0^+$, (1.2)

$$\begin{aligned} x'' - x' - x = 0, &\qquad t \in (0,1), \\ -\alpha x(0) + \beta x'(0) = A, &\qquad ax(1) + bx'(1) = B. \end{aligned}$$

It has a unique solution if

$$\begin{vmatrix} \beta s_1 - \alpha & \beta s_2 - \alpha \\ (a + b s_1) e^{s_1} & (a + b s_2) e^{s_2} \end{vmatrix} \neq 0, \tag{3.5}$$

where $s_1 = \frac{1-\sqrt{5}}{2}$ and $s_2 = \frac{1+\sqrt{5}}{2}$ are the roots of the characteristic equation.

Theorem 3.2. *Let* **(A_1)** *hold,* **(B_2)** *hold for*

$$M_0 = \max\{|A/\alpha|, |B/a|, M\}, \qquad G_m = (B - aM_0)b^{-1} \quad \text{and} \quad G_M = (B + aM_0)b^{-1},$$

(C) *hold for the same constant M_0 and for $m_0 = -M_0, m_1 = F'_1$, $M_1 = L'_1$ and (3.5) be satisfied. Then BVP (1.1), (1.2) has at least one solution in $C^2[0,1]$.*

Proof. It is similar to the proof of the previous theorem. Now we will apply Theorem 1.1 on the family of BVPs $(1.1)_\lambda^+$, (1.2).

First step. Again assume firstly that the inequality in **(A_1)** is strong, that is, **(A)** holds.

The condition *(i)* follows from (3.5). The condition *(ii)* is again obvious, and the verification of *(iii)* is as in Theorem 3.1. Now, the bounds

$$m_0 \le x(t) \le M_0 \quad \text{and} \quad m_1 \le x'(t) \le M_1, \qquad t \in [0,1],$$

for each solution $x(t) \in C^2[0,1]$ to $(1.1)_\lambda^+$, (1.2), follow by Lemmas 2.4 and 2.8, respectively, and the bound

$$m_2 \le x''(t) \le M_2 \quad \text{for } t \in [0,1]$$

follows by arguments similar to those in the proof of Theorem 3.1. Thus, *(iv)* holds. Finally, *(v)* follows again from **(C)**. So, the assertion is true by Theorem 1.1.

Second step. Now, **(A_1)** is in the form given. Consider the family of BVPs

$$\begin{cases} x'' = f_n(t, x, x'), \ t \in (0,1), \\ -\alpha x(0) + \beta x'(0) = A, \ ax(1) + bx'(1) = B, \end{cases} \tag{3.6}$$

where $f_n(t,x,x') = f(t,x,x') + (x+x') : n,\ n = 1,2,3,\dots$. Clearly,

$$xf_n(t,x,0) = xf(t,x,0) + \frac{x^2}{n} > 0 \quad \text{for } t \in [0,1], x \in D_x \setminus [-M,M],$$

that is, **(A)** is satisfied for each $n = 1,2,3,\dots$. Besides,

$$\frac{x+p}{n} \geq 0 \quad \text{for } x \in [-M_0, M_0] \text{ and } p \geq L_1' \geq M_0,$$
$$\frac{x+p}{n} \leq 0 \quad \text{for } x \in [-M_0, M_0] \text{ and } p \leq F_1' \leq -M_0,$$

and **(B$_2$)** imply

$$f_n(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0, M_0] \times [L_1', L_2'],$$

and

$$f_n(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0, M_0] \times [F_2', F_1'].$$

Thus, family (3.6) satisfies **(B$_2$)** for all $n = 1,2,3,\dots$. Obviously, (3.6) satisfies and **(C)**. As in the proof of Theorem 3.1, we notice that (3.5) is also fulfilled. Consequently, according to what we proved in the *First step*, each boundary value problem arising from the family (3.6) for $n = 1,2,3,\dots$ has a solution $x_n \in C^2[0,1]$.

Further, just as in the proof of Lemma 2.4, we find that

$$|x_n(t)| \leq M_0 \quad \text{for } t \in [0,1],$$

and just as in the proof of Lemma 2.8 establish that

$$F_1' \leq x_n'(t) \leq L_1' \quad \text{for } t \in [0,1]$$

for each $n = 1,2,3,\dots$ Besides, the continuity of $|f(t,x,p)| + |x+p|$ on the compact set $[0,1] \times [-M_0, M_0] \times [F_1', L_1']$ implies that there is a constant M_2 independent of n such that

$$|x_n''(t)| \leq M_2 \quad \text{for } t \in [0,1].$$

By the Arzelà–Ascoli theorem the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$, $k = 1,2,3,\dots$, converging uniformly on $[0,1]$ to a function $x \in C^1[0,1]$. Next, arguing as in the proof of Theorem 3.1, establish that $x(t)$ is a $C^2[0,1]$-solution to BVP (1.1), (1.2). □

It is standardly found that the BVP

$$x'' + x' - x = 0, \quad t \in (0,1),$$
$$x'(0) = A, \quad ax(1) + bx'(1) = B,$$

in fact this is $(1.1)_0^-$, (1.3), has a unique solution in $C^2[0,1]$ if

$$\begin{vmatrix} r_1 & r_2 \\ (a+br_1)e^{r_1} & (a+br_2)e^{r_2} \end{vmatrix} \neq 0, \tag{3.7}$$

where r_1 and r_2 are as in (3.1).

Theorem 3.3. *Let* **(A$_2$)** *hold for* $J = [m_v, M_v]$ *and* $C = A$, **(B$_1$)** *hold for constants*

$$M_0 = \frac{L_-}{\min\{1,K\}} + \max\{|m_v|, |M_v|\}, \qquad G_m = \min\{-M_0, A\} \quad \text{and} \quad G_M = \max\{M_0, A\},$$

(C) *hold for the same constant* M_0 *and for* $m_0 = -M_0, m_1 = F_1$ *and* $M_1 = L_1$ *and* (3.7) *be satisfied. Then BVP* (1.1), (1.3) *has at least one solution in* $C^2[0,1]$.

Proof. We will apply Theorem 1.1 on family $(1.1)^-_\lambda$, (1.3). The hypothesis *(i)* follows immediately from (3.7). Besides, *(ii)* is obvious. The map $\Lambda_h : C^2_{\mathcal{B}_0}[0,1] \to C[0,1]$, defined by $\Lambda_h x = x''$ is one-to-one, because for an arbitrary $y(t) \in C[0,1]$ the BVP

$$x'' = y(t)$$
$$x'(0) = 0, \qquad ax(1) + bx'(1) = 0,$$

has a unique solution in $C^2[0,1]$, *(iii)* is also fulfilled. For each solution $x(t) \in C^2[0,1]$ to $(1.1)^-_\lambda$, (1.3) we have

$$m_0 \le x(t) \le M_0, \qquad t \in [0,1], \quad \text{by Lemma 2.5},$$
$$m_1 \le x'(t) \le M_1, \qquad t \in [0,1], \quad \text{by Lemma 2.9},$$

and the bound

$$m_2 \le x''(t) \le M_2 \quad \text{for } t \in [0,1]$$

follows as in the proof of Theorem 3.1. So, *(iv)* also holds. Finally, **(C)** ensures the validity of *(v)*. Therefore, we can apply Theorem 1.1 to conclude that the assertion is true. □

It is easily established that BVP $(1.1)^+_0$, (1.4), namely,

$$x'' - x' - x = 0, \qquad t \in (0,1),$$
$$x'(0) = A, \qquad ax(1) + bx'(1) = B,$$

has a unique solution if

$$\begin{vmatrix} \beta s_1 - \alpha & \beta s_2 - \alpha \\ s_1 e^{s_1} & s_2 e^{s_2} \end{vmatrix} \neq 0, \tag{3.8}$$

where s_1 and s_2 are as in (3.5).

Theorem 3.4. *Let* **(A$_2$)** *hold for* $J = [m_w, M_w]$ *and* $C = B$, **(B$_2$)** *hold for constants*

$$M_0 = \frac{L_+}{\min\{1,K\}} + \max\{|m_w|, |M_w|\}, \qquad G_m = \min\{-M_0, B\} \quad \text{and} \quad G_M = \max\{M_0, B\},$$

(C) *hold for the same constant* M_0 *and for* $m_0 = -M_0, m_1 = F_1'$, $M_1 = L_1'$ *and* (3.8) *be satisfied. Then BVP* (1.1), (1.4) *has at least one solution in* $C^2[0,1]$.

Proof. The proof is virtually the same as that of Theorem 3.3. Now, apply Theorem 1.1 on family $(1.1)^+_\lambda$, (1.4). *(i)* follows from (3.8), to check *(iii)* show that for an arbitrary $y(t) \in C[0,1]$ the BVP

$$x'' = y(t)$$
$$-\alpha x(0) + \beta x'(0) = 0, \qquad x'(1) = 0,$$

has a unique solution in $C^2[0,1]$, and the bounds from *(iv)*,

$$m_0 \le x(t) \le M_0, \qquad t \in [0,1],$$
$$m_1 \le x'(t) \le M_1, \qquad t \in [0,1],$$

for each solution $x(t) \in C^2[0,1]$ to $(1.1)^+_\lambda$, (1.4) follow from Lemmas 2.6 and 2.10, respectively. □

4 Examples

The following examples illustrate the application of the obtained results.

Example 4.1. Consider the BVPs for the equation

$$\begin{gathered} x'' = x'(x'-10) + (x-2)^2(x+2), \qquad t \in (0,1), \\ -3x(0) + 2x'(0) = 6, \qquad 2x(1) + x'(1) = -3. \end{gathered}$$

We easily check that **(A_1)** holds for $M = 2$. Calculate $M_0 = 2, G_m = 0$ and $G_M = 6$. Next, keeping in mind that

$$0 \le (x-2)^2(x+2) < 1 \quad \text{for } x \in [-2,2],$$

we can choose, for example, $F_2 = -4, F_1 = -3, L_1 = 7$ and $L_2 = 8$ to see that **(B_1)** also hold. **(C)** is obvious, and (3.1) is also easily verifiable. So, we can apply Theorem 3.1 to conclude that the considered BVP has at least one solution in $C^2[0,1]$.

Example 4.2. Consider the BVP

$$\begin{gathered} x'' = x'^3 + (x-1)\sqrt{x+10}, \qquad t \in (0,1), \\ -x(0) + 4x'(0) = 5, \qquad 2x(1) + x'(1) = 4. \end{gathered}$$

First observe that

$$xf(t,x,0) = x(x-1)\sqrt{x+10} \ge 0 \text{ for } x \in [-10,\infty) \setminus [-1,1],$$

which means that **(A_1)** is satisfied for $M = 1$. Then $M_0 = 5, G_m = -6, G_M = 14$. Besides,

$$-6\sqrt{5} \le (x-1)\sqrt{x+10} \le 4\sqrt{15} \quad \text{for } x \in [-5,5].$$

So, we can choose $F_2' = -8, F_1' = -7, L_1' = 15$ and $L_2' = 16$ to check **(B_2)**. In addition $f(t,x,p) = p^3 + (x-1)\sqrt{x+10}$ is continuous on the set $[0,1] \times [-5-\delta, 5+\delta] \times [-7-\delta, 15+\delta]$, where $\delta > 0$ is sufficiently small, to say $\delta = 0.1$, that is, **(C)** is satisfied. Checking (3.5) also presents no difficulty. So, we can apply Theorem 3.2 to conclude that the considered BVP has a solution in $C^2[0,1]$.

Example 4.3. Consider the BVP

$$\begin{gathered} x'' = \phi(t,x) + P_n(x'), \qquad t \in (0,1), \\ x'(0) = A, \qquad ax(1) + bx'(1) = B, \end{gathered}$$

where the function $\phi\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, differentiable with respect to x and there is a constant $K > 0$ for which

$$\phi_x(t,x) \ge K > 0 \quad \text{for } (t,x) \in [0,1] \times \mathbb{R},$$

the polynomial $P_n(p) = \sum_{k=0}^{n} a_k p^k, n = 2s+1, s \in \mathbf{N}$, is such that $a_n < 0$, and a and b are such that (3.7) holds.

Clearly, **(A_2)** is satisfied. Further, if

$$f_{\max} := \max |\phi(t,x) + P_n(A)| \quad \text{for } (t,x) \in [0,1] \times [m_v, M_v],$$

where m_v and M_v are the constants used in Theorem 3.3, and

$$g_{\max} := \max |x - A| \quad \text{for } (t, x) \in [0,1] \times [m_v, M_v],$$

we determine first

$$L_- = \max\{f_{\max}, g_{\max}\}$$

and then M_0. Because the continuity of ϕ, there are constants $\phi_m, \phi_M \in \mathbb{R}$ such that

$$\phi_m \le \phi(t,x) \le \phi_M \quad \text{for } (t,x) \in [0,1] \times [-M_0, M_0].$$

Then, for sufficiently large $p > \max\{M_0, A\}$ we have

$$P_n(p) < -\max\{|\phi_m|, |\phi_M|\},$$

which implies that the constants L_1 and L_2 of **(B$_1$)** exist. Besides, for sufficiently small $p < \min\{-M_0, A\}$ we have

$$P_n(p) > \max\{|\phi_m|, |\phi_M|\},$$

which means that the constants F_1 and F_2 of **(B$_1$)** also exist. Finally, **(C)** is obvious and so the considered BVP has a solution in $C^2[0,1]$ by Theorem 3.3.

Example 4.4. Consider the BVP

$$x'' = \phi(t,x) + P_n(x'), \qquad t \in (0,1),$$
$$-\alpha x'(0) + \beta x'(0) = A, \qquad x'(1) = B,$$

where the function $\phi[0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, differentiable with respect to x and there is a constant $K > 0$ for which

$$\phi_x(t,x) \ge K > 0 \quad \text{for } (t,x) \in [0,1] \times \mathbb{R},$$

the polynomial $P_n(p) = \sum_{k=0}^{n} a_k p^k, n = 2s+1, s \in \mathbf{N}$, is such that $a_n > 0$, and α and β are such that (3.8) holds.

An analysis similar to that of Example 4.3 shows that we can apply Theorem 3.4 to conclude that this BVP has a solution in $C^2[0,1]$.

Example 4.5. Consider the BVP

$$x'' = x + 10^{-1}x'\sqrt{(x'+14)(11-x')}, \qquad t \in (0,1),$$
$$-x(0) + x'(0) = 3, \qquad x'(1) = 2.$$

We will check the conditions of Theorem 3.4. Here $w(t) = 2t - 1$ with $m_w = -1$ and $M_w = 1$. Then **(A$_2$)** is satisfied for $K = 1$, $J = [-1,1]$ and $C = 2$, because $f_x(t,x,2) = 1$ for $(t,x) \in [0,1] \times \mathbb{R}$. Form $f(t,x,2) = x + 2.4$. Then, for $(t,x) \in [0,1] \times [-1,1]$ we have $\max |f(t,x,2)| = 3.4$ and $\max |x+2| = 3$. We calculate $L_+ = 3.4, M_0 = 4.4, G_m = -4.4$ and $G_M = 4.4$. We are now ready to check that **(B$_2$)** is satisfied for $F_2' = -6, F_1' = -5, L_1' = 5$ and $L_2' = 6$. Keeping in mind that $f(t,x,p)$ is defined and continuous for $(t,x,p) \in [0,1] \times \mathbb{R} \times [-14, 11]$, we easily conclude that **(C)** also holds for $m_0 = -4.4, M_0 = 4.4, m_1 = -5, M_1 = 5$ and, to say, $\delta = 0.01$. Finally, to check that (3.8) is satisfied, we establish that the determinant

$$\begin{vmatrix} \frac{1-\sqrt{5}}{2} - 1 & \frac{1+\sqrt{5}}{2} - 1 \\ \frac{1-\sqrt{5}}{2} e^{\frac{1-\sqrt{5}}{2}} & \frac{1+\sqrt{5}}{2} e^{\frac{1+\sqrt{5}}{2}} \end{vmatrix}$$

is different from zero. So, we can apply Theorem 3.4 to conclude that the considered BVP has a solution in $C^2[0,1]$.

5 Conclusions

Here we will comment on conditions (3.1), (3.5), (3.7) and (3.8). In fact, they are not essential. If, for example, (3.1) is not fulfilled, we can replace $(1.1)^{-}_{\lambda}$ with

$$x'' = \lambda f(t, x, x') + (1 - \lambda)(x - kx'), \qquad t \in (0, 1),$$

where $\lambda \in [0, 1]$ and $k > 0$. Now $(1.1)^{-}_{0}$ has the form

$$x'' + kx' - x = 0$$

and we can choose k such that its characteristic equation has roots r_1 and r_2 for which (3.1) is satisfied. This necessitates a slight change in **(B_1)**, namely

$$F_2 < F_1 \le \min\{-M_0/k, G_m\}, \qquad \max\{M_0/k, G_M\} \le L_1 < L_2,$$

References

[1] R. P. Agarwal, P. Kelevedjiev, On the solvability of fourth-order two-point boundary value problems, *Mathematics* **8**(2020), 603. https://doi.org/10.3390/math8040603

[2] R. P. Agarwal, T. Zh. Todorov, P. Kelevedjiev, Existence for two-point nth order boundary value problems under barrier strips, *Symmetry* **15**(2023), 1394. https://doi.org/10.3390/sym15071394

[3] M. Dobkevich, F. Sadyrbaev, Types and multiplicity of solutions to Sturm–Liouville boundary value problem, *Math. Model. Anal.* **20**(2015), No. 1, 1–8. https://doi.org/10.3846/13926292.2015.996259

[4] A. M. A. El-Sayed, E. M. A. Hamdallah, H. M. A. Alama, Multiple solutions of Sturm–Liouville boundary value problem of nonlinear differential inclusion with nonlocal integral conditions, *AIMS Math.* **7**(2022), No. 6, 11150–11164. https://doi.org/10.3934/math.2022624

[5] A. M. A. El-Sayed, M. Sh. Mohamed, R. E. M. Embia, On the multiple solutions of a nonhomogeneous Sturm–Liouville equation with nonlocal boundary conditions, *Internat. J. Appl. Math.* **32**(2019), No. 1, 35–43. https://doi.org/10.12732/ijam.v32i1.3

[6] A. Granas, R. B. Guenther, J. W. Lee, Nonlinear boundary value problems for ordinary differential equations, Instytut Matematyczny Polskiej Akademi Nauk, Warszawa, 1985, 133 pp. Published in: *Dissertationes Math. (Rozprawy Mat.)* **244**(1985). MR808227

[7] P. Kelevedjiev, T. Zh. Todorov, Existence of solutions of nonlinear third-order two-point boundary value problems, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 23, 1–15. https://doi.org/10.14232/ejqtde.2019.1.23

[8] P. Kelevedjiev, Existence of solutions for two-point boundary value problems, *Nonlinear Anal.* **22** (1994), No. 2, 217–224. https://doi.org/10.1016/0362-546X(94)90035-3

[9] Y. Liu, S. Chen, L. Ou, Solvability of Sturm–Liouville boundary value problems for multiple order fractional differential equations, *Arab. J. Math. Sci.* **22**(2016), No. 2, 207–231. https://doi.org/10.1016/j.ajmsc.2015.08.001

[10] C. C. Tisdell, Existence of solutions to second-order boundary value problems without growth restrictions, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 92, 1–14. https://doi.org/10.14232/ejqtde.2016.1.92

[11] F. H. Wong, T. G. Chen, S. P. Wang, Existence of positive solutions for various boundary value problems, *Comput. Math. Appl.* **56**(2008), No. 4, 953–958.https://doi.org/10.1016/j.camwa.2007.11.050

[12] T. Xue, F. Kong, L. Zhang, Research on Sturm–Liouville boundary value problems of fractional p-Laplacian equation, *Adv. Differ. Equ.* **2021**, No. 177. https://doi.org/10.1186/s13662-021-03339-3

[13] L. Zhang, X. Huang, W. Ge, Positive solutions of Sturm–Liouville boundary value problems in presence of upper and lower solutions, *Internat. J. Differ. Equ.* **2011**, ID 383294, pp. 11. https://doi.org/10.1155/2011/383294