Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **65**, 1–21; https://doi.org/10.14232/ejqtde.2024.1.65 www.math.u-szeged.hu/ejqtde/

Mirroring in lattice equations and a related functional equation

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Received 7 June 2024, appeared 21 November 2024
Communicated by Mihály Pituk

Abstract. We use a functional form of the mirroring technique to fully characterize equivalence classes of unbounded stationary solutions of lattice reaction-diffusion equations with eventually negative and decreasing nonlinearities. We show that solutions which connect a stable fixed point of the nonlinearity with infinity can be characterized by a single parameter from a bounded interval. Within a two-dimensional parametric space, these solutions form a boundary to an existence region of solutions which diverge in both directions. Additionally, we reveal a natural relationship of lattice equations with an interesting functional equation which involves an unknown function and its inverse.

Keywords: Nagumo equation, lattice differential equation, patterns, unbounded solutions, equivalence class.

2020 Mathematics Subject Classification: 34A33, 39A12, 39A22, 34B22.

1 Introduction

We study a special class of unbounded stationary solutions to reaction-diffusion lattice differential equations (LDE)

$$u_i'(t) = d(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) + g(u_i(t)), \quad i \in \mathbb{Z}, \quad t > 0, \tag{1.1}$$

in which $d > 0$ is a diffusion rate and g is a reaction function. We assume that g is a C^1-function and satisfies the following assumptions:

(g_1) $g(\ell) = 0$ for some $\ell \in \mathbb{R}$,

(g_2) $g'(u) < 0$ for all $u \in [\ell, \infty)$.

Let us immediately note that general assumptions (g_1)–(g_2) cover well-known and widely studied prototypes of monostable and bistable dynamics – the Fisher lattice equation (with logistic reaction) and the Nagumo lattice equation (with cubic reaction) as well as many others reactions, their modifications, and caricatures which have been commonly used in numerous studies on lattice equations [2,5,12,15,21]. See Section 4 for detailed examples.

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The primary object of our interest is a class of unbounded stationary solutions which connect the stationary point ℓ with infinity. Therefore, we refer to them as onesided unbounded solutions.

Bounded stationary and traveling patterns

The LDE (1.1) serves as a discrete-space counterpart of the reaction-diffusion partial differential equation (PDE)

$$u_t(x,t) = du_{xx}(x,t) + g(u(x,t)), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

which occurs naturally as a model to many biological and chemical processes and has inspired many mathematical concepts and techniques – traveling wave solutions $u(x,t) = \Phi(x - ct)$, perturbation techniques, stability of waves, etc. [13].

Recent interest in the LDE (1.1) stems from its natural applications (see, e.g., [15]) and the fact that the discrete space provides new or richer dynamic phenomena in comparison with the PDE (1.2). The most notable among those is the pinning of traveling waves for sufficiently small diffusion parameters. The Nagumo lattice equation is a prototype of wave pinning [12, 16, 21]. In other words, for sufficiently small $d > 0$, solutions of the form $u_i(t) = \Phi(i - ct)$ of the LDE (1.1) with the bistable nonlinearity with a monotone profile Φ satisfy $c = 0$. The phenomenon is general and has been studied in various discrete-time models [11] and systems of lattice equations [4, 6].

The presence of pinning regions in discrete-space models is naturally related to spatial topological chaos, the existence of large number of bounded heterogeneous stationary patterns of the LDE (1.1), [2, 15]. Stationary solutions of the LDE (1.1) are double-sequences $u = (u_i)$, $i \in \mathbb{Z}$, satisfying difference equations

$$d(u_{i-1} - 2u_i + u_{i+1}) + g(u_i) = 0, \quad i \in \mathbb{Z}. \tag{1.3}$$

The structure of large number of solutions for small diffusion $0 < d \ll 1$ is still not fully understood. Partial results are related to localized pulses and their bifurcations [1] or the ordering and symmetry of exponential number of k-periodic patterns [9]. Explicit forms of specific solutions have been found for piecewise linear nonlinearities [3, 5, 18, 20]. Connections of stable periodic patterns then lead to existence of nonmonotone waves, [8, 10]. However, many fascinating open questions remain unanswered. These are related, for example, to bifurcations of pulses [1] finite-dimensional graph reaction-diffusion equations which are connected to k-periodic patterns [19] but also to the broader picture, e.g., coexistence of bounded and unbounded patterns and a related ambition to describe all types of nonnegative patterns of the LDE (1.1). The goal of this paper is to contribute by describing onesided unbounded patterns and as a by-product describe mirroring functional iterations and a relationship to a functional equation.

Unbounded stationary patterns

In [7] we fully characterized equivalence classes of generally asymmetric twosided unbounded stationary solutions of (1.1) with (g_1)–(g_2) being satisfied such that $u_i > \ell$ for every $i \in \mathbb{Z}$ and

$$\lim_{i \to \pm\infty} u_i = \infty, \tag{1.4}$$

see Figure 1.1. In contrast to twosided unbounded solutions of the PDE (1.2) we have shown that the twosided unbounded solutions (i) form a two-parametric family of equivalence classes,

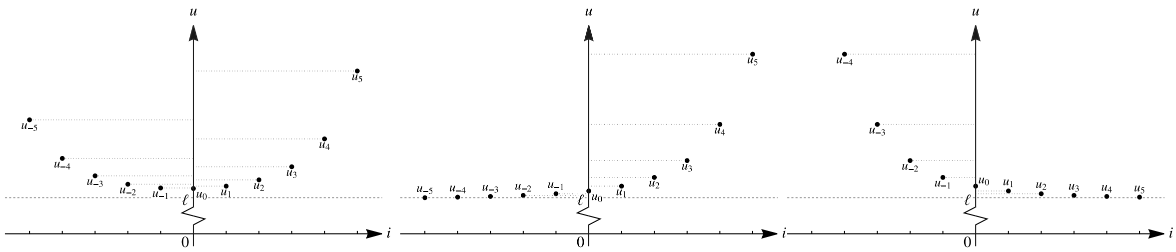

Figure 1.1: A twosided unbounded stationary solution of (1.1) satisfying (1.4) (left panel) and onesided unbounded stationary solutions satisfying either (1.5) or (1.6) (center and right panel).

(ii) are generally asymmetric, and (iii) exist on the whole unbounded integer lattice $\mathbb{Z}$ (i.e., they do not blow up at the ends of bounded spatial interval), see Theorem 3.1 below. Finally, in contrast to bounded patterns, twosided unbounded patterns of the LDE (1.1) exist for all diffusion values $d > 0$.

Onesided unbounded stationary solutions

Motivated by miscellaneous types of stationary solutions of (1.1), we primarily focus on the characterization of other type of unbounded stationary solutions, specifically, onesided unbounded stationary solutions which satisfy $u_i > \ell$ for all $i \in \mathbb{Z}$ and either (see Figure 1.1)

$$\lim_{i\to-\infty} u_i = \ell \quad \text{and} \quad \lim_{i\to\infty} u_i = \infty, \tag{1.5}$$

or

$$\lim_{i\to-\infty} u_i = \infty \quad \text{and} \quad \lim_{i\to\infty} u_i = \ell. \tag{1.6}$$

The equations (1.1) and (1.3) are autonomous in the spatial variable $i \in \mathbb{Z}$. Thus, every solution generates an equivalence class of another solutions which are only shifted in i. For this purpose, we say that stationary solutions u, u^* of (1.1) are *equivalent* (denoted $u \sim u^*$) if there exists an $s \in \mathbb{Z}$ such that $u_{i+s} = u_i^*$ for every $i \in \mathbb{Z}$. The equivalence class represented by a solution u^* is denoted by $[u^*] = \left\{u \in \mathbb{R}^{\mathbb{Z}} : \; u \sim u^*\right\}$.

The first main result of this manuscript characterizes the onesided stationary solutions of (1.1).

Theorem 1.1. *Let g be a C^1-function and satisfy (g_1)–(g_2). There exists a unique function $f : [\ell, \infty) \to [\ell, \infty)$ which is continuous, strictly increasing with $f(\ell) = \ell$, $\lim_{u\to\infty} f(u) = \infty$, and $f(u) > u$ for all $u > \ell$ such that for arbitrary $\xi > \ell$ every $\alpha \in [\xi, f(\xi))$ determines an equivalence class $\left[u^{\alpha,I}\right]$ of strictly increasing stationary solutions u of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.5); and an equivalence class $\left[u^{\alpha,D}\right]$ of strictly decreasing stationary solutions u of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.6). The representatives $u^{\alpha,I}$ and $u^{\alpha,D}$ satisfy*

$$u_0^{\alpha,I} = \alpha \quad \textit{and} \quad u_1^{\alpha,I} = f(\alpha); \tag{1.7}$$

and

$$u_0^{\alpha,D} = \alpha \quad \textit{and} \quad u_1^{\alpha,D} = f^{-1}(\alpha), \tag{1.8}$$

respectively. Moreover, for every $\tilde{\alpha} \in [\xi, f(\xi))$ there is $\left[u^{\alpha,I}\right] \neq \left[u^{\tilde{\alpha},I}\right]$ and $\left[u^{\alpha,D}\right] \neq \left[u^{\tilde{\alpha},D}\right]$ provided $\alpha \neq \tilde{\alpha}$.

On the contrary, every stationary solution u of (1.1) satisfying $u_i > \ell$ for every $i \in \mathbb{Z}$ and (1.5)

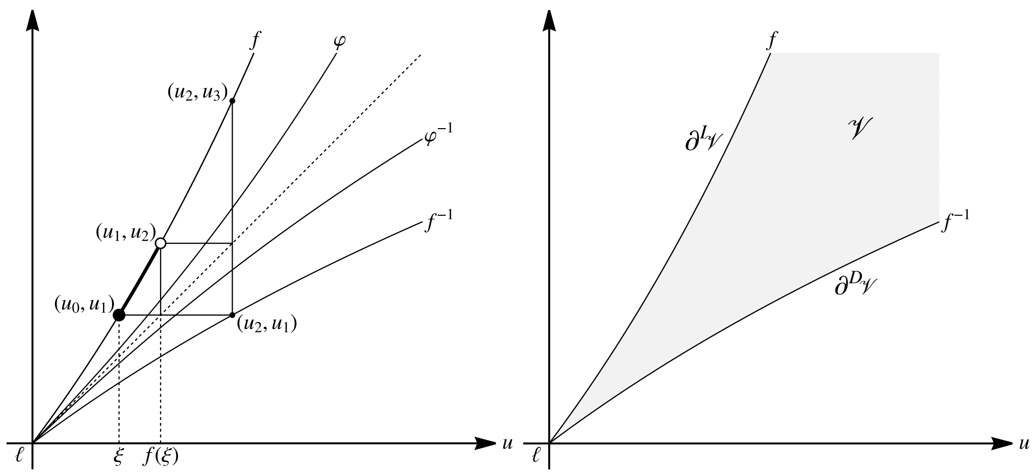

Figure 1.2: Illustration of two main results. Theorem 1.1 implies the existence of a function f and the corresponding interval $[\xi, f(\xi))$. Each value from this interval characterizes an equivalence class of increasing (1.5) or decreasing (1.6) onesided unbounded solutions. Mirroring symmetry via φ from (1.11) and cobwebbing which are used to construct the solutions are indicated (left panel). Theorem 1.2 then shows that the curves f and f^{-1} form a boundary to a set $\mathscr{V}$. All pairs $(u_i, u_{i+1}) \in \mathscr{V}$ generate twosided unbounded solutions with (1.4), whereas all pairs $(u_i, u_{i+1}) \in \partial^I\mathscr{V}$ or $(u_i, u_{i+1}) \in \partial^D\mathscr{V}$ lead to onesided unbounded solutions with (1.5) or (1.6) (right panel).

is strictly increasing and belongs into one of the above described equivalence classes $[u^{\alpha,I}]$ *for some* $\alpha \in [\xi, f(\xi))$*; and every stationary solution* u *of* (1.1) *satisfying* $u_i > \ell$ *for every* $i \in \mathbb{Z}$ *and* (1.6) *is strictly decreasing and belongs into one of the above described equivalence classes* $[u^{\alpha,D}]$ *for some* $\alpha \in [\xi, f(\xi))$*.*

In other words, we are able to characterize the equivalence classes by a single value $\alpha \in [\xi, f(\xi))$, see Figure 1.2. In Sections 2 and 3 we provide the proof of Theorem 1.1 which relies on an iterative construction of function f. The onesided unbounded stationary solutions can then also be iteratively constructed via mirroring or cobwebbing as indicated in Figure 1.2 as well.

Characterization of onesided and twosided unbounded stationary solutions

Combining Theorem 1.1 and results from [7] we obtain a full characterization of unbounded stationary solutions of (1.1) which satisfy $u_i > \ell$ for all $i \in \mathbb{Z}$. We define the following open set using the function f from Theorem 1.1:

$$\mathscr{V} = \left\{ (\xi, \zeta) \in \mathbb{R}^2 : \ \xi > \ell \ \text{ and } \ f^{-1}(\xi) < \zeta < f(\xi) \right\}, \tag{1.9}$$

and upper and lower parts of its boundary (see Figure 1.2)

$$\begin{aligned} \partial^I\mathscr{V} &= \left\{ (\xi, \zeta) \in \mathbb{R}^2 : \ \xi > \ell \ \text{ and } \ \zeta = f(\xi) \right\}, \\ \partial^D\mathscr{V} &= \left\{ (\xi, \zeta) \in \mathbb{R}^2 : \ \xi > \ell \ \text{ and } \ \zeta = f^{-1}(\xi) \right\}. \end{aligned} \tag{1.10}$$

Obviously, there is $\partial^I\mathscr{V} \cap \partial^D\mathscr{V} = \emptyset$ and the boundary $\partial\mathscr{V}$ of $\mathscr{V}$ satisfies

$$\partial\mathscr{V} = \partial^I\mathscr{V} \cup \partial^D\mathscr{V} \cup \{(\ell,\ell)\}.$$

Our second main result states that the sets $\mathscr{V}$, $\partial^I\mathscr{V}$, and $\partial^D\mathscr{V}$ fully describe all values of all twosided and onesided unbounded stationary solutions of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$, respectively.

Theorem 1.2. *Let g be a C^1-function, satisfy (g_1)–(g_2), and u be a stationary solution of* (1.1) *such that $u_i > \ell$ for all $i \in \mathbb{Z}$. Then:*

(i) (1.4) *holds if and only if $(u_i, u_{i+1}) \in \mathscr{V}$ for all $i \in \mathbb{Z}$,*

(ii) (1.5) *holds if and only if $(u_i, u_{i+1}) \in \partial^I\mathscr{V}$ for all $i \in \mathbb{Z}$,*

(iii) (1.6) *holds if and only if $(u_i, u_{i+1}) \in \partial^D\mathscr{V}$ for all $i \in \mathbb{Z}$.*

Mirroring

Our main tool to study asymmetric and symmetric twosided unbounded solutions in [7] was the mirroring technique. The second order difference equation (1.3) for finding stationary solutions of (1.1) can be transformed into

$$u_{i+1} - \left(u_i - \tfrac{1}{2d}g(u_i)\right) = \left(u_i - \tfrac{1}{2d}g(u_i)\right) - u_{i-1}, \quad i \in \mathbb{Z}.$$

If we define an auxiliary function (which we call a mirroring function)

$$\varphi(u) = u - \tfrac{1}{2d}g(u), \tag{1.11}$$

we obtain a mirroring symmetry with respect to φ for all stationary solutions (1.3), since

$$u_{i+1} - \varphi(u_i) = \varphi(u_i) - u_{i-1}, \quad i \in \mathbb{Z}. \tag{1.12}$$

In this paper we go one step further and study mirroring of functions and their sequences in Section 2.

Functional equation

The mirroring (1.12) in the proof of Theorem 1.1 closely relate the problem of finding stationary solutions (1.3) of the LDE (1.1) with an interesting functional equation

$$\frac{f(u) + f^{-1}(u)}{2} = \varphi(u), \quad u \in [0,\infty), \tag{1.13}$$

in which φ is a given function and f is an unknown function to be found, see Figure 1.2. To our best knowledge, this challenging problem has not been studied itself and its analysis can have deep consequences for other stationary patterns of the LDE (1.1), most notably classes of bounded patterns. See Section 5 for more details.

Paper structure

In Section 2 we generalize the mirroring technique (1.12) to a functional iterative scheme and study the monotonicity and convergence of generated function sequences. In Section 3 we then use these results to prove Theorems 1.1 and 1.2 and show that the onesided unbounded solutions (1.5) or (1.6) are generated by a value from a single interval and form a boundary to the two-parametric domain which generate twosided unbounded solutions satisfying (1.4). We then illustrate our results by several examples with different nonlinearities g in Section 4 and discuss the functional equation (1.13) and its solvability in a special case connected to our analysis in Section 5. We conclude in Section 6 by final remarks which connect our study to the solutions of the PDE (1.2), topological chaos of the LDE (1.1), and further possible applications of mirroring schemes and the functional equation (1.13).

2 Mirroring idea and functional iterative scheme

To describe the functional generalization of the mirroring (1.12) and establish that generated iterations are well-defined, we need some functions (for now specifically φ) satisfy some desired properties. For this purpose, we say that a function $p : [\ell, \infty) \to \mathbb{R}$ satisfies *(p_1)* or *(p_2)* provided:

(p_1) $p(\ell) = \ell$,

(p_2) $p'(u) > 1$ for all $u \in [\ell, \infty)$ (which also yields that $p(u)$ is strictly increasing and thus invertible),

respectively. The next lemma states that the function φ given by (1.11) satisfies *(p_1)*–*(p_2)* provided *(g_1)*–*(g_2)* hold.

Lemma 2.1. *Let (g_1)–(g_2) be satisfied. The function φ defined by* (1.11) *is of class C^1 and satisfies (p_1)–(p_2).*

Proof. The statements follow immediately from the definition (1.11) of φ. □

Now we are able to make the following considerations. We call the relation (1.12) the mirroring scheme, since for given initial conditions $u_0 > \ell$, $u_1 > \ell$ the point (u_2, u_1) as a point in the $\mathbb{R}^2$-plane is by (1.12) the horizontal mirror image of (u_0, u_1) with respect to the graph of φ^{-1}. Then, the point (u_2, u_3) is by (1.12) the vertical mirror image of (u_2, u_1) with respect to the graph of function φ, etc. (see Figure 1.2). Therefore, the forward solution u_i of (1.3) for $i = 2, 3, \dots$ can be generated from the initial conditions u_0, u_1 by mirroring of the points with respect to φ^{-1} horizontally and with respect to φ vertically, respectively, in the switching manner.

Analogically, the backward solution u_i of (1.3) for $i = -1, -2, \dots$ can be generated from the initial conditions u_0, u_1 by mirroring of the points with respect to φ vertically and with respect to φ^{-1} horizontally, respectively.

At this stage, we generalize the mirroring scheme (1.12) to functions. Let φ satisfy *(p_1)*–*(p_2)* and consider the following functional iterative scheme:

$$f_{n+1}(u) = 2\varphi(u) - f_n^{-1}(u), \quad n \in \mathbb{N}_0, \quad u \in [\ell, \infty). \tag{2.1}$$

Generally, the iterates do not have to be well-defined because of the inverses. This fundamentally depends on the properties of the initial function f_0. We focus on two special sequences given by specific initial iterates f_0 for which we establish that all iterates given by (2.1) are well-defined.

Definition 2.2. Let φ be a C^1-function and satisfy *(p_1)*–*(p_2)*. We define functional sequences $(\underline{f}_n)$ and $(\bar{f}_n)$, $n \in \mathbb{N}_0$, as follows:

(i) $(\underline{f}_n)$ are the iterates of (2.1) initiated by $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$,

(ii) $(\bar{f}_n)$ are the iterates of (2.1) initiated by $\bar{f}_0(u) = 2\varphi(u) - \ell$.

Let us note that the inverse φ^{-1} is well-defined thanks to *(p₂)*. Further, let us verify that the iterates $\underline{f}_n$ and $\bar{f}_n$ in Definition 2.2 are correctly defined for every $n \in \mathbb{N}_0$ as well. This (besides others) follows from the following lemma.

Lemma 2.3. *Let φ be a C^1-function and satisfy (p_1)–(p_2). If f_n is a C^1-function and satisfies (p_1)–(p_2), then f_{n+1} defined by (2.1) is well-defined C^1-function and satisfies (p_1)–(p_2) as well.*

In particular, $\underline{f}_n$ and $\bar{f}_n$ are well-defined C^1-functions for every $n \in \mathbb{N}_0$ and all satisfy (p_1)–(p_2).

Proof. Since $f_n'(u) > 1$ for all $u \in [\ell, \infty)$ by (p_2), the inverse f_n^{-1} is well-defined and of class C^1 by the inverse function rule

$$(f_n^{-1})'(u) = \frac{1}{f_n'(f_n^{-1}(u))}. \tag{2.2}$$

Hence, f_{n+1} defined by (2.1) is also well-defined and of class C^1. If $f_n(\ell) = \ell$ by (p_1), then $f_{n+1}(\ell) = \ell$ immediately from (2.1) and thanks to $\varphi(\ell) = \ell$ (the function φ satisfies (p_1) as well by the assumption). Moreover, if $f_n'(u) > 1$ for all $u \in [\ell, \infty)$, then $(f_n^{-1})'(u) < 1$ for all $u \in [\ell, \infty)$ again by (2.2). Since also $\varphi'(u) > 1$ for all $u \in [\ell, \infty)$ (φ satisfies (p_2)), then

$$f_{n+1}'(u) = 2\varphi'(u) - (f_n^{-1})'(u) > 2 \cdot 1 - 1 = 1 \quad \text{for all} \quad u \in [\ell, \infty).$$

Finally, one can similarly show that $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$ and $\bar{f}_0(u) = 2\varphi(u) - \ell$ satisfy (p_1)–(p_2). Then (p_1)–(p_2) hold for all $\underline{f}_n$ and $\bar{f}_n$ as well by induction. □

In the next lemma we show that the sequence $(\underline{f}_n)$ is increasing, $(\bar{f}_n)$ is decreasing, and whole sequence $(\underline{f}_n)$ lies below $(\bar{f}_n)$, see Figure 2.1.

Lemma 2.4. *Let φ be a C^1-function and satisfy (p_1)–(p_2). Then for every $m, n \in \mathbb{N}_0$ and all $u \in [\ell, \infty)$ the following hold:*

(i) $\varphi(u) \le \underline{f}_n(u) \le \underline{f}_{n+1}(u)$,

(ii) $\bar{f}_{n+1}(u) \le \bar{f}_n(u)$,

(iii) $\underline{f}_n(u) \le \bar{f}_m(u)$.

Moreover, the equalities hold if and only if $u = \ell$.

Proof. Firstly, there is $u \le \varphi(u) \le \underline{f}_0(u) \le \bar{f}_0(u)$ for all $u \in [\ell, \infty)$. Indeed, the first and last inequalities follow immediately from (p_1)–(p_2) (Lemma 2.1). The middle one is verified again by (p_1)–(p_2) and by the following:

$$\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u) = \varphi(u) + (\varphi(u) - \varphi^{-1}(u)) \ge \varphi(u).$$

For the inverses, the reversed inequalities $\ell \le \bar{f}_0^{-1}(u) \le \underline{f}_0^{-1}(u) \le \varphi^{-1}(u)$ hold for all $u \in [\ell, \infty)$.

Let us prove *(i)*, i.e., that $\varphi(u) \le \underline{f}_n(u) \le \underline{f}_{n+1}(u)$ for all $n \in \mathbb{N}_0$ and $u \in [\ell, \infty)$ by induction. For $n = 0$ there is $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$, i.e., $2\varphi(u) = \underline{f}_0(u) + \varphi^{-1}(u)$, and thus

$$\underline{f}_1(u) = 2\varphi(u) - \underline{f}_0^{-1}(u) = \underline{f}_0(u) + \varphi^{-1}(u) - \underline{f}_0^{-1}(u) \ge \underline{f}_0(u) \ge \varphi(u),$$

since $\underline{f}_0(u) \ge \varphi(u) \ge \varphi^{-1}(u)$ for all $u \in [\ell, \infty)$. Assume that $\varphi(u) \le \underline{f}_{n-1}(u) \le \underline{f}_n(u)$ for some $n \in \mathbb{N}$ and all $u \in [\ell, \infty)$. Then, $\ell \le \underline{f}_n^{-1}(u) \le \underline{f}_{n-1}^{-1}(u) \le \varphi^{-1}(u)$ for all $u \in [\ell, \infty)$ by inversion. Further, for $n + 1$ there is

$$\underline{f}_{n+1}(u) = 2\varphi(u) - \underline{f}_n^{-1}(u) \ge 2\varphi(u) - \underline{f}_{n-1}^{-1}(u) = \underline{f}_n(u) \ge \varphi(u),$$

which concludes the induction step.

The inequality in *(ii)*, i.e., $\bar{f}_{n+1}(u) \le \bar{f}_n(u)$ for every $n \in \mathbb{N}_0$ and $u \in [\ell, \infty)$, and also that

$$\underline{f}_n(u) \le \bar{f}_n(u) \quad \text{for every} \quad n \in \mathbb{N}_0 \quad \text{and all} \quad u \in [\ell, \infty) \tag{2.3}$$

can be proved similarly by induction.

Hence, let us finally show *(iii)*, i.e., that $\underline{f}_n(u) \le \bar{f}_m(u)$ for all $m, n \in \mathbb{N}$ and $u \in [\ell, \infty)$. Assume by contradiction that there are some $m_c, n_c \in \mathbb{N}$, $m_c \neq n_c$, such that $\underline{f}_{n_c}(u_c) > \bar{f}_{m_c}(u_c)$ for some $u_c > \ell$ (note that for $u = \ell$ there is $\underline{f}_n(\ell) = \bar{f}_m(\ell) = \ell$ for all $m_c, n_c \in \mathbb{N}_0$). If $n_c \ge m_c$ then by *(ii)*

$$\underline{f}_{n_c}(u_c) > \bar{f}_{m_c}(u_c) \ge \bar{f}_{m_c+1}(u_c) \ge \bar{f}_{m_c+2}(u_c) \ge \ldots \ge \bar{f}_{n_c}(u_c),$$

a contradiction with (2.3). If otherwise $m_c \ge n_c$ then by *(i)*

$$\bar{f}_{m_c}(u_c) < \underline{f}_{n_c}(u_c) \le \underline{f}_{n_c+1}(u_c) \le \underline{f}_{n_c+2}(u_c) \le \ldots \le \underline{f}_{m_c}(u_c),$$

again a contradiction with (2.3).

One can easily check that all the verified inequalities are strict if and only if $u > \ell$. □

As a by-product, Lemma 2.4 guarantees the existence of limit functions for both $(\underline{f}_n)$ and $(\bar{f}_n)$. We show their existence in the next corollary and provide several properties of these limit functions, see Figure 2.1.

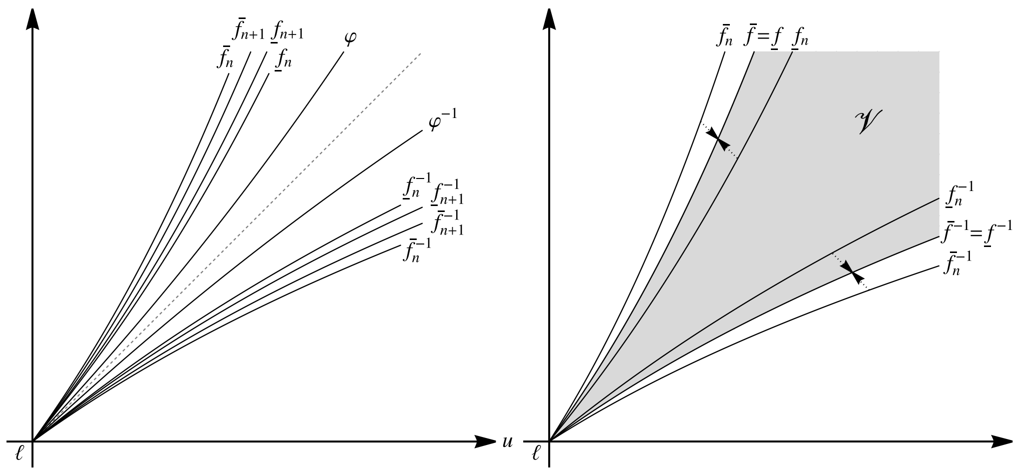

Figure 2.1: Illustration of the mirroring functional iterative scheme (2.1) and functional sequences $(\underline{f}_n)$ and $(\bar{f}_n)$ from Lemma 2.4 (left panel). Convergence of these sequences is implied by Corollaries 2.5 and 2.7 (right panel).

Corollary 2.5. *Let φ be a C^1-function and satisfy (p₁)–(p₂). There exist continuous limit functions*

$$\underline{f}(u) = \lim_{n\to\infty} \underline{f}_n(u) \quad \textit{and} \quad \bar{f}(u) = \lim_{n\to\infty} \bar{f}_n(u), \qquad u \in [\ell, \infty),$$

which satisfy:

(i) $\varphi(u) \le \underline{f}_n(u) \le \underline{f}(u) \le \bar{f}(u) \le \bar{f}_n(u)$ *for all* $n \in \mathbb{N}_0$ *and* $u \in [\ell, \infty)$,

(ii) $\underline{f}$ *and* $\bar{f}$ *are strictly increasing, i.e., invertible, on* $[\ell, \infty)$ *and*

$$\underline{f}^{-1}(u) = \lim_{n\to\infty} \underline{f}_n^{-1}(u) \quad \textit{and} \quad \bar{f}^{-1}(u) = \lim_{n\to\infty} \bar{f}_n^{-1}(u) \quad \textit{for all} \quad u \in [\ell, \infty),$$

(iii) $\underline{f}(u) = 2\varphi(u) - \underline{f}^{-1}(u)$ *and* $\bar{f}(u) = 2\varphi(u) - \bar{f}^{-1}(u)$ *for all* $u \in [\ell, \infty)$.

Proof. The sequence $(\underline{f}_n(u))$, $n \in \mathbb{N}_0$, for given $u \in [\ell, \infty)$ is increasing and bounded from above by all $\bar{f}_n(u)$, $n \in \mathbb{N}_0$, (see Lemma 2.4) which guarantees the existence of pointwise limit $\underline{f}(u)$. The existence of $\bar{f}(u)$, $u \in [\ell, \infty)$, follows similarly.

Lemma 2.3 yields that the iterates $\underline{f}_n$ and $\bar{f}_n$ (and also their inverses $\underline{f}_n^{-1}$ and $\bar{f}_n^{-1}$), $n \in \mathbb{N}_0$, are strictly increasing C^1-functions and for all $n \in \mathbb{N}_0$ there is

$$1 < \underline{f}_{n+1}'(u) = 2\varphi'(u) - (\underline{f}_n^{-1})'(u) \le 2\varphi'(u) \quad \text{(analogically for } \bar{f}_{n+1}'(u)),$$

by (2.1). Since φ is a C^1-function, the functions $\underline{f}_n$ and $\bar{f}_n$ have uniformly bounded derivatives on every compact subinterval of $[\ell, \infty)$. This implies that they converge to their pointwise limits $\underline{f}$ and $\bar{f}$ uniformly on such intervals and thus, $\underline{f}$ and $\bar{f}$ are continuous on whole $[\ell, \infty)$.

The limits have to satisfy that $\varphi(u) \le \underline{f}_n(u) \le \underline{f}(u) \le \bar{f}(u) \le \bar{f}_n(u)$ for all $n \in \mathbb{N}_0$ and $u \in [\ell, \infty)$ by Lemma 2.4 again which proves *(i)*.

The limit functions $\underline{f}$ and $\bar{f}$ are strictly increasing and therefore invertible on $[\ell, \infty)$. Indeed, for every $u_1, u_2 \in [\ell, \infty)$ such that, e.g., $u_1 < u_2$ the mean value theorem and Lemma 2.3 implies

$$\underline{f}_n(u_2) - \underline{f}_n(u_1) = \underline{f}_n'(\xi)(u_2 - u_1) > u_2 - u_1.$$

Passing $n \to \infty$ we obtain

$$\underline{f}(u_2) - \underline{f}(u_1) \ge u_2 - u_1 > 0,$$

which implies that $\underline{f}$ (and analogously $\bar{f}$) is strictly increasing. By the reflection with respect to the axis of the first quadrant we obtain that the inverse functions $\underline{f}^{-1}$ and $\bar{f}^{-1}$ satisfy $\underline{f}^{-1}(u) = \lim_{n\to\infty} \underline{f}_n^{-1}(u)$ and $\bar{f}^{-1}(u) = \lim_{n\to\infty} \bar{f}_n^{-1}(u)$, $u \in [\ell, \infty)$, which proves *(ii)*.

Finally, both iterates $(\underline{f}_n)$ and $(\bar{f}_n)$, $n \in \mathbb{N}_0$, are consistent with the iterative scheme (2.1), specifically,

$$f_{n+1}(u) = 2\varphi(u) - f_n^{-1}(u), \quad u \in [\ell, \infty).$$

Taking $n \to \infty$ in this equality together with *(i)* and *(iii)* we obtain that both limit functions $\underline{f}$ and $\bar{f}$ satisfy

$$f(u) = 2\varphi(u) - f^{-1}(u), \quad u \in [\ell, \infty),$$

which concludes the proof of *(iii)*. □

In the rest of this section we show that $\underline{f} = \bar{f}$ on $[\ell, \infty)$, i.e., both sequences $(\underline{f}_n)$ and $(\bar{f}_n)$ converge to a common limit function (specifically, $(\underline{f}_n)$ from below and $(\bar{f}_n)$ from above). We build our argument on the following technical lemma.

Lemma 2.6. *Let φ be a C^1-function and satisfy (p_1)–(p_2). Then*

$$\bar{f}\left(\underline{f}^{-1}\left(\bar{f}(u)\right)\right) - \bar{f}(u) \ge \underline{f}^{-1}(\bar{f}(u)) - u \quad \text{for all} \quad u \in [\ell, \infty).$$

Proof. Let $u \in [\ell, \infty)$ be arbitrary but fixed. Lemma 2.3 guarantees that $\bar{f}_n'(s) > 1$ for $s \in \left[u, \underline{f}^{-1}(\bar{f}(u))\right]$. Therefore, the mean value theorem yields that for some $\xi \in \left(u, \underline{f}^{-1}(\bar{f}(u))\right)$ there is

$$\bar{f}_n\left(\underline{f}^{-1}\left(\bar{f}(u)\right)\right) - \bar{f}_n(u) = \bar{f}_n'(\xi) \cdot (\underline{f}^{-1}(\bar{f}(u)) - u) > \underline{f}^{-1}(\bar{f}(u)) - u.$$

Taking $n \to \infty$ we obtain the statement of the lemma. □

Now we are able to show that the limit functions $\underline{f}$ and $\bar{f}$ of iterates $(\underline{f}_n)$ and $(\bar{f}_n)$ are the same, see again Figure 2.1.

Corollary 2.7. *Let φ be a C^1-function and satisfy (p1)–(p2). Then $\underline{f}(u) = \bar{f}(u)$ for all $u \in [\ell, \infty)$.*

Proof. Assume by contradiction that there exists $u_1 > \ell$ such that $\underline{f}(u_1) < \bar{f}(u_1)$, i.e., $\bar{f}(u_1) - \underline{f}(u_1) > 0$ (note that for $u = \ell$ there is $\underline{f}(\ell) = \bar{f}(\ell) = \ell$). Corollary 2.5 *(iii)* yields that

$$\begin{aligned}\bar{f}(u_1) - \underline{f}(u_1) &= (2\varphi(u_1) - \bar{f}^{-1}(u_1)) - (2\varphi(u_1) - \underline{f}^{-1}(u_1)) \\ &= \underline{f}^{-1}(u_1) - \bar{f}^{-1}(u_1).\end{aligned} \tag{2.4}$$

Let $u_1 = \bar{f}(\bar{u})$ for some $\bar{u} > \ell$ (recall that the limit functions are homeomorphisms of $[\ell, \infty)$, see Corollary 2.5 *(i)–(ii)*) and denote $u_2 = \underline{f}^{-1}(u_1) = \underline{f}^{-1}(\bar{f}(\bar{u}))$. Then

$$\begin{aligned}\bar{f}(u_2) - \underline{f}(u_2) &= \bar{f}(\underline{f}^{-1}(\bar{f}(\bar{u}))) - \underline{f}(\underline{f}^{-1}(\bar{f}(\bar{u}))) \\ &= \bar{f}(\underline{f}^{-1}(\bar{f}(\bar{u}))) - \bar{f}(\bar{u}) \\ &\geq \underline{f}^{-1}(\bar{f}(\bar{u})) - \bar{u} \\ &= \underline{f}^{-1}(\bar{f}(\bar{u})) - \bar{f}^{-1}(\bar{f}(\bar{u})) \\ &= \underline{f}^{-1}(u_1) - \bar{f}^{-1}(u_1) \\ &= \bar{f}(u_1) - \underline{f}(u_1),\end{aligned}$$

in which the inequality follows from Lemma 2.6 and the last equality from (2.4). Thus, there exists $u_2 = \underline{f}^{-1}(u_1) < u_1$ such that $\bar{f}(u_2) - \underline{f}(u_2) \geq \bar{f}(u_1) - \underline{f}(u_1) > 0$. By induction we construct a sequence (u_n), $n \in \mathbb{N}$, such that

$$\ell < u_{n+1} = \underline{f}^{-1}(u_n) < u_n$$

and

$$\bar{f}(u_{n+1}) - \underline{f}(u_{n+1}) \geq \bar{f}(u_n) - \underline{f}(u_n) > \bar{f}(u_1) - \underline{f}(u_1) > 0$$

for all $n \in \mathbb{N}$. Since (u_n) is decreasing, bounded, and satisfies $u_{n+1} = \underline{f}^{-1}(u_n)$, it has to converge to the unique fixed point of $\underline{f}^{-1}$, i.e., $u_n \to \ell$ for $n \to \infty$. The continuity of limit functions $\underline{f}$ and $\bar{f}$ (see Corollary 2.5) then yields

$$0 < \bar{f}(u_1) - \underline{f}(u_1) \leq \bar{f}(u_n) - \underline{f}(u_n) \to \bar{f}(\ell) - \underline{f}(\ell) = \ell - \ell = 0$$

for $n \to \infty$, a contradiction which concludes the proof. □

3 Proofs of main theorems

In this section we go back to the problem (1.3) for stationary solutions of (1.1) and prove our main result Theorem 1.1 with the help of the mirroring technique and related functional iterative scheme (2.1) for the specific mirroring function (1.11).

Firstly, let us note that using the mirroring scheme (1.12), the authors proved in [7] the following result on twosided unbounded stationary solutions of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.4). Specifically, we showed that these solutions are uniquely characterised and indexed by points of two-dimensional set

$$\mathscr{U} = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \ \alpha > \ell \ \text{ and } \ \varphi^{-1}(\alpha) \leq \beta \leq \varphi(\alpha)\right\}, \tag{3.1}$$

in contrast to Theorem 1.1 on onesided unbounded stationary solutions in which the characterizing set is one-dimensional and even bounded. Note that while $\mathscr{U}$ consists of all unique characteristic pairs generating twosided unbounded stationary solutions, the set $\mathscr{V}$ consists of all pairs generated by initial conditions from $\mathscr{U}$, see Figure 3.1. Theorem 1.1 implies the same relationship between the curve

$$\mathscr{C} = \{(\alpha, f(\alpha)) \in \mathbb{R}^2 : \xi \le \alpha < f(\xi))\}$$

and the boundaries $\partial^I \mathscr{V}$, $\partial^D \mathscr{V}$, see the left panel of Figure 3.1.

Theorem 3.1 ([7, Theorem 5]). *Let (g1)–(g2) be satisfied. Then every point $(\alpha, \beta) \in \mathscr{U}$ determines an equivalence class $[u^{\alpha,\beta}]$ of stationary solutions u of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.4) represented by a solution $u^{\alpha,\beta}$ such that $u_0^{\alpha,\beta} = \alpha$, $u_1^{\alpha,\beta} = \beta$, and:*

(i) if $\varphi^{-1}(\alpha) < \beta < \varphi(\alpha)$, then $[u^{\alpha,\beta}] \ne [u^{\tilde{\alpha},\tilde{\beta}}]$ for every $(\tilde{\alpha}, \tilde{\beta}) \ne (\alpha, \beta)$, $(\tilde{\alpha}, \tilde{\beta}) \in \mathscr{U}$,

(ii) if either $\varphi^{-1}(\alpha) = \beta$, or $\beta = \varphi(\alpha)$, then $[u^{\alpha,\beta}] = [u^{\beta,\alpha}]$.

Moreover, every stationary solution u of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.4) is an element of an equivalence class $[u^{\alpha,\beta}]$ determined by a point $(\alpha, \beta) \in \mathscr{U}$.

In order to establish relationship of twosided unbounded solutions characterized by Theorem 3.1 with iterative schemes from Section 2, we focus on initial conditions outside the set $\mathscr{U}$ given by (3.1). First, let us characterize solutions which eventually also have values $u_i \notin [\ell, \infty)$.

Lemma 3.2. *Let (g1)–(g2) be satisfied and u be a stationary solution of (1.1). If there exists $i_0 \in \mathbb{Z}$ such that*

$$u_{i_0+1} \ge \bar{f}_{n_0}(u_{i_0}) \quad \text{or} \quad u_{i_0} \ge \bar{f}_{n_0}(u_{i_0+1}) \quad \text{for some} \quad n_0 \in \mathbb{N}_0,$$

then there has to exists $j_0 \in \mathbb{Z}$ such that $u_{j_0} \le \ell$.

Proof. Let $u_{i_0+1} \ge \bar{f}_{n_0}(u_{i_0})$ for some $i_0 \in \mathbb{Z}$ and $n_0 \in \mathbb{N}_0$ (the other case $u_{i_0} \ge \bar{f}_{n_0}(u_{i_0+1})$ is similar). Then we obtain by (2.1) that $u_{i_0+1} \ge \bar{f}_{n_0}(u_{i_0}) = 2\varphi(u_{i_0}) - \underline{f}_{n_0-1}^{-1}(u_{i_0})$ and therefore, by (1.3) that

$$u_{i_0-1} = 2u_{i_0} - u_{i_0+1} - \tfrac{1}{d} g(u_{i_0}) = 2\varphi(u_{i_0}) - u_{i_0+1} \le \underline{f}_{n_0-1}^{-1}(u_{i_0}).$$

Applying the increasing function $\underline{f}_{n_0-1}$ to this inequality (note that Lemma 2.1 verifies that φ defined by (1.11) satisfies the needed hypotheses (p1)–(p2) from Section 2, i.e., Lemma 2.3 *(iii)* holds), we get

$$u_{i_0} \ge \underline{f}_{n_0-1}(u_{i_0-1}).$$

Repeating this procedure n_0-times and applying Definition 2.2 we obtain that

$$u_{i_0-n_0+1} \ge \underline{f}_0(u_{i_0-n_0}) = 2\varphi(u_{i_0-n_0}) - \ell.$$

Then (1.3) yields that

$$u_{i_0-n_0-1} = 2u_{i_0-n_0} - u_{i_0-n_0+1} - \tfrac{1}{d} g(u_{i_0-n_0}) = 2\varphi(u_{i_0-n_0}) - u_{i_0-n_0+1} \le \ell,$$

i.e., we obtain the statement for $j_0 = i_0 - n_0 - 1$. □

Our next auxiliary lemma characterize initial conditions which generate $u_i > \ell$ and do not lead to onesided unbounded solutions but to twosided ones from Theorem 3.1 and are thus part of solutions characterized by a pair $(\alpha, \beta) \in \mathscr{U}$.

Lemma 3.3. *Let (g1)–(g2) be satisfied and u be a stationary solution of (1.1). If there exists $i_0 \in \mathbb{Z}$ such that*

$$u_{i_0} \le u_{i_0+1} \le \underline{f}_{n_0}(u_{i_0}) \quad \text{or} \quad u_{i_0+1} \le u_{i_0} \le \underline{f}_{n_0}(u_{i_0+1}) \quad \text{for some} \quad n_0 \in \mathbb{N}_0,$$

then there has to exists $j_0 \in \mathbb{Z}$ such that $(u_{j_0}, u_{j_0+1}) \in \mathscr{U}$ or $(u_{j_0+1}, u_{j_0}) \in \mathscr{U}$, respectively, and thus, (1.4) holds.

Proof. If $u_{i_0} \le u_{i_0+1} \le \varphi(u_{i_0})$, then $(u_{i_0}, u_{i_0+1}) \in \mathscr{U}$ by the definition (3.1) of $\mathscr{U}$. Hence, (1.4) holds by Theorem 3.1. Otherwise, if $\varphi(u_{i_0}) < u_{i_0+1} \le \underline{f}_{n_0}(u_{i_0})$, then one can proceed analogically as in the proof of Lemma 3.2 to verify the statement.

For $u_{i_0} \le \underline{f}_{n_0}(u_{i_0+1})$ it is again similar. □

In other words, Lemma 3.2 characterizes initial conditions outside $\overline{\mathscr{V}}$ and Lemma 3.3 inside $\mathscr{V}$. We have thus collected all tools to prove the first main result of the manuscript, Theorem 1.1.

Proof of Theorem 1.1. Let us note that the function φ defined by (1.11) is of class C^1 and satisfies *(p1)–(p2)*, since g satisfies *(g1)–(g2)* (see Lemma 2.1). Thus, all results from Section 2 hold. Let $f(u) = \underline{f}(u) = \bar{f}(u)$, $u \in [\ell, \infty)$, be the limit function of iterative scheme (2.1) and $\xi > \ell$. Let us prove the existence of equivalence class $[u^{\alpha,I}]$ of strictly increasing solutions satisfying (1.7). Firstly, put $u_0^{\alpha,I} = \alpha \in [\xi, f(\xi))$ and for $i \neq 0$ define

$$u_{i+1}^{\alpha,I} = f(u_i^{\alpha,I}), \quad \text{or equivalently,} \quad u_{i-1}^{\alpha,I} = f^{-1}(u_i^{\alpha,I}), \tag{3.2}$$

since f is invertible by Corollary 2.5 *(ii)*. Then, $(u_i^{\alpha,I})$ is defined for all $i \in \mathbb{Z}$ and is strictly increasing, because $f(u) > u$, resp. $f^{-1}(u) < u$, for all $u > \ell$. Since $u = \ell$ is the only fixed point of the mapping f (and of f^{-1} as well) on $[\ell, \infty)$ and again $f(u) > u$, resp. $\ell < f^{-1}(u) < u$, for all $u \in [\ell, \infty)$, then

$$u_i^{\alpha,I} > \ell \quad \text{for all} \quad i \in \mathbb{Z}, \quad \lim_{i\to-\infty} u_i^{\alpha,I} = \ell, \quad \text{and} \quad \lim_{i\to\infty} u_i^{\alpha,I} = \infty.$$

Let us verify that such a sequence $(u_i^{\alpha,I})$, $i \in \mathbb{Z}$, complies with (1.3) and thus forms a stationary solution of (1.1). One can compute for arbitrary $i \in \mathbb{Z}$

$$\begin{aligned} d(u_{i-1}^{\alpha,I} - 2u_i^{\alpha,I} + u_{i+1}^{\alpha,I}) + g(u_i^{\alpha,I}) &= d\left(u_{i+1}^{\alpha,I} + u_{i-1}^{\alpha,I} - 2\left(u_i^{\alpha,I} - \tfrac{1}{2d} g(u_i^{\alpha,I})\right)\right) \\ &= d\left(f(u_i^{\alpha,I}) + f^{-1}(u_i^{\alpha,I}) - 2\varphi(u_i^{\alpha,I})\right) \\ &= 0, \end{aligned}$$

which is verified by Corollary 2.5 *(iii)*.

Let $\tilde{\alpha} \in [\xi, f(\xi))$ be such that $\tilde{\alpha} < \alpha$ (for $\tilde{\alpha} > \alpha$ it is similar) and assume that $[u^{\alpha,I}] = [u^{\tilde{\alpha},I}]$, i.e., there exists $s_0 \in \mathbb{N}$ such that $u_{s_0}^{\tilde{\alpha},I} = u_0^{\alpha,I} = \alpha$. Then Corollary 2.5 *(ii)* implies

$$f(\xi) \le f(\tilde{\alpha}) \le f^{s_0}(\tilde{\alpha}) = u_{s_0}^{\tilde{\alpha},I} = u_0^{\alpha,I} = \alpha$$

(the symbol f^{s_0} denotes s_0-multiple composition of f), which is a contradiction.

On the contrary, let u be a stationary solution of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.5). This implies that (u_i), $i \in \mathbb{Z}$, is strictly increasing, since otherwise there exists $i \in \mathbb{Z}$ such that $u_{i-1} < u_i$ and $u_{i+1} \le u_i$, i.e., $u_{i-1} - 2u_i + u_{i+1} < 0$. Therefore Eq. (1.3) implies $g(u_i) > 0$, a contradiction. Hence, there is $u_i < u_{i+1}$ for all $i \in \mathbb{Z}$. If there exists $i_0 \in \mathbb{Z}$ such that $u_{i_0+1} = f(u_{i_0})$, then the uniqueness of solution of (1.3) with given initial conditions $u_{i_0} = \alpha$ and $u_{i_0+1} = f(\alpha)$ yields that the solution lies in the equivalence class $[u^{\alpha,I}]$. If $\alpha \in [\xi, f(\xi))$, we are done. Thus, let us assume that $\alpha < \xi$ (for $\alpha \ge f(\xi)$ it is similar). Then there exists $s_0 \in \mathbb{N}$ such that $f^{s_0}(\alpha) \in [\xi, f(\xi))$ and thus, $[u^{\alpha,I}] = [u^{\gamma,I}]$ for $\gamma = f^{s_0}(\alpha)$. Indeed, since $f^s(\alpha) \to \infty$ for $s \to \infty$ there has to exist $s_0 \in \mathbb{Z}$ such that $f^{s_0-1}(\alpha) < \xi$ and $f^{s_0}(\alpha) \ge \xi$. Since f is strictly increasing on $[\ell, \infty)$ by Corollary 2.5 *(iii)*, there is

$$f^{s_0}(\alpha) = f(f^{s_0-1}(\alpha)) < f(\xi), \quad \text{i.e.,} \quad f^{s_0}(\alpha) \in [\xi, f(\xi)).$$

Finally, we show by Lemma 3.2 and Lemma 3.3 that other cases cannot occur. Indeed, if $u_{i+1} \neq f(u_i)$ for every $i \in \mathbb{Z}$, there is either $u_{i+1} > f(u_i)$, or $u_i < u_{i+1} < f(u_i)$ for

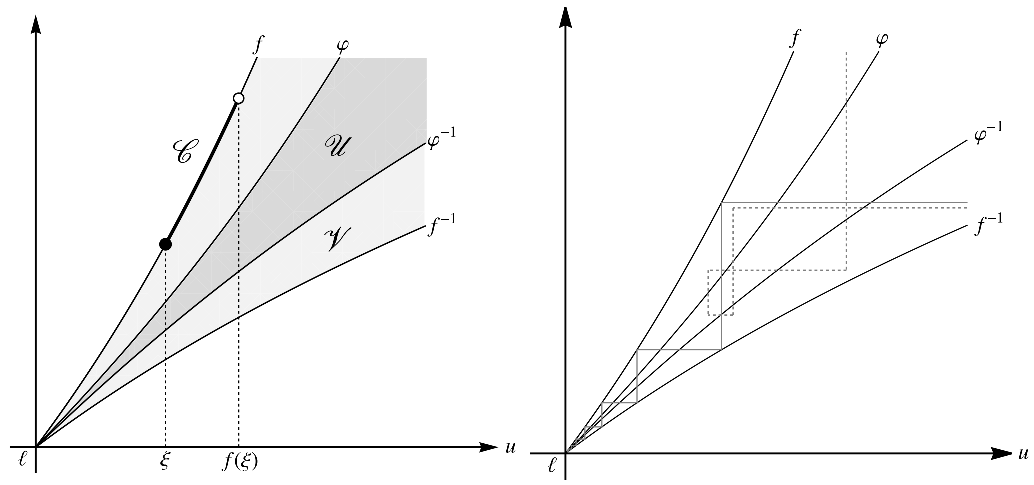

Figure 3.1: The left panel refines Figure 1.2 by including the set $\mathscr{U}$ from (3.1) which characterizes all twosided solutions (Theorem 3.1). The right panel then shows the mirroring scheme (1.12) for onesided (full line) and twosided unbounded solutions (dashed line).

some $i \in \mathbb{Z}$. If $u_{i+1} > f(u_i)$, then Corollary 2.5 *(i)* implies that there exists $n_0 \in \mathbb{N}_0$ such that $u_{i+1} > \bar{f}_{n_0}(u_i)$. Then Lemma 3.2 yields that there has to exists $j_0 \in \mathbb{Z}$ such that $u_{j_0} \leq \ell$, a contradiction. If $u_i < u_{i+1} < f(u_i)$, then Corollary 2.5 *(i)* implies that there exists $n_0 \in \mathbb{N}_0$ such that $u_i < u_{i+1} < \underline{f}_{n_0}(u_i)$. Now Lemma 3.3 implies that (1.4) holds, which is a contradiction with (1.5).

The statement of Theorem 1.1 for equivalence classes of decreasing solutions $[u^{\alpha,D}]$ can be proved similarly. □

To conclude, we prove the second main result – Theorem 1.2 which characterizes all unbounded stationary solutions of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$.

Proof of Theorem 1.2. Let u be an unbounded stationary solution of (1.1) which satisfies $u_i > \ell$ for $i \in \mathbb{Z}$. Then $(u_i, u_{i+1}) \in \mathscr{V} \cup \partial^I\mathscr{V} \cup \partial^D\mathscr{V}$ for all $i \in \mathbb{Z}$. Indeed, assuming by contradiction that $u_{i_0+1} > f(u_{i_0})$ for some $i_0 \in \mathbb{N}_0$ (and similarly if $u_{i_0+1} < f^{-1}(u_{i_0})$) there has to exists an index $n_0 \in \mathbb{N}_0$ such that $u_{i_0+1} > \bar{f}_{n_0}(u_{i_0})$ ($\bar{f}_n \to f$). Then Lemma 3.2 yields that $u_{j_0} \leq \ell$ for some $j_0 \in \mathbb{N}_0$, a contradiction.

Let us prove *(i)* and assume that $(u_i, u_{i+1}) \in \mathscr{V}$ for all $i \in \mathbb{Z}$. Let $i_0 \in \mathbb{Z}$ be arbitrary, denote $\alpha = u_{i_0}$, and assume $u_{i_0} \leq u_{i_0+1}$ (for $u_{i_0} \geq u_{i_0+1}$ similarly). Since $(u_{i_0}, u_{i_0+1}) \in \mathscr{V}$, there is $u_{i_0} \leq u_{i_0+1} < f(u_{i_0})$ and thus $u_{i_0} \leq u_{i_0+1} \leq \underline{f}_{n_0}(u_{i_0})$ for an index $n_0 \in \mathbb{N}_0$. Consequently, Lemma 3.3 yields that (1.4) holds.

On the contrary, assuming (1.4) there has to be $(u_i, u_{i+1}) \in \mathscr{V}$ for all $i \in \mathbb{Z}$. Indeed, if $(u_{i_0}, u_{i_0+1}) \in \partial^I\mathscr{V}$ for some $i_0 \in \mathbb{N}_0$ (and similarly if $(u_{i_0}, u_{i_0+1}) \in \partial^D\mathscr{V}$), i.e., $u_{i_0+1} = f(u_{i_0})$ by definition of $\partial^I\mathscr{V}$, then Theorem 1.1 implies that $u \in [u^{\alpha,I}]$ with $\alpha = u_{i_0}$ and $\lim_{i\to-\infty} u_i = \ell$, a contradiction with (1.4).

Let us prove *(ii)* and suppose $(u_i, u_{i+1}) \in \partial^I\mathscr{V}$ for all $i \in \mathbb{Z}$. For arbitrary $i_0 \in \mathbb{Z}$ there is $u_{i_0+1} = f(u_{i_0})$ and Theorem 1.1 yields that $u \in [u^{\alpha,I}]$ with $\alpha = u_{i_0}$, i.e., (1.5) holds.

Assuming (1.5) there has to be $(u_i, u_{i+1}) \in \partial^I\mathscr{V}$ for all $i \in \mathbb{Z}$. Indeed, if $(u_{i_0}, u_{i_0+1}) \in \mathscr{V}$ for some $i_0 \in \mathbb{N}_0$ then $\lim_{i\to\pm\infty} u_i = \infty$ similarly as above which is a contradiction with (1.5).

In the same way, if $(u_{i_0}, u_{i_0+1}) \in \partial^D \mathscr{V}$, then $u_{i_0+1} = f^{-1}(u_{i_0})$ and Theorem 1.1 yields that $u \in [u^{\alpha,D}]$ and $\lim_{i\to+\infty} u_i = \ell$, again a contradiction with (1.5).

The third item *(iii)* can be proved similarly as *(ii)*. □

4 Examples of specific reaction-diffusion LDEs

In this section we illustrate Theorems 1.1 and 1.2 for specific reaction functions g in (1.1) satisfying the key assumptions *(g1)*–*(g2)*. Let us start with two most common reactions.

Example 4.1 (Fisher and Nagumo equation)**.** Considering the logistic (monostable) or cubic (bistable) reaction functions

$$g(u) = u(1-u) \qquad \text{or} \qquad g(u) = u(1-u)(u-a), \quad a \in (0,1), \tag{4.1}$$

the LDE (1.1) becomes the well-known Fisher or Nagumo lattice equation, respectively, [2, 15]. Both reaction functions in (4.1) satisfy *(g1)*–*(g2)* with $\ell = 1$. We can therefore apply Theorem 1.1 and 1.2 to characterize onesided and twosided unbounded stationary solutions u of the corresponding LDE (1.1) with $u_i > 1$ for all $i \in \mathbb{Z}$ via the function f and the set $\mathscr{V}$ (which are qualitatively the same in both cases, see Figure 3.1).

Our next example contains simple piecewise linear reaction functions for which we can analytically express boundary of $\mathscr{V}$ and explicit formulas for $u^{\alpha,I}$ and $u^{\alpha,D}$.

Example 4.2 (Sawtooth and McKean's caricatures of bistability)**.** For simplicity, let us consider the LDE (1.1) and piecewise linear caricatures of the standard cubic bistable nonlinearity

$$g(u) = \begin{cases} -u & \text{for } u \in \left[0, \frac{a}{2}\right), \\ u-a & \text{for } u \in \left[\frac{a}{2}, \frac{1+a}{2}\right], \\ 1-u & \text{for } u \in \left(\frac{1+a}{2}, \infty\right), \end{cases}$$

or

$$g(u) \begin{cases} = -u & \text{for } u \in [0,a), \\ \in [-a, 1-a] & \text{for } u = a, \\ = 1-u & \text{for } u \in (a,\infty), \end{cases} \qquad a \in (0,1),$$

(proposed by [17] and nicknamed as sawtooth and McKean's caricature, respectively, [5, 14, 20]). The functions are smooth on (a,∞) and the assumptions *(g1)*–*(g2)* are satisfied with $\ell = 1$ in both cases and the mirroring function φ is for $u \in [1,\infty)$ defined by

$$\varphi(u) = \tfrac{2d+1}{2d}(u-1) + 1$$

(i.e., $\varphi^{-1}(u) = \frac{2d}{2d+1}(u-1)+1$). Thus, all iterates $\underline{f}_n$ or $\bar{f}_n$, $n \in \mathbb{N}_0$, are linear functions which yields that the limit function f is linear as well, i.e., $f(u) = k(u-1)+1$ and $f^{-1}(u) = \frac{1}{k}(u-1)+1$ for some $k > 1$. Then Corollary 2.5 *(iii)* implies

$$k = k(d) = \frac{2d+1+\sqrt{4d+1}}{2d}.$$

The corresponding sets $\mathscr{U}$ and $\mathscr{V}$ are therefore cones in this case (see Figure 4.1) and

$$\lim_{d\to\infty} k(d) = 1 \qquad \text{and} \qquad \lim_{d\to 0+} k(d) = \infty.$$

Then, we obtain from (3.2) explicit formulas for the representatives $u^{\alpha,I}$ and $u^{\alpha,D}$ of equivalence

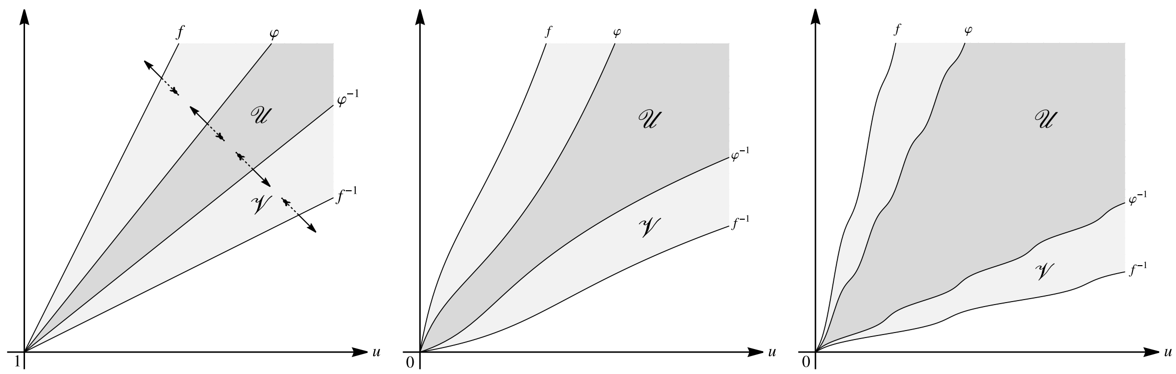

Figure 4.1: The left panel shows linear functions φ and f induced by piecewise linear bistable caricatures from Ex. 4.2. The arrows indicate widening (full arrows) and shrinking (dashed arrows) of conical sets $\mathscr{U}$ and $\mathscr{V}$ as d decreases or increases, respectively. The center and right panels then provide examples of nonconvex functions φ and f. We depict numerically obtained sets $\mathscr{U}$ and $\mathscr{V}$ for the Holling functional response of type II (4.2) with $a = 0.6$ and $b = 0.2$ from Ex. 4.3 (center panel) and the wavy reaction (4.4) with $a = 1.1$ from Ex. 4.4 (right panel).

classes $[u^{\alpha,I}]$, $[u^{\alpha,D}]$ of onesided unbounded stationary solutions, respectively, for $\alpha > 1$ as

$$u_i^{\alpha,I} = (\alpha-1)k^i + 1 \qquad \text{and} \qquad u_i^{\alpha,D} = (\alpha-1)k^{-i} + 1.$$

In contrast, our next example considers a reaction leading to more complicated sets $\mathscr{U}$ and $\mathscr{V}$, which we obtain only numerically. Logistic reaction with a predation term leads to sets $\mathscr{U}$ and $\mathscr{V}$ with nonconvex upper boundaries φ and f.

Example 4.3 (Holling functional response II)**.** Let us modify the Fisher equation and consider the LDE (1.1) with reaction function g consisting of the logistic term (describing the intraspecific competition) and of an external predation term determined by Holling functional response of type II (describing the interspecific competition), specifically,

$$g(u) = u(1-u) - \frac{au}{b+u}, \quad a, b > 0. \tag{4.2}$$

We discuss two distinct situations. The largest root of g is $\ell = \frac{1}{2}(1 - b + \sqrt{b^2 + 2b + 1 - 4a}) > 0$ provided

$$(a,b) \in \mathscr{P} = \left\{(a,b) \in \mathbb{R}_+^2 : \ (a \in (0,1) \wedge b > 2\sqrt{a} - 1) \ \vee \ (a \geq 1 \wedge b > a)\right\}.$$

It is possible to show that in this case

$$g''(u) < 0 \quad \text{for all} \quad u \in [\ell, \infty), \tag{4.3}$$

which implies that *(g_1)*–*(g_2)* hold for every such pair $(a,b) \in \mathscr{P}$. Thus, by application of Theorems 1.1 and 1.2 we obtain the function f and the set $\mathscr{V}$ describing onesided and twosided unbounded stationary solutions u of (1.1) with $u_i > \ell > 0$ for all $i \in \mathbb{Z}$. Moreover, (4.3) yields that $\varphi''(u) > 0$ for all $u \in [\ell, \infty)$ and hence, the function f satisfies $f''(u) > 0$ for all $u \in [\ell, \infty)$ from which we deduce that f and $\mathscr{V}$ have qualitatively same shape as in Ex. 4.1 and Figure 3.1.

For

$$(a,b) \in \mathscr{Z} = \left\{(a,b) \in \mathbb{R}_+^2 : \ (a \in (0,1) \wedge b \leq 2\sqrt{a} - 1) \ \vee \ (a \geq 1 \wedge b \leq a)\right\}$$

the largest root of g is $\ell = 0$. In this case the situation is more intricate and there are values of

$(a,b) \in \mathscr{X}$ for which *(g_2)* holds as well as values $(a,b) \in \mathscr{X}$ for which *(g_2)* is not satisfied. For example, *(g_2)* does not hold for $a = 0.4$ and $b = 0.2$. However, the assumption *(g_2)* is valid, e.g., for $a = 0.6$ and $b = 0.2$ and consequently, Theorems 1.1 and 1.2 provide the function f and the set $\mathscr{V}$ characterizing onesided and twosided unbounded stationary solutions u of (1.1) with $u_i > 0$ for all $i \in \mathbb{Z}$. Interestingly, the function φ is not convex for $u \in [\ell, \infty)$ in this case which implies that the limit function f has inflection as well, see Figure 4.1.

We conclude with an illustrative example which provides an interesting wavy shape of the corresponding function f and underlying sets $\mathscr{U}$ and $\mathscr{V}$.

Example 4.4. Considering the LDE (1.1) with

$$g(u) = \sin(u) - au, \quad a > 1, \tag{4.4}$$

the assumptions *(g_1)*–*(g_2)* are satisfied with $\ell = 0$ for all $a > 1$. The limit function f has infinitely many inflection points in this case. For the corresponding sets $\mathscr{U}$ and $\mathscr{V}$, see Figure 4.1.

5 Arithmetic mean of function with its inverse

The iterative scheme (2.1) is motivated by the mirroring form (1.12) of the equation (1.3). Focusing on Corollary 2.5 *(iii)*, we interestingly reveal a connection between stationary solutions of (1.1), iterative scheme (2.1), and the following functional equation with an unknown function f:

$$\frac{f(u) + f^{-1}(u)}{2} = \varphi(u), \quad u \in [0, \infty), \tag{5.1}$$

in which φ is a given C^1-function on $[0, \infty)$ which satisfies *(p_1)*–*(p_2)* with $\ell = 0$. In other words, the unknown function f should be such that the arithmetic mean of f and its inverse f^{-1} gives the prescribed function φ.

Remark 5.1. First of all, we point out that the functional equation (5.1) has in principle infinitely many solution pairs provided at least one exists. Indeed, let f be a solution of (5.1), $u_0 \in (0, \infty)$ be given, and (u_i), $i \in \mathbb{Z}$, be defined iteratively by

$$u_{i+1} = f(u_i) \quad \text{and} \quad u_{i-1} = f^{-1}(u_i).$$

Considering the following function:

$$g(u) = \begin{cases} f(u), & \text{if } u \neq u_i \text{ for all } i \in \mathbb{Z}, \\ f^{-1}(u), & \text{if } u = u_i \text{ for some } i \in \mathbb{Z}, \end{cases} \tag{5.2}$$

one can verify that g is also a solution of (5.1), different from f and f^{-1} (note that $f(u) \neq f^{-1}(u)$ for all $u \in (0, \infty)$ because of *(p_1)*–*(p_2)*), although it still uses only values of either f or f^{-1} (it only interchanges them at u_i, $i \in \mathbb{Z}$). This motivates the following definition.

Taking a solution f of (5.1), it has to satisfy for every $u \in [0, \infty)$ that either $f(u) > \varphi(u)$, or $f^{-1}(u) > \varphi(u)$, or $f(u) = f^{-1}(u) = \varphi(u)$. Define the mapping $P : f \mapsto P(f)$, where $P(f) : [0, \infty) \to \mathbb{R}$, as

$$P(f)(u) = \begin{cases} f(u), & \text{if } f(u) \geq \varphi(u), \\ f^{-1}(u), & \text{if } f^{-1}(u) > \varphi(u). \end{cases} \tag{5.3}$$

We immediately see that $P(f)(u) \geq \varphi(u)$ for all $u \in [0, \infty)$.

Consequently, we define an equivalence relation $f \backsim g$ between two solutions f and g of (5.1) saying that $f \backsim g$ provided $P(f) = P(g)$ (e.g., the function g defined by (5.2) is equivalent to the original solution f, also to its inverse f^{-1}, and also to its own inverse g^{-1}).

The following lemma claims that every equivalence class of solutions of (5.1) containing f and using the same values (as f and g above) has a unique representative $P(f)$ which satisfies $P(f)(u) \geq \varphi(u)$ for all $u \in [0, \infty)$.

Lemma 5.2. *Let φ be a C^1-function and satisfy (p1)–(p2) with $\ell = 0$. Let f be a solution of* (5.1) *and $P(f)$ be defined by* (5.3)*. Then $P(f)$ is also a solution of* (5.1)*.*

Proof. The function $P(f)$ is injective and thus invertible on $[0, \infty)$. Indeed, let us assume by contradiction that $P(f)(u_1) = P(f)(u_2)$ for some $u_1 < u_2$. If $f(u_1) \geq \varphi(u_1)$ and $f(u_2) \geq \varphi(u_2)$ (analogically for $f^{-1}(u_1) \geq \varphi(u_1)$ and $f^{-1}(u_2) \geq \varphi(u_2)$), then

$$f(u_1) = P(f)(u_1) = P(f)(u_2) = f(u_2),$$

a contradiction, since f is invertible. If $f(u_1) \geq \varphi(u_1)$ and $f^{-1}(u_2) \geq \varphi(u_2)$ (analogically for $f^{-1}(u_1) \geq \varphi(u_1)$ and $f(u_2) \geq \varphi(u_2)$), then

$$f(u_1) = P(f)(u_1) = P(f)(u_2) = f^{-1}(u_2) = w.$$

Since $w = f(u_1) = P(f)(u_1) \geq \varphi(u_1)$, $w = P(f)(u_2) = f^{-1}(u_2) \geq \varphi(u_2)$, and φ is strictly increasing and $\varphi^{-1}(u) \leq u \leq \varphi(u)$ for all $u \in [0, \infty)$ by (p1)–(p2), then

$$f^{-1}(w) < f(w) \leq \varphi^{-1}(w) \leq \varphi(w),$$

a contradiction with (5.1). Thus, $P(f)$ is invertible, $(P(f))^{-1}(u) \leq \varphi^{-1}(u)$ for all $u \in [0, \infty)$, and by definition of $P(f)$ there has to be

$$(P(f))^{-1}(u) = \begin{cases} f^{-1}(u), & \text{if } f(u) \geq \varphi(u), \\ f(u), & \text{if } f^{-1}(u) > \varphi(u). \end{cases}$$

Therefore, $P(f)$ is also a solution of (5.1), since

$$\frac{P(f)(u) + (P(f))^{-1}(u)}{2} = \frac{f(u) + f^{-1}(u)}{2} = \varphi(u)$$

in both cases $f(u) \geq \varphi(u)$, or $f^{-1}(u) > \varphi(u)$ for a $u \in [0, \infty)$. □

Finally, as a byproduct of our previous considerations from Section 2 we obtain the following result which states that the functional equation (5.1) has a unique equivalence class of nonnegative solutions f and characterizes its representative $P(f)$.

Theorem 5.3. *Let φ be a C^1-function and satisfy (p1)–(p2) with $\ell = 0$. Then there exists a unique equivalence class of* nonnegative *solutions f of functional equation* (5.1) *(with respect to the relation $\backsim$) and its representative $P(f)$ is given by*

$$P(f)(u) = \lim_{n \to \infty} \underline{f}_n(u) = \lim_{n \to \infty} \bar{f}_n(u), \quad u \in [0, \infty),$$

in which $\underline{f}_n$ and $\bar{f}_n$ are given by the iterative scheme (2.1) *with $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$ and $\bar{f}_0(u) = 2\varphi(u)$, respectively (cf. Definition 2.2). In particular, the representative $P(f)$ is continuous.*

Proof. The existence and properties of a common limit function $f(u) = \lim_{n\to\infty} \underline{f}_n(u) = \lim_{n\to\infty} \bar{f}_n(u)$, $u \in [0, \infty)$, of (2.1) follows from Corollary 2.5 *(i)–(ii)*. Moreover, f (and also f^{-1}) solves the functional equation (5.1) by Corollary 2.5 *(iii)*. Since the limit function f satisfies $f(u) \geq \varphi(u)$ for all $u \in [0, \infty)$, then $P(f)(u) = f(u)$ for all $u \in [0, \infty)$ which concludes the proof of existence and representation of the equivalence class.

Let g be another nonnegative solution of (5.1) such that $P(g) \neq P(f)$ and without loss of generality (see Lemma 5.2) assume that $P(g) = g$. Let us show that

$$\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u) \leq g(u) \leq 2\varphi(u) = \bar{f}_0(u) \quad \text{for all} \quad u \in [0, \infty) . \tag{5.4}$$

For the former inequality, assume by contradiction that $g(u_c) < 2\varphi(u_c) - \varphi^{-1}(u_c)$ for some $u_c > 0$ (for $u = 0$ there has to be $g(0) = g^{-1}(0) = 0$, since both g and g^{-1} are assumed to be nonnegative and $\varphi(0) = 0$ by *(p_1)*). Then (5.1) yields that

$$2\varphi(u_c) - \varphi^{-1}(u_c) > g(u_c) = 2\varphi(u_c) - g^{-1}(u_c), \quad \text{i.e.,} \quad g^{-1}(u_c) > \varphi^{-1}(u_c).$$

Thus, since φ^{-1} is a strictly increasing function, there has to exist a $v_c > 0$ such that $\varphi(v_c) > g(v_c)$, which is a contradiction because $g(u) = P(g)(u) \geq \varphi(u)$ for all $u \in [0, \infty)$.

For the latter inequality in (5.4), assume again by contradiction that $g(u_c) > 2\varphi(u_c)$ for some $u_c > 0$. Then (5.1) implies that

$$2\varphi(u_c) < g(u_c) = 2\varphi(u_c) - g^{-1}(u_c), \quad \text{i.e.,} \quad g^{-1}(u_c) < 0,$$

which is a contradiction with the nonnegativeness of g^{-1}.

Finally, if two initial functions of the iterative scheme (2.1) are ordered for all $u \in [0, \infty)$, then all iterates of (2.1) are ordered in the same fashion for all $u \in [0, \infty)$, this can be proved similarly as in Lemma 2.4. Therefore, the inequalities (5.4) and the fact that g is the fixed element of (2.1) yield that

$$\underline{f}_n(u) \leq g(u) \leq \bar{f}_n(u) \quad \text{for all} \quad n \in \mathbb{N}_0 \quad \text{and} \quad u \in [0, \infty) .$$

The squeeze argument then implies that

$$P(f)(u) \leftarrow \underline{f}_n(u) \leq g(u) \leq \bar{f}_n(u) \rightarrow P(f)(u) \quad \text{for all} \quad u \in [0, \infty) \quad \text{and} \quad n \to \infty,$$

i.e., $g(u) = P(g)(u) = P(f)(u)$ for all $u \in [0, \infty)$, a contradiction. This concludes the proof of the uniqueness of the equivalence class of nonnegative solutions of (5.1). □

Remark 5.4. Note that besides the existence, uniqueness, and several properties of the class of nonnegative solutions of (5.1), Theorem 5.3 presents the procedure how the continuous representative $P(f)$ can be approximated by the iterations (2.1) with $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$ (from below) and $\bar{f}_0(u) = 2\varphi(u)$ (from above).

6 Discussion

In this paper we showed that onesided unbounded stationary solutions of the LDE (1.1) form a one-parametric family of equivalence classes, Theorem 1.1, and bound the region of unbounded twosided solutions in a two-parametric space, Theorem 1.2 and Figures 3.1 and 4.1.

Continuous counterpart

Let us emphasize the behaviour of corresponding solutions of the PDE (1.2). The simple phase plane analysis (e.g., [7, Section 4]) yields that there is only a unique equivalence class of strictly increasing solutions with (1.5) and a unique class of strictly decreasing onesided solutions satisfying (1.6). Moreover, these continuous solutions exist only on a bounded spatial interval and blow up to infinity at its ends.

Topological chaos and unbounded solutions

Let us also highlight the fact that both onesided and twosided lattice stationary solutions characterized by Theorem 1.2 exist for any diffusion parameters. This fact and a simple look at the white regions in Figure 3.1 lead to an intriguing problem. In the case of Nagumo lattice equation (1.1) with $g(u) = u(1-u)(u-a)$, $a \in (0,1)$, stationary solutions which are represented by

$$(u_i, u_{i+1}) \in \mathscr{W} = \left\{ (\xi, \zeta) \in \mathbb{R}^2 : \ \xi, \zeta > \ell \ \text{ and } \ (\xi, \zeta) \notin \overline{\mathscr{V}} \right\},$$

where $\mathscr{V}$ is defined in (1.9), can be very difficult to characterize fully because the iterations lead to the domain of topological chaos. For example, which initial conditions lead to positive stationary solutions? How does the set of such initial conditions depend on the value of $d > 0$ in the LDE (1.1)? Twosided and onesided lattice stationary solutions satisfying (1.4), (1.5), or (1.6) correspond to continuous counterparts in the PDE (1.2) and are more numerous, generally asymmetric in the twosided case (1.4) and do not blow up to infinity in finite spatial interval. Intuitively, solutions with $(u_i, u_{i+1}) \in \mathscr{W}$ may have richer behaviour and, most importantly, could generate qualitatively new types of stationary solutions.

Applications of mirroring

In this paper we generalized the mirroring technique to functional mirroring scheme and connected it to the functional equation (5.1). It is possible that this geometric approach could contribute to one of the many problems related to the topological chaos, e.g., explicit solutions for special reaction functions.

Functional equation

Our final remark is related to the functional equation (5.1). Note that φ in our case is the specific mirroring function defined by (1.11). The functional equation (5.1) represent an interesting problem itself once any function φ is considered. In principle, the solvability of functional equations is nontrivial and depend for example on the domain. Theorem 5.3 provides a specific existence and uniqueness result in the case in which φ satisfies *(p_1)*–*(p_2)*.

Acknowledgments

The authors acknowledge the support of the project GA22-18261S by the Czech Science Foundation. We also thank to an anonymous reviewer for detailed comments which helped to improve the text to its current form.

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