Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **62**, 1–15; https://doi.org/10.14232/ejqtde.2024.1.62 www.math.u-szeged.hu/ejqtde/

Limit cycles bifurcations of a Liénard system with a hyperelliptic Hamiltonian of degree five

Yi Shao ✉ and **Chunxiang A**

School of Mathematics and Statistics, Zhaoqing University, Guangdong, 526061, P. R. China

Received 1 June 2024, appeared 7 November 2024
Communicated by Armengol Gasull

Abstract. We deal with limit cycles bifurcating from the period annulus of Liénard system with a hyperelliptic Hamiltonian of degree five under quartic perturbation, where Liénard system has a normal form $\dot{x} = y$, $\dot{y} = x(x-1)(x^2+ax+b)$, $a^2-4b<0$. It is proved that the perturbation of this system can produce at most six limit cycles for $a=b=2$.

Keywords: Liénard system, Poincaré bifurcation, limit cycles.

2020 Mathematics Subject Classification: 34C05, 34C08, 34C23.

1 Introduction

In the qualitative theory of real planar differential systems, one of research focus is the number and configuration of limit cycles, which belong to the context of the second part of Hilbert's 16th Problem. Until now the problem still remains to be unsolved even though a lot of works have been done. As well known, Arnold [1] proposed a weaker version of this problem, the so-called infinitesimal Hilbert's 16th problem, that is to study the number of isolated zeros of the Abelian integrals obtained from integrating polynomial 1-forms over ovals of polynomial Hamiltonian.

Consider perturbations of the Hamiltonian system

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon P(x,y), \\ \dot{y} = -H_x(x,y) + \varepsilon Q(x,y), \end{cases} \tag{1.1}$$

where $H(x,y)$ is a polynomial of degree $n+1$, $P(x,y)$ and $Q(x,y)$ are polynomials of degree m in x, y, and ε is a small parameter.

We assume that there is a family of ovals $\Gamma_h \subset \{(x,y) \mid H(x,y) = h\}$, continuously depending on a parameter $h \in (h_1, h_2)$, then the Abelian integral of system (1.1) is defined as

$$I(h) = \oint_{\Gamma_h} P(x,y)dy - Q(x,y)dx, \tag{1.2}$$

✉ Corresponding author. Email: mathsyishao@126.com

where Γ_h is the open punctured neighborhood foliated by periodic orbits of system (1.1) as $\varepsilon = 0$. The displacement function $d(h, \varepsilon)$ of system (1.1) is defined on a segment transversal to the flow, which is parameterized by the Hamiltonian value h, then

$$d(h, \varepsilon) = \oint_{\Gamma_h} dH = \varepsilon(I(h) + O(\varepsilon)). \tag{1.3}$$

Hence, if $I(h)$ is not identically zero, then the number of isolated zeros of the Abelian integral $I(h)$ (or be called the first order Melnikov function) gives an upper bound for the number of limit cycles of system (1.1) in any compact region of period annulus. It is well known that the limit cycles bifurcating from the period annulus is called Poincaré bifurcation.

The generalized Liénard system $\dot{x} = y, \dot{y} = f(x) + \varepsilon y g(x)$ of type (m, n) has rich dynamic behavior, where m and n are degrees of polynomials $f(x)$ and $g(x)$, respectively. If $m = 2, 3$ and $\varepsilon = 0$, then Hamiltonian functions of this system are called elliptic. Many authors studied the bifurcations of limit cycles on this system. Dumortier and Li have made a complete investigate for Liénard system of type $(3, 2)$ in a series of papers (see [3–6]), and they proved that the upper bound of number of isolated zeroes of Abelian integrals is five. In [11], the authors also investigated some Liénard systems of type $(3, 2)$ with symmetry, which exist at most two limit cycles.

If $m \geq 4$ and $\varepsilon = 0$, then Hamiltonian functions of the above Liénard system are called hyperelliptic. In [7], Gavrilov and Iliev given the topological classification of hyperelliptic Hamiltonian system of degree five, its normal form of Hamiltonian function is

$$H(x, y) = \frac{1}{2}y^2 + \frac{\lambda\mu}{2}x^2 - \frac{\lambda + \mu + \lambda\mu}{3}x^3 + \frac{1 + \lambda + \mu}{4}x^4 - \frac{1}{5}x^5, \tag{1.4}$$

where there are eleven cases having period annulus. J. Wang (see [15,16]) studied the number of limit cycles of two classes of Liénard systems of type $(4, 3)$ and $(4, 2)$, in which unperturbed systems has a saddle and degenerated polycycle, respectively. The authors of [20] obtained lower bounds of the number of limit cycles for a Liénard system of type $(4, n)$ having two elementary centers, where $20 \leq n \leq 24$.

In this paper, we choose one of eleven cases in [7], that is, we investigation Poincaré bifurcation for a Liénard system of type $(4, 3)$ with hyperelliptic Hamiltonian $H(x, y) = h$, $h \in (h_1, h_2)$ in (1.4) having a pair of conjugated complex critical points. The perturbation system is as follows

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(x-1)(x^2 + ax + b) + \varepsilon(\alpha + \beta x + \gamma x^2 + x^3)y, \quad a^2 - 4b < 0. \end{cases} \tag{1.5}$$

It is easy to know that the unperturbed system of (1.5) has a bounded period annulus surrounding the elementary center $(0, 0)$, corresponding to endpoint h_1, and a homoclinic loop (bounder of period annulus) passing through hyperbolic saddle $(1, 0)$, corresponding to endpoint h_2. By (1.2), we know that the Abelian integral of system (1.5) is

$$I(h) = \int_{\Gamma_h} (\alpha + \beta x + \gamma x^2 + x^3) y dx = \alpha I_0(h) + \beta I_1(h) + \gamma I_2(h) + I_3(h), \tag{1.6}$$

where $I_i(h) = \int_{\Gamma_h} x^i y dx, i = 0, 1, 2, 3$ and Γ_h is the compact component of $H(x, y) = h$, defined by (1.5).

There are many techniques and arguments to tackle the problem of bounding the number of zeroes of Abelian integrals, lots of them are very long and non-trivial, see [2]. Since

Hamiltonian function $H(x,y)$ has the higher degree, a purely algebraic criterion proposed in [8] and [12] are usual methods. These methods can transfer the estimation the number of zeros of Abelian integrals to that of the number of real roots of linear combinations of a tuple $(I_0(h), I_1(h), I_2(h), I_3(h))$ of associated semi-algebraic systems (SAS for short). This criterion can reduce the difficulties of qualitative analysis in limit cycle bifurcating from a center. Nonetheless, it is challenging and very difficult to obtain the cyclicity of this family of period annuli that depends on the variables a, b indeed, which need to verify the problem whether the collection of Abelian integrals is an ECT-system or a Chebyshev system with accuracy k (see [8]). We attempt several values of the variables a, b by using software *Maple*, which lead to a desktop computer to a dead end due to the huge polynomials with huge coefficients and thousands of terms.

In the present paper, we take $a = b = 2$, and use approaches of *real root isolation* and interval analysis to get the number of roots of huge polynomial, as a result, we can obtain the number of zeros of Abelian integrals of system (1.5) for $a = b = 2$. We rewrite system (1.5) as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(x-1)(x^2+2x+2) + \varepsilon(\alpha + \beta x + \gamma x^2 + x^3)y, \end{cases} \tag{1.7}$$

where α, β, γ are arbitrary real constants and $\varepsilon > 0$ is a small parameter, and the first integral of (1.7) is

$$H(x,y) = \frac{1}{2}y^2 + x^2 - \frac{1}{4}x^4 - \frac{1}{5}x^5 = h, \qquad h \in \left(0, \frac{11}{20}\right) \tag{1.8}$$

as $\varepsilon = 0$. The projection interval of the period annulus Γ_h on the x-axis is $(x_0, 1)$, where $x_0 \approx -0.763592319985$, and it is an intersection of the homoclinic loop with the negative half axis of the x-axis. Phase portrait of the unperturbed system of (1.7) see Figure 1.1.

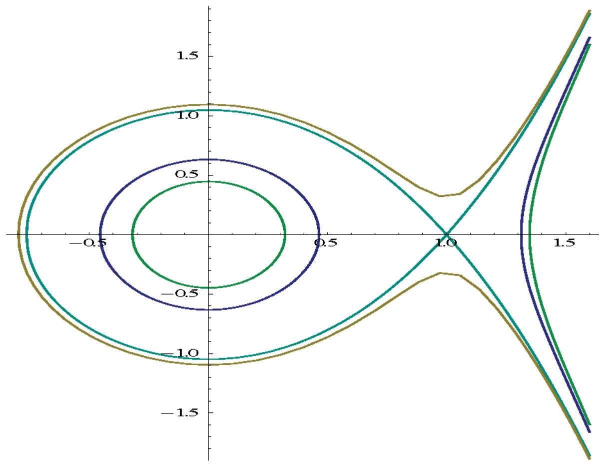

Figure 1.1: Phase portrait of system (1.5) when $\varepsilon = 0$.

The main purpose in this paper is to show that system (1.7) can undergo Poincaré bifurcation from the period annulus surrounding the origin. We can prove that the Abelian integral $I(h)$ has at most six zeros (taken into account muitiplicity), see Proposition 3.1 in Section 3. Proposition 3.1 and the equation (1.3) imply that system (1.7) can produce at most six limit cycles. The main results of this paper as follows.

Theorem 1.1. *The number of limit cycles of system* (1.7) *bifurcating from period annulus surrounding the center is at most six for arbitrary value of parameters* α, β, γ.

Note that unperturbed system of (1.7) has a period annulus surrounding the elementary center, its outer boundary is a saddle loop. According to Roussarie's theorem [14], the upper bound of number of isolated zeros of the Abelian integral $I(h)$ covers the number of limit cycles from center, from period annulus and from the homoclinic loop, therefore we have the following Theorem.

Theorem 1.2. *System (1.7) could give rise to at most six limit cycles in the finite plane surrounding the origin for sufficiently small ε and any parameters α, β, γ.*

The paper is organized as follows. In Section 2, we introduce some definitions and properties of Chebyshev systems. In Section 3, we study the number of zeros of Abelian integral $I(h)$ and obtain the maximal number of limit cycles bifurcating from period annulus by using Chebyshev criterion. Hence Proposition 3.1 is main result of this paper.

2 Preliminary properties

In order to study the number of isolated zeros of Abelian integral $I(h)$ in $h \in (0, \frac{11}{20})$, Grau et al. in [8] give a Chebyshev criterion, which check whether (I_0, I_1, I_2, I_3) in (1.6) is an extended complete Chebyshev system or Chebyshev system with accuracy k. Hence we introduce some preliminary definitions and properties, the reader can refer to [8, 12] or the recent paper [13] for more details.

Definition 2.1. Let $\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x)$ be analytic functions on an open interval L of $\mathbb{R}$.

(i) The set of functions $(\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x))$ is a Chebyshev system (T-system) with accuracy k on L if any nontrivial linear combination

$$\alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \cdots + \alpha_{n-1}\varphi_{n-1}(x)$$

has at most $n + k - 1$ isolated zeros for $x \in L$.

(ii) The set of functions $(\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x))$ is an extended complete Chebyshev system (ECT-system) on L if for all $m = 1, 2, \ldots, n$, any nontrivial linear combination

$$\alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \cdots + \alpha_{m-1}\varphi_{m-1}(x)$$

has at most $m - 1$ isolated zeros on L counted with multiplicities.

(iii) The continuous Wronskian of $(\varphi_0(x), \varphi_1(x), \ldots, \varphi_{m-1}(x))$ at $x \in L$ is

$$W[\varphi_0, \varphi_1, \ldots, \varphi_{m-1}](x) = \mathrm{Det}(\varphi_j^{(i)}(x))_{0 \le i,j \le m-1} = \begin{vmatrix} \varphi_0(x) & \cdots & \varphi_{m-1}(x) \\ \varphi_0'(x) & \cdots & \varphi_{m-1}'(x) \\ \vdots & \ddots & \vdots \\ \varphi_0^{(m-1)}(x) & \cdots & \varphi_{m-1}^{(m-1)}(x) \end{vmatrix},$$

where $\varphi_j'(x)$ and $\varphi_j^{(i)}(x) (i \ge 2)$ represent the derivative of one order and the ith order of $\varphi_j(x)$, respectively.

Lemma 2.2 ([10] or [8]). *$(\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x))$ is an extended complete Chebyshev system on L if and only if, for each $m = 1, 2, \ldots, n$,*

$$W[\varphi_0, \varphi_1, \ldots, \varphi_{m-1}](x) \ne 0 \quad \textit{for all } x \in L.$$

Now we rewrite the first integral (1.8) as

$$H(x,y) = A(x) + B(x)y^2, \tag{2.1}$$

where

$$A(x) = x^2 - \frac{1}{4}x^4 - \frac{1}{5}x^5, \qquad B(x) = \frac{1}{2}$$

and $H(x,y)$ is an analytic function in open interval. There exists a period annulus filled by the set of ovals $\Gamma_h \in \{(x,y)|H(x,y) = h, h \in (0, \frac{11}{20})\}$ and $H(0,0) = 0$ is a local minimum. It is easy to verify that $xA'(x) = x^2(1-x)(x^2+2x+2) > 0$ for any $x \in (x_0, 1) \setminus 0$ ($x_0 \approx -0.763592319985$). Thus, there exists an analytic involution $z = \sigma(x)$ ($\sigma \circ \sigma = \mathrm{Id}$ and $\sigma \neq \mathrm{Id}$) with $x_0 < z < 0$ such that

$$A(x) = A(\sigma(x)) \quad \text{for all } x \in (0,1)$$

and $\sigma(0) = 0$. Using Theorem A in [12], we get the following lemma.

Lemma 2.3. *Assume that $g_i(x)$ are an analytic function on the interval $(x_0, 1)$, $i = 0,1,2,3$. Denote*

$$\bar{I}_i(h) = \int_{\Gamma_h} g_i(x) y^{2s-1} dx,$$

where Γ_h is the set of periodic orbit surrounding the origin inside the level curve $\{A(x)+B(x)y^{2m} = h\}$ for each $h \in (0, \frac{11}{20})$. Let

$$\varphi_i(x) = \frac{g_i(x)}{A'(x)(B(x))^{\frac{2s-1}{2m}}} - \frac{g_i(\sigma(x))}{A'(\sigma(x))(B(\sigma(x)))^{\frac{2s-1}{2m}}}.$$

If the following statements hold:

(i) $W[\varphi_0, \varphi_1, \ldots, \varphi_m](x)$ is not vanish on $(0,1)$ for $m = 1, 2, \ldots, n-2$;

(ii) $W[\varphi_0, \varphi_1, \ldots, \varphi_{n-1}](x)$ has k zeroes on $(0,1)$ counted with multiplicities, and

(iii) $s > m(n+k-2)$,

then $(\bar{I}_0(h), \bar{I}_1(h), \ldots, \bar{I}_{n-1}(h))$ has at most $n+k-1$ isolated zeros on $(0, \frac{11}{20})$ counted with multiplicities.

To prove Proposition 3.1 in Section 3, note that $I_i(h) = \int_{\Gamma_h} x^i y dx$, $s = 1, m = 1$ and $n = 4$, even if $k = 0$, but the condition $s > 2$ in Lemma 2.3 would not be satisfied. Hence we can not apply Lemma 2.3 directly. We need promote the power y in the integrand of $I_i(h)$ such that the conditions $s > m(n+k-2) = k+2$ hold. By using Lemma 4.1 in [8], we have

Lemma 2.4. *Let Γ_h be an oval inside the level curve $\{A(x) + B(x)y^2 = h\}$. If there exists a function $U(x)$ such that $\frac{U(x)}{A'(x)}$ is analytic at $x = 0$. Then, for any $s \in \mathbb{N}$,*

$$\int_{\Gamma_h} U(x) y^{s-2} dx = \int_{\Gamma_h} V(x) y^s dx,$$

where $V(x) = \frac{2}{s}\left(\frac{B(x)U(x)}{A'(x)}\right)' - \frac{B'(x)U(x)}{A'(x)}$.

In order to get the number of real roots of linear combinations of a tuple $(I_0(h), I_1(h), I_2(h), I_3(h))$ in (1.6), Lemma 2.3 is a main criterion in our paper. By applying this criterion, we can transfer the estimation of the number of real zeroes of Abelian integral to that of the number of real roots of a tuple, which reduce a semi-algebraic systems(SAS) and the reader is referred to see [18,19] for more details. The key to solve of SAS question is to solve polynomial equations.

We suppose that the polynomial equation $W(x,z) = 0$ has a real root (x^*, z^*) at the rectangle domain $D = \{(x,z) \mid (x,z) \in (0,1) \times (x_0, 0)\}$, where the variables x, z also satisfy the equation $q(x,z) = 0$, where $z = \sigma(x)$ is an involution and $\sigma'(x) < 0$. Solving this SAS question is to solve common root of systems of equations of two unknowns $W(x,z) = 0, q(x,z) = 0$. We divide analytic techniques into three steps.

Step 1 (Elimination variable by resultant): We can elimination variable x (or z) by the theory of resultant, that is, variables x, z satisfy the resultant equation

$$R(z) = \text{res}(W(x,z), q(x,y), x) = 0 \quad \text{or} \quad \bar{R}(x) = \text{res}(W(x,z), q(x,y), z) = 0.$$

Step 2 (Interval isolation of real root): Without loss of generality, Assume that the resultant equation $\bar{R}(x) = 0$ has a real root x^* in $[x_1, x_2] \subset [0,1]$ (corresponding z^* in $[z_1, z_2] \subset [x_0, 0]$) by using command realroot in *Maple*. We substitute $x = x_1$ and $x = x_2$ into $p(x,z) = 0$, and let maximal interval of isolation real root of equation $p(x,z) = 0$ be $[z_{11}, z_{12}] \subset [z_1, z_2]$. Thus we minimize the possible existing the rectangle domain of the common roots of $W(x,z) = 0$ and $q(x,y) = 0$.

However, it should be noted that if we solve $\bar{R}(x) = 0$ directly without using interval isolation of real root, then we can only get a approximation of x^*. But the results of these numerical calculation are sometimes unreliable due to the thousands of terms of polynomials with huge coefficients, a famous cautionary example see [17], and example of numerical calculation see Lemma 3.4 [iv] of [9].

Step 3 (Analysis of common real roots): Let the above the rectangle domain be $ABCD$, where $A(x_1, z_{12})$, $B(x_1, z_{11})$, $C(x_2, z_{11})$ and $D(x_2, z_{12})$. We can analysize whether the rectangle domain has a common root or not by positive and negative values of $W(x,z)$ and $q(x,z)$ at vertices A, B, C, D, the involution and the intermediate value theorem of continuous function. The detailed application skills see proof of Lemma 3.2.

3 Poincaré bifurcations of system (1.7)

In this Section, to prove Theorem 1.1, firstly, we will study the number of isolated zeros of Abelian integral $I(h)$ in $h \in (0, \frac{11}{20})$, that is the following Proposition 3.1.

Proposition 3.1. *For arbitrary value of parameters (α, β, γ), the Abelian integral $I(h)$ of system* (1.7) *has at most six zeros (counting multiplicities) in $h \in (0, \frac{11}{20})$.*

The main tools of proving Proposition 3.1 are Lemmas 2.2–2.4. Hence we need to check the Chebyshev property of the Abelian integral (1.6). Proof of Proposition 3.1 will be given at the end of this section.

Now applying Lemma 2.4, we rewrite $I_i(h)$ in (1.7) as

$$\begin{aligned} I_i(h) &= \frac{1}{h} \int_{\Gamma_h} \left(A(x) + \frac{1}{2} y^2 \right) x^i y dx = \frac{1}{h} \int_{\Gamma_h} \left[x^i A(x) y + \frac{1}{2} x^i y^3 \right] dx \\ &= \frac{1}{h} \int_{\Gamma_h} V_i(x) y^3 dx, \qquad i = 0, 1, 2, 3, \end{aligned}$$

where

$$V_i(x) = \frac{x^i v_i(x)}{60(x-1)^2(x^2+2x+2)^2}$$

with

$$\begin{aligned} v_i(x) = &(160+40i) - (130+30i)x^2 - (112+28i)x^3 \\ &+ (35+5i)x^4 + (68+9i)x^5 + (34+4i)x^6. \end{aligned}$$

To promote the power y such that the condition $s > 2$ (suppose $k = 0$) is satisfied, by using Lemma 2.4 again, we obtain that

$$\begin{aligned} I_i(h) &= \frac{1}{h^2}\int_{\Gamma_h}\left(A(x)+\frac{1}{2}y^2\right)V_i(x)y^3dx \\ &= \frac{1}{h^2}\int_{\Gamma_h}\left[V_i(x)A(x)y^3+\frac{1}{2}V_i(x)y^5\right]dx = \frac{1}{h^2}\int_{\Gamma_h} g_i(x)y^5dx, \end{aligned}$$

where

$$g_i(x) = \frac{x^i \tau_i(x)}{6000(x-1)^4(x^2+2x+2)^4} \tag{3.1}$$

with

$$\begin{aligned} \tau_i(x) = &(38400+16000i+1600i^2) - (62400+24800i+2400i^2)x^2 \\ &- (51840+21920i+2240i^2)x^3 + (44300+15200i+1300i^2)x^4 \\ &+ (81120+27600i+2400i^2)x^5 + (23992+8324i+804i^2)x^6 \\ &- (41480+11990i+820i^2)x^7 - (37939+10782i+719i^2)x^8 \\ &- (5354+1851i+134i^2)x^9 + (11096+2375i+121i^2)x^{10} \\ &+ (7344+1492i+72i^2)x^{11} + (1836+352i+16i^2)x^{12}. \end{aligned}$$

Denote

$$\bar{I}_i(h) = h^2 I_i(h) = \int_{\Gamma_h} g_i(x)y^5dx, \qquad h \in \left(0, \frac{11}{20}\right).$$

We can see that $g_i(x)$ is analytic on $(x_0, 1)$ and $(I_0(h), I_1(h), I_2(h), I_3(h))$ is an ECT-system or T-system on $(0, \frac{11}{20})$ if and only if so is $(\bar{I}_0(h), \bar{I}_1(h), \bar{I}_2(h), \bar{I}_3(h))$. Therefore, applying Lemma 2.4 to $(\bar{I}_0(h), \bar{I}_1(h), \bar{I}_2(h), \bar{I}_3(h))$ with $s = 3$, we have

$$\bar{\varphi}_i(x,z) = \bar{g}_i(x) - \bar{g}_i(z) = \left(\frac{4\sqrt{2}g_i}{A'}\right)(x) - \left(\frac{4\sqrt{2}g_i}{A'}\right)(z), \qquad i = 0,1,2,3, \tag{3.2}$$

where

$$\bar{g}_i(x) = \frac{\sqrt{2}x^{i-1}\tau_i(x)}{1500(1-x)^5(x^2+2x+2)^5},$$

$$\bar{g}_i(z) = \frac{\sqrt{2}z^{i-1}\tau_i(z)}{1500(1-z)^5(z^2+2z+2)^5}$$

and $z = \sigma(x)$ is an involution function.

On the other hand, due to

$$A(x) - A(z) = \frac{1}{20}(x-z)p(x,z) = 0,$$

where

$$p(x,z) = -20(x+z) + 5(x^3 + x^2z + xz^2 + z^3) + 4(x^4 + x^3z + x^2z^2 + xz^3 + z^4).$$

Since $\sigma(0) = 0$, It turns out that $z = \sigma(x)$ is defined by means of $p(x,z) = 0$. Moreover, we get that

$$\sigma'(x) = \frac{dz}{dx} = -\frac{p'_x(x,z)}{p'_z(x,z)}, \tag{3.3}$$

where

$$p'_x(x,z) = -20 + 15x^2 + 10xz + 5z^2 + 16x^3 + 12x^2z + 8xz^2 + 4z^3,$$
$$p'_z(x,z) = -20 + 5x^2 + 10xz + 15z^2 + 4x^3 + 8x^2z + 12xz^2 + 16z^3.$$

By Lemma 2.2, we find that the Wronskian $W[(\bar{\varphi}_0, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)](x)$ has two zeros on $(0,1)$ by interval analysis, which shows that $(\bar{\varphi}_0, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ is not an ECT-system on $(0,1)$. According to Lemma 2.3(iii), we need to take $s > m(n+k-2) = 4$. Hence we lift the power of y in the integrand of $I_i(h)$ to $2s-1 = 9$. But we find that the associated Wronskian $W[(\varphi_0, \varphi_1, \varphi_2, \varphi_3)](x)$ has three zeros for $x \in (0,1)$. So we further take $s = 6$ and lift the power of y of integrand of $I_i(h)$ to $2s-1 = 11$, fortunately, the associated Wronskian $W[(\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)](x)$ has still three zeros, all of the first order Wronskians $W[(\tilde{\varphi}_i)](x)(i = 0,1,2,3)$, the second order Wronskians $W[(\tilde{\varphi}_2, \tilde{\varphi}_1)](x)$ and the third order Wronskians $W[(\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0)](x)$ have all no zero for $x \in (0,1)$, which give us hope to obtain the number of zeros of linear combinations of the tuple of functions $(\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$ by Lemma 2.3.

Repeating the above procedures, it follows from Lemma 2.4 that

$$\tilde{I}_i(h) = h^5 I_i(h) = \int_{\Gamma_h} F_i(x) y^{11} dx, \qquad h \in \left(0, \frac{11}{20}\right). \tag{3.4}$$

where

$$F_i(x) = \frac{x^i G_i(x)}{33264000000(x-1)^{10}(x^2+2x+2)^{10}},$$

and $G_i(x)$ is a polynomial of degree 30 in x, we omit it here because the polynomial has a longer expression. Let

$$\tilde{\varphi}_i(x,z) = \left(\frac{32\sqrt{2}F_i}{A'}\right)(x) - \left(\frac{32\sqrt{2}F_i}{A'}\right)(z), \qquad i = 0,1,2,3, \tag{3.5}$$

where $z = \sigma(x)$ is an involution and $x \in (0,1)$.

According to Lemma 2.3, we need to compute the number of zeros of many Wronskians, moreover, each of Wronskians is a the huge polynomial with huge coefficients and hundreds of items. After many attempts to the ordered linear combinations of associated criterion function $(\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$, finally we find that the tuple of functions $(\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3)$ satisfy the statements in Lemma 2.3. Then we get the following lemma.

Lemma 3.2. $(\tilde{I}_2(h), \tilde{I}_1(h), \tilde{I}_0(h), \tilde{I}_3(h))$ *has at most six isolated zeros on* $(0, \frac{11}{20})$ *counted with multiplicities.*

Proof. It follows from Lemma 2.3 that we need to verify the statements (i) and (ii). For this purpose, we divide the proof into four cases.

Case 1: The fourth Wronskian $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has three zeros for $(x,z) \in (0,1) \times (x_0, 0)$.

With the help of the computer algebraic system *Maple*, by direct computation, we get that

$$\begin{aligned}&W[\tilde{\varphi}_2,\tilde{\varphi}_1,\tilde{\varphi}_0,\tilde{\varphi}_3](x,z)\\&=-\frac{(x-z)^{10}\omega_4(x,z)}{834198140917968750000000000000x^4z^4q_4(x)q_4(z)(p'_z(x,z))^6},\end{aligned}\tag{3.6}$$

where $\omega_4(x,z)$ is a polynomial of degree 228 in (x,z), $p'_z(x,z)$ as in (3.3),

$$q_4(x)=(x-1)^{38}(x^2+2x+1)^{38},\qquad q_4(z)=(z-1)^{38}(z^2+2z-1)^{38},$$

and $z=\sigma(x)$ is an implicit function determined by the polynomial equation $p(x,z)=0$. Meanwhile, we find that $\omega_4(x,z)$ is symmetric polynomial with respect to x,z.

We assert that $p'_z(x,z)\neq 0$ for any $(x,z)\in(0,1)\times(x_0,0)$. In fact, by computing the resultant $R(p'_z,p,z)$ with respect to z between $p'_z(x,z)$ and $p(x,z)$, we have

$$\begin{aligned}R(p'_z,p,z)=&\,8000(4x^3+13x^2+22x+11)(4x^3+5x^2-20)\\&\times(16x^6-24x^5-7x^4-68x^3+256x^2-296x+148).\end{aligned}$$

It is easy to verify that three polynomial factors in $R(p'_z,p,z)$ have not zeros in $x\in(0,1)$. This implies $W[\tilde{\varphi}_2,\tilde{\varphi}_1,\tilde{\varphi}_0,\tilde{\varphi}_3](x,z)$ is well defined in $(0,1)\times(x_0,0)$.

Next we calculate the resultant with respect to z between $\omega_4(x,z)$ and $p(x,z)$ and obtain

$$R(\omega_4,p,z)=396154108207169536000000000000(x-1)^{28}(x^2+20+2)^{28}\phi_4(x),$$

where $\phi_4(x)$ is a polynomial of degree 828 in x.

Note that $\omega_4(x,z)$ and $p(x,z)$ are symmetric polynomials with respect to x,z and $z=\sigma(x)$ is an involution. Therefore the resultant $R(\omega_4,p,z)$(or $R(\omega_4,p,x)$) between $\omega_4(x,z)$ and $p(x,z)$ with respect to x (or z) suffices that the statement $R(\omega_4,q,x)=R(\omega_4,q,z)|_{z=x}$. By command realroot in *Maple*, we get that $\phi_4(x)$ has five isolate zeros $x_i^*(i=1,2,3,4,5)$ in $(0,1)$ and three isolate zeros $z_i^*(i=1,2,3)$ in $(x_0,0)(x_0\approx-0.763592319985)$. List of the isolation intervals of these zeros are as follows

$$\begin{aligned}x_1^*\in[\hat{x}_{11},\hat{x}_{12}]&=\left[\frac{73134709858141}{140737488355328},\frac{36567354929071}{70368744177664}\right],\\x_2^*\in[\hat{x}_{21},\hat{x}_{22}]&=\left[\frac{42527490909395}{70368744177664},\frac{85054981818791}{140737488355328}\right],\\x_3^*\in[\hat{x}_{31},\hat{x}_{32}]&=\left[\frac{29671532711011}{35184372088832},\frac{118686130844045}{140737488355328}\right],\\x_4^*\in[\hat{x}_{41},\hat{x}_{42}]&=\left[\frac{122355652851087}{140737488355328},\frac{7647228303193}{8796093022208}\right],\\x_5^*\in[\hat{x}_{51},\hat{x}_{52}]&=\left[\frac{137735701579791}{140737488355328},\frac{8608481348737}{8796093022208}\right]\end{aligned}$$

and

$$\begin{aligned}z_1^*\in[\hat{z}_{11},\hat{z}_{12}]&=\left[-\frac{53678146230419}{70368744177664},-\frac{107356292460837}{140737488355328}\right],\\z_2^*\in[\hat{z}_{21},\hat{z}_{22}]&=\left[-\frac{102328736960905}{140737488355328},-\frac{12791092120113}{17592186044416}\right],\\z_3^*\in[\hat{z}_{31},\hat{z}_{32}]&=\left[-\frac{80910761056793}{140737488355328},-\frac{10113845132099}{17592186044416}\right].\end{aligned}$$

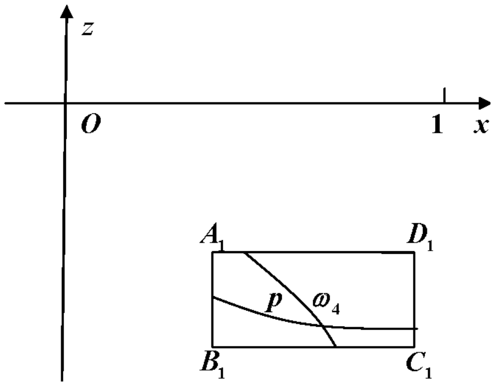

Figure 3.1: The sketch graph of $\omega_4(x,z) = 0$ and $p(x,z) = 0$ intersecting in the rectangle $A_1B_1C_1D_1$.

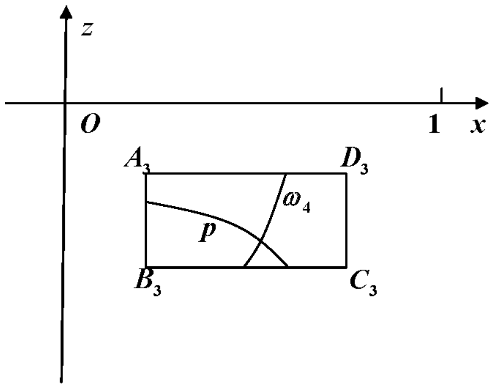

Figure 3.2: The sketch graph of $\omega_4(x,z) = 0$ and $p(x,z) = 0$ intersecting in the rectangle $A_3B_3C_3D_3$.

Since $z = \sigma(x)$ is an involution determined by $p(x,z) = 0$ and satisfy $\sigma'(x) < 0$ for $x \in (0,1)$, which imply that $\omega_4(x,z)$ and $p(x,z) = 0$ have at most three common real roots. Firstly, we substitute $z = \hat{z}_{11}$ and $z = \hat{z}_{12}$ into $p(x,z)$, then equation $p(x,z) = 0$ has one root in interval

$$[x_{11}, x_{12}] = \left[\frac{68867850789901}{70368744177664}, \frac{137735701579803}{140737488355328}\right],$$

and

$$[x_{21}, x_{22}] = \left[\frac{34433925394947}{35184372088832}, \frac{137735701579789}{140737488355328}\right],$$

respectively. It is easy to verify that interval $[x_{11}, x_{12}] \supset [\hat{x}_{51}, \hat{x}_{52}]$ and $[x_{21}, x_{22}] \supset [\hat{x}_{51}, \hat{x}_{52}]$, which shows that $\omega_4(x,z)$ and $p(x,z) = 0$ possibly have common real root.

Let the four vertices of closed rectangle containing the point (x_5^*, z_1^*) be $A_1(\hat{x}_{51}, \hat{z}_{12})$, $B_1(\hat{x}_{51}, \hat{z}_{11})$, $C_1(\hat{x}_{52}, \hat{z}_{11})$ and $D_1(\hat{x}_{52}, \hat{z}_{12})$, see Figure 3.1. Substituting coordinates of four vertices A_1, B_1, C_1, D_1 into $\omega_4(x,z)$, we get that

$$\omega_4(A_1) = -\frac{1103261\cdots6171875}{1000814\cdots7682176}, \qquad \omega_4(B_1) = -\frac{2177946\cdots3395375}{1973759\cdots7196544},$$
$$\omega_4(C_1) = \frac{3939512\cdots4495625}{2780712\cdots3723776}, \qquad \omega_4(D_1) = \frac{3353939\cdots1640625}{2365568\cdots5694464},$$

here we omit digits using dots for brevity because their numerators and denominators are all huge numbers.

Note that $\omega_4(A_1) < 0, \omega_4(B_1) < 0, \omega_4(C_1) > 0$ and $\omega_4(D_1) > 0$, by command fsolve in *Maple*, we find that $\omega_4(x,z)$ has not zero at the sides A_1B_1 and C_1D_1 of rectangle $A_1B_1C_1D_1$ as $x = \hat{x}_{51}$ and $x = \hat{x}_{52}$, respectively, which implies that $\omega_4(x,z) < 0$ at the side A_1B_1 and $\omega_4(x,z) > 0$ at the side C_1D_1. Meanwhile, the zero set of polynomial $\omega_4(x,z)$ intersecting with the sides B_1C_1 and A_1D_1 of rectangle forms a simple curve ended by points $(0.9786710221234176135, \hat{z}_{11})$ and $(0.9786710221234176104, \hat{z}_{12})$, respectively.

Using the same sequence, substituting coordinates of four vertices A_1, B_1, C_1, D_1 into $p(x,z)$, we have

$$p(A_1) = -\frac{1656260\cdots5869779}{9807971\cdots0539264}, \qquad p(B_1) = \frac{9711472\cdots0200531}{9807971\cdots0539264},$$
$$p(C_1) = \frac{5556451\cdots8475385}{6129982\cdots4408704}, \qquad p(D_1) = -\frac{2477410\cdots4753759}{9807971\cdots0539264}.$$

By direct computation, we get that $p(x,z) < 0$ at the side A_1D_1, $p(x,z) > 0$ at the side B_1C_1, a simple curve $p(x,z) = 0$ intersect the side A_1B_1 with point $(\hat{x}_{51}, -0.76281233746184644)$ and the side C_1D_1 with point $(\hat{x}_{52}, -0.76281233746184695)$. According to the Intermediate Value Theorem, curves $\omega_4(x,z) = 0$ intersect transversely with $p(x,z) = 0$ in the rectangle $A_1B_1C_1D_1$, which implies that $\omega_4(x,z) = 0$ and $p(x,z) = 0$ have and only have one common solution in the associated rectangle. Hence $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has one zero in the rectangle $A_1B_1C_1D_1$.

Secondly, we substitute $z = \hat{z}_{21}$ and $z = \hat{z}_{22}$ into $p(x,z)$, we find that $p(x,z) = 0$ has one root in interval

$$[\bar{x}_{11}, \bar{x}_{12}] = \left[\frac{59343065422023}{70368744177664}, \frac{118686130844047}{140737488355328}\right],$$

and

$$[\bar{x}_{21}, \bar{x}_{22}] = \left[\frac{118686130844043}{140737488355328}, \frac{29671532711011}{35184372088832}\right],$$

respectively. Note that

$$[\min(\bar{x}_{11}, \bar{x}_{21}), \max(\bar{x}_{12}, \bar{x}_{22})] = [\bar{x}_{21}, \bar{x}_{12}] \supset [\hat{x}_{31}, \hat{x}_{32}].$$

By similar discussion and computation to the above procedure and omitting the details for the sake of brevity, we get that $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has also one zero in the rectangle $A_2B_2C_2D_2$, where the vertices $A_2(\hat{x}_{31}, \hat{z}_{22})$, $B_2(\hat{x}_{31}, \hat{z}_{21})$, $C_2(\hat{x}_{32}, \hat{z}_{21})$ and $D_2(\hat{x}_{32}, \hat{z}_{22})$.

Finally, we substitute $z = \hat{z}_{31}$ into $p(x,z) = 0$ and get that it has one root just in interval $[\hat{x}_{21}, \hat{x}_{22}]$. Using the same sequence, let $A_3(\hat{x}_{21}, \hat{z}_{32})$, $B_3(\hat{x}_{21}, \hat{z}_{31})$, $C_3(\hat{x}_{22}, \hat{z}_{31})$ and $D_3(\hat{x}_{22}, \hat{z}_{32})$, see Figure 3.2. In the rectangle $A_3B_3C_3D_3$, we obtain that

$$\omega_4(A_3) < 0, \qquad \omega_4(B_3) < 0, \qquad \omega_4(C_3) > 0, \qquad \omega_4(D_3) > 0,$$

and

$$p(A_3) < 0, \qquad p(B_3) > 0, \qquad p(C_3) < 0, \qquad p(D_3) < 0,$$

here we omit the specific values of $\omega_4(x,z)$ and $p(x,z)$ at A_3, B_3, C_3, D_3 for brevity. By similar analysis, we find that one simple curve $\omega_4(x,z) = 0$ in the rectangle $A_3B_3C_3D_3$ intersect the side A_3D_3 with point $(0.604351994715489, \hat{z}_{32})$ and the side B_3C_3 with point $(0.604351994715488, \hat{z}_{31})$, the other simple curve $p(x,z) = 0$ intersect the side A_3B_3 with point $(\hat{x}_{21}, -0.574905535137251)$ and the side B_3C_3 with point $(0.604351994715491, \hat{z}_{31})$. Since $0.604351994715491 > 0.604351994715488$, it is obvious that curves $\omega_4(x,z) = 0$ and $p(x,z) = 0$ has one intersection in the rectangle $A_3B_3C_3D_3$, which shows that $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has one zero in the rectangle $A_3B_3C_3D_3$.

To sum up, we prove that $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has three zeros for $(x,z) \in (0,1) \times (x_0, 0)$.

Case 2: The third Wronskian $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0](x,z)$ has no zero for $(x,z) \in (0,1) \times (x_0, 0)$.

Applying the same method as the fourth Wronskian, we obtain the third Wronskian

$$W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0](x,z) = \frac{\sqrt{2}(x-z)^6\omega_3(x,z)}{5777995781250000000000000x^3z^3q_3(x)q_3(z)(p_z'(x,z))^3}, \tag{3.7}$$

where $\omega_3(x,z)$ is a asymmetric polynomial of degree 177 in (x,z), $p_z'(x,z)$ as in (3.3),

$$q_3(x) = (x-1)^{30}(x^2+2x+2)^{30} \quad \text{and} \quad q_3(z) = (z-1)^{30}(z^2+2z+2)^{30}.$$

The resultant with respect to x between $\omega_3(x,z)$ and $p(x,z)$ is

$$R(\omega_3, p, x) = 629407744000000z^2(z-1)^{24}(z^2+2z+2)^{24}\phi_3(z),$$

here $\phi_3(z)$ is a polynomial of degree 636 in z. We find three isolate zeros of $\phi_3(z)$ in $(x_0, 0)$ and one isolate zero in $(0,1)$. These real root isolation intervals are as follows

$$\tilde{z}_1 \in [\tilde{z}_{11}, \tilde{z}_{12}] = \left[-\frac{771}{2024}, -\frac{6167}{8192}\right], \qquad \tilde{z}_2 \in [\tilde{z}_{21}, \tilde{z}_{22}] = \left[-\frac{39}{64}, -\frac{4991}{8192}\right],$$
$$\tilde{z}_3 \in [\tilde{z}_{31}, \tilde{z}_{32}] = \left[-\frac{3753}{16384}, -\frac{7505}{32768}\right], \qquad \tilde{x} \in [\tilde{x}_{11}, \tilde{x}_{12}] = \left[\frac{6279}{8192}, \frac{785}{1024}\right].$$

We take interval end points $\tilde{x}_{11}$ and $\tilde{x}_{12}$ into interval polynomial equation $p(x,z)) = 0$ and get two real roots intervals in $(x_0, 0)$

$$[\bar{z}_{11}, \bar{z}_{12}] = \left[-\frac{5637}{8192}, \frac{1409}{2408}\right], \qquad [\bar{z}_{21}, \bar{z}_{22}] = \left[-\frac{2819}{4096}, \frac{5637}{8192}\right],$$

respectively. We find that each of intervals $[\bar{z}_{i,1}, \bar{z}_{i,2}] (i = 1,2)$ has not intersection with any of the intervals $[\tilde{z}_{i,1}, \tilde{z}_{i,2}]$ for $i = 1,2,3$, which implies that there does not exist value of (x,z) in plane area $D = (0,1) \times (x_0, 0)$ such that both $\omega_3(x,z) = 0$ and $p(x,z) = 0$ hold simultaneously. This shows that $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0](x,z) \neq 0$ for $(x,z) \in (0,1) \times (x_0, 0)$.

Case 3: The second Wronskian $W[\tilde{\varphi}_2, \tilde{\varphi}_1](x,z)$ has no zero for $(x,z) \in (0,1) \times (x_0, 0)$.

By similar calculation to Case 2, we get that

$$W[\tilde{\varphi}_2, \tilde{\varphi}_1](x,z) = \frac{(x-z)^3 \omega_2(x,z)}{10005187500000000 q_2(x) q_2(z) p_z'(x,z)}, \tag{3.8}$$

where $\omega_2(x,z)$ is a asymmetric polynomial of degree 120 in (x,z),

$$q_1(x) = (x-1)^{21}(x^2+2x+2)^{21} \quad \text{and} \quad q_1(z) = (z-1)^{21}(z^2+2z+2)^{21}. \tag{3.9}$$

The resultant with respect to x between $\omega_2(x,z)$ and $p(x,z)$ is

$$R(\omega_1, p, x) = 6400(z-1)^{18}(z^2+2z+2)^{18}\phi_2(z),$$

where $\phi_2(z)$ is a polynomial of degree 426 in z.

Using command realroot in *Maple*, we find that $\phi_2(z)$ has two zero in the interval $(x_0, 0)$ and one zero in $(0,1)$, which are

$$z_1 \in [z_{11}, z_{12}] = \left[-\frac{93303}{131072}, -\frac{46651}{65536}\right], \qquad z_2 \in [z_{21}, z_{22}] = \left[-\frac{85549}{262144}, -\frac{21387}{65536}\right],$$
$$x \in [x_1, x_2] = \left[\frac{39715}{65536}, \frac{79431}{131073}\right].$$

By solving polynomial equations $p([x_i, z]) = 0$ $(i = 1,2)$, we get the following the isolation intervals of real root z

$$[\bar{z}_{11}, \bar{z}_{12}] = \left[-\frac{75527}{131072}, -\frac{37763}{65536}\right] \quad \text{and} \quad [\bar{z}_{21}, \bar{z}_{22}] = \left[-\frac{9441}{16384}, -\frac{75527}{131072}\right].$$

We find that the intervals $[\bar{z}_{11}, \bar{z}_{12}]$ and $[\bar{z}_{21}, \bar{z}_{22}]$ have not intersection with intervals $[z_{11}, z_{12}]$ and $[z_{21}, z_{22}]$. This shows that $\omega_2(x,z) = 0$ and $p(x,z) = 0$ have no common root for $x_0 < z < 0 < x < 1$. Therefore, $W[\bar{\varphi}_2, \bar{\varphi}_1](x,z) \neq 0$ for all $(x,z) \in (0,1) \times (x_0, 0)$.

Case 4: The first Wronskian $W[\tilde{\varphi}_2](x,z)$ has no zero for $(x,z) \in (0,1) \times (x_0, 0)$.

We get easily that the first Wronskian

$$W[\tilde{\varphi}_2](x,z) = \tilde{\varphi}_2(x,z) = \frac{\sqrt{2}(x-z)\omega_1(x,z)}{346500000 q_1(x) q_1(z)}, \tag{3.10}$$

where $\omega_1(x,z)$ is a polynomial of degree 63 in (x,z),

$$q_1(x) = (x-1)^{11}(x^2+2x+2)^{11} \quad \text{and} \quad q_1(z) = (z-1)^{11}(z^2+2z+2)^{11}.$$

The resultant between $\omega_1(x,z)$ and $p(x,z)$ with respect to x is

$$R(\omega_1, p, x) = (z-1)^{10}(z^2+2z+2)^{10}\phi_1(z),$$

where $\phi_1(z)$ is a polynomial of degree 222 in z. It is easy to know that $\phi_1(z) \neq 0$ for $z \in (x_0, 0)$ by command realroot in *Maple* , which implies that $\omega_1(x,z) = 0$ and $p(x,z) = 0$ have no common root. Hence $W[\tilde{\varphi}_2](x,z) \neq 0$ for all $(x,z) \in (0,1) \times (x_0, 0)$.

Summarizing the above cases 1-4, we have verified that the statements (i) and (ii) in Lemma 2.3 are hold. It follows from Lemma 2.3 that Lemma 3.2 holds, thus we finish proof of Lemma 3.2. □

Proof of Proposition 3.1. Since $h^5 I_i(h) = \tilde{I}_i(h)$, $i = 0,1,2,3$, any linear combination of $(\tilde{I}_0, \tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ has the same number of zeros as that occurs with (I_0, I_1, I_2, I_3). It is easy to see that Abelian integral in (1.6) together with Hamiltonian (1.8) is the linear span of generators (I_0, I_1, I_2, I_3). By Lemma 3.2, we know that any linear combination of $(\tilde{I}_0, \tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ has at most six zeros on $(0, \frac{11}{20})$ counted with multiplicities, this implies that any linear combination of (I_0, I_1, I_2, I_3) has also at most six zeros. Therefore, the Abelian integral $I(h)$ has at most six zeros for arbitrary parameters (α, β, γ). The proof of Proposition 3.1 is completed. □

Proof of Theorem 1.1. By Proposition 3.1 and the equation (1.3), if $I(h)$ is not identically zero, then system (1.7) has at most six limit cycles bifurcating from the period annulus of unperturbed system of (1.7) by Poincaré bifurcation. Thus we complete proof of Theorem 1.1. □

Acknowledgements

The first author is partially supported by NSF of China (No. 12271212 and 71801186).

References

[1] V. I. Arnold, M. I. Vishik, Yu. S. Ilyashenko, A. S. Kalashnikov, V. A. Kondratev et al., Some unsolved problems in the theory of differential equations and mathematical physics, *Russian Math. Surveys* **44**(1989), No. 4, 157–171. https://doi.org/10.1070/RM1989v044n04ABEH002139; MR1023106; Zbl 0703.35002

[2] C. Christopher, C. Li, *Limit cycles of differential equations*, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Cham, 2007. https://doi.org/10.1007/978-3-7643-8410-4; MR4701038; Zbl 1359.34001

[3] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four: I. Saddle loop and two saddle cycle, *J. Differential Equations* **176**(2001), No. 1, 114–157. https://doi.org/10.1006/jdeq.2000.3977; MR1861185; Zbl 1004.34018

[4] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four: II. Cuspidal loop, *J. Differential Equations* **175**(2001), No. 2, 209–243. https://doi.org/10.1006/jdeq.2000.3978; MR1855970; Zbl 1034.34036

[5] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four—III Global center, *J. Differential Equations* **188**(2003), No. 2 , 473–511. https://doi.org/10.1016/S0022-0396(02)00110-9; MR1954291; Zbl 1056.34044

[6] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four—IV Figure eight-loop, *J. Differential Equations* **188**(2003), No. 2, 512–554. https://doi.org/10.1016/S0022-0396(02)00111-0; MR1954292; Zbl 1057.34015

[7] L. Gavrilov, I. D. Iliev, Complete hyperelliptic integrals of the first kind and their non-oscillation, *Trans. Amer. Math. Soc.* **356**(2003), No. 3, 1185–1207. https://doi.org/10.1090/S0002-9947-03-03432-9; MR2021617; Zbl 1043.34031

[8] M. Grau, F. Mañosas, J. Villadelprat, A Chebyshev criterion for Abelian integrals, *Trans. Amer. Math. Soc.* **363**(2011), No. 1, 109–129. https://doi.org/10.1090/S0002-9947-2010-05007-X; MR2719674; Zbl 1217.34052

[9] R. Kazemi, H. R. Z. Zangeneha, A. Atabaigia, On the number of limit cycles in small perturbations of a class of hyper-elliptic Hamiltonian systems, *Nonlinear Anal.* **75**(2012), No. 2, 574–587. https://doi.org/10.1016/j.na.2011.08.060; MR2847441; Zbl 1236.34038

[10] S. Karlin, W. Studden, *Tchebycheff systems: with applications in analysis and statistics*, Pure and Applied Mathematics, Vol. 15, Interscience Publishers, New York, 1966. https://doi.org/10.1137/1009050; MR0204922; Zbl 0153.38902

[11] C. Li, P. Mardešić, R. Roussarie, Perturbations of symmetric elliptic Hamiltonians of degree four, *J. Differential Equations* **231**(2011), No. 1, 78–91. https://doi.org/10.1016/j.jde.2006.07.016; MR2287878; Zbl 1105.37034

[12] F. Mañosas, J. Villadelprat, Bounding the number of zeroes of certain Abelian integrals, *J. Differential Equations* **251**(2011), No. 6, 1656–1669. https://doi.org/10.1016/j.jde.2011.05.026; MR2813894; Zbl 1237.37039

[13] D. D. Novaes, J. Torregrosa, On extended Chebyshev systems with positive accuracy, *J. Math. Anal. Appl.* **448**(2017), No. 1, 171–186. https://doi.org/10.1016/j.jmaa.2016.10.076; MR3579878; Zbl 1357.37064

[14] R. Roussarie, On the number of limit cycles which appear by perturbation of separatrix loop of planar vector field, *Bol. Soc. Bras. Mat.* **17**(1986), No. 2, 67–101. https://doi.org/10.1007/BF02584827; MR0901596; Zbl 0628.37032

[15] J. Wang, Bound the number of limit cycles bifurcating from center of polynomial Hamiltonian system via interval analysis, *Chaos Solitons Fractals* **87**(2016), 30–38. https://doi.org/10.1016/j.chaos.2016.03.007; MR3501954; Zbl 1355.34058

[16] J. Wang, D. Xiao, M. Han, The number of zeros of Abelian integrals for a hyper-elliptic Hamiltonian systems with degenerated polycycle, *Int. J. Bifur. Chaos* **23**(2013), No. 3, 1350047[18 pages]. https://doi.org/10.1142/S0218127413500478; MR3047962; Zbl 1270.34057

[17] J. Wilkinson, The evaluation of the zeroes of ill-conditioned polynomials, Part 1., *Numer. Math.* **1**(1959), 150–166. MR0109435; Zbl 0202.43701

[18] B. Xia, X. Hou, A complete algorithm for counting real solutions of polynomial systems of equations and inequalities, *Comput. Math. Appl.* **44**(2002), No. 1, 633–642. https://doi.org/10.1016/S0898-1221(02)00178-5; MR1925808; Zbl 1035.62054

[19] B. Xia, L. Yang, An algorithm for isolating the real solutions of semi-algebraic systems, *J. Symb. Comput.* **34**(2002), No. 5, 461–477. https://doi.org/10.1006/jsco.2002.0572; MR1937470; Zbl 1027.68150

[20] J. Yang, L. Zhou, Limit cycle bifurcations in a kind of perturbed Liénard system, *Nonlinear Dynam.* **85**(2016), 1695–1704. https://doi.org/10.1007/s11071-016-2787-0; MR3520148; Zbl 1349.37052