Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **76**, 1–14; https://doi.org/10.14232/ejqtde.2024.1.76 www.math.u-szeged.hu/ejqtde/

Multiplicity of solutions for $p(x)$-curl systems arising in electromagnetism

Fariba Gharehgazlouei and **Ghasem Alizadeh Afrouzi** ✉

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar 47416-13534, Iran

Received 27 May 2024, appeared 24 December 2024
Communicated by Gabriele Bonanno

Abstract. We are interested in the existence of multiple solutions for a class of $p(x)$-curl systems arising in electromagnetism. We work on variable exponent Sobolev spaces and by using critical point theory and the variational method, we investigate the existence of at least one, two, and three solutions to the problem.

Keywords: $p(x)$-curl systems, electromagnetic problems, variational methods, variable exponent Sobolev spaces.

2020 Mathematics Subject Classification: 35J50, 47J20, 78A30, 78M30.

1 Introduction

The study of partial differential equations or systems with variable exponents is a recent research topic that developed quickly. It started when it was understood that variable exponents give better descriptions of the behavior of certain materials or phenomena.

Let $\Omega \subset \mathbb{R}^3$, is a bounded simply connected domain with a $C^{1,1}$ boundary denoted by $\partial\Omega$. In what follows, vector functions and spaces of vector functions will be denoted by boldface symbols. We will use n to denote the outward unitary normal vector to $\partial\Omega$ and ∂_x to denote the partial derivative of a function with respect to the variable x.

The divergence of a vector function $\mathbf{v} = (v_1, v_2, v_3)$ is denoted by

$$\nabla.\mathbf{v} = \partial_{x_1} v_1 + \partial_{x_2} v_2 + \partial_{x_3} v_3$$

and the curl of $\mathbf{v}$ by

$$\nabla \times \mathbf{v} = (\partial_{x_2} v_3 - \partial_{x_3} v_2, \partial_{x_3} v_1 - \partial_{x_1} v_3, \partial_{x_1} v_2 - \partial_{x_2} v_1).$$

We recall the identity

$$-\Delta \mathbf{v} = \nabla \times (\nabla \times \mathbf{v}) - \nabla.(\nabla.\mathbf{v}),$$

where $\Delta \mathbf{v} = (\Delta v_1, \Delta v_2, \Delta v_3)$ and $\Delta v_i = \nabla.(\nabla v_i), i = 1, 2, 3$.

✉ Corresponding author. Email: afrouzi@umz.ac.ir

In this article, We are interested in the existence of multiple solutions for the following intriguing system

$$\begin{cases} \nabla \times (|\nabla \times u|^{p(x)-2}\nabla \times u) + a(x)|u|^{p(x)-2}u = \lambda f(x,u), \ \nabla . u = 0 & \text{in } \Omega, \\ |\nabla \times u|^{p(x)-2}\nabla \times u \times n = 0, \ u.n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in (0, +\infty)$, $\Omega \subset \mathbb{R}^3$, is a bounded simply connected domain with a $C^{1,1}$ boundary denoted by $\partial\Omega$. We will use n to denote the outward unitary normal vector to $\partial\Omega$. a is a functional in L^∞ and there exist $a_0, a_1 > 0$ such that

$$a_0 < a(x) < a_1, \qquad \forall x \in \Omega.$$

$p \in C(\bar{\Omega})$, with

$$3 < p^- = \min_{x\in\Omega} p(x) \le p^+ = \max_{x\in\Omega} p(x) < \infty,$$

and $p(x)$ satisfies logarithmic continuity: there exists a function $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\forall x, y \in \bar{\Omega}, |x-y| < 1, \quad |p(x) - p(y)| \le \omega(|x-y|), \quad \text{and} \quad \lim_{\tau \to 0^+} \omega(\tau) \log \frac{1}{\tau} = C < \infty. \tag{1.2}$$

The interest in transposing the problems into new problems with variable exponents is linked to a large scale of applications that are involving some nonhomogeneous materials. It was established that for appropriate treatment of these materials, we can not rely on the classical Sobolev space and that we have to allow the exponent to vary instead. Working with variable exponents, hence working in the framework of variable exponent spaces, opens the door for multiple applications. The variable exponent problems arise in many different applications, such as nonlinear elastic [26], electrorheological fluids [22], image processing [13] and other physics phenomena [2, 27]. The literature on variable exponent Sobolev spaces and their applications is quite large, here we just quote a few, see [5,6,12,13,19,20,23] and the references therein. For the basic properties of variable exponent Sobolev spaces and their applications to partial differential equations, we refer the readers to [14,21].

In [4], Antontsev, Miranda, and Santos studied the qualitative properties of solutions for the following $p(x,t)$-curl systems:

$$\begin{cases} \partial_t u + \nabla \times (|\nabla \times u|^{p(x,t)-2}\nabla \times u) = \lambda f(u), \ \nabla . u = 0 & \text{in } \Omega \times (0,T), \\ |\nabla \times u|^{p(x,t)-2}\nabla \times u \times n = 0, \ u.n = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $\nabla \times (|\nabla \times u|^{p(x,t)-2}\nabla \times u)$ is the $p(x,t)$-curl operator, $f(u) = \lambda u(\int_\Omega |u|^2 dx)^{\frac{\rho-2}{2}}$ with $\lambda \in \{-1, 0, 1\}$ and $\rho > 0$ is constant. The authors introduced a suitable variable exponent Sobolev space and obtained the existence of local or global weak solutions for system (1.3) by using Galerkin's method. The authors also studied the blow-up and finite-time extinction properties of solutions. When $p(x,t) \equiv p$, then problem (1.3) turns into a model from the generalized Maxwell's equations in the electromagnetic field theory. More precisely, u denotes the magnetic field, $\nabla \times u$ denotes the total current density, f denotes an internalmagnetic current, and $\nabla \times (|\nabla \times u|^{p-2}\nabla \times u)$ denotes the electric field.

Motivated by the above works, we study the existence and multiplicity of solutions for systems (1.1) with general nonlinearities. To the best of our knowledge, this is the first time to deal with the existence of steady-state solutions for systems (1.1) involving the $p(x)$-curl operator by applying variational methods different from that used in [24].

Xianga, Wanga, and Zhangc [24] investigated the existence and multiplicity of solutions to problem (1.1) in the case $\lambda = 1$. They studied the existence of ground state solutions and infinitely many solutions for (1.1) in the case $\lambda = 1$ with the nonlinearity f satisfying superlinear growth conditions is obtained by combining the mountain pass theorem with the Nehari manifold method, and a variant of the mountain pass theorem.

In this paper, we obtain three different results about the existence of weak solutions to the problem (1.1) by using critical point theorems established in [8, 9, 11]. The first aim of this paper is to provide an estimate of the positive interval for the parameter λ in which the problem (1.1) possesses at least one nontrivial weak solution. We also wish to consider the existence of two solutions to our problem by using a result of Bonanno [9, Theorem 3.2]. In a recent paper, Bonanno and Chinnì [10] studied the existence of at least two distinct weak solutions to a problem involving a p(x)-Laplacian by applying critical point theory. Our first main result will require the $(P.S.)^{[r]}$ condition, while in our second one, we will ask that the (AR)-condition holds and use it to ensure that the (usual) (PS)-condition is satisfied. We refer the reader to the papers [7, 10, 17, 18] where this approach was applied successfully. Finally, our third goal is to obtain the existence of three solutions to (1.1); this problem is less studied by researchers. In this case, we consider problem (1.1) where the nonlinearity f has subcritical growth, and we apply variational methods and critical point theory. The main tool used is the critical point theorem of Bonanno and Marano [11, Theorem 3.6].

The remainder of this paper is organized as follows. First, in Section 2, we recall briefly some basic results for fractional Sobolev spaces. In Section 3, we obtain the existence of at least one, two, or three nontrivial weak solutions to the problem (1.1) provided the parameter λ belongs to a positive interval to be determined.

2 Preliminaries

In this section, we introduce some definitions and results of Sobolev spaces with variable exponents.

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a $C^{1,1}$ boundary denoted by $\partial\Omega$. Let $p \in C(\bar{\Omega})$. Set

$$p^- = \min_{x\in\bar{\Omega}} p(x), \quad \text{and} \quad p^+ = \max_{x\in\bar{\Omega}} p(x), \quad \text{with } 1 < p^- \leq p^+ < \infty.$$

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx,$$

is finite. We define a norm, the so-called Luxemburg norm, on this space by the formula

$$\|u\|_{p(\cdot)} = \inf\left\{\gamma > 0 : \rho_{p(\cdot)}\left(\frac{u}{\gamma}\right) \leq 1\right\}.$$

The space $(L^{p(\cdot)}(\Omega), \|.\|_{L^{p(\cdot)}(\Omega)})$ is a separable and reflexive Banach space. Moreover, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uvdx\right| \le \left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$. An important role in manipulating the generalized Lebesgue spaces is played by the $\rho_{p(\cdot)}$-modular of the space $L^{p(\cdot)}(\Omega)$, we have the following result.

Proposition 2.1 (See [16]). *If $u \in L^{p(\cdot)}(\Omega)$, $u_n \in L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then*

(i) if $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$, then $\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$;

(ii) if $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, then $\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$;

(iii) $\lim_{n\to\infty}\|u_n-u\|_{p(\cdot)} = 0 \Leftrightarrow \lim_{n\to\infty}\rho_{p(\cdot)}(u_n-u) = 0$.

Define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|.\|_{W^{1,p(\cdot)}(\Omega)})$ is a separable and reflexive Banach space. We consider also

$$W_0^{1,p(\cdot)}(\Omega) = \left\{u \in W^{1,p(\cdot)}(\Omega) : u\,|_{\partial\Omega)} = 0\right\},$$

with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

Remark 2.2. Assuming (1.2), we have $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and this last space can be defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{1,p(\cdot)}(\Omega)}$. The density of smooth functions in the space $W_0^{1,p(\cdot)}(\Omega)$ is crucial for the understanding of these spaces. The condition of log-continuity of $p(\cdot)$ is the best known and the most frequently used sufficient condition for the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ (see [3, 14]). Although this condition is not necessary and can be substituted by other conditions (see [14, Chapter 9] for a discussion of this question) we keep it throughout the paper for the sake of simplicity of presentation.

Also, we observe that $W_0^{1,p(\cdot)}(\Omega) \subseteq W_0^{1,p^-}(\Omega)$, the Sobolev inequality

$$\|u\|_{L^{q(\cdot)}(\Omega)} \le C\|u\|_{W^{1,p(\cdot)}(\Omega)},$$

holds, with $1 \le q(x) < \frac{3p(x)}{3-p(x)}$ if $p^- < 3$, any q if $p^- = 3$, and $q = \infty$ if $p^- > 3$ Here $C = C(p^-, \Omega)$ is a positive constant.

Now, we define the space $\mathbf{w}^{p(x)}(\Omega)$

Let $\mathbf{L}^{p(x)}(\Omega) = L^{p(x)}(\Omega) \times L^{p(x)}(\Omega) \times L^{p(x)}\Omega$ and define

$$\mathbf{W}^{p(x)}(\Omega) = \{\mathbf{v} \in \mathbf{L}^{p(x)}(\Omega) : \nabla\times\mathbf{v} \in \mathbf{L}^{p(x)}(\Omega), \nabla.\mathbf{v}, \mathbf{v}.\mathbf{n}|_{\partial\Omega} = 0\}$$

where $\mathbf{n}$ denotes the outward unitary normal vector to $\partial\Omega$. Equip $\mathbf{W}^{p(x)}(\Omega)$ with the norm

$$\|\mathbf{v}\|_{\mathbf{W}^{p(x)}(\Omega)} = \|\mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)} + \|\nabla \times \mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)}.$$

If $p^- > 1$, by [4, Theorem 2.1], $\mathbf{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathbf{W}_n^{1,p(x)}(\Omega)$, where

$$\mathbf{W}_n^{1,p(x)}(\Omega) = \{\mathbf{v} \in \mathbf{W}^{p(x)}(\Omega) : \mathbf{v}.\mathbf{n}|_{\partial\Omega} = 0\}$$

and

$$\mathbf{W}^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega) \times W^{1,p(x)}\Omega.$$

Thus, we have the following theorem.

Theorem 2.3 (see [4, Theorem 2.1])**.** *Assume that $1 < p^- \leq p^+ < \infty$ and p satisfies (1.2). Then $\mathbf{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathbf{W}_n^{1,p(x)}(\Omega)$. Moreover, if $p^- > \frac{6}{5}$, then $\|\nabla \times .\|_{\mathbf{L}^{p(x)}(\Omega)}$ is a norm in $\mathbf{W}^{p(x)}(\Omega)$ and there exists $C = C(N, p^-, p^+) > 0$ such that*

$$\|\mathbf{v}\|_{\mathbf{W}^{p(x)}(\Omega)} \leq C\|\nabla \times \mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)}.$$

Remark 2.4. By Remark 2.2 and Theorem 2.3, we know the embedding $\mathbf{W}^{p(x)}(\Omega) \hookrightarrow C_0^\infty(\Omega)$ is compact, with $3 < p^- \leq p^+ < \infty$, for all $x \in \bar{\Omega}$. Moreover, $(\mathbf{W}^{p(x)}(\Omega), \|.\|_{\mathbf{W}^{p(x)}(\Omega)})$ is a reflexive Banach space. We set

$$c_0 = \sup_{u \in \mathbf{W}^{p(x)}(\Omega)} \frac{\|u\|_\infty}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}}.$$

Definition 2.5 ([8, p. 2993], [9, p. 210])**.** Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais-Smale condition cut off upper at r, denoted by $(P.S.)^{[r]}$ if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

(1) $\{I(u_n)\}$ is bounded;

(2) $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$;

(3) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$

has a convergent subsequence.

If only conditions (1) and (2) hold, then $I = \Phi - \Psi$ is said to satisfy the (usual) Palais–Smale $(P.S.)$ condition.

We next wish to define what is meant by a weak solution to our problem.

Definition 2.6. We say that a function $u \in \mathbf{W}^{p(x)}(\Omega)$ is a weak solution of the problem (1.1) if

$$\int_\Omega |\nabla \times u|^{p(x)-2} \nabla \times u.\nabla \times v dx + \int_\Omega a(x)|u|^{p(x)-2}u.v dx = \int_\Omega f(x,u).v dx,$$

holds for all $v \in \mathbf{W}^{p(x)}(\Omega)$.

Remark 2.7. Let u be a classical solution of (1.1). Let $e = |\nabla \times u|^{p(x)-2}\nabla \times u$ and v be a smooth function in Ω, then we obtain

$$\nabla(e \times v) = v.\nabla \times e - e.\nabla \times v. \tag{2.1}$$

Multiplying the first equation of (1.1) by v and integrating over Ω, we get

$$\int_\Omega \nabla \times e.vdx + \int_\Omega a(x)|u|^{p(x)-2}u.vdx = \int_\Omega f(x,u).vdx.$$

Using (2.1) and the boundary conditions in (1.1) and integrating by parts, we have

$$\int_\Omega e.\nabla \times vdx + \int_\Omega a(x)|u|^{p(x)-2}u.vdx = \int_\Omega f(x,u).vdx,$$

which means that Definition 2.6 is correct.

Assume that $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathédory function. We set

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}^3.$$

The variational structure of this problem leads us to introduce We define the functionals $\Phi, \Psi : \mathbf{W}^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) := \int_\Omega \frac{|\nabla \times u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)}dx \tag{2.2}$$

and

$$\Psi(u) := \int_\Omega F(x,u)dx. \tag{2.3}$$

Lemma 2.8 ([24, Lemmas 3.1. and 3.2.]). *The functional Φ is of class C^1 and*

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla \times u|^{p(x)-2}\nabla \times u.\nabla \times vdx + \int_\Omega a(x)|u|^{p(x)-2}u.vdx$$

for every $u, v \in \mathbf{W}^{p(x)}(\Omega)$. For each $u \in \mathbf{W}^{p(x)}(\Omega)$, $\Phi'(u) \in (\mathbf{W}^{p(x)}(\Omega))^*$, where $(\mathbf{W}^{p(x)}(\Omega))^*$ is the dual space of $\mathbf{W}^{p(x)}(\Omega)$. Moreover, Φ is a convex functional in $\mathbf{W}^{p(x)}(\Omega)$.

The functional Ψ is of class C^1 and

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u).vdx$$

for every $u, v \in \mathbf{W}^{p(x)}(\Omega)$.

By Lemma 2.8, we know that $I_\lambda = \Phi - \lambda\Psi$ is of class C^1 and

$$\langle I'_\lambda(u), v\rangle = \int_\Omega |\nabla \times u|^{p(x)-2}\nabla \times u.\nabla \times vdx + \int_\Omega a(x)|u|^{p(x)-2}u.vdx - \lambda\int_\Omega f(x,u).vdx,$$

for all $u, v \in \mathbf{W}^{p(x)}(\Omega)$. Hence a critical point of I_λ is a (weak) solution of (1.1).

3 Main results

We begin by presenting a result that guarantees the existence of at least one solution to problem (1.1).

Theorem 3.1. *Let $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Carathédory function and assume that there exist two positive constants τ and δ, such that:*

(H_1) $\quad p^+ c_0^{p^-} a_1 \operatorname{meas}(\Omega) \max\{\delta^{p^-}, \delta^{p^+}\} < p^- \min\{1, a_0\} \tau^{p^-}$;

(H_2) $\quad \frac{\int_\Omega \sup_{|t|\leq\tau} F(x,t)dx}{\tau^{\underline{p}}} < \frac{p^- \min\{1,a_0\} \int_\Omega F(x,\delta)dx}{p^+ c_0^{p^-} a_1 \operatorname{meas}(\Omega) \max\{\delta^{p^-},\delta^{p^+}\}}$;

(H_3) $\quad \inf_{x\in\Omega, t\in\mathbb{R}^3, |t|=1} F(x,t) > 0$.

Then, for each

$$\lambda \in \Lambda_w := \left] \frac{a_1 \operatorname{meas}(\Omega) \max\{\delta^{p^-}, \delta^{p^+}\}}{p^- \int_\Omega F(x,\delta)dx}, \frac{\min\{1, a_0\}\tau^{p^-}}{p^+ c_0^{p^-} \int_\Omega \sup_{|t|\leq\tau} F(x,t)dx} \right[, \tag{3.1}$$

problem (1.1) admits at least one nontrivial solution $u_\lambda \in \mathbf{W}^{p(x)}(\Omega)$ such that $\|u_\lambda\|_\infty \leq \tau$.

Proof. Our goal is to apply [9, Theorem 2.3] to problem (1.1). To this end, take the real Banach space $\mathbf{W}^{p(x)}(\Omega)$ with the norm as defined in Section 2, Φ, Ψ be the functionals defined in (2.2) and (2.3). We can see Φ, Ψ are of C^1 in Lemma 2.8. For each $u \in \mathbf{W}^{p(x)}(\Omega)$ we have

$$\frac{\min\{1, a_0\}}{p^+} \|u\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)} \leq \Phi(u) \leq \frac{\max\{1, a_1\}}{p^-} \|u\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)}. \tag{3.2}$$

From the first inequality in (3.2), it follows that Φ is coercive. To show that Φ' admits a continuous inverse, in view of [25, Theorem 26.A(d)], it suffices to show that Φ' is coercive, hemicontinuous, and uniformly monotone. For any $u \in \mathbf{W}^{p(x)}(\Omega)$ we have

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}} = \frac{\int_\Omega |\nabla \times u|^{p(x)}dx + \int_\Omega a(x)|u|^{p(x)}dx}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}} \geq \frac{\min\{1, a_0\}\|u\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)}}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}}.$$

By Proposition 2.1 for any $u \in \mathbf{W}^{p(x)}(\Omega)$

$$\lim_{\|u\|_{\mathbf{W}^{p(x)}(\Omega)} \to \infty} \frac{\langle \Phi'(u), u\rangle}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}} \geq \lim_{\|u\|_{\mathbf{W}^{p(x)}(\Omega)} \to \infty} \left(\min\{1, a_0\}\|u\|^{p^- -1}_{\mathbf{W}^{p(x)}(\Omega)}\right) = \infty,$$

i.e. Φ' is coercive. The fact that Φ' is hemicontinuous can be verified using standard arguments. (see, for example, [18]).

Finally, we show that Φ' is uniformly monotone. First, recall the inequality that for any $\xi, \psi \in \mathbb{R}$,

$$(|\xi|^{r-2}\xi - |\psi|^{r-2}\psi)(\xi - \psi) \geq 2^{-r}|\xi - \psi|^r, \quad \text{for all } r > 2. \tag{3.3}$$

Thus, for every $u, v \in X$, we deduce that

$$\begin{aligned}
&\langle \Phi'(u) - \Phi'(v), u - v\rangle \\
&= \int_\Omega (|\nabla \times u|^{p(x)-2}\nabla \times u - |\nabla \times v|^{p(x)-2}\nabla \times v)(\nabla \times u - \nabla \times v)dx \\
&\quad + \int_\Omega a(x)(|u|^{p(x)-2}u - |v|^{p(x)-2}v)(u-v)dx \\
&\geq 2^{-p^+}(\int_\Omega |\nabla \times u - \nabla \times v|^{p(x)}dx + \int_\Omega a(x)|u-v|^{p(x)}dx) \\
&\geq \min\{2^{-p^+}, a_0 2^{-p^+}\}(\int_\Omega |\nabla \times u - \nabla \times v|^{p(x)}dx + \int_\Omega |u-v|^{p(x)}dx) \\
&\geq \begin{cases} c_1\|u-v\|^{p^-}_{\mathbf{W}^{p(x)}(\Omega)} & \text{if } \|u-v\|_{\mathbf{L}^{p(x)}(\Omega)}, \|\nabla \times (u-v)\|_{\mathbf{L}^{p(x)}(\Omega)} > 1, \\ c_2\|u-v\|^{p^+}_{\mathbf{W}^{p(x)}(\Omega)} & \text{if } \|u-v\|_{\mathbf{L}^{p(x)}(\Omega)}, \|\nabla \times (u-v)\|_{\mathbf{L}^{p(x)}(\Omega)} < 1, \end{cases}
\end{aligned}$$

the last inequality is obtained from Proposition 2.1. It is easy to check that Φ' is uniformly monotone. Moreover, Ψ' is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on $\mathbf{W}^{p(x)}(\Omega)$. For this end, for fixed $u \in \mathbf{W}^{p(x)}(\Omega)$, let $u_n \to u$ weakly in $\mathbf{W}^{p(x)}(\Omega)$ as $n \to \infty$, then $u_n(x)$ converges uniformly to $u(x)$ on Ω as $n \to \infty$; see [25]. Since f is continuous in $\mathbb{R}^3$ for every $x \in \Omega$, so

$$f(x, u_n) \to f(x, u),$$

as $n \to \infty$. Thus $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Hence we proved that Ψ' is a compact operator by [25, Proposition 26.2]. This ensures that the functional $I_\lambda = \Phi - \lambda\Psi$ verifies $(P.S.)^{[r]}$ condition for each $r > 0$ (see [8, Proposition 2.1]). To apply [9, Theorem 2.3] to the functional I_λ, first note that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. We need to show that there is an $r > 0$ and $w \in X$ with $0 < \Phi(w) < r$ such that $\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{\Psi(w)}{\Phi(w)}$. To this end, set

$$r := \frac{\min\{1, a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-},$$

and define $w \in \mathbf{W}^{p(x)}(\Omega)$ by

$$w(x) = \begin{cases} \delta, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

One has

$$\Phi(w) = \int_\Omega \frac{a(x)}{p(x)}|w(x)|^{p(x)}dx \leq \frac{\text{meas}(\Omega)a_1}{p^-}\max\{\delta^{p^+}, \delta^{p^-}\}. \tag{3.5}$$

Hence, it follows from (H_1) that $0 < \Phi(w) < r$. If $u \in \Phi^{-1}([0, r])$, by Proposition 2.1 (*i*) and (3.2), for any $u \in \mathbf{W}^{p(x)}(\Omega)$ with $\|u\|_{\mathbf{W}^{p(x)}(\Omega)} > 1$, we obtain

$$\frac{\min\{1, a_0\}}{p^+}\|u\|^{p^-}_{\mathbf{W}^{p(x)}(\Omega)} \leq \Phi(u) \leq \frac{\min\{1, a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-}.$$

Similarly, by Proposition 2.1 (*ii*) and (3.2), for any $u \in \mathbf{W}^{p(x)}(\Omega)$ with $\|u\|_{\mathbf{W}^{p(x)}(\Omega)} < 1$, we obtain

$$\frac{\min\{1, a_0\}}{p^+}\|u\|^{p^+}_{\mathbf{W}^{p(x)}(\Omega)} \leq \Phi(u) \leq \frac{\min\{1, a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-}.$$

Then

$$\|u\|_{\mathbf{W}^{p(x)}(\Omega)} \leq \frac{\tau}{c_0}.$$

Hence, we obtain

$$|u(x)| \leq \|u\|_{L^\infty(\Omega)} \leq c_0 \|u\|_{\mathbf{W}^{p(x)}(\Omega)} \leq \tau \qquad \forall x \in \Omega.$$

Hence, for each $u \in \Phi^{-1}((-\infty, r])$

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} = \frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \int_\Omega F(x,u)dx}{\frac{\min\{1,a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-}} \leq \frac{c_0^{p^-}\, p^+ \int_\Omega \sup_{|t|\leq\tau} F(x,t)dx}{\min\{1,a_0\}\tau^{p^-}}. \tag{3.6}$$

Moreover, thanks to (H_2) and (3.5), one has

$$\begin{aligned}
\frac{\Psi(w)}{\Phi(w)} &\geq \frac{p^- \int_\Omega F(x,\delta)dx}{a_1 \operatorname{meas}(\Omega) \max\{\delta^{p^-}, \delta^{p^+}\}} \\
&\geq \frac{c_0^{p^-}\, p^+ \int_\Omega \sup_{|t|\leq\tau} F(x,t)dx}{\min\{1,a_0\}\tau^{p^-}} \\
&\geq \frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r},
\end{aligned}$$

which means that $\frac{\Phi(\bar{v})}{\Psi(\bar{v})} \geq \frac{r}{\sup_{\Phi(x)\leq r}\Psi(u)}$ holds for some $\bar{v} \in \mathbf{W}^{p(x)}(\Omega)$. Hence, for each $\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(u)}\right[$, the functional I_λ admits at least one critical point u_λ with

$$0 < \Phi(u_\lambda) < r$$

which in turn is a nontrivial solution of problem (1.1) such that $\|u_\lambda\|_\infty < \tau$. □

Our second aim in this paper is to obtain a result on the existence of two distinct solutions to problem (1.1). The following theorem is obtained by applying [9, Theorem 3.2].

Theorem 3.2. *Let $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Carathédory function, and assume that*

(H_4) *(Ambrosetti–Rabinowitz Condition) there exist $\mu > p^+$ and $R > 0$ such that*

$$0 < \mu F(x,t) \leq t f(x,t) \qquad \forall x \in \Omega \text{ and } t \in \mathbb{R}^3 \backslash \{0\}, \text{ with } |t| \geq R.$$

Then, for each

$$\lambda \in \Lambda_r := \left]0, \frac{\min\{1,a_0\}\tau^{p^-}}{p^+ c_0^{p^-} \int_\Omega \sup_{|t|\leq\tau} F(x,t)dx}\right[,$$

the problem (1.1) admits at least two nontrivial solutions.

Proof. Let Φ, Ψ be the functionals defined in Theorem (2.2) and (2.3). Notice that they satisfy all regularity assumptions required in [9, Theorem 3.2]). Arguing as in the proof of Theorem 3.1, choosing

$$r = \frac{\min\{1,a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-},$$

for each $\lambda \in \Lambda_r$ we obtain

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} \le \frac{c_0^{p^-}\, p^+ \int_\Omega \sup_{|t|\le\tau} F(x,t)dx}{\min\{1,a_0\}\tau^{p^-}} < \frac{1}{\lambda}$$

(see (3.6)). Now, from condition (H_4), a straight forward calculation shows that there are positive constants m and C such that

$$F(x,t) \ge m|t|^\mu - C \quad \text{for all } x \in \Omega, t \in \mathbb{R}^3.$$

Hence, for every $\lambda \in \Lambda_r$, $u \in \mathbf{W}^{p(x)}(\Omega) \setminus \{0\}$ and $t > 1$, we obtain

$$\begin{aligned} I_\lambda(tu) &= \Phi(tu) - \lambda \int_\Omega F(x,tu)dx \\ &\le \frac{t^{p^+}}{p^-}\int_\Omega (|\nabla \times u|^{p(x)} + a(x)|u|^{p(x)})dx \\ &\quad - m\lambda t^\mu \int_\Omega |u|^\mu dx + C\lambda\,\mathrm{meas}(\Omega). \end{aligned}$$

Since $\mu > p^+$, this condition guarantees that I_λ is unbounded from below. To show that I_λ satisfies the (PS)-condition, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{W}^{p(x)}(\Omega)$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_{\lambda,\mu}(u_n) \to 0$ in $(\mathbf{W}^{p(x)}(\Omega))^*$ as $n \to +\infty$. Then, there exists a positive constant s_0 such that

$$|I_\lambda(u_n)| \le s_0, \qquad \|I'_\lambda(u_n)\| \le s_0 \qquad \forall n \in \mathbb{N}.$$

Using also the condition (H_4), and the definition of I'_λ, we see that, for all $n \in \mathbb{N}$, there exists $D > 0$ such that

$$\begin{aligned} \mu s_0 + s_0\|u_n\|_{\mathbf{W}^{p(x)}(\Omega)} &\ge \mu I_\lambda(u_n) - I'_\lambda(u_n)u_n \\ &\ge \left(\frac{\mu}{p^+} - 1\right)\int_\Omega |\nabla \times u|^{p(x)} + a_0\left(\frac{\mu}{p^+} - 1\right)\int_\Omega |u|^{p(x)}dx \\ &\quad + \lambda \int_\Omega (f(x,u_n)u_n - \mu F(x,u_n))\,dx \\ &\ge \left(\frac{\mu}{p^+} - 1\right)\min\{1,a_0\}\|u_n\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)} - D. \end{aligned}$$

Since $\mu > p^+$ it follows $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, since $\mathbf{W}^{p(x)}(\Omega)$ is a reflexive Banach space we have, up to taking a subsequence if necessary,

$$u_n \rightharpoonup u \quad \text{in } \mathbf{W}^{p(x)}(\Omega).$$

By the fact that $I'_\lambda(u_n) \to 0$ and $u_n \rightharpoonup u$ in $\mathbf{W}^{p(x)}(\Omega)$, we obtain

$$(I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) \to 0.$$

Furthermore,

$$\int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx \to 0, \quad \text{as } n \to +\infty.$$

An easy computation shows that

$$
\begin{aligned}
&\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \\
&= \int_\Omega (|\nabla \times u_n|^{p(x)-2}\nabla \times u_n - |\nabla \times u|^{p(x)-2}\nabla \times u)(\nabla \times u_n - \nabla \times u)dx \\
&\quad + \int_\Omega a(x)(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx \\
&\quad - \lambda \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\geq 2^{-p^+}\|\nabla \times (u_n - u)\|^{p(x)}_{\mathbf{L}^{p(x)}(\Omega)} + a_0 2^{-p^+}\|u_n - u\|^{p(x)}_{\mathbf{L}^{p(x)}(\Omega)} \\
&\quad - \lambda \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\geq \min\{2^{-p^+}, a_0 2^{-p^+}\}\|u_n - u\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)} - \lambda \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx.
\end{aligned}
$$

The last of the above inequality is obtained by using (3.3). Combining the last relation with Proposition 2.1 (iii), we find that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $\mathbf{W}^{p(x)}(\Omega)$. Therefore, $I_{\lambda,\mu}$ satisfies the (PS)-condition and so all hypotheses of [9, Theorem 3.2], are verified. Hence, for each $\lambda \in \Lambda_r$ the function I_λ admits at least two distinct critical points that are solutions of the problem (1.1). □

In our final result, we discuss the existence of at least three solutions to the problem (1.1).

Theorem 3.3. *Let $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Carathédory function, and let (H_2), (H_3) in Theorem 3.1 hold. Moreover, assume that there exist two positive constants τ and δ, such that*

(H_5) $a_0 c_0^{p^-} \operatorname{meas}(\Omega)\min\{\delta^{p^-}, \delta^{p^+}\} > \tau^{p^-}\min\{1, a_0\}$,

(H_6) *there exist constants $c > 0$, $q \in C(\bar{\Omega})$ and $1 < q(x) \leq q^+ < p^-$ in $\bar{\Omega}$ such that*

$$|F(x,t)| \leq c(1 + |t|^{q(x)}) \;\; \forall (x,t) \in \Omega \times \mathbb{R}^3.$$

Then for every $\lambda \in \Lambda_w$ as in (3.1), the problem (1.1) admits at least three distinct solutions.

Proof. Our aim is to apply [11, Theorem 3.6]. We consider the functionals Φ and Ψ, defined in (2.2) and (2.3). Once again, they satisfy the regularity assumptions needed in [11, Theorem 3.6]. Now, we argue as in the proof of Theorem 3.1 with $w(x)$ defined in (3.4), and

$$r = \frac{\min\{1, a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-}.$$

In view of (H_5) we have $\Phi(w) > r > 0$. Therefore, from (H_2), inequality (3.6) holds, and so

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{\Psi(\bar{v})}{\Phi(\bar{v})}$$

holds for some $\bar{v} \in \mathbf{W}^{p(x)}(\Omega)$.

Now, we prove that, for each $\lambda \in \Lambda_w$ the functional I_λ is coercive. By using inequality (3.2), conditions (H_6), and Sobolev embedding theorem, we easily obtain for all $u \in \mathbf{W}^{p(x)}(\Omega)$:

$$
\begin{aligned}
I_\lambda(u) &\geq \frac{\min\{1, a_0\}}{p^+}\|u\|^{p^-}_{\mathbf{W}^{p(x)}(\Omega)} - \lambda \int_\Omega F(x,u)dx \\
&\geq \frac{\min\{1, a_0\}}{p^+}\|u\|^{p^-}_{\mathbf{W}^{p(x)}(\Omega)} - \lambda c\|u\|^{q^+}_{\mathbf{W}^{p(x)}(\Omega)} - \lambda c \operatorname{meas}(\Omega).
\end{aligned}
$$

Since $p^- > q^+$ we see that $I_\lambda \to +\infty$ as $\|u\|_{\mathbf{W}^{p(x)}(\Omega)} \to +\infty$, so the functional I_λ is coercive. Thus, for each $\lambda \in \Lambda_w$, [11, Theorem 3.6] implies that the functional I_λ admits at least three critical points in $\mathbf{W}^{p(x)}(\Omega)$ that are solutions of the problem (1.1). □

References

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14**(1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7; Zbl 0273.49063

[2] S. N. Antontsev, S. I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, *Nonlinear Anal.* **60**(2005), 515–545. https://doi.org/10.1016/j.na.2004.09.026

[3] S. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions. Existence, uniqueness, localization, blow-up*, Atlantis Studies in Differential Equations, Atlantis Press, 2015. https://doi.org/10.2991/978-94-6239-112-3; Zbl 1410.35001

[4] S. Antontsev, F. Mirandac, L. Santos, Blow-up and finite time extinction for $p(x,t)$-curl systems arising in electromagnetism, *J. Math. Anal. Appl.* **440**(2016), 300–322. https://doi.org/10.1016/j.jmaa.2016.03.045; Zbl 1339.35060

[5] S. N. Antontsev, J. F. Rodrigues, On stationary thermo-rheological viscous flows, *Ann. Univ. Ferrara, Sez. VII Sci. Mat.* **52**(2006), 19–36. https://doi.org/10.1007/s11565-006-0002-9; Zbl 1117.76004

[6] M. Bendahmane, M. Chrif, S. El Manouni, An approximation result in generalized anisotropic Sobolev spaces and applications, *Z. Anal. Anwend.* **30**(2011), No. 3, 341–353. https://doi.org/10.4171/ZAA/3445; Zbl 1231.35065

[7] M. Bohner, G. Caristi, F. Gharehgazlouei, S. Heidarkhani, Existence and multiplicity of weak solutions for a Neumann elliptic problem with $\vec{p}(x)$-Laplacian, *Nonauton. Dyn. Syst.* **7**(2020), 53–64. https://doi.org/10.1515/msds-2020-0108; Zbl 1465.35255

[8] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.* **75**(2012), 2992–3007. https://doi.org/10.1016/j.na.2011.12.003; Zbl 1239.58011

[9] G. Bonanno, Relations between the mountain pass theorem and local minima, *Adv. Nonlinear Anal.* **1**(2012), 205–220. https://doi.org/10.1515/anona-2012-0003; Zbl 1277.35170

[10] G. Bonanno, A. Chinnì, Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent, *J. Math. Anal. Appl.* **418**(2014), 812–827. https://doi.org/10.1016/j.jmaa.2014.04.016; Zbl 1312.35111

[11] G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.* **89**(2010), 1–10. https://doi.org/10.1080/00036810903397438; Zbl 1194.58008

[12] M. M. Boureanu, V. D. Rădulescu, Anisotropic Neumann problems in Sobolev spaces with variable exponent, *Nonlinear Anal.* **75**(2012), No. 12, 4471–4482. https://doi.org/10.1016/j.na.2011.09.033; Zbl 1262.35090

[13] Y. Chen, S. Levine, R. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66**(2006), 1383–1406. https://doi.org/10.1137/050624522; Zbl 1102.49010

[14] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Springer, Berlin, 2011. https://doi.org/10.1007/978-3-642-18363-8; Zbl 1222.46002

[15] X. L. Fan, C. Ji, Existence of infinitely many solutions for a Neumann problem involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.* **334**(2007), 248–260. https://doi.org/10.1016/j.jmaa.2006.12.055

[16] X. L. Fan, D. Zhao, On the generalised Orlicz–Sobolev space $W^{k,p(x)}(\Omega)$, *J. Gansu Educ. College* **12**(1998), No. 1, 1–6.

[17] F. Gharehgazlouei, J. R. Graef, S. Heidarkhani, L. Kong, Existence and multiplicity of solutions to a fractional p-Laplacian elliptic Dirichlet problem, *Electron. J. Differential Equations* **2023**, No. 46, 1–15. https://doi.org/10.58997/ejde.2023.46; Zbl 1529.35547

[18] S. Heidarkhani, F. Gharehgazlouei, M. Imbesi, Existence and multiplicity of homoclinic solutions for a difference equation, *Electron. J. Differential Equations* **2020**, No. 115, 1–12. https://doi.org/10.58997/ejde.2020.115; Zbl 1461.39002

[19] B. Kone, S. Ouaro, S. Traore, Weak solutions for anisotropic nonlinear elliptic equations with variable exponents, *Electron. J. Differential Equations* **2009**, No. 144, 1–11. Zbl 1182.35092

[20] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Anal.* **121**(2015), 336–369. https://doi.org/10.1016/j.na.2014.11.007; Zbl 1321.35030

[21] V. Rădulescu, D. Repovš, *Partial differential equations with variable exponents: variational methods and qualitative analysis*, CRC Press, Boca Raton, FL, 2015. https://doi.org/10.1201/b18601; Zbl 1343.35003

[22] K. R. Rajagopal, M. Růžička, On the modeling of electrorheological materials, *Mech. Res. Commun.* **23**(1996), No. 4, 401–407. https://doi.org/10.1016/0093-6413(96)00038-9; Zbl 0890.76007

[23] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Springer-Verlag, Berlin, 2002. https://doi.org/10.1007/BFb0104029; Zbl 0968.76531

[24] M. Xiang, F. Wang, B. Zhang, Existence and multiplicity of solutions for $p(x)$-curl systems arising in electromagnetism, *J. Math. Anal. Appl.* **448**(2017), 1600–1617. https://doi.org/10.1016/j.jmaa.2016.11.086; Zbl 1358.35181

[25] E. Zeidler, *Nonlinear functional analysis and its applications, Vol. II/B*, Springer New York, NY, 1985. https://doi.org/10.1007/978-1-4612-0981-2

[26] V. V. Zhikov, Averaging of functionals in the calculus of variations and elasticity theory, *Math. USSR, Izv.* **29**(1987), 675–710. https://doi.org/10.1070/IM1987v029n01ABEH000958; Zbl 0599.49031

[27] V. V. Zhikov, On Lavrentiev's phenomenon, *Russ. J. Math. Phys.*, **3**(1995), No. 2, 249–269. Zbl 0910.49020