Electronic Journal of Qualitative Theory of Differential Equations
2024, No. **64**, 1–3; https://doi.org/10.14232/ejqtde.2024.1.64 www.math.u-szeged.hu/ejqtde/

Addendum to Higher order stroboscopic averaged functions: a general relationship with Melnikov functions

Douglas D. Novaes ✉

Departamento de Matemática – Instituto de Matemática, Estatística e Computação Científica (IMECC) – Universidade Estadual de Campinas (UNICAMP), Rua Sérgio Buarque de Holanda, 651, Cidade Universitária Zeferino Vaz, 13083-859, Campinas, SP, Brazil

Received 27 May 2024, appeared 12 November 2024
Communicated by Armengol Gasull

Abstract. This addendum presents a relevant stronger consequence of the main theorem of the paper "Higher order stroboscopic averaged functions: a general relationship with Melnikov functions", *Electron. J. Qual. Theory Differ. Equ.* **2021**, No. 77.

Keywords: averaging theory, Melnikov method, averaged functions, Melnikov functions, higher order analysis.

2020 Mathematics Subject Classification: 34C29, 34E10, 34C25.

This addendum addresses the findings presented in the paper [1] titled "Higher order stroboscopic averaged functions: a general relationship with Melnikov functions" published in *Electron. J. Qual. Theory Differ. Equ.* **2021**, No. 77.

The main result of the referred paper, [1, Theorem A], establishes a general relationship between averaged functions $\mathbf{g}_i$ and Melnikov functions $\mathbf{f}_i$. As a direct consequence of this general relationship, [1, Corollary A] states that if, for some $\ell \in \{2, \dots, k\}$, either $\mathbf{f}_1 = \cdots = \mathbf{f}_{\ell-1} = 0$ or $\mathbf{g}_1 = \cdots = \mathbf{g}_{\ell-1} = 0$, then $\mathbf{f}_i = T\,\mathbf{g}_i$ for $i \in \{1, \dots, \ell\}$. This consequence was somewhat expected based on existing results in the literature within more restricted contexts. Here, we will demonstrate that under the same conditions, the relationship $\mathbf{f}_i = T\,\mathbf{g}_i$ actually holds for every $i \in \{1, \dots, 2\ell - 1\}$, which represents a more unexpected outcome. The expression for $\mathbf{g}_{2\ell}(z)$ will also be provided.

Proposition 1. *Let $\ell \in \{2, \dots, k\}$. If either $\mathbf{f}_1 = \cdots = \mathbf{f}_{\ell-1} = 0$ or $\mathbf{g}_1 = \cdots = \mathbf{g}_{\ell-1} = 0$, then $\mathbf{f}_i = T\,\mathbf{g}_i$ for $i \in \{1, \dots, 2\ell - 1\}$ and*

$$\mathbf{g}_{2\ell}(z) = \frac{1}{T}\left(\mathbf{f}_{2\ell}(z) - \frac{1}{2} d\mathbf{f}_\ell(z) \cdot \mathbf{f}_\ell(z)\right) \text{ or, equivalently, } \mathbf{f}_{2\ell}(z) = T\mathbf{g}_{2\ell}(z) + \frac{T^2}{2} d\mathbf{g}_\ell(z) \cdot \mathbf{g}_\ell(z).$$

✉Corresponding author. Email: ddnovaes@unicamp.br

Proof. Given $\ell \in \{2,\dots,k\}$, assume that either $\mathbf{f}_1 = \cdots = \mathbf{f}_{\ell-1} = 0$ or $\mathbf{g}_1 = \cdots = \mathbf{g}_{\ell-1} = 0$. From [1, Corollary A], we have that

$$\mathbf{g}_i = \mathbf{f}_i = 0, \quad \text{for } i \in \{1,\dots,\ell-1\}, \quad \text{and} \quad \mathbf{g}_\ell = \frac{1}{T}\mathbf{f}_\ell. \tag{1}$$

For any i, [1, Theorem A] provides

$$\mathbf{g}_i(z) = \frac{1}{T}\left(\mathbf{f}_i(z) - \sum_{j=1}^{i-1}\sum_{m=1}^{j}\frac{1}{j!}d^m\mathbf{g}_{i-j}(z)\int_0^T B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,z)ds\right), \tag{2}$$

where $\tilde{y}_i(t,z)$, for $i \in \{1,\dots,k\}$, are polynomial in the variable t recursively defined as follows:

$$\begin{aligned} \tilde{y}_1(t,z) &= t\,\mathbf{g}_1(z) \\ \tilde{y}_i(t,z) &= i!t\,\mathbf{g}_i(z) + \sum_{j=1}^{i-1}\sum_{m=1}^{j}\frac{i!}{j!}d^m\mathbf{g}_{i-j}(z)\int_0^t B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,z)ds. \end{aligned} \tag{3}$$

Taking (1) into account, the function $\mathbf{g}_{i-j}$ in (2) vanishes for $i-j \leq \ell-1$, that is, for $j \geq i-\ell+1$. Thus,

$$\mathbf{g}_i(z) = \frac{1}{T}\left(\mathbf{f}_i(z) - \sum_{j=1}^{i-\ell}\sum_{m=1}^{j}\frac{1}{j!}d^m\mathbf{g}_{i-j}(z)\int_0^T B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,z)ds\right). \tag{4}$$

Also, from (1) and (3), one has

$$\tilde{y}_1 = \cdots \tilde{y}_{\ell-1} = 0 \quad \text{and} \quad \tilde{y}_\ell(t,z) = \ell!t\mathbf{g}_\ell(z) = \frac{\ell!}{T}t\mathbf{f}_\ell(z). \tag{5}$$

Now, let $i \in \{\ell+1,\dots,2\ell-1\}$. Thus, for $j \leq i-\ell$ and $m \geq 1$, one has that

$$j-m+1 \leq i-\ell \leq 2\ell-1-\ell = \ell-1,$$

which implies, from (5), that $\tilde{y}_1 = \cdots = \tilde{y}_{j-m+1} = 0$ in (4). Consequently, $\mathbf{f}_i(z) = T\mathbf{g}_i(z)$.

Finally, from (4),

$$\mathbf{g}_{2\ell}(z) = \frac{1}{T}\left(\mathbf{f}_{2\ell}(z) - \sum_{j=1}^{\ell}\sum_{m=1}^{j}\frac{1}{j!}d^m\mathbf{g}_{2\ell-j}(z)\int_0^T B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,z)ds\right). \tag{6}$$

Notice that, for $1 \leq j \leq \ell$ and $1 \leq m \leq j$, the relationship $j-m+1 \geq \ell$ implies that $m=1$ and $j=\ell$, which are the only possible values for m and j for which $\tilde{y}_{j-m+1}$ in (6) is not vanishing. In this case, from (5), $\tilde{y}_1 = \cdots = \tilde{y}_{j-m} = 0$ and $\tilde{y}_{j-m+1} = \tilde{y}_\ell = \ell!t\mathbf{g}_\ell(z)$. Thus,

$$\begin{aligned} \mathbf{g}_{2\ell}(z) &= \frac{1}{T}\left(\mathbf{f}_{2\ell}(z) - \frac{1}{\ell!}d\mathbf{g}_\ell(z)\int_0^T B_{\ell,1}(0,\dots,0,\ell!t\mathbf{g}_\ell(z))\,ds\right) \\ &= \frac{1}{T}\left(\mathbf{f}_{2\ell}(z) - \frac{1}{\ell!}d\mathbf{g}_\ell(z)\int_0^T \ell!t\mathbf{g}_\ell(z)\,ds\right) \\ &= \frac{1}{T}\left(\mathbf{f}_{2\ell}(z) - \frac{T^2}{2}d\mathbf{g}_\ell(z)\cdot\mathbf{g}_\ell(z)\right) = \frac{1}{T}\left(\mathbf{f}_{2\ell}(z) - \frac{1}{2}d\mathbf{f}_\ell(z)\cdot\mathbf{f}_\ell(z)\right). \end{aligned}$$

Equivalently,

$$\mathbf{f}_{2\ell}(z) = T\mathbf{g}_{2\ell}(z) + \frac{T^2}{2}d\mathbf{g}_\ell(z)\cdot\mathbf{g}_\ell(z).$$

□

Acknowledgements

The author is partially supported by São Paulo Research Foundation (FAPESP) grants 2021/10606-0, 2018/13481-0, and 2019/10269-3, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) grant 309110/2021-1.

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