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Normalized solutions for a fractional coupled critical Hartree system

Shengbing Deng✉ and **Wenshan Luo**

School of Mathematics and Statistics, Southwest University, Chongqing, 400715, P.R. China

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Abstract. We consider the existence of normalized solutions for a fractional coupled Hartree system, with the upper critical exponent in the sense of the Hardy–Littelwood–Sobolev inequality. Particularly, in an L^2-subcritical regime or an L^2-supercritical regime, we establish the existence of positive normalized solutions for the two cases, respectively. Furthermore, we prove the nonexistence of positive normalized solutions, under the nonlinearities satisfying the Sobolev critical growth.

Keywords: fractional Hartree system, normalized solutions, Hardy–Littlewood–Sobolev critical exponent.

2020 Mathematics Subject Classification: 35J50, 35B33, 35R11, 58E05.

1 Introduction

This paper is concerned with the existence of solutions $(\lambda_1, \lambda_2, u, v) \in \mathbb{R}^2 \times H^s(\mathbb{R}^N, \mathbb{R}^2)$ to the following fractional critical Hartree system:

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} + (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{p-2}v + \beta r_2 |v|^{r_2-2}v|u|^{r_1} + (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

satisfying the additional conditions

$$\int_{\mathbb{R}^N} u^2 dx = a^2, \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 dx = b^2. \tag{1.2}$$

The masses $a, b > 0$ are prescribed and the parameters $\mu_1, \mu_2, \beta > 0$. Here $(-\Delta)^s$ is the fractional Laplacian, $s \in (0,1)$, $2s < N \leq 4s$, $\alpha \in (0, N)$, $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$ is the upper critical exponent due to the Hardy–Littlewood–Sobolev inequality, $2^*_s = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, $r_1, r_2 > 1$, $p, r_1 + r_2 \in (2, 2^*_s]$ with $p < r_1 + r_2$ and $*$ stands for the convolution on $\mathbb{R}^N$ with $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is the Riesz potential,

$$I_\alpha(x) = \frac{A_{N,\alpha}}{|x|^\alpha}, \quad \text{with } A_{N,\alpha} = \frac{\Gamma(\frac{\alpha}{2})}{2^{N-\alpha}\pi^{\frac{N}{2}}\Gamma(\frac{N-\alpha}{2})}.$$

✉Corresponding author. Email: shbdeng@swu.edu.cn

The fractional Laplacian operator $(-\Delta)^s$ is defined for any $u : \mathbb{R}^N \to \mathbb{R}$ sufficiently smooth by

$$(-\Delta)^s u(x) = C(N,s)P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where $P.V.$ stands for the Cauchy principal value and $C(N,s)$ is a positive constant depending only on N and s. Recently, a great attention has been devoted to study the nonlinear problems involving fractional elliptic operators, both for the pure mathematical research and applications. We refer to [3,11,16,37] for a simple introduction to basic properties of the fractional Laplacian operator and concrete applications based on variational methods. Moreover, fractional Choquard type equation with critical growth has been studied by many researchers, see [1,22,23,35,36] and references therein.

The problem under investigation comes from the research of solitary waves for the following physical model:

$$\begin{cases} (-\Delta)^s \phi_1 = -i\frac{\partial \phi_1}{\partial t} + \mu_1 |\phi_1|^{p-2}\phi_1 + \beta r_1 |\phi_1|^{r_1-2}\phi_1 |\phi_2|^{r_2} + (I_\alpha * |\phi_2|^{2^*_{\alpha,s}})|\phi_1|^{2^*_{\alpha,s}-2}\phi_1, \\ (-\Delta)^s \phi_2 = -i\frac{\partial \phi_2}{\partial t} + \mu_2 |\phi_2|^{p-2}\phi_2 + \beta r_2 |\phi_2|^{r_2-2}\phi_2 |\phi_1|^{r_1} + (I_\alpha * |\phi_1|^{2^*_{\alpha,s}})|\phi_2|^{2^*_{\alpha,s}-2}\phi_2, \end{cases} \tag{1.3}$$

where $i^2 = -1$ and $\phi_j (j = 1,2)$ is the wave function of the j_{th} component, and μ_j, β denote the intra-species and intra-species scattering lengths. In particular, the interaction of states is attractive if $\beta > 0$, while the interaction of states is repulsive when $\beta < 0$. Solitary wave solutions of system (1.3) are solutions having the form

$$\phi_1(x,t) = e^{i\lambda_1 t} u(x), \qquad \phi_2(x,t) = e^{i\lambda_1 t} v(x),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are the chemical potentials and (u,v) solves (1.1). Since $\phi_1(x,t), \phi_2(x,t)$ retain their masses over time, we consider this problem from two aspects: one can either regard the frequencies λ_1, λ_2 as fixed, or include them in the unknown and prescribe the masses.

Fixing the parameters λ_1, λ_2 in (1.1), we call it the fixed frequency problem. The two-component system with Hartree-type nonlinearities describes the boson stars in mean-field theory [18, 27], which appears naturally in optical systems [30] and is known to influence the propagation of electromagnetic waves in plasmas [7]. Moreover, the non-locality of the critical term also plays an important role in the theory of Bose-Einstein condensation, where it accounts for the finite-range many-body interaction [15]. The Hartree type systems, mainly on λ_1, λ_2 are prescribed, have been widely studied. We refer to [20] and references therein. However, much less is known when the masses are prior prescribed. In this case, $\lambda_1, \lambda_2 \in \mathbb{R}$ are unknown quantities arising as Lagrange multipliers. In recent years, since physicists are interested in normalized solutions (which L^2-norms of solutions are prescribed), mathematical researchers began to investigate the solutions of various classes of Schrödinger equations or systems having a prescribed L^2-norm, that is a solution which satisfies $\int_{\mathbb{R}^N} |u|^2 dx = c$ for a priori given c.

When $s = 1$, i.e. the fractional Laplace operator $(-\Delta)^s$ reduces to the local differential operator $-\Delta$, the literature for the normalized solutions of Schrödinger equations or systems is abundant. Starting from the seminal paper by Jeanjean in [25], he firstly studied L^2-supercritical case, and dealt with the existence of normalized solutions when the energy functional is unbounded from below, by using the mountain pass lemma and a skillful compactness argument. Furthermore, for the particular case of a combined nonlinearity of power type, in [38], Soave considered the existence of normalized solutions and orbitally stable for the following problem:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $N \geq 1$, $q, p \in (2, 2^*)$ and $q < p$. Moreover, when $p = 2^*$ in (1.4), in [39], the Sobolev critical case was studied by Soave, where he considered the energy level less than a certain number to get the compactness, and obtained the existence and nonexistence of normalized solutions. For the system case, Bartsch, Jeanjean and Soave investigated the following elliptic system

$$\begin{cases} -\Delta u = \lambda_1 u + \mu_1 u^3 + \beta u v^2, & \text{in } \mathbb{R}^3, \\ -\Delta v = \lambda_2 v + \mu_2 u^3 + \beta v u^2, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = a, \int_{\mathbb{R}^3} |v|^2 dx = b, \end{cases} \tag{1.5}$$

where $\mu_1, \mu_2, a, b > 0$. In [4], Bartsch, Jeanjean and Soave obtained the existence results for different ranges of $\beta > 0$ and stability properties of (1.5). Furthermore, in [6], Bartsch and Soave considered the case $\beta < 0$ of (1.5) and showed phase separation occurs for the solutions as $\beta \to -\infty$. In particular, Bartsch, Li and Zou [5] studied the normalized solutions for a Schrödinger systems with Sobolev critical nonlinearities. Specifically, in [5], they proved the existence and nonexistence results and obtained the asymptotic behavior as $\beta \to 0^+$ or $\beta \to +\infty$. When $3 \leq N \leq 4$, in [29], Li and Zou obtained the existence of positive normalized ground state for (1.5). For more researches of the normalized solutions of the Laplacian systems, we refer to [31,34] and references therein.

The situation is different when $s \in (0,1)$, and few results are available. We note that the L^2-critical exponent for fractional case is $\bar{p} := 2 + \frac{4s}{N}$. In [32], Luo and Zhang studied the existence and nonexistence of normalized solutions for the following fractional problem

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $q, p \in (2, 2_s^*)$, $q < p$ and $\mu \in \mathbb{R}$. Moreover, when $p = 2_s^*$ in (1.6), Zhen and Zhang [44] proved the existence and nonexistence results of the normalized solutions by using the Jeanjean's skill in [25], and they also considered the behavior of the ground state obtained as $\mu \to 0^+$. Furthermore, in [24], He, Rădulescu and Zou showed the existence and nonexistence of solutions for a fractional equation with the upper critical exponent, among 3 cases: L^2-subcritical, L^2-critical and L^2-supercritical. In the case of fractional systems, Zuo and Rădulescu studied the following problem

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + |u|^{2_s^*-2}u + \gamma\alpha |u|^{\alpha-2}u|v|^{\beta}, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{q-2}v + |v|^{2_s^*-2}v + \gamma\beta |v|^{\alpha-2}v|u|^{\beta}, & \text{in } \mathbb{R}^N. \\ \int_{\mathbb{R}^N} |u|^2 dx = a, \int_{\mathbb{R}^N} |v|^2 dx = b, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $s \in (0,1)$, $p, q, \alpha + \beta \in (\bar{p}, 2_s^*)$. In [45], Zuo and Rădulescu showed the existence of positive normalized solutions when γ is big enough, and obtained the nonexistence of positive normalized solutions if $p = q = \alpha + \beta = 2_s^*$. Li [28] studied the existence of positive radial solutions for a fractional Hartree–Fock type system in L^2-subcritical case, L^2-critical case and L^2-supercritical case, but without the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality (see Lemma 2.1).

Inspired by the above mentioned works, in the present paper, our goal is two-fold. On one hand, we show the existence of normalized ground states for $p \in (2, 2_s^*)$, and $r_1 + r_2 \in (p, 2_s^*)$; on the other hand, we obtain the nonexistence result for $p = r_1 + r_2 = 2_s^*$. Compared to the Laplace operator, the fractional Laplacian problems are nonlocal and more challenging. Moreover, since the compactness of system is closely related to (see Proposition 4.9) the following problem

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu |u|^{p-2} u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c^2, \end{cases} \tag{1.8}$$

we may be more careful to the energy level and solutions of (1.8). However, for $p = \bar{p}$ and $c > 0$, the Pohožaev manifold related to (1.8) is indefinite (see Lemma 5.2), which makes it difficult to construct the geometry for the related energy functional.

Before we state our main results, we introduce some notations for the fractional Sobolev space $H^s(\mathbb{R}^N)$. Let $s \in (0,1)$. We denote by $D^s(\mathbb{R}^N)$ the completion of $C_c^\infty(\mathbb{R}^N)$ with

$$[u]^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy.$$

The fractional Sobolev space is defined by

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \ [u] < \infty\},$$

with the standard norm and inner product

$$\|u\|^2 = [u]^2 + \int_{\mathbb{R}^N} |u|^2 dx, \quad \text{and} \quad \langle u, \varphi \rangle = \int_{\mathbb{R}^N} \Big((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + u\varphi \Big) dx.$$

It is well known (see [2]) that the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for all $q \in [2, 2_s^*]$, locally compact for all $q \in [1, 2_s^*)$ and $D^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous. Then we define the working space H as

$$H := \{(u,v) : u \in H^s(\mathbb{R}^N), v \in H^s(\mathbb{R}^N)\},$$

endowed with the norm

$$\|(u,v)\|_H^2 := \|u\|^2 + \|v\|^2,$$

and related inner product is, for any $(\varphi, \psi) \in H$:

$$\langle (u,v), (\varphi, \psi) \rangle_H := \langle u, \varphi \rangle + \langle v, \psi \rangle.$$

By using the variational methods, a classical way for studying the normalized solutions of system (1.1) is to look for critical points of the following C^1-functional

$$J(u,v) = \frac{1}{2}([u]^2 + [v]^2) - \frac{1}{p}(\mu_1 |u|_p^p + \mu_2 |v|_p^p) - \beta \int_{\mathbb{R}^N} |u|^{r_1} |v|^{r_2} dx - \frac{1}{2_{\alpha,s}^*} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*}) |v|^{2_{\alpha,s}^*} dx,$$

constrained on the set

$$S := \{(u,v) \in H : (u,v) \in S_a \times S_b\},$$

where $|u|_r = \big(\int_{\mathbb{R}^N} |u|^r dx \big)^{\frac{1}{r}}$ and $S_a := \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = a^2\}$. The main results of this paper can be stated as follows:

Theorem 1.1. *When $2s < N \le 4s$, $p \in (2, \bar{p})$ and $r_1 + r_2 \in (p, 2_s^*)$, there exists $\beta^* > 0$ such that for $0 < \beta < \beta^*$, there exist $\mu_1^* = \mu_1^*(\beta), \mu_2^* = \mu_2^*(\beta)$, such that for any $\mu_1 \in (0, \mu_1^*), \mu_2 \in (0, \mu_2^*)$, (1.1)–(1.2) has a normalized ground state (u, v), which is a positive and radially symmetric function, for some $\lambda_1, \lambda_2 < 0$. Moreover, (u, v) is an interior local minimizer on the set*

$$B_r(a, b) := \{(u, v) \in S : ([u]^2 + [v]^2)^{\frac{1}{2}} < r\},$$

for a suitable $r > 0$ small enough; and any other ground state solution of $J(u, v)|_S$ is a local minimizer of $J(u, v)$ on $B_r(a, b)$.

Theorem 1.2. *When $2s < N \le 4s$, $p \in (\bar{p}, 2_s^*)$ and $r_1 + r_2 \in (p, 2_s^*)$, there exists $\beta_0 > 0$, such that for any $\beta > \beta_0$, (1.1)–(1.2) has a normalized ground state (u, v), which is a positive and radially symmetric function, for some $\lambda_1, \lambda_2 < 0$, and (u, v) is a Mountain Pass type solution.*

Theorem 1.3. *When $2s < N \le 4s$, suppose $p = r_1 + r_2 = 2_s^*$, then the system (1.1)–(1.2) has no positive normalized solutions.*

Remark 1.4.

- (I) In Theorem 1.1, we consider 3 cases: $r_1 + r_2 \in (2, \bar{p})$, $r_1 + r_2 = \bar{p}$ and $r_1 + r_2 \in (\bar{p}, 2_s^*)$. These different situations are mainly reflected in Lemmas 4.3 and 5.3.
- (II) From the processes in our proof, one difference between Theorems 1.1 and 1.2 lies in their respective geometric structures. In fact, when p changes from L^2-subcritical to L^2-supercritical, it changes the geometry of $J(u, v)|_S$ and prevents the existence of a local minimizer in Theorem 1.2.
- (III) Compared with the result in [24], we need an elementary inequality (see Proposition 4.9), which combined the single case (1.8) with the coupling case (1.1), to ensure compactness result. Theorems 1.1, 1.2, 1.3 seem to be the first results of normalized solutions for a fractional coupling systems with the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality.

The paper is organized as follows. In Section 2, we give some preliminaries for the functional space. In Section 3, we will briefly introduce the properties of a single case (1.8), which plays an important role to the proof of Palais–Smale condition in our problem. In Section 4, we prove Theorem 1.1. In Section 5, we obtain Theorem 1.2. At last, we show the nonexistence for Theorem 1.3 in Section 6.

2 Preliminaries

Following, for the convenience of the reader, we recall some basic properties, which we shall need in the sequel. Let us first recall the well-known Hardy–Littlewood–Sobolev inequality.

Lemma 2.1 ([30])**.** *Let $t, r > 1$, $0 < \alpha < N$, with $\frac{1}{t} + \frac{\alpha}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(N, t, \alpha, r)$ independent of f and h, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x - y|^\alpha} \le C(N, t, \alpha, r)|f|_t |h|_r, \tag{2.1}$$

where $|\cdot|_q$ *stands for the* $L^q(\mathbb{R}^N)$*-norm for* $q \in [1, +\infty)$*. If* $t = r = \frac{2N}{2N-\alpha}$*, then*

$$C(N,t,\alpha,r) = C(N,\alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma(\frac{N}{2}-\frac{\alpha}{2})}{\Gamma(N-\frac{\alpha}{2})}.$$

Besides, there is a equality in (2.1) *if and only if* $f \equiv (constant.)h$ *and*

$$h(x) = C(\gamma^2 + |x-a|^2)^{-\frac{2N-\alpha}{2}},$$

for some $C \in \mathbb{C}$*,* $\gamma \neq 0$ *and* $a \in \mathbb{R}^N$*.*

According to Lemma 2.1, the functional

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^p(x)u^p(y)}{|x-y|^\alpha} dydx,$$

is well defined in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ if $\frac{2N-\alpha}{N} \leq p \leq \frac{2N-\alpha}{N-2s}$. We often call $\frac{2N-\alpha}{N}$ is the lower Hardy–Littlewood–Sobolev critical exponent and $\frac{2N-\alpha}{N-2s}$ is the upper Hardy–Littlewood–Sobolev critical exponent. From Lemma 2.1, we define the best constant

$$S_{h,l} = \inf_{D^s(\mathbb{R}^N\setminus\{0\})} \frac{[u]^2}{\left(\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}dx\right)^{\frac{1}{2^*_{\alpha,s}}}},$$

and from [23], we know $S_{h,l}$ is attained by the function

$$\tilde{u}_{\varepsilon,y} = \tilde{C}_{N,\alpha,s} u_{\varepsilon,y}, \quad x, y \in \mathbb{R}^N, \quad \text{and } \varepsilon > 0,$$

such that

$$[\tilde{u}_{\varepsilon,y}]^2 = S_{h,l}^{\frac{2N-\alpha}{N-\alpha+2s}},$$

with $\tilde{u}_{\varepsilon,y}$ satisfying this equation

$$(-\Delta)^s u = (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, \quad x \in \mathbb{R}^N.$$

The function $u_{\varepsilon,y} = \kappa(\varepsilon^2 + |x-y|^2)^{-\frac{N-2s}{2}}$ solves

$$(-\Delta)^s u = |u|^{2^*_s-2}u, \quad \text{in } \mathbb{R}^N,$$

and achieves the infimum of

$$S := \inf_{D^s(\mathbb{R}^N\setminus\{0\})} \frac{[u]^2}{|u|^2_{2^*_s}},$$

with

$$S_{h,l} = S C_{N,\alpha,s}^{-\frac{1}{2^*_{\alpha,s}}} \quad \text{and} \quad \kappa = \left(\frac{S^{\frac{N}{2s}}\Gamma(N)}{\pi^{\frac{N}{2}}\Gamma(\frac{N}{2})}\right)^{\frac{N-2s}{2N}}.$$

In order to prove our problem, we shall make use of the following infimum

$$S^* := \inf_{(u,v)\in D^s(\mathbb{R}^N)\times D^s(\mathbb{R}^N)} \frac{[u]^2+[v]^2}{\left(\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx\right)^{\frac{1}{2^*_{\alpha,s}}}}, \tag{2.2}$$

and from [43, Lemma 2.2], we know

Lemma 2.2. *We have*

$$S^* = 2S_{h,l},$$

and S^* is achieved if and only if, for $C > 0$,

$$u = v = Cu_{\varepsilon,y}.$$

Then we recall the fractional Gagliardo–Nirenberg–Sobolev inequality, which can be seen in [19].

Lemma 2.3. *Let $N > 2s$ and $p \in (2, 2_s^*)$, then there exists a constant $C(N,p,s) > 0$, such that for all $u \in H^s(\mathbb{R}^N)$,*

$$|u|_p^p \le C(N,p,s)|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{N(p-2)}{2s}}|u|_2^{p-\frac{N(p-2)}{2s}}. \tag{2.3}$$

Defining $\gamma_p := \frac{N(p-2)}{2ps}$, it is easy to see

$$p\gamma_p \begin{cases} < 2, & \text{if } 2 < p < \bar{p}, \\ = 2, & \text{if } p = \bar{p}, \text{ and } \gamma_{2_s^*} = 1, \\ > 2, & \text{if } \bar{p} < p < 2_s^*. \end{cases}$$

and

$$|u|_p^p \le C(N,p,s)|(-\Delta)^{\frac{s}{2}}u|_2^{p\gamma_p}|u|_2^{p(1-\gamma_p)}. \tag{2.4}$$

Following, we obtain the corresponding Pohožaev type identity for system (1.1). Before the statement of this result, we introduce the s-harmonic extension (see [11]) techniques. Denote $\mathbb{R}^{N+1} = \{(x,y) : x \in \mathbb{R}^N, y \in \mathbb{R}\}$ and define $X = X^s(\mathbb{R}_+^{N+1}) \times X^s(\mathbb{R}_+^{N+1})$ under the norms

$$\|(U,V)\|_X = \left(\kappa_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla U|^2 dxdy + \kappa_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla V|^2 dxdy\right)^{\frac{1}{2}},$$

where $X^s(\mathbb{R}_+^{N+1})$ is the completion of $C_0^\infty(\mathbb{R}_+^{N+1})$ with the norm

$$\|U\|_{X^s(\mathbb{R}_+^{N+1})} = \left(\kappa_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla U|^2 dxdy\right)^{\frac{1}{2}}.$$

Let $(u,v) \in H$ be a solution of (1.1) and define $(U,V) \in X$ be its s-harmonic extension to the upper half space $\mathbb{R}_+^{N+1}$, then $u = U(x,0)$, $v = V(x,0)$ and (U,V) is a solution to the following problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla U) = 0; -\operatorname{div}(y^{1-2s}\nabla V) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ -\frac{\partial U}{\partial y^{1-2s}} = \lambda_1 u + \mu_1|u|^{p-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2} + (I_\alpha * |v|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*-2}u, & \text{on } \mathbb{R}^N, \\ -\frac{\partial V}{\partial y^{1-2s}} = \lambda_2 v + \mu_2|v|^{p-2}v + \beta r_2|v|^{r_2-2}v|u|^{r_1} + (I_\alpha * |u|^{2_{\alpha,s}^*})|v|^{2_{\alpha,s}^*-2}v, & \text{on } \mathbb{R}^N. \end{cases} \tag{2.5}$$

From [8, Proposition A.1] and [42, Lemma 4.1], we have the following result.

Proposition 2.4. *Let $(u,v) \in H$ be a weak solution of (1.1), that is (u,v) satisfies:*

$$\begin{aligned} 0 = &\langle u,\varphi\rangle + \langle v,\psi\rangle - \int_{\mathbb{R}^N}(\lambda_1 u\varphi + \lambda_2 v\psi)dx \\ &- \int_{\mathbb{R}^N}(\mu_1|u|^{p-2}u\varphi + \mu_2|v|^{p-2}v\varphi)dx - \beta\int_{\mathbb{R}^N}(r_1|u|^{r_1-2}u|v|^{r_2}\varphi + r_2|u|^{r_1}|v|^{r_2-2}v\psi)dx \\ &- \int_{\mathbb{R}^N}[(I_\alpha * |v|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*-2}u\varphi + (I_\alpha * |u|^{2_{\alpha,s}^*})|v|^{2_{\alpha,s}^*-2}v\psi]dx, \quad \forall(\varphi,\psi) \in H, \end{aligned}$$

then we have (u,v) satisfies

$$\begin{aligned}&\frac{N-2s}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2+|(-\Delta)^{\frac{s}{2}}v|^2)dx\\&=\frac{N}{2}\int_{\mathbb{R}^N}(\lambda_1|u|^2+\lambda_2|v|^2)dx+\frac{N}{p}\int_{\mathbb{R}^N}(\mu_1|u|^p+\mu_2|v|^p)dx\\&\quad+\beta N\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx+\frac{2N-\alpha}{2^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.\end{aligned}$$

Proof. If $(u,v)\in H$ is a weak solution of (1.1), from [2, Proposition 3.2.14], we have $(u,v)\in L^\infty(\mathbb{R}^N)\times L^\infty(\mathbb{R}^N)$. Using the same arguments as in [13, Proposition 4.1], we get $(u,v)\in C^{2,\tau}(\mathbb{R}^N)\times C^{2,\tau}(\mathbb{R}^N)$ with τ depending on s. Let (U,V) be its s-harmonic extension and satisfy (2.5), then $(U,V)\in C^2(\mathbb{R}^{N+1}_+)\times C^2(\mathbb{R}^{N+1}_+)$.

Set $D_m:=\{(x,y)\in\mathbb{R}^{N+1}:|(x,y)|\le m\}$ and $Q_r=D_r^+\cup(D_r\cap(\mathbb{R}^N\times\{0\}))$, where $D_r^+=D_r\cap\mathbb{R}^{N+1}_+$. Let $\varphi\in C_0^\infty(\mathbb{R}^{N+1})$ with $0\le\varphi\le 1$, $\varphi=1$ in D_1, $\varphi=0$ outside D_2 and $|\nabla\varphi|\le 2$. For $R>0$, define

$$\psi_R(x,y)=\psi\left(\frac{(x,y)}{R}\right),\quad \text{where } \psi=\varphi|_{\overline{\mathbb{R}^{N+1}_+}}.$$

Multiplying (2.5) by $((x,y)\cdot\nabla U)\psi_R$ and $((x,y)\cdot\nabla V)\psi_R$ respectively, we obtain from [8, Proposition A.1],

$$\begin{aligned}\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}\times\{0\})}|u|^{p-2}u\cdot(x,y)\cdot\nabla U\psi_R dx&=-\frac{N}{p}\int_{\mathbb{R}^N}|u|^pdx.\\ \lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}\times\{0\})}u\cdot(x,y)\cdot\nabla U\psi_R dx&=-\frac{N}{2}\int_{\mathbb{R}^N}|u|^2dx.\end{aligned}$$

and

$$\begin{aligned}\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}^N\times\{0\})}|v|^{p-2}v\cdot(x,y)\cdot\nabla V\psi_R dx&=-\frac{N}{p}\int_{\mathbb{R}^N}|v|^pdx.\\ \lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}^N\times\{0\})}v\cdot(x,y)\cdot\nabla V\psi_R dx&=-\frac{N}{2}\int_{\mathbb{R}^N}|v|^2dx.\end{aligned}$$

Moreover,

$$\begin{aligned}&\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}^N\times\{0\})}\big(r_1|v|^{r_2}|u|^{r_1-2}u\cdot(x,y)\cdot\nabla U\psi_R+r_2|u|^{r_1}|v|^{r_2-2}v\cdot(x,y)\cdot\nabla V\psi_R\big)dx\\&=-N\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx.\end{aligned}$$

and

$$\begin{aligned}\lim_{R\to\infty}\int_{Q_{2R}}y^{1-2s}\nabla U\nabla[((x,y)\cdot\nabla U)\psi_R]dxdy&=-\frac{N-2s}{2}\int_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla U|^2dxdy,\\ \lim_{R\to\infty}\int_{Q_{2R}}y^{1-2s}\nabla V\nabla[((x,y)\cdot\nabla V)\psi_R]dxdy&=-\frac{N-2s}{2}\int_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla V|^2dxdy.\end{aligned}$$

Furthermore, combining with [42, Lemma 4.1], we have

$$\begin{aligned}&\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}^N\times\{0\})}\big((I_\alpha*|v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}\cdot(x,y)\cdot\nabla U\psi_R\\&\qquad+(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}\cdot(x,y)\cdot\nabla V\psi_R\big)dx\\&=\frac{\alpha-2N}{2^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.\end{aligned}$$

Multiplying (2.5) by $U\psi_R$ and $V\psi_R$ respectively, and using the same techniques of [8, Proposition A.1], we firstly obtain

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla U|^2 dxdy = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx,$$
$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V|^2 dxdy = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx,$$

and then we finish this proof. □

Lemma 2.5. *Let $(u,v) \in H$ be a weak solution of (1.1), then we have Pohožaev manifold*

$$\mathcal{P}_{\mu_1,\mu_2} = \{(u,v) \in S : P_{\mu_1,\mu_2}(u,v) = 0\},$$

where

$$\begin{aligned} P_{\mu_1,\mu_2}(u,v) = s([u]^2 + [v]^2) - s\gamma_p(\mu_1|u|_p^p + \mu_2|v|_p^p) - s\beta(r_1+r_2)\gamma_{(r_1+r_2)} \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ - 2s \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx. \end{aligned} \tag{2.6}$$

Proof. Since Proposition 2.4, we have (u,v) satisfies

$$\begin{aligned} &\frac{N-2s}{2} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) dx \\ &= \frac{N}{2} \int_{\mathbb{R}^N} (\lambda_1|u|^2 + \lambda_2|v|^2) dx + \frac{N}{p} \int_{\mathbb{R}^N} (\mu_1|u|^p + \mu_2|v|^p) dx \\ &\quad + \beta N \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx + \frac{2N-\alpha}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) dx = &\int_{\mathbb{R}^N} (\lambda_1|u|^2 + \lambda_2|v|^2) dx + \int_{\mathbb{R}^N} (\mu_1|u|^p + \mu_2|v|^p) dx \\ &+ \beta(r_1+r_2) \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx + 2 \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx. \end{aligned}$$

Thus,

$$\begin{aligned} &s \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) dx \\ &= s\gamma_p \int_{\mathbb{R}^N} (\mu_1|u|^p + \mu_2|v|^p) dx + \beta s(r_1+r_2)\gamma_{(r_1+r_2)} \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ &\quad + 2s \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx, \end{aligned}$$

and the conclusions follows. □

Under the L^2-invariant scaling introduced by Jeanjean in [25],

$$t * u := e^{\frac{Nt}{2}} u(e^t x), \quad \text{and} \quad t * (u,v) := (t * u, t * v),$$

it is natural to study the fiber maps

$$\begin{aligned} \Psi_{\mu_1,\mu_2}(t) := J(t * (u,v)) = &\frac{e^{2st}}{2}([u]^2 + [v]^2) - \frac{e^{sp\gamma_p t}}{p}(\mu_1|u|_p^p + \mu_2|v|_p^p) \\ &- \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)} t} \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ &- \frac{e^{22^*_{\alpha,s} st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx, \end{aligned} \tag{2.7}$$

satisfying $\Psi'_{\mu_1,\mu_2}(t) = P_{\mu_1,\mu_2}(t * u, t * v)$, that is

$$\mathcal{P}_{\mu_1,\mu_2} = \{(u, v) \in S : \Psi'_{\mu_1,\mu_2}(0) = 0\}.$$

We decompose $\mathcal{P}_{\mu_1,\mu_2}$ into 3 disjoint unions $\mathcal{P}_{\mu_1,\mu_2} = \mathcal{P}^+_{\mu_1,\mu_2} \cup \mathcal{P}^0_{\mu_1,\mu_2} \cup \mathcal{P}^-_{\mu_1,\mu_2}$, defined by

$$\begin{aligned}
\mathcal{P}^+_{\mu_1,\mu_2} &:= \{u \in \mathcal{P}_{\mu_1,\mu_2} : \Psi''_{\mu_1,\mu_2}(0) > 0\};\\
\mathcal{P}^0_{\mu_1,\mu_2} &:= \{u \in \mathcal{P}_{\mu_1,\mu_2} : \Psi''_{\mu_1,\mu_2}(0) = 0\};\\
\mathcal{P}^-_{\mu_1,\mu_2} &:= \{u \in \mathcal{P}_{\mu_1,\mu_2} : \Psi''_{\mu_1,\mu_2}(0) < 0\}.
\end{aligned}$$

Set $m(a, b) = \inf_{\mathcal{P}_{\mu_1,\mu_2}} J(u, v)$ and $m^\pm(a, b) = \inf_{(u,v)\in\mathcal{P}^\pm_{\mu_1,\mu_2}} J(u, v)$, respectively. The main idea of this paper is to show whether $m(a, b)$ is achieved.

3 The relevant results

Before solving problem (1.1) and (1.2), we study the following problem:

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu |u|^{p-2} u, & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N} u^2 dx = c^2, & u \in H^s(\mathbb{R}^N), \end{cases} \tag{3.1}$$

where $\mu, c > 0$, $p \in (2, 2^*_s) \setminus \{\bar{p}\}$. The standard method obtaining the normalized solutions of (3.1) is to search for the critical points of

$$I_{\mu,c}(u) = \frac{1}{2}[u]^2 - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p dx,$$

constrained on $S_c := \{u \in H^s(\mathbb{R}^N) : |u|_2^2 = c^2\}$. By the same arguments as in Section 2, the Pohožaev identity related to (3.1) is

$$P_{\mu,c}(u) = s[u]^2 - \mu\gamma_p s |u|_p^p,$$

and the corresponding Pohožaev manifold is

$$\mathcal{P}_{\mu,c} := \{u \in S_c : [u]^2 = \mu\gamma_p |u|_p^p\}.$$

Moreover, we have

$$\Psi_{\mu,c}(t) := I_{\mu,c}(t * u) = \frac{e^{2st}}{2}[u]^2 - \frac{\mu e^{p\gamma_p st}}{p} \int_{\mathbb{R}^N} |u|^p dx,$$

and $\mathcal{P}_{\mu,c}$ can also be divided into 3 disjoint unions $\mathcal{P}_{\mu,c} = \mathcal{P}^+_{\mu,c} \cup \mathcal{P}^0_{\mu,c} \cup \mathcal{P}^-_{\mu,c}$, where

$$\begin{aligned}
\mathcal{P}^+_{\mu,c} &:= \{u \in \mathcal{P}_{\mu,c} : \Psi''_{\mu,c}(0) > 0\};\\
\mathcal{P}^0_{\mu,c} &:= \{u \in \mathcal{P}_{\mu,c} : \Psi''_{\mu,c}(0) = 0\};\\
\mathcal{P}^-_{\mu,c} &:= \{u \in \mathcal{P}_{\mu,c} : \Psi''_{\mu,c}(0) < 0\}.
\end{aligned}$$

Define $m_\mu(c) = \inf_{u\in\mathcal{P}_{\mu,c}} I_{\mu,c}(u)$ and let $m(a, 0) = m_{\mu_1}(a)$, $m(0, b) = m_{\mu_2}(b)$. From Lemma 2.3, for any $u \in S_a$, there is $C_1 := C_1(N, p, a, s) > 0$, such that

$$\int_{\mathbb{R}^N} |u|^p dx \le C(N, p, s) |u|_2^{p(1-\gamma_p)} [u]^{p\gamma_p} = C_1 [u]^{p\gamma_p} \le C_1([u]^2 + [v]^2)^{\frac{p\gamma_p}{2}}. \tag{3.2}$$

In particular, when $p < \bar{p}$, from (3.2) we get

$$I_{\mu_1,a}(u) = \frac{1}{2}[u]^2 - \frac{\mu_1}{p}\int_{\mathbb{R}^N} |u|^p dx \geq \frac{1}{2}[u]^2 - \frac{C_1\mu_1}{p}[u]^{p\gamma_p} =: h([u]),$$

where

$$h(\rho) := \frac{1}{2}\rho^2 - \frac{C_1\mu_1}{p}\rho^{p\gamma_p}. \tag{3.3}$$

Setting

$$\rho_* := (C_1\mu_1\gamma_p)^{\frac{1}{2-p\gamma_p}},$$

we have that $h(\rho_*) < 0$, $h(\rho)$ is strictly decreasing in $(0, \rho_*)$, and is strictly increasing in (ρ_*, ∞). If we denote $R_0 = (\frac{2C_1\mu_1}{p})^{\frac{1}{2-p\gamma_p}}$, then $h(R_0) = 0$ and $h(\rho) < 0$ iff $\rho \in (0, R_0)$.

From [44], we have the following already known results. For the mass subcritical case:

Theorem 3.1 ([44, Theorem 1.1])**.** *When $2s < N \leq 4s$, $p \in (2, \bar{p})$ and $\mu, c > 0$ in (3.1), there is $\hat{\mu} > 0$, for any $\mu \in (0, \hat{\mu})$, then $I_{\mu,c}|_{S_c}$ has a ground state solution u_μ for some $\lambda < 0$. Moreover,*

$$m_\mu(c) = \inf_{u\in S_c} I_{\mu,c}(u) = I_{\mu,c}(u_\mu) < 0,$$

and u_μ is an interior local minimizer of $I_{\mu,c}$ on the set

$$\hat{B}_{R_0} := \{u \in S_c : [u] < R_0\}.$$

Besides, any other normalized ground state solution is a minimizer of $I_{\mu,c}$ on B_{R_0}.

Remark 3.2. We set $\hat{\mu}_1, \hat{\mu}_2$ to obtain Theorem 3.1, under $\mu = \mu_1, c = a$ and $\mu = \mu_2, c = b$ in (3.1), respectively.

For the mass supercritical case:

Theorem 3.3 ([44, Theorem 1.3])**.** *When $2s < N \leq 4s$, $p \in (\bar{p}, 2_s^*)$ and $\mu, c > 0$ in (3.1), then $I_{\mu,c}|_{S_c}$ has a ground state solution u_μ for some $\lambda < 0$. Moreover u_μ is a critical point of Mountain Pass type and*

$$m_\mu(c) = \inf_{u\in S_c}\max_{t\in\mathbb{R}} I_{\mu,c}(t * u) = \max_{t\in\mathbb{R}} I_{\mu,c}(t * u_\mu) = I_{\mu,c}(u_\mu) > 0.$$

In order to proceed our proof, we also need the following monotonicity result which is essential for Lemmas 4.6 and 5.7.

Lemma 3.4. *$m_{\mu_1}(a)$ is non-increasing with respect to a, that is*

$$m_{\mu_1}(a) \leq m_{\mu_1}(a_1), \quad \textit{for any } 0 < a_1 \leq a.$$

Proof. We will prove for any $0 < a_1 \leq a$ and an arbitrary $\varepsilon > 0$,

$$m_{\mu_1}(a) \leq m_{\mu_1}(a_1) + \varepsilon.$$

We divide this proof into two cases.

Case 1: $2 < p < \bar{p}$. For this case, from the definition of R_0 in (3.3), we see R_0 is increasing as a is increasing. Hence, by Theorem 3.1 and $a_1 \leq a$, there exists a $\hat{R}_0$ with $\hat{R}_0 < R_0$, such that

$$m_{\mu_1}(a_1) = \inf_{u\in\hat{B}_{\hat{R}_0}} I_{\mu_1}(u).$$

Let $u \in \hat{B}_{\hat{R}_0} \subset \hat{B}_{R_0}$ be such that $I_{\mu_1}(u) \le m_{\mu_1}(a_1) + \frac{\varepsilon}{2}$. Setting $\phi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function satisfies $0 \le \phi \le 1$ and

$$\phi(x) = \begin{cases} 0, & \text{if } |x| \ge 2, \\ 1, & \text{if } |x| \le 1. \end{cases}$$

For $\delta > 0$, defined $u_\delta(x) = u(x)\phi(\delta x)$, we get $u_\delta \to u$ in $H^s(\mathbb{R}^N)$ as $\delta \to 0$. Thus, for $\eta = \frac{\varepsilon}{6} > 0$, there exists $\delta > 0$ such that

$$I_{\mu_1}(u_\delta) \le I_{\mu_1}(u) + \frac{\varepsilon}{6}, \quad \text{and} \quad [u_\delta] < R_0 - \frac{\eta}{R_0}. \tag{3.4}$$

Taking $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfies $\operatorname{supp}(\varphi) \subset O_{1+\frac{3}{\delta}}(0) \setminus O_{\frac{3}{\delta}}(0)$, where $O_m(n)$ means a ball in $\mathbb{R}^N$ with radius m and centered at n. Let

$$w(x) = \frac{(a^2 - |u_\delta|_2^2)^{\frac{1}{2}}}{|\varphi|_2}\varphi,$$

then for $t < 0$,

$$\operatorname{supp}(u_\delta) \cap \operatorname{supp}(t * w) = \emptyset.$$

Therefore, we get $u_\delta + t * w \in S_a$. Moreover, as $t \to -\infty$, we have

$$I_{\mu_1}(t * w) \le \frac{\varepsilon}{6}, \quad \text{and} \quad [t * w] \le \frac{\eta}{R_0}. \tag{3.5}$$

By the Hölder inequality, we obtain

$$\begin{aligned} [u_\delta + t * w]^2 &= \iint_{\mathbb{R}^{2N}} \frac{|(u_\delta + t * w)(x) - (u_\delta + t * w)(y)|^2}{|x-y|^{N+2s}} dxdy \\ &= \iint_{\mathbb{R}^{2N}} \frac{|u_\delta(x) - u_\delta(y)|^2}{|x-y|^{N+2s}} dxdy + \iint_{\mathbb{R}^{2N}} \frac{|(t * w)(x) - (t * w)(y)|^2}{|x-y|^{N+2s}} dxdy \\ &\quad + 2\iint_{\mathbb{R}^{2N}} \frac{(u_\delta(x) - u_\delta(y))((t * w)(x) - (t * w)(y))}{|x-y|^{N+2s}} dxdy \\ &\le [u_\delta]^2 + [t * w]^2 + 2[u_\delta][t * w] \\ &= ([u_\delta] + [t * w])^2, \end{aligned}$$

then $[u_\delta + t * w] < R_0$. Now from Theorem 3.1, $m_{\mu_1}(a) = \inf_{u \in \hat{B}_{R_0}} I_{\mu_1}(u)$, by (3.4)–(3.5), we obtain

$$\begin{aligned} m_{\mu_1}(a) &\le I_{\mu_1}(u_\delta + t * w) \le I_{\mu_1}(u_\delta) + I_{\mu_1}(t * w) + [u_\delta][t * w] \\ &\le m_{\mu_1}(a_1) + \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \le m_{\mu_1}(a_1) + \varepsilon. \end{aligned}$$

Case 2: $\bar{p} < p < 2_s^*$. In this case, $p\gamma_p > 2$, and by the definition of $m_{\mu_1}(a_1)$, there exists $u \in \mathcal{P}_{\mu_1, a_1}$, such that

$$I_{\mu_1}(u) \le m_{\mu_1}(a_1) + \frac{\varepsilon}{2}.$$

From Theorem 3.3, we have u is bounded in $H^s(\mathbb{R}^N)$ and

$$[u]^2 = \mu_1 \gamma_p |u|_p^p.$$

Since (3.2) and $a_1 \le a$, we get

$$[u] \ge \left(\frac{1}{\mu_1 \gamma_p C(N,p,s) a_1^{p(1-\gamma_p)}} \right)^{\frac{1}{p\gamma_p - 2}} \ge \left(\frac{1}{\mu_1 \gamma_p C(N,p,s) a^{p(1-\gamma_p)}} \right)^{\frac{1}{p\gamma_p - 2}}.$$

Hence there are $\hat{C}, \tilde{C} > 0$, which are independent with a_1, such that $[u] \geq \hat{C}$, and $|u|_p^p \geq \tilde{C}$. Later we may assume $\varepsilon < \tilde{C}$. Same definitions as in Case 1, we have $u_\delta \to u$ in $H^s(\mathbb{R}^N)$ as $\delta \to 0$ and from Theorem 3.3, $t_{u_\delta} * u_\delta \to t_u * u$ in $H^s(\mathbb{R}^N)$ as $\delta \to 0$, where t_u means strict maximum point of $\Psi_{\mu_1, a_1}(t)$ and the map $u \to t_u$ is of C^1 class. Then, for fixed $\delta > 0$, there exists $C > 0$ such that

$$I_{\mu_1}(t_{u_\delta} * u_\delta) \leq I_{\mu_1}(u) + \frac{\varepsilon}{6}, \quad [u_\delta] \leq C, \quad \text{and} \quad |u_\delta|_p^p \geq \tilde{C} - \frac{\varepsilon}{2}. \tag{3.6}$$

Choose $\psi \in C_0^\infty(\mathbb{R}^N)$ with $\text{supp}(\psi) \subset O_{1+\frac{3}{\delta}}(0) \setminus O_{\frac{3}{\delta}}(0)$, where $O_m(n)$ means a ball as defined on the previous page. Set

$$\kappa = \frac{(a^2 - |u_\delta|_2^2)^{\frac{1}{2}}}{|\psi|_2} \psi.$$

Then for $\tau < 0$, we have

$$\text{supp}(u_\delta) \cap \text{supp}(\tau * \kappa) = \emptyset.$$

Let $u_\tau := u_\delta + \tau * \kappa \in S_a$, and as $\tau \to -\infty$,

$$\begin{aligned} \int_{\mathbb{R}^N} |u_\tau|^p dx &= \int_{\mathbb{R}^N} |u_\delta|^p dx + \int_{\mathbb{R}^N} |\tau * \kappa|^p dx \\ &= \int_{\mathbb{R}^N} |u_\delta|^p dx + e^{p\gamma_p s\tau} \int_{\mathbb{R}^N} |\kappa|_p^p dx \to |u_\delta|_p^p. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} [u_\tau]^2 &\leq [u_\delta]^2 + [\tau * \kappa]^2 + 2[u_\delta][\tau * \kappa] \\ &= [u_\delta]^2 + e^{2s\tau}[\kappa]^2 + 2e^{s\tau}[u_\delta][\kappa] \to [u_\delta]^2. \end{aligned}$$

From Theorem 3.3, there exists t_τ such that $P_{\mu_1, a}(t_\tau * u_\tau) = 0$, i.e.

$$\frac{1}{e^{(p\gamma_p - 2)st_\tau}} [u_\tau]^2 = \gamma_p \mu_1 |u_\tau|_p^p.$$

Then as $\tau \to -\infty$,

$$e^{(p\gamma_p - 2)st_\tau} = \frac{[u_\tau]^2}{\gamma_p \mu_1 |u_\tau|_p^p} \leq \frac{[u_\delta]^2}{\gamma_p \mu_1 |u_\delta|_p^p}.$$

Combining with (3.6), we get t_τ is bounded from above as $\tau \to -\infty$. Hence, for $\tau < -1$ sufficiently small, there exists $C^* > 0$ such that

$$[t_\tau * u_\delta] \leq C^*, \quad I_{\mu_1}((t_\tau + \tau) * \kappa) \leq \frac{\varepsilon}{6}, \quad \text{and} \quad [(t_\tau + \tau) * \kappa] < \frac{\varepsilon}{6C^*}. \tag{3.7}$$

Thus from (3.6) and (3.7), we obtain

$$\begin{aligned} m_{\mu_1}(a) &\leq I_{\mu_1}(t_\tau * u_\tau) \leq I_{\mu_1}(t_\tau * u_\delta) + I_{\mu_1}((t_\tau + \tau) * \kappa) + [t_\tau * u_\delta][(t_\tau + \tau) * \kappa] \\ &\leq I_{\mu_1}(t_{u_\delta} * u_\delta) + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \\ &\leq I_{\mu_1}(u) + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \leq m_{\mu_1}(a_1) + \varepsilon. \end{aligned}$$

Then, we complete this proof. □

4 The case: $2 < p < \bar{p}$, $p < (r_1 + r_2) < 2_s^*$

In this section, we consider the mixed exponent case. For any $(u, v) \in S$, from (2.2), the Hölder inequality and (2.4), there are $C_2 = C_2(N, p, b, s) > 0$, $C_3 = C_3(N, (r_1 + r_2), s, a, b)$ and $C_4 = (S^*)^{-2^*_{\alpha,s}}$, such that

$$\int_{\mathbb{R}^N} |v|^p dx \leq C(N, p, s)|v|_2^{p(1-\gamma_p)}[v]^{p\gamma_p} = C_2[v]^{p\gamma_p} \leq C_2([v]^2 + [u]^2)^{\frac{p\gamma_p}{2}}, \tag{4.1}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}dx &\leq |u|^{r_1}_{(r_1+r_2)}|v|^{r_2}_{(r_1+r_2)} \\ &\leq C(N, (r_1 + r_2), s)|u|_2^{r_1(1-\gamma_{(r_1+r_2)})}[u]^{r_1\gamma_{(r_1+r_2)}}|v|_2^{r_2(1-\gamma_{(r_1+r_2)})}[v]^{r_2\gamma_{(r_1+r_2)}} \\ &\leq C_3([u]^2 + [v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}},\end{aligned} \tag{4.2}$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}dx \leq (S^*)^{-2^*_{\alpha,s}}([u]^2 + [v]^2)^{2^*_{\alpha,s}} = C_4([u]^2 + [v]^2)^{2^*_{\alpha,s}}. \tag{4.3}$$

Hence, substituting (3.2), (4.1)–(4.3) into $J(u, v)$, we obtain

$$\begin{aligned}J(u, v) \geq{}& \frac{1}{2}([u]^2 + [v]^2) - \frac{\mu_1 C_1 + \mu_2 C_2}{p}([u]^2 + [v]^2)^{\frac{p\gamma_p}{2}} - \beta C_3([u]^2 + [v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}} \\ &- \frac{C_4}{2^*_{\alpha,s}}([u]^2 + [v]^2)^{2^*_{\alpha,s}}.\end{aligned} \tag{4.4}$$

Then we introduce the function $k : \mathbb{R}^+ \to \mathbb{R}$ by

$$k(t) := \frac{1}{2}t^2 - \frac{\mu_1 C_1 + \mu_2 C_2}{p}t^{p\gamma_p} - \beta C_3 t^{(r_1+r_2)\gamma_{(r_1+r_2)}} - \frac{C_4}{2^*_{\alpha,s}}t^{22^*_{\alpha,s}}, \tag{4.5}$$

and $k(0^+) = 0^-$, and $k(+\infty) = -\infty$.

Lemma 4.1. *There exists $\beta_* > 0$, such that for any $\beta \in (0, \beta_*)$, there exist $\mu_{1,*} = \mu_{1,*}(\beta) > 0$ and $\mu_{2,*} = \mu_{2,*}(\beta) > 0$, for any $\mu_1 \in (0, \mu_{1,*})$, $\mu_2 \in (0, \mu_{2,*})$, the function $k(t)$ has exactly two critical points, one is a local strict minimum at a negative level, and the other one is a global maximum at a positive level. Further, there exist $0 < R_2 < R_3$ such that $k(R_2) = k(R_3) = 0$, $k(t) > 0$ if and only if $t \in (R_2, R_3)$.*

Proof. Since the monotonicity of $k(t)$ will be strongly affected by the comparison of p and $r_1 + r_2$, we may divide this proof into 3 different situations.

Case 1: $2 < p < (r_1 + r_2) < \bar{p}$. In this case, we have $p\gamma_p < (r_1 + r_2)\gamma_{(r_1+r_2)} < 2$ and

$$k'(t) = t^{p\gamma_p-1}[t^{2-p\gamma_p} - C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}t^{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p} - \gamma_p(\mu_1 C_1 + \mu_2 C_2)].$$

Denote

$$\tilde{k}(t) := t^{2-p\gamma_p} - C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}t^{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p},$$

then

$$\begin{aligned}\tilde{k}'(t) = t^{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p-1}[&(2 - p\gamma_p)t^{2-(r_1+r_2)\gamma_{(r_1+r_2)}} - 2C_4(22^*_{\alpha,s} - p\gamma_p)t^{22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}} \\ &- C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}((r_1 + r_2)\gamma_{(r_1+r_2)} - p\gamma_p)].\end{aligned}$$

Let

$$\hat{k}(t) := (2-p\gamma_p)t^{2-(r_1+r_2)\gamma_{(r_1+r_2)}} - 2C_4(22^*_{\alpha,s}-p\gamma_p)t^{22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}}, \tag{4.6}$$

then

$$\begin{aligned}\hat{k}'(t) = t^{1-(r_1+r_2)\gamma_{(r_1+r_2)}}[&(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)}) \\ &- 2C_4(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})t^{22^*_{\alpha,s}-2}].\end{aligned}$$

We see from the definition of $\hat{k}'(t)$ that $\hat{k}(t)$ has a unique critical point t_0 in $(0,+\infty)$ satisfying

$$t_0^{22^*_{\alpha,s}-2} = \frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}.$$

Moreover, since $p\gamma_p < (r_1+r_2)\gamma_{(r_1+r_2)} < 2$, we have $\tilde{k}(+\infty) = -\infty$, $\tilde{k}(0^+) = 0^-$. If

$$\begin{aligned}&\hat{k}(t_0) > C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p];\\ &\tilde{k}(t_0) > \gamma_p(\mu_1C_1+\mu_2C_2), \quad \text{and} \quad k(t_0) > 0,\end{aligned} \tag{4.7}$$

i.e.

$$\begin{cases}\left[\frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{2-(r_1+r_2)\gamma_{(r_1+r_2)}}{22^*_{\alpha,s}-2}} \frac{(2-p\gamma_p)(22^*_{\alpha,s}-2)}{22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}} \\ \quad > C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}((r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p); \\ \left[\frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{p\gamma_p}{22^*_{\alpha,s}-2}}\left[1-\frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right] \\ \quad > \gamma_p(\mu_1C_1+\mu_2C_2) + C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)} \\ \quad\quad \times \left[\frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p}{22^*_{\alpha,s}-2}}; \\ \left[\frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{22^*_{\alpha,s}}{22^*_{\alpha,s}-2}} C_4\left[\frac{(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_p)}{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})} - \frac{1}{2^*_{\alpha,s}}\right] \\ \quad > \frac{\mu_1C_1+\mu_2C_2}{p} \times \left[\frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{p\gamma_p}{22^*_{\alpha,s}-2}} \\ \quad\quad + \beta C_3\left[\frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{22^*_{\alpha,s}-2}},\end{cases} \tag{4.8}$$

then the function $k(t)$ has exactly two critical points, one is a local minimum at a negative level, the other one is a global maximum at a positive level. Therefore, there exist R_2, R_3 with $0 < R_2 < R_3$ such that $k(R_2) = k(R_3) = 0$, $k(t) > 0$ if and only if $t \in (R_2, R_3)$.

Case 2: $2 < p < r_1 + r_2 = \bar{p}$. This implies $p\gamma_p < (r_1+r_2)\gamma_{(r_1+r_2)} = 2$. We choose β such that $C_3\beta < \frac{1}{2}$ and $k(t)$ turns to be

$$k(t) = \left(\frac{1}{2} - C_3\beta\right)t^2 - \frac{\mu_1C_1+\mu_2C_2}{p}t^{p\gamma_p} - \frac{C_4}{2^*_{\alpha,s}}t^{22^*_{\alpha,s}}.$$

Taking a similar argument as in *Case 1*, first we have

$$k'(t) = t^{p\gamma_p-1}[(1-2C_3\beta)t^{2-p\gamma_p} - 2C_4t^{22^*_{\alpha,s}-p\gamma_p} - \gamma_p(\mu_1C_1+\mu_2C_2)].$$

Denote

$$\tilde{k}(t) = (1-2C_3\beta)t^{2-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p},$$

and

$$\tilde{k}'(t) = t^{1-p\gamma_p}[(1-2C_3\beta)(2-p\gamma_p) - 2C_4(22^*_{\alpha,s}-p\gamma_p)t^{22^*_{\alpha,s}-2}].$$

Thus there exists $t_1 \in (0,+\infty)$ satisfying

$$t_1^{22^*_{\alpha,s}-2} = \frac{(1-2C_3\beta)(2-p\gamma_p)}{2C_4(22^*_{\alpha,s}-p\gamma_p)},$$

and if

$$\tilde{k}(t_1) > \gamma_p(\mu_1 C_1 + \mu_2 C_2), \quad \text{and} \quad k(t_1) > 0, \tag{4.9}$$

that is

$$\begin{cases} \left(\dfrac{2-p\gamma_p}{2C_4}\right)^{\frac{(2-p\gamma_p)}{22^*_{\alpha,s}-2}} (22^*_{\alpha,s}-2)(22^*_{\alpha,s}-p\gamma_p)^{\frac{p\gamma_p-22^*_{\alpha,s}}{22^*_{\alpha,s}-2}} > \gamma_p(\mu_1C_1+\mu_2C_2)(1-2C_3\beta)^{\frac{p\gamma_p-22^*_{\alpha,s}}{22^*_{\alpha,s}-2}}; \\ \left(\dfrac{22^*_{\alpha,s}-p\gamma_p}{2-p\gamma_p} - \dfrac{1}{2^*_{\alpha,s}}\right)C_4 > \dfrac{\mu_1C_1+\mu_2C_2}{p}\left[\dfrac{(1-2C_3\beta)(2-p\gamma_p)}{2C_4(22^*_{\alpha,s}-p\gamma_p)}\right]^{\frac{p\gamma_p-22^*_{\alpha,s}}{22^*_{\alpha,s}-2}}, \end{cases} \tag{4.10}$$

then we get the same conclusions as *Case 1*.

Case 3: $2 < p < \bar{p} < r_1 + r_2 < 2^*_s$. In this case, $p\gamma_p < 2 < (r_1+r_2)\gamma_{(r_1+r_2)}$. Similarily, we have

$$\tilde{k}(t) := t^{2-p\gamma_p} - C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)} t^{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p},$$

and

$$\begin{aligned}\tilde{k}'(t) = t^{1-p\gamma_p}[&(2-p\gamma_p) - C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)t^{(r_1+r_2)\gamma_{(r_1+r_2)}-2} \\ &- 2C_4(22^*_{\alpha,s}-p\gamma_p)t^{22^*_{\alpha,s}-2}].\end{aligned}$$

Therefore, $\tilde{k}(t)$ has a unique critical point $t_2 \in (0,+\infty)$. If

$$\tilde{k}(t_2) > \gamma_p(\mu_1C_1+\mu_2C_2), \quad \text{and} \quad k(t_2) > 0, \tag{4.11}$$

we obtain the same conclusions as *Case 1*. Following, we get an estimate at t_2. Let

$$t_* = \left[\frac{(\mu_1C_1+\mu_2C_2)d((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)}{2-p\gamma_p}\right]^{\frac{1}{2-p\gamma_p}},$$

where d will be fixed later. If $t_2 > t_*$ and $d > \frac{\gamma_p(2-p\gamma_p)}{(r_1+r_2)\gamma_{(r_1+r_2)}-2}$, we get

$$\begin{aligned}&(\mu_1C_1+\mu_2C_2)\gamma_p t_2^{p\gamma_p} + C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)} t_2^{(r_1+r_2)\gamma_{(r_1+r_2)}} + 2C_4 t_2^{22^*_{\alpha,s}} \\ &\le (\mu_1C_1+\mu_2C_2)\gamma_p t_*^{p\gamma_p-2} t_2^2 + \frac{2-p\gamma_p}{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p} t_2^2 < t_2^2,\end{aligned}$$

and if $d > \frac{2(r_1+r_2)\gamma_{(r_1+r_2)}}{p[(r_1+r_2)\gamma_{(r_1+r_2)}-2]}$,

$$\begin{aligned}&\frac{\mu_1C_1+\mu_2C_2}{p} t_2^{p\gamma_p} + C_3\beta t_2^{(r_1+r_2)\gamma_{(r_1+r_2)}} + \frac{C_4}{2^*_{\alpha,s}} t_2^{22^*_{\alpha,s}} \\ &\le \frac{2-p\gamma_p}{(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]} t_2^2 + \frac{\mu_1C_1+\mu_2C_2}{p} t_*^{p\gamma_p-2} t_2^2 < \frac{1}{2} t_2^2.\end{aligned}$$

Therefore, if we choose $d > \frac{2(r_1+r_2)\gamma_{(r_1+r_2)}}{p((r_1+r_2)\gamma_{(r_1+r_2)}-2)}$, we get (4.11). Hence we only need $t_2 > t_*$. By the definition of t_2, we need

$$(2-p\gamma_p) > C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}((r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p)t_*^{(r_1+r_2)\gamma_{(r_1+r_2)}-2} + 2C_4(22^*_{\alpha,s} - p\gamma_p)t_*^{22^*_{\alpha,s}-2},$$

that is,

$$\begin{aligned}(2-p\gamma_p) > & \left[\frac{(\mu_1 C_1 + \mu_2 C_2)d((r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p)}{2-p\gamma_p}\right]^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}-2}{2-p\gamma_p}} \\ & \times \Bigg[C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}((r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p) \\ & + 2C_4(22^*_{\alpha,s} - p\gamma_p)\left(\frac{(\mu_1 C_1 + \mu_2 C_2)d((r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p)}{2-p\gamma_p}\right)^{\frac{22^*_{\alpha,s}-2}{2-p\gamma_p}}\Bigg].\end{aligned} \tag{4.12}$$

To sum up, there exists $\beta_* > 0$, such that for any $\beta \in (0, \beta_*)$, there exist $\mu_{1,*} = \mu_{1,*}(\beta) > 0$ and $\mu_{2,*} = \mu_{2,*}(\beta) > 0$, for any $\mu_1 \in (0, \mu_{1,*})$, $\mu_2 \in (0, \mu_{2,*})$, then (4.8), (4.10) and (4.12) are satisfied. We complete this lemma. □

We now study the structure of Pohožaev manifold. Recalling the decomposition of $\mathcal{P}_{\mu_1,\mu_2} = \mathcal{P}^+_{\mu_1,\mu_2} \cup \mathcal{P}^0_{\mu_1,\mu_2} \cup \mathcal{P}^-_{\mu_1,\mu_2}$, we have:

Lemma 4.2. *There exists $\tilde{\beta}_* > 0$, such that for any $\beta \in (0, \tilde{\beta}_*)$, there exist $\tilde{\mu}_{1,*} = \tilde{\mu}_{1,*}(\beta) > 0$ and $\tilde{\mu}_{2,*} = \tilde{\mu}_{2,*}(\beta) > 0$, for every $\mu_1 \in (0, \tilde{\mu}_{1,*})$, $\mu_2 \in (0, \tilde{\mu}_{2,*})$, then $\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$ and $\mathcal{P}_{\mu_1,\mu_2}$ is a C^1-submanifold in H with codimension 3.*

Proof. Firstly, assume by contradiction that there exists a $(u, v) \in \mathcal{P}^0_{\mu_1,\mu_2}$ satisfying

$$\begin{aligned}([u]^2 + [v]^2) = & \gamma_p(\mu_1|u|_p^p + \mu_2|v|_p^p) + \beta(r_1+r_2)\gamma_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx \\ & + 2\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}2([u]^2 + [v]^2) = & p\gamma_p^2(\mu_1|u|_p^p + \mu_2|v|_p^p) + \beta(r_1+r_2)^2\gamma^2_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx \\ & + 42^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.\end{aligned} \tag{4.14}$$

Following we define

$$\begin{aligned}\hbar(\rho) := & \rho\Psi'_{\mu_1,\mu_2}(0) - \Psi''_{\mu_1,\mu_2}(0) \\ = & (\rho-2)([u]^2 + [v]^2) - \gamma_p(\rho - p\gamma_p)(\mu_1|u|_p^p + \mu_2|v|_p^p) \\ & - 2(\rho - 22^*_{\alpha,s})\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx \\ & - \beta(r_1+r_2)\gamma_{(r_1+r_2)}(\rho - (r_1+r_2)\gamma_{(r_1+r_2)})\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx = 0,\end{aligned} \tag{4.15}$$

and let

$$\eta := ([u]^2 + [v]^2)^{\frac{1}{2}}.$$

Case 1: When $p < r_1 + r_2 < \bar{p}$, we have $p\gamma_p < (r_1+r_2)\gamma_{(r_1+r_2)} < 2$. From (4.15) and (4.3), we have $\hbar((r_1+r_2)\gamma_{(r_1+r_2)}) = 0$ and

$$\begin{aligned} \left[2-(r_1+r_2)\gamma_{(r_1+r_2)}\right]\eta^2 &\le 2\left[22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}\right]\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx \\ &\le 2C_4\left[22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}\right]\eta^{22^*_{\alpha,s}}. \end{aligned} \tag{4.16}$$

It follows $\eta \ge \left[\frac{2-(r_1+r_2)\gamma_{(r_1+r_2)}}{2C_4(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{1}{22^*_{\alpha,s}-2}}$. Moreover, by $\hbar(22^*_{\alpha,s}) = 0$, from (3.2), (4.1) and (4.2),we obtain

$$\begin{aligned} (22^*_{\alpha,s}-2)\eta^2 =& \gamma_p(22^*_{\alpha,s}-p\gamma_p)(\mu_1|u|_p^p+\mu_2|v|_p^p) \\ &+\beta(r_1+r_2)\gamma_{(r_1+r_2)}\left[22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}\right]\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx \\ \le& \gamma_p(22^*_{\alpha,s}-p\gamma_p)(\mu_1C_1+\mu_2C_2)t^{p\gamma_p} \\ &+C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}\left[22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}\right]t^{(r_1+r_2)\gamma_{(r_1+r_2)}}, \end{aligned}$$

that is

$$\begin{aligned} 22^*_{\alpha,s}-2 \le& \gamma_p(22^*_{\alpha,s}-p\gamma_p)(\mu_1C_1+\mu_2C_2)\left[\frac{2-(r_1+r_2)\gamma_{(r_1+r_2)}}{2C_4(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{p\gamma_p-2}{22^*_{\alpha,s}-2}} \\ &+C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}\left[\frac{2-(r_1+r_2)\gamma_{(r_1+r_2)}}{2C_4(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}-2}{22^*_{\alpha,s}-2}}. \end{aligned} \tag{4.17}$$

Hence, we can choose $\tilde{\beta}_* > 0$, such that for any $\beta \in (0, \tilde{\beta}_*)$, there exist $\tilde{\mu}_{1,*} = \tilde{\mu}_{1,*}(\beta) > 0$ and $\tilde{\mu}_{2,*} = \tilde{\mu}_{2,*}(\beta) > 0$, for every $\mu_1 \in (0, \tilde{\mu}_{1,*})$, $\mu_2 \in (0, \tilde{\mu}_{2,*})$, such that (4.17) can not happen. Therefore, $\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$.

Case 2: As in $p < r_1 + r_2 = \bar{p}$, we get $p\gamma_p < (r_1+r_2)\gamma_{(r_1+r_2)} = 2$. Similarly as in *Case 1*, from (4.1)–(4.3) and (4.15), we have $\hbar(p\gamma_p) = 0$, i.e.

$$\begin{aligned} (2-p\gamma_p)t^2 =& \beta(r_1+r_2)\gamma_{(r_1+r_2)}\left[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p\right]\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx \\ &+2(22^*_{\alpha,s}-p\gamma_p)\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx \\ \le& 2C_3\beta(2-p\gamma_p)\eta^2+2C_4(22^*_{\alpha,s}-p\gamma_p)\eta^{22^*_{\alpha,s}}. \end{aligned} \tag{4.18}$$

From $\hbar(22^*_{\alpha,s}) = 0$ we get

$$(22^*_{\alpha,s}-2)\eta^2 \le \gamma_p(\mu_1C_1+\mu_2C_2)(22^*_{\alpha,s}-p\gamma_p)\eta^{p\gamma_p}+2C_3\beta(22^*_{\alpha,s}-2)\eta^2. \tag{4.19}$$

Combining with (4.18), we first suppose $1-2C_3\beta > 0$ and then

$$\left[\frac{(1-2C_3\beta)(2-p\gamma_p)}{2C_4(22^*_{\alpha,s}-p\gamma_p)}\right]^{\frac{1}{22^*_{\alpha,s}-2}} \le \left[\frac{\gamma_p(\mu_1C_1+\mu_2C_2)(22^*_{\alpha,s}-p\gamma_p)}{(22^*_{\alpha,s}-2)(1-2C_3\beta)}\right]^{\frac{1}{2-p\gamma_p}},$$

that is

$$\left(\frac{2-p\gamma_p}{2C_4}\right)^{2-p\gamma_p}\left(\frac{22^*_{\alpha,s}-2}{\gamma_p}\right)^{22^*_{\alpha,s}-2}\left(\frac{1}{22^*_{\alpha,s}-p\gamma_p}\right)^{22^*_{\alpha,s}-p\gamma_p}$$
$$\leq(\mu_1C_1+\mu_2C_2)^{22^*_{\alpha,s}-2}\left(\frac{1}{1-2C_3\beta}\right)^{22^*_{\alpha,s}-p\gamma_p}.$$

Similar argument as in Case 1, choose appropriate $\tilde{\beta}_*, \tilde{\mu}_{1,*}=\tilde{\mu}_{1,*}(\beta), \tilde{\mu}_{2,*}=\tilde{\mu}_{2,*}(\beta)$, such that the last inequality may not happen. Therefore $\mathcal{P}^0_{\mu_1,\mu_2}=\emptyset$.

Case 3: If $p<\bar{p}<r_1+r_2$, then $p\gamma_p<2<(r_1+r_2)\gamma_{(r_1+r_2)}$. Also by (3.2), (4.1)–(4.3) and (4.15), since $\hbar(p\gamma_p)=0$ we have

$$(2-p\gamma_p)\eta^2\leq C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}\left[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p\right]\eta^{(r_1+r_2)\gamma_{(r_1+r_2)}} + 2C_4(22^*_{\alpha,s}-p\gamma_p)\eta^{22^*_{\alpha,s}}. \tag{4.20}$$

By the definition of t_2 and t_* in Lemma 4.1, we need

$$\eta\geq t_2>t_*:=\left[\frac{(\mu_1C_1+\mu_2C_2)d((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)}{2-p\gamma_p}\right]^{\frac{1}{2-p\gamma_p}}.$$

Besides, from $\hbar((r_1+r_2)\gamma_{(r_1+r_2)})=0$ we have

$$((r_1+r_2)\gamma_{(r_1+r_2)}-2)\eta^2\leq\gamma_p(\mu_1C_1+\mu_2C_2)\left[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p\right]\eta^{p\gamma_p}. \tag{4.21}$$

i.e.

$$\eta\leq\left[\frac{\gamma_p(\mu_1C_1+\mu_2C_2)((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)}{((r_1+r_2)\gamma_{(r_1+r_2)}-2)}\right]^{\frac{1}{2-p\gamma_p}}.$$

This is a contradiction with $d>\frac{2(r_1+r_2)\gamma_{(r_1+r_2)}}{p((r_1+r_2)\gamma_{(r_1+r_2)}-2)}$ in Lemma 4.1. Hence, we can fix $\tilde{\beta}_*=\beta_*$, $\tilde{\mu}_{1,*}:=\tilde{\mu}_{1,*}(\beta)=\mu_{1,*}$ and $\tilde{\mu}_{2,*}:=\tilde{\mu}_{2,*}(\beta)=\mu_{2,*}$, to make sure $t_2>t_*$ and $\mathcal{P}^0_{\mu_1,\mu_2}=\emptyset$, where $\beta_*, \mu_{1,*}$ and $\mu_{2,*}$ are from Lemma 4.1.

Following, we prove $\mathcal{P}_{\mu_1,\mu_2}$ is a C^1-submanifold in H with codimension 3. For any $(u,v)\in\mathcal{P}_{\mu_1,\mu_2}$, we have $P_{\mu_1,\mu_2}(u,v)=0$, $G(u)=0$ and $F(v)=0$, where

$$G(u):=\int_{\mathbb{R}^N}u^2-a^2dx,\quad\text{and}\quad F(v):=\int_{\mathbb{R}^N}v^2-b^2dx.$$

Then we need to prove

$$d(P_{\mu_1,\mu_2}(u,v),G(u),F(v)):H\mapsto\mathbb{R}^3\quad\text{is surjective.}$$

If not, there exist $\nu_1,\nu_2\in\mathbb{R}$, for every $(\varphi,0)$ and $(0,\psi)$ in H such that

$$2s\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi dx=sp\gamma_p\int_{\mathbb{R}^N}\mu_1|u|^{p-2}u\varphi dx+s\beta(r_1+r_2)\gamma_{(r_1+r_2)}r_1\int_{\mathbb{R}^N}|u|^{r_1-2}u\varphi dx + 2s2^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha*|v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u\varphi dx+2\nu_1\int_{\mathbb{R}^N}u\varphi dx;$$

and

$$2s\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}v(-\Delta)^{\frac{s}{2}}\psi dx=sp\gamma_p\int_{\mathbb{R}^N}\mu_2|v|^{p-2}v\psi dx+s\beta(r_1+r_2)\gamma_{(r_1+r_2)}r_2\int_{\mathbb{R}^N}|v|^{r_2-2}v\psi dx + 2s2^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v\psi dx+2\nu_2\int_{\mathbb{R}^N}v\psi dx.$$

From which (u,v) is a weak solution of the system in $\mathbb{R}^N$

$$\begin{cases} 2s(-\Delta)^s u = 2\nu_1 u + sp\gamma_p\mu_1|u|^{p-2}u + s\beta(r_1+r_2)\gamma_{(r_1+r_2)}r_1|u|^{r_1-2}u|v|^{r_2} \\ \qquad\qquad + 2s2^*_{\alpha,s}(I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, \\ 2s(-\Delta)^s v = 2\nu_2 v + sp\gamma_p\mu_2|v|^{p-2}v + s\beta(r_1+r_2)\gamma_{(r_1+r_2)}r_2|v|^{r_2-2}|u|^{r_1} \\ \qquad\qquad + 2s2^*_{\alpha,s}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v, \\ \int_{\mathbb{R}^N}|u|^2dx = a^2, \quad \int_{\mathbb{R}^N}|v|^2dx = b^2. \end{cases}$$

The related Pohožaev identity of the above system is

$$2([u]^2+[v]^2) = p\gamma_p^2(\mu_1|u|_p^p + \mu_2|v|_p^p) + \beta(r_1+r_2)^2\gamma^2_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx$$
$$+ 42^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,$$

thus $P^0_{\mu_1,\mu_2}(u,v) = 0$, which contradicts with $\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$. We complete this lemma. □

From Lemmas 4.1 and 4.2, we can have the geometry of Ψ_{μ_1,μ_2}.

Lemma 4.3. *For every $(u,v) \in S$, the function $\Psi_{\mu_1,\mu_2}(t)$ has exactly two critical points $s_{u,v} < t_{u,v} \in \mathbb{R}$ and two zeros $c_{u,v} < d_{u,v} \in \mathbb{R}$ with $s_{u,v} < c_{u,v} < t_{u,v} < d_{u,v}$. Moreover,*

(i) $s_{u,v} * (u,v) \in \mathcal{P}^+_{\mu_1,\mu_2}$ and $t_{u,v} * (u,v) \in \mathcal{P}^-_{\mu_1,\mu_2}$, and if $t * (u,v) \in \mathcal{P}_{\mu_1,\mu_2}$, then either $t = s_{u,v}$ or $t = t_{u,v}$.

(ii) $([t*u]^2 + [t*v]^2)^{\frac{1}{2}} \le R_2$ (R_2 is from Lemma 4.1) for every $t \le c_{u,v}$ and

$$J(s_{u,v} * (u,v)) = \min\{J(t*(u,v)) : t \in \mathbb{R} \quad and \quad ([t*u]^2+[t*v]^2)^{\frac{1}{2}} \le R_2\}.$$

(iii) We get $J(t_{u,v} * (u,v)) = \max\{J(t*(u,v)) : t \in \mathbb{R}\} > 0$ and $\Psi_{\mu_1,\mu_2}(t)$ is strictly decreasing and concave on $(t_{u,v}, \infty)$. In particular, if $t_{u,v} < 0$, then $P_{\mu_1,\mu_2}(u,v) < 0$.

(iv) The maps $(u,v) \mapsto s_{u,v}$, and $(u,v) \mapsto t_{u,v}$ for any $(u,v) \in S$ are of class C^1.

Proof. Let $(u,v) \in S$, then $t*(u,v) \in \mathcal{P}_{\mu_1,\mu_2}$ if and only if $\Psi'_{\mu_1,\mu_2}(t) = 0$. By (4.4)-(4.5),

$$\Psi_{\mu_1,\mu_2}(t) = J(t*(u,v)) \ge k(e^{st}([u]^2+[v]^2)^{\frac{1}{2}}),$$

thus from Lemma 4.1, $\Psi_{\mu_1,\mu_2}(t)$ is positive on

$$\left(s^{-1}\ln\frac{R_2}{([u]^2+[v]^2)^{\frac{1}{2}}}, s^{-1}\ln\frac{R_3}{([u]^2+[v]^2)^{\frac{1}{2}}}\right).$$

Since $p\gamma_p < 2$, we see $\Psi_{\mu_1,\mu_2}(-\infty) = 0^-$ and $\Psi_{\mu_1,\mu_2}(+\infty) = -\infty$. Then $\Psi_{\mu_1,\mu_2}(t)$ has at least two critical points. Therefore, $\Psi_{\mu_1,\mu_2}(t)$ has a local minimum point $s_{u,v}$ at a negative level in $\big(-\infty, s^{-1}\ln\frac{R_2}{([u]^2+[v]^2)^{\frac{1}{2}}}\big)$, and has a global maximum point $t_{u,v}$ at a positive level in $\big(s^{-1}\ln\frac{R_2}{([u]^2+[v]^2)^{\frac{1}{2}}}, s^{-1}\ln\frac{R_3}{([u]^2+[v]^2)^{\frac{1}{2}}}\big)$. We claim $\Psi_{\mu_1,\mu_2}(t)$ has exactly two critical points. Let $\Psi'_{\mu_1,\mu_2}(t) = 0$, namely

$$\Psi'_{\mu_1,\mu_2}(t) = se^{2st}([u]^2+[v]^2) - s\gamma_p e^{sp\gamma_p t}(\mu_1|u|_p^p + \mu_2|v|_p^p) - 2se^{22^*_{\alpha,s}st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx$$
$$- s\beta(r_1+r_2)\gamma_{(r_1+r_2)}e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx. \tag{4.22}$$

Case 1: $2 < p < r_1 + r_2 < \bar{p}$. From (4.22) we have

$$\Psi'_{\mu_1,\mu_2}(t) = e^{sp\gamma_p t}\big[se^{(2-p\gamma_p)st}([u]^2+[v]^2) - s\beta(r_1+r_2)\gamma_{(r_1+r_2)}e^{s[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx$$
$$-2se^{(22^*_{\alpha,s}-p\gamma_p)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx - s\gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p)\big].$$

Denote

$$g_1(t) := se^{(2-p\gamma_p)st}([u]^2+[v]^2) - 2se^{(22^*_{\alpha,s}-p\gamma_p)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx$$
$$- s\beta(r_1+r_2)\gamma_{(r_1+r_2)}e^{s[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx,$$

then

$$g_1'(t) = e^{[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]st}\Big[(2-p\gamma_p)s^2e^{(2-(r_1+r_2)\gamma_{(r_1+r_2)})st}([u]^2+[v]^2)$$
$$-2(22^*_{\alpha,s}-p\gamma_p)s^2e^{(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx$$
$$-[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]s^2(r_1+r_2)\gamma_{(r_1+r_2)}\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx\Big].$$

Now define

$$f_1(t) := (2-p\gamma_p)s^2e^{[2-(r_1+r_2)\gamma_{(r_1+r_2)}]st}([u]^2+[v]^2)$$
$$-2(22^*_{\alpha,s}-p\gamma_p)s^2e^{(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,$$

thus

$$f_1'(t) = e^{(2-(r_1+r_2)\gamma_{(r_1+r_2)})st}\Big[(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})s^3([u]^2+[v]^2)$$
$$-2(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})s^3e^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx\Big].$$

We see $f_1(t)$ has only one critical point $\bar{t}$, which is also a maximum point. Therefore if

$$f_1(\bar{t}) \le [(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]s^2(r_1+r_2)\gamma_{(r_1+r_2)}\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx,$$

we have $g_1'(t) < 0$ and $g_1(t)$ is strictly decreasing in $\mathbb{R}\setminus\{\bar{t}\}$. Since $g_1(-\infty) = 0^-$ and $g_1(+\infty) = -\infty$, we get

$$g_1(t) < 0 < s\gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p),$$

and hence $\Psi'_{\mu_1,\mu_2}(t) < 0$, which means $\Psi_{\mu_1,\mu_2}(t)$ has no critical points. On the other hand, if

$$f_1(\bar{t}) > [(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]s^2(r_1+r_2)\gamma_{(r_1+r_2)}\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx,$$

then by $f_1(-\infty) = 0^+$, $f_1(+\infty) = -\infty$, there exist two constants $\bar{t}_1 < \bar{t} < \bar{t}_2$, such that

$$f_1(\bar{t}_1) = f_1(\bar{t}_2) = [(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]s^2(r_1+r_2)\gamma_{(r_1+r_2)}\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx.$$

Therefore, we find from the definitions of $g_1(t)$ and $\Psi_{\mu_1,\mu_2}(t)$ that

$$g_1(t) = s\gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p),$$

has at most two critical points, which implies $\Psi_{\mu_1,\mu_2}(t)$ has at most two critical points.

Case 2: $2 < p < (r_1 + r_2) = \bar{p}$. In this case, (4.22) becomes

$$\Psi'_{\mu_1,\mu_2}(t) = s\left([u]^2 + [v]^2 - 2\beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}dx\right)e^{2st} - s\gamma_p e^{sp\gamma_p t}(\mu_1|u|_p^p + \mu_2|v|_p^p)$$
$$- 2se^{22^*_{\alpha,s}st} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx. \tag{4.23}$$

If

$$[u]^2 + [v]^2 - 2\beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}dx \leq 0,$$

we see $\Psi'_{\mu_1,\mu_2}(t) < 0$ and $\Psi_{\mu_1,\mu_2}(t)$ has no critical points. Now we suppose

$$[u]^2 + [v]^2 - 2\beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}dx > 0.$$

Then similarly as in *Case 1*, we conclude $\Psi_{\mu_1,\mu_2}(t)$ has at most two critical points.

Case 3: $2 < p < \bar{p} < r_1 + r_2$. From the definition of $g_1(t)$, we have

$$g_1'(t) = e^{(2-p\gamma_p)st}\Big[(2-p\gamma_p)s^2([u]^2+[v]^2) - 2s^2(22^*_{\alpha,s} - p\gamma_p)e^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx$$
$$- s^2\beta(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p]e^{((r_1+r_2)\gamma_{(r_1+r_2)}-2)st}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx\Big]$$
$$=: e^{(2-p\gamma_p)st}[(2-p\gamma_p)s^2([u]^2+[v]^2) - Q(t)],$$

where

$$Q(t) = 2s^2(22^*_{\alpha,s} - p\gamma_p)e^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx$$
$$+ s^2\beta(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p]e^{((r_1+r_2)\gamma_{(r_1+r_2)}-2)st}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx.$$

Moreover by

$$p\gamma_p < 2 < \min\{(r_1+r_2)\gamma_{(r_1+r_2)}, 22^*_{\alpha,s}\},$$

we see $Q(t)$ is strictly increasing in $\mathbb{R}$. Therefore $g_1(t)$ has a unique critical point $\hat{t}$, which is also a maximum point and $g_1(t)$ is strictly increasing in $(-\infty, \hat{t})$, strictly decreasing in $(\hat{t}, +\infty)$. On one hand, if

$$g_1(\hat{t}) \leq s\gamma_p(\mu_1|u|_p^p + \mu_2|v|_p^p),$$

we have $\Psi_{\mu_1,\mu_2}(t)$ has no critical points. Besides, if

$$g_1(\hat{t}) > s\gamma_p(\mu_1|u|_p^p + \mu_2|v|_p^p),$$

then $\Psi'_{\mu_1,\mu_2}(t) = 0$ has at most two solutions, that is $\Psi_{\mu_1,\mu_2}(t)$ has at most two critical points. Hence, $\Psi_{\mu_1,\mu_2}(t)$ has exactly two critical points $s_{u,v} < t_{u,v}$.

By the definitions of $P_{\mu_1,\mu_2}(u,v)$ in (2.6) and $\Psi_{\mu_1,\mu_2}(t)$ in (2.7), we find $\Psi'_{\mu_1,\mu_2}(t) = P_{\mu_1,\mu_2}(t * u, t * v)$. Therefore we know $P_{\mu_1,\mu_2}(t * u, t * v) = 0$ if and only if t is a critical point of $\Psi_{\mu_1,\mu_2}(t)$. From above we find $\Psi_{\mu_1,\mu_2}(t)$ has exactly two critical points $s_{u,v}, t_{u,v}$, then we have $P_{\mu_1,\mu_2}(t * u, t * v) = 0$ if and only if $t = s_{u,v}$ or $t = t_{u,v}$. Moreover from Lemma 2.5 the definition of $\mathcal{P}_{\mu_1,\mu_2}$ here, by $(t * u, t * v) \in S$ we see that $(t * u, t * v) \in \mathcal{P}_{\mu_1,\mu_2}$ if and only if $t = s_{u,v}$ or $t = t_{u,v}$. Noticing $\Psi''_{\mu_1,\mu_2}(s_{u,v}) \geq 0$, $\Psi''_{\mu_1,\mu_2}(t_{u,v}) \leq 0$ and $\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$, we obtain $s_{u,v} * (u,v) \in$

$\mathcal{P}^{+}_{\mu_1,\mu_2}$ and $t_{u,v} * (u,v) \in \mathcal{P}^{-}_{\mu_1,\mu_2}$. By the monotonicity and the behavior of $\Psi_{\mu_1,\mu_2}(t)$, we see $\Psi_{\mu_1,\mu_2}(t)$ has exactly two zeros $c_{u,v} < d_{u,v} \in \mathbb{R}$ with $s_{u,v} < c_{u,v} < t_{u,v} < d_{u,v}$, and $\Psi_{\mu_1,\mu_2}(t)$ has exactly two inflection points. Moreover, $\Psi_{\mu_1,\mu_2}(t)$ is concave on $[t_{u,v}, \infty)$, and if $t_{u,v} < 0$, then $P_{\mu_1,\mu_2}(u,v) = \Psi'_{\mu_1,\mu_2}(0) < 0$. Finally, we apply implicit function theorem on the C^1 function $\Phi(t,u,v) = \Psi'_{\mu_1,\mu_2}(t)$, then $\Phi(s_{u,v},u,v) = \Psi'_{\mu_1,\mu_2}(s_{u,v}) = 0$, $\partial_t\Phi(s_{u,v},u,v) = \Psi''_{\mu_1,\mu_2}(s_{u,v}) > 0$. Therefore we know $(u,v) \to s_{u,v}$ is of class C^1. Similarly, $(u,v) \to t_{u,v}$ is also of class C^1. □

For $r > 0$, define

$$B_r(a,b) := \{(u,v) \in S : ([u]^2 + [v]^2)^{\frac{1}{2}} < r\}, \quad \text{and} \quad \hat{m}(a,b) := \inf_{(u,v)\in B_{R_2}(a,b)} J(u,v).$$

From Lemma 4.3, we can deduce the following conclusion directly.

Corollary 4.4. *The set $\mathcal{P}^{+}_{\mu_1,\mu_2} \subset B_{R_2}(a,b)$, and*

$$\sup_{(u,v)\in\mathcal{P}^{+}_{\mu_1,\mu_2}} J(u,v) \leq 0 \leq \inf_{(u,v)\in\mathcal{P}^{-}_{\mu_1,\mu_2}} J(u,v).$$

Lemma 4.5. *We have $\hat{m}(a,b) \in (-\infty, 0)$, moreover*

$$\hat{m}(a,b) = m(a,b) = m^{+}(a,b) \quad \text{and} \quad \hat{m}(a,b) < \inf_{\overline{B_{R_2}(a,b)\setminus B_{R_2-\delta}(a,b)}} J(u,v),$$

for $\delta > 0$ small enough.

Proof. For any $(u,v) \in B_{R_2}(a,b)$, by (4.4) and (4.5), we get

$$J(u,v) \geq k(([u]^2 + [v]^2)^{\frac{1}{2}}) \geq \min_{t\in[0,R_2]} k(t) > -\infty.$$

Hence $\hat{m}(a,b) > -\infty$. Moreover, for any $(u,v) \in S$, when $t \ll -1$, we have $([t * u]^2 + [t * v]^2)^{\frac{1}{2}} < R_2$ and $J(t * (u,v)) < 0$. Hence $\hat{m}(a,b) < 0$. From Corollary 4.4, $\mathcal{P}^{+}_{\mu_1,\mu_2} \subset B_{R_2}(a,b)$, then $\hat{m}(a,b) \leq m^{+}(a,b)$. On the other hand, for any $(u,v) \in B_{R_2}(a,b)$, from Lemma 4.3 we get

$$m^{+}(a,b) \leq J(s_{u,v} * (u,v)) \leq J(u,v).$$

Thus $m^{+}(a,b) = \hat{m}(a,b)$. Since $J(u,v) > 0$ on $\mathcal{P}^{-}_{\mu_1,\mu_2}$, we know $m(a,b) = m^{+}(a,b)$. Finally, by the continuity of $k(t)$ and $k(R_2) = 0$, we see from $-\infty < \hat{m}(a,b) < 0$ that there is $\delta > 0$ satisfying $k(t) \geq \frac{\hat{m}(a,b)}{2}$ if $t \in [R_2 - \delta, R_2]$. Thus

$$J(u,v) \geq k(([u]^2 + [v]^2)^{\frac{1}{2}}) \geq \frac{\hat{m}(a,b)}{2} \geq \hat{m}(a,b),$$

for any $(u,v) \in S$ with $R_2 - \delta \leq ([u]^2 + [v]^2)^{\frac{1}{2}} \leq R_2$. This completes the proof. □

Similarly from *Case 1* in Lemma 3.4, we obtain the monotonicity for this problem (1.1)–(1.2).

Lemma 4.6. *There exists $\hat{\beta}_* > 0$, for $\beta \in (0, \hat{\beta}_*)$, there are $\hat{\mu}_{1,*} := \hat{\mu}_{1,*}(\beta), \hat{\mu}_{2,*} := \hat{\mu}_{2,*}(\beta) > 0$, for any $\mu_1 \in (0, \hat{\mu}_{1,*})$ and $\mu_2 \in (0, \hat{\mu}_{2,*})$, the level satisfies $m(a,b) \leq m(a_1,b_1)$ for any $0 < a_1 \leq a$, $0 < b_1 \leq b$.*

Proof. We also divide this proof into 3 cases.

Case 1: $2 < p < r_1 + r_2 < \bar{p}$. From Lemmas 4.1 and 4.3, we have $m(a,b) = \inf_{B_{t_0}(a,b)} J(u,v)$ and

$$t_0 = \left[\frac{(2 - p\gamma_p)(2 - (r_1 + r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1 + r_2)\gamma_{(r_1+r_2)})} \right]^{\frac{1}{22^*_{\alpha,s}-2}},$$

which is independent with a, b. Besides, from (3.2), (4.1) and (4.2), we get C_1, C_2 and C_3 are increasing when a, b are increasing. Hence, we can choose $\hat{\beta}_* = \min\{\beta_*, \tilde{\beta}_*\}$, $\hat{\mu}_{1,*} = \min\{\mu_{1,*}, \tilde{\mu}_{1,*}\}$ and $\hat{\mu}_{2,*} = \min\{\mu_{1,*}, \tilde{\mu}_{1,*}\}$, such that there is $(u,v) \in B_{t_0}(a_1, b_1)$ with $J(u,v) \leq m(a_1, b_1) + \frac{\varepsilon}{2}$, for ε is arbitrarily small. Using the same argument as Case 1 in Lemma 3.4, we get this result.

Case 2: $2 < p < r_1 + r_2 = \bar{p}$. Similarly, we have $m(a,b) = \inf_{B_{t_1}(a,b)} J(u,v)$ with

$$t_1 = \left[\frac{(1 - 2C_3\beta)(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{1}{22^*_{\alpha,s}-2}} \leq \left[\frac{(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{1}{22^*_{\alpha,s}-2}} := t_{1,*},$$

which is independent with a, b. If there exists $\breve{\beta}_* > 0$. Then for any $\beta \in (0, \breve{\beta}_*)$, there are $\breve{\mu}_{1,*} = \breve{\mu}_{1,*}(\beta), \breve{\mu}_{2,*} = \breve{\mu}_{2,*}(\beta) > 0$, for any $\mu_1 \in (0, \breve{\mu}_{1,*})$ and $\mu_2 \in (0, \breve{\mu}_{2,*})$, such that $k(t_{1,*}) \geq 0$, that is

$$\frac{1}{2} - \frac{2 - p\gamma_p}{22^*_{\alpha,s}(22^*_{\alpha,s} - p\gamma_p)} \geq C_3\beta + \frac{\mu_1 C_1 + \mu_2 C_2}{p} \left[\frac{(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{p\gamma_p - 2}{22^*_{\alpha,s}-2}}. \tag{4.24}$$

Then from Lemmas 4.1 and 4.3, we can have $m(a,b) = \inf_{B_{t_{1,*}}(a,b)} J(u,v)$. Hence, there exists $\hat{\beta}_* = \min\{\beta_*, \tilde{\beta}_*, \breve{\beta}_*\}$, for $\beta \in (0, \hat{\beta}_*)$, there are $\hat{\mu}_{1,*}(\beta) = \min\{\mu_{1,*}, \tilde{\mu}_{1,*}, \breve{\mu}_{1,*}\}, \hat{\mu}_{2,*}(\beta) = \min\{\mu_{2,*}, \tilde{\mu}_{2,*}, \breve{\mu}_{2,*}\}$, for any $\mu_1 \in (0, \hat{\mu}_{1,*})$ and $\mu_2 \in (0, \hat{\mu}_{2,*})$, there is $(u,v) \in B_{t_{1,*}}(a_1, b_1)$ with $J(u,v) \leq m(a_1, b_1) + \frac{\varepsilon}{2}$. The remainder of the proof is similar to Lemma 3.4, and so we omit the details here.

Case 3: $2 < p < \bar{p} < (r_1 + r_2) < 2^*_s$. First we have $m(a,b) = \inf_{B_{t_2}(a,b)} J(u,v)$ and

$$2 - p\gamma_p = C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}[(r_1 + r_2)\gamma_{(r_1+r_2)} - p\gamma_p] t_2^{(r_1+r_2)\gamma_{(r_1+r_2)}-2} + 2C_4(22^*_{\alpha,s} - p\gamma_p) t_2^{22^*_{\alpha,s}-2}.$$

If we choose

$$t_{2,*} = \left[\frac{2 - p\gamma_p}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{1}{22^*_{\alpha,s}-2}},$$

which is independent with a, b and satisfies

$$2 - p\gamma_p < C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}[(r_1 + r_2)\gamma_{(r_1+r_2)} - p\gamma_p] t_{2,*}^{(r_1+r_2)\gamma_{(r_1+r_2)}-2} + 2C_4(22^*_{\alpha,s} - p\gamma_p) t_{2,*}^{22^*_{\alpha,s}-2},$$

then $t_2 \leq t_{2,*}$. Furthermore, if $k(t_{2,*}) \geq 0$, that is

$$\frac{1}{2} - \frac{2 - p\gamma_p}{22^*_{\alpha,s}(22^*_{\alpha,s} - p\gamma_p)} \geq \frac{\mu_1 C_1 + \mu_2 C_2}{p} \left[\frac{(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{p\gamma_p - 2}{22^*_{\alpha,s}-2}} + C_3\beta \left[\frac{(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{p\gamma_p - 2}{(r_1+r_2)\gamma_{(r_1+r_2)}-2}}.$$

Hence $m(a,b) = \inf_{B_{t_{2,*}}(a,b)} J(u,v)$. Like the same argument as before, choosing appropriate $\hat{\beta}_*, \hat{\mu}_{1,*}, \hat{\mu}_{2,*}$, using the same techniques in Lemma 3.4, we finish this problem. □

Lemma 4.7. *We have*

$$m(a,b) < \min\{m(a,0), m(0,b)\}.$$

Proof. From Theorem 3.1, we get $m(a,0)$ can be achieved by $\hat{u} \in S_a$. We choose a proper test function $\hat{v} \in S_b$ such that $(\hat{u}, t * \hat{v}) \in S$. By (3.3) and (4.5), we obtain $h(t) > k(t)$ for $t \in (0, +\infty)$. Hence, from Lemma 4.1, we have $R_0 < R_2$. By Theorem 3.1, we get

$$m(a,0) = \inf_{\mathcal{P}_{\mu_1,a}} I_{\mu_1,a}(u) = \inf_{B_{R_0}} I_{\mu_1,a}(u).$$

Therefore $[\hat{u}] \leq R_0 < R_2$. Thus, for $t \ll -1$, we have $(\hat{u}, t * \hat{v}) \in B_{R_2}(a,b)$ and

$$\begin{aligned} m(a,b) = & \inf_{(u,v)\in B_{R_2}(a,b)} J(u,v) \leq J(\hat{u}, t * \hat{v}) \\ &= \frac{1}{2}[\hat{u}]^2 - \frac{\mu_1}{p} \int_{\mathbb{R}^N} |\hat{u}|^p dx + \left(\frac{e^{2st}}{2}[\hat{v}]^2 - \frac{\mu_2 e^{p\gamma_p st}}{p} \int_{\mathbb{R}^N} |\hat{v}|^p dx \right. \\ &\quad \left. - \beta \int_{\mathbb{R}^N} |\hat{u}|^{r_1} |t * \hat{v}|^{r_2} dx - \frac{1}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}|^{2^*_{\alpha,s}}) |t * \hat{v}|^{2^*_{\alpha,s}} dx \right) \\ &< \frac{1}{2}[\hat{u}]^2 - \frac{\mu_1}{p} \int_{\mathbb{R}^N} |\hat{u}|^p dx = m(a,0). \end{aligned}$$

Analogously, we have $m(a,b) < m(0,b)$. Hence, the proof is completed. □

To obtain the compact result, we prove the boundedness first.

Lemma 4.8. *Let $2 < p < \bar{p}$, $p < r_1 + r_2 < 2^*_s$ and $\mu_1, \mu_2, a, b > 0$. Let $\{(u_n, v_n)\} \subset S_r$ be a Palais–Smale sequence, such that*

$$J(u_n, v_n) \to c; \quad J'(u_n, v_n)|_S \to 0 \quad \text{and} \quad P_{\mu_1,\mu_2}(u_n, v_n) \to 0,$$

where $S_r = S \cap H_r$ and H_r is the space of radially symmetric functions in H. Then $\{(u_n, v_n)\}$ is bounded in H.

Proof. We divide this proof into two cases. *Case 1*: $2 < p < r_1 + r_2 < \bar{p}$. This implies $p\gamma_p < (r_1 + r_2)\gamma_{(r_1+r_2)} < 2$. Since (3.2), (4.1)–(4.3),

$$\begin{aligned} c + o_n(1) &= J(u_n, v_n) - \frac{1}{22^*_{\alpha,s}} P_{\mu_1,\mu_2}(u_n, v_n) \\ &= \frac{N+2s-\alpha}{2(2N-\alpha)}([u_n]^2 + [v_n]^2) - \left(\frac{1}{p} - \frac{\gamma_p}{22^*_{\alpha,s}} \right) (\mu_1 |u_n|_p^p + \mu_2 |v_n|_p^p) \\ &\quad - \beta \left[1 - \frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{22^*_{\alpha,s}} \right] \int_{\mathbb{R}^N} |u_n|^{r_1} |v_n|^{r_2} dx \\ &\geq \frac{N+2s-\alpha}{2(2N-\alpha)}([u_n]^2 + [v_n]^2) - \left(\frac{1}{p} - \frac{\gamma_p}{22^*_{\alpha,s}} \right) (\mu_1 C_1 + \mu_2 C_2)([u_n]^2 + [v_n]^2)^{\frac{p\gamma_p}{2}} \\ &\quad - \beta \left[1 - \frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{22^*_{\alpha,s}} \right] C_3([u_n]^2 + [v_n]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}}. \end{aligned}$$

Then, $\{(u_n, v_n)\}$ is bounded in H.

Case 2: $2 < p < \bar{p} \le r_1 + r_2 < 2_s^*$. From (3.2), (4.1)–(4.3) and $\alpha < N$, we can obtain

$$
\begin{aligned}
c + o_n(1) &= J(u_n, v_n) - \frac{1}{(r_1+r_2)\gamma_{(r_1+r_2)}} P_{\mu_1,\mu_2}(u_n, v_n) \\
&= \left[\frac{1}{2} - \frac{1}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right]([u_n]^2 + [v_n]^2) - \left[\frac{1}{p} - \frac{\gamma_p}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right](\mu_1 |u_n|_p^p + \mu_2 |v_n|_p^p) \\
&\quad - \left[\frac{1}{2_{\alpha,s}^*} - \frac{2}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right] \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_{\alpha,s}^*}) |v_n|^{2_{\alpha,s}^*} dx \\
&\ge \left[\frac{1}{2} - \frac{1}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right]([u_n]^2 + [v_n]^2) \\
&\quad - \left[\frac{1}{p} - \frac{\gamma_p}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right](\mu_1 C_1 + \mu_2 C_2)([u_n]^2 + [v_n]^2)^{\frac{p\gamma_p}{2}}.
\end{aligned}
$$

From this, we have $\{(u_n, v_n)\}$ is bounded in H. □

In what follows, we discuss the convergence of a special Palais–Smale sequence, satisfying suitable additional conditions.

Proposition 4.9. *Let* $\{(u_n, v_n)\} \subset S_r$ *such that as* $n \to \infty$,

$$
\begin{aligned}
J'(u_n, v_n) - \lambda_{1,n} u_n - \lambda_{2,n} v_n \to 0, &\quad \text{for some } \lambda_{1,n}, \lambda_{2,n} \in \mathbb{R}; \\
J(u_n, v_n) \to m(a,b), &\quad P_{\mu_1,\mu_2}(u_n, v_n) \to 0; \\
u_n^-, v_n^- \to 0, &\quad \text{a.e. in } \mathbb{R}^N,
\end{aligned}
\tag{4.25}
$$

with

$$
m(a,b) \neq 0, \quad \text{and} \quad m(a,b) < \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}.
$$

Then there exist $(u, v) \in H_r$ *with* $u, v > 0$ *and* $\lambda_1, \lambda_2 < 0$, *such that up to a subsequence,* $(u_n, v_n) \to (u, v)$ *in* H *and* $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2)$ *in* $\mathbb{R}^2$.

Proof. From Lemma 4.8, we get $\{(u_n, v_n)\}$ is bounded in H_r. Moreover, by (4.25) we get

$$
\lambda_{1,n} = \frac{1}{a^2} J'(u_n, v_n)(u_n, 0) + o_n(1), \quad \text{and} \quad \lambda_{2,n} = \frac{1}{b^2} J'(u_n, v_n)(0, v_n) + o_n(1),
$$

thus $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ are bounded in $\mathbb{R}$. Therefore, there exist $(u, v) \in H_r$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that up to a subsequence,

$$
\begin{aligned}
(u_n, v_n) &\rightharpoonup (u, v), && \text{in } H_r, \\
(u_n, v_n) &\to (u, v), && \text{in } L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N), \quad \text{for } 2 < q < 2_s^*, \\
(u_n, v_n) &\to (u, v), && \text{a.e. in } \mathbb{R}^N, \\
(\lambda_{1,n}, \lambda_{2,n}) &\to (\lambda_1, \lambda_2), && \text{in } \mathbb{R}^2.
\end{aligned}
$$

Since

$$
|v_n|^{2_{\alpha,s}^*} \rightharpoonup |v|^{2_{\alpha,s}^*}, \quad \text{in } L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N),
$$

and the map $T : L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N) \mapsto L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ defined by $T(w) = I_\alpha * w$ is well defined , linear and continuous, we have

$$
I_\alpha * |v_n|^{2_{\alpha,s}^*} \rightharpoonup I_\alpha * |v|^{2_{\alpha,s}^*}, \quad \text{in } L^{\frac{2N}{\alpha}}(\mathbb{R}^N).
$$

Besides, by

$$|u_n|^{2^*_{\alpha,s}-2}u_n \rightharpoonup |u|^{2^*_{\alpha,s}-2}u, \quad \text{in } L^{\frac{2N}{N+2s-\alpha}}(\mathbb{R}^N),$$

we get

$$(I_\alpha * |v_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n \rightharpoonup (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, \quad \text{in } L^{\frac{2N}{N+2s}}(\mathbb{R}^N).$$

Hence for any $\phi, \psi \in C_0^\infty(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n\phi dx \to \int_{\mathbb{R}^N}(I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u\phi dx,$$

and

$$\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2}v_n\psi dx \to \int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v\psi dx.$$

Therefore from (4.25), (u, v) satisfies

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1|u|^{p-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2} + (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2|v|^{p-2}v + \beta r_2|u|^{r_1}|v|^{r_2-2}v + (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v, & \text{in } \mathbb{R}^N, \\ u \geq 0,\ v \geq 0, \end{cases} \tag{4.26}$$

and $P_{\mu_1,\mu_2}(u, v) = 0$.

Next, we will show $u \not\equiv 0$ and $v \not\equiv 0$. If not, we assume $u \equiv 0$. We claim $v \not\equiv 0$. Otherwise, from $P_{\mu_1,\mu_2}(u_n, v_n) \to 0$ and $u_n, v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*_s)$, we get

$$[u_n]^2 + [v_n]^2 = 2\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}dx + o_n(1).$$

Since $\{(u_n, v_n)\}$ is bounded in H, we may assume $[u_n]^2 + [v_n]^2 \to l \in \mathbb{R}$. Then from (2.2), we have

$$l = 0 \quad \text{or } l \geq 2\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

On one hand, if $l = 0$, we have $(u_n, v_n) \to (0, 0)$ in $D^s(\mathbb{R}^N) \times D^s(\mathbb{R}^N)$. Consequently $J(u_n, v_n) \to 0$ which gives a contradiction with $m(a, b) \neq 0$. On the other hand, if $l \geq 2(\frac{S^*}{2})^{\frac{2N-\alpha}{N+2s-\alpha}}$, from $P_{\mu_1,\mu_2}(u_n, v_n) \to 0$, we obtain

$$m(a, b) = J(u_n, v_n) + o_n(1) = J(u_n, v_n) - \frac{1}{2}P_{\mu_1,\mu_2}(u_n, v_n) + o_n(1) \geq \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}},$$

which can not happen since $m(a, b) < \frac{N+2s-\alpha}{2N-\alpha}(\frac{S^*}{2})^{\frac{2N-\alpha}{N+2s-\alpha}}$. Therefore $v \not\equiv 0$. From (4.26) and $u \equiv 0$, we have v satisfies

$$\begin{cases} (-\Delta)^s v = \lambda_2 v + \mu_2|v|^{p-2}v, & \text{in } \mathbb{R}^N, \\ v \geq 0. \end{cases}$$

Then we obtain from $|v|_2 \leq b$ and Lemma 3.4,

$$\begin{aligned} m(a, b) &= J(u_n, v_n) - \frac{1}{2}P_{\mu_1,\mu_2}(u_n, v_n) + o_n(1) \\ &= \left(\frac{\gamma_p}{2} - \frac{1}{p}\right)\mu_2|v|_p^p + (1 - \frac{1}{2^*_{\alpha,s}})\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}dx + o_n(1) \\ &\geq \left(\frac{\gamma_p}{2} - \frac{1}{p}\right)\mu_2|v|_p^p \geq m(0, |v|_2) \geq m(0, b). \end{aligned}$$

From Lemma 4.7, which contradicts with $m(a,b) < m(0,b)$. Similar to [45, Lemma 3.7] and [32, Section 3], by the strong maximum principle [37, Proposition 2.17], we have $u > 0$. Analogously $v > 0$.

We claim $(u_n, v_n) \to (u, v)$ in $D^s(\mathbb{R}^N) \times D^s(\mathbb{R}^N)$. Indeed, if we let $(\hat{u}_n, \hat{v}_n) := (u_n - u, v_n - v)$, by [12, Lemma 2.2],

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} dx - \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}_n|^{2^*_{\alpha,s}})|\hat{v}_n|^{2^*_{\alpha,s}} dx + o_n(1) = \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx,$$

and [45, Lemma 2.4],

$$\int_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2} - |\hat{u}_n|^{r_1}|\hat{v}_n|^{r_2} - |u|^{r_1}|v|^{r_2} dx = o_n(1).$$

Therefore by the Brézis–Lieb Lemma [9], we have

$$P_{\mu_1,\mu_2}(\hat{u}_n, \hat{v}_n) = P_{\mu_1,\mu_2}(u_n, v_n) - P_{\mu_1,\mu_2}(u, v) + o_n(1) = o_n(1).$$

We deduce by the strong embedding in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*_s)$ that,

$$\lim_{n\to\infty} ([\hat{u}_n]^2 + [\hat{v}_n]^2) = \lim_{n\to\infty} 2 \int_{\mathbb{R}^N} (I_\mu * |\hat{u}_n|^{2^*_{\alpha,s}})|\hat{v}_n|^{2^*_{\alpha,s}}.$$

Same argument as before, from (2.2) we can have

$$([\hat{u}_n]^2 + [\hat{v}_n]^2) \to 0,$$

or

$$([\hat{u}_n]^2 + [\hat{v}_n]^2) \geq 2 \left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

If the latter happens, we obtain from $|u|_2 \leq a$, $|v|_2 \leq b$ and Lemma 4.6,

$$\begin{aligned} m(a,b) + o_n(1) = J(u,v) + J(\hat{u}_n, \hat{v}_n) &= J(u,v) + J(\hat{u}_n, \hat{v}_n) - \frac{1}{22^*_{\alpha,s}} P_{\mu_1,\mu_2}(\hat{u}_n, \hat{v}_n) \\ &\geq m(|u|_2, |v|_2) + \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}} \\ &\geq m(a,b) + \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}, \end{aligned}$$

this can not happen. Therefore we have $([\hat{u}_n]^2 + [\hat{v}_n]^2) \to 0$, and we finish this claim.

Following, we claim $\lambda_1, \lambda_2 < 0$. If not, we may assume $\lambda_1 \geq 0$. From $u \geq 0$ we have

$$(-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1-2} u |v|^{r_2} + (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2} u \geq 0.$$

From [29, Lemma 2.3] and $2s < N \leq 4s$, we have $u \equiv 0$, which is a contradiction. Hence, we obtain $\lambda_1 < 0$, and analogously $\lambda_2 < 0$. Then we deduce from taking $(u_n - u, v_n - v)$ into

(4.26) and the first formula of (4.25),

$$
\begin{aligned}
&[u_n-u]^2+[v_n-v]^2+o_n(1)\\
&=\int_{\mathbb{R}^N}(\lambda_{1,n}u_n-\lambda_1u)(u_n-u)+(\lambda_{2,n}v_n-\lambda_2v)(v_n-v)dx\\
&\quad+\mu_1\int_{\mathbb{R}^N}(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)dx\\
&\quad+\mu_2\int_{\mathbb{R}^N}(|v_n|^{p-2}v_n-|v|^{p-2}v)(v_n-v)dx\\
&\quad+\beta r_1\int_{\mathbb{R}^N}(|u_n|^{r_1-2}u_n|v_n|^{r_2}-|u|^{r_1-2}u|v|^{r_2})(u_n-u)dx\\
&\quad+\beta r_2\int_{\mathbb{R}^N}(|u_n|^{r_1}|v_n|^{r_2-2}v_n-|u|^{r_1}|v|^{r_2-2}v)(v_n-v)dx\\
&\quad+\int_{\mathbb{R}^N}[(I_\alpha*|v_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n-(I_\alpha*|v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u](u_n-u)dx\\
&\quad+\int_{\mathbb{R}^N}[(I_\alpha*|u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2}v_n-(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v](v_n-v)dx.
\end{aligned}
$$

Since $(u_n,v_n)\to(u,v)$ in $D^s(\mathbb{R}^N)\times D^s(\mathbb{R}^N)$ and the embedding $D^s(\mathbb{R}^N)\hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ is continuous, we have

$$
\begin{aligned}
0&=\lim_{n\to\infty}\int_{\mathbb{R}^N}(\lambda_{1,n}u_n-\lambda_1u)(u_n-u)+(\lambda_{2,n}v_n-\lambda_2v)(v_n-v)dx\\
&=\lim_{n\to\infty}\int_{\mathbb{R}^N}\lambda_1(u_n-u)^2+\lambda_2(v_n-v)^2dx,
\end{aligned}
$$

by $\lambda_1,\lambda_2<0$, then $(u_n,v_n)\to(u,v)$ in H and we complete this proof. □

Proof of Theorem 1.1. Taking $\beta^*=\hat{\beta}_*$, there exist $\mu_1^*(\beta)=\min\{\hat{\mu}_{1,*},\hat{\mu}_1\}$ and $\mu_2^*(\beta)=\min\{\hat{\mu}_{2,*},\hat{\mu}_2\}$, for any $\mu_1\in(0,\mu_1^*)$ and $\mu_2\in(0,\mu_2^*)$, such that Lemmas 4.1, 4.2 and 4.7 are satisfied. Then, from Proposition 4.9, to finish this proof, it is sufficient to prove the existence of a sequence which satisfies Proposition 4.9. Let $\{(u_n,v_n)\}$ be a minimizing sequence for $m(a,b)=\inf_{B_{R_2}(a,b)}J(u,v)$, and assume that $\{(u_n,v_n)\}\subset S_r$ is radially decreasing, symmetry and non-negative for every $n\in\mathbb{N}$ (Firstly, due to $|(-\Delta)^{\frac{s}{2}}|u||\le|(-\Delta)^{\frac{s}{2}}u|$, we can have (u_n,v_n) is non-negative. Secondly, we replace $|u_n|$ with $|u_n|^*$ and $|v_n|$ with $|v_n|^*$, where $|\cdot|^*$ is the Schwarz symmetrization rearrangement, then we can obtain another function in $B_{R_2}(a,b)$ with $J(|u_n|^*,|v_n|^*)\le J(u_n,v_n)$). Moreover by Lemma 4.3, $s_{u_n,v_n}*(u_n,v_n)\in\mathcal{P}^+_{\mu_1,\mu_2}$ such that

$$([u_n]^2+[v_n]^2)^{\frac{1}{2}}<R_2,$$

and

$$
\begin{aligned}
J(s_{u_n,v_n}*(u_n,v_n))&=\min\{J(t*(u_n,v_n)):t\in\mathbb{R}\text{ and }([t*u_n]^2+[t*v_n]^2)^{\frac{1}{2}}<R_2\}\\
&\le J(u_n,v_n).
\end{aligned}
$$

Thus, we get another minimizing sequence $\{\tilde{u}_n:=s_{u_n,v_n}*u_n,\tilde{v}_n:=s_{u_n,v_n}*v_n\}$ with $\{(\tilde{u}_n,\tilde{v}_n)\}\subset S_r$. By Lemma 4.5, we have $([\tilde{u}_n]^2+[\tilde{v}_n]^2)^{\frac{1}{2}}\le R_2-\delta$. Then, from Ekeland's Variational Principle [17], we know there exists a radially Palais–Smale sequence $\{(w_n,z_n)\}$ for $J|_S$ satisfying $\|(w_n,z_n)-(\tilde{u}_n,\tilde{v}_n)\|_H\to0$ as $n\to\infty$. Following, we claim $P_{\mu_1,\mu_2}(w_n,z_n)=P(\tilde{u}_n,\tilde{v}_n)+o_n(1)=o_n(1)$. Firstly, by the Brézis–Lieb Lemma and Sobolev's embedding Theorem, we have

$$[w_n]^2=[w_n-\tilde{u}_n]^2+[\tilde{u}_n]^2+o_n(1)=[\tilde{u}_n]^2+o_n(1),$$

and

$$\int_{\mathbb{R}^N} |w_n|^p dx = \int_{\mathbb{R}^N} |w_n - \tilde{u}_n|^p dx + \int_{\mathbb{R}^N} |\tilde{u}_n|^p dx + o_n(1) = \int_{\mathbb{R}^N} |\tilde{u}_n|^p dx + o_n(1).$$

Moreover, by the Hölder inequality and Lemma 2.1, we get

$$\int_{\mathbb{R}^N} |w_n|^{r_1} |z_n|^{r_2} dx = \int_{\mathbb{R}^N} |\tilde{u}_n|^{r_1} |\tilde{v}_n|^{r_2} dx + o_n(1),$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |w_n|^{2^*_{\alpha,s}}) |z_n|^{2^*_{\alpha,s}} dx = \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_n|^{2^*_{\alpha,s}}) |\tilde{v}_n|^{2^*_{\alpha,s}} dx + o_n(1).$$

The same relationship happen to z_n and $\tilde{v}_n$. Therefore, we obtain

$$P_{\mu_1,\mu_2}(w_n, z_n) = P_{\mu_1,\mu_2}(\tilde{u}_n, \tilde{v}_n) + o_n(1) = o_n(1), \quad \text{and} \quad w_n^-, z_n^- \to 0, \quad \text{a.e. in } \mathbb{R}^N.$$

Thus from Proposition 4.9, we obtain there is $(u, v) \in H_r$ and $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ with $\lambda_1, \lambda_2 < 0$, such that $(w_n, z_n) \to (u, v)$ in H and $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2)$ in $\mathbb{R}^2$. Hence, $(u, v) \in \mathcal{P}_{\mu_1,\mu_2}$ is a solution for (1.1)–(1.2), which is a normalized ground state with $J(u, v) = m(a, b)$. Moreover, for any ground state solution (u, v), from $m(a, b) < 0$ and Lemma 4.2, we have

$$J(u, v) = m(a, b) = \inf_{B_{R_2}(a,b)} J(u, v), \quad \text{and} \quad ([u]^2 + [v]^2)^{\frac{1}{2}} < R_2,$$

i.e. (u, v) is a local minimizer for $J(u, v)$ on $B_{R_2}(a, b)$. □

5 The case: $\bar{p} < p < r_1 + r_2 < 2^*_s$

Firstly, we show the boundedness result for this case.

Lemma 5.1. *Let $\bar{p} < p < r_1 + r_2 < 2^*_s$ and $\mu_1, \mu_2, a, b > 0$. Let $\{(u_n, v_n)\} \subset S_r$ be a Palais–Smale sequence such that*

$$J(u_n, v_n) \to c; \quad J'(u_n, v_n)|_S \to 0, \quad \text{and} \quad P_{\mu_1,\mu_2}(u_n, v_n) \to 0.$$

Then $\{(u_n, v_n)\}$ is bounded in H.

Proof. In this case, we have $2 < p\gamma_p < (r_1 + r_2)\gamma_{(r_1+r_2)}$. Then

$$\begin{aligned} c + o_n(1) &= J_{\mu_1,\mu_2}(u_n, v_n) - \frac{1}{2} P_{\mu_1,\mu_2}(u_n, v_n) \\ &= \left(\frac{\gamma_p}{2} - \frac{1}{p}\right)(\mu_1 |u_n|_p^p + \mu_2 |v_n|_p^p) + \beta \left[\frac{(r_1 + r_2)\gamma_{(r_1+r_2)}}{2} - 1\right] \int_{\mathbb{R}^N} |u_n|^{r_1} |v_n|^{r_2} dx \\ &\quad + \left(1 - \frac{1}{2^*_{\alpha,s}}\right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}}) |v_n|^{2^*_{\alpha,s}} dx, \end{aligned}$$

by each coefficient is positive, we get $\{(u_n, v_n)\}$ is bounded in H. The proof is completed. □

Recalling the decomposition of $\mathcal{P}_{\mu_1,\mu_2} = \mathcal{P}^+_{\mu_1,\mu_2} \cup \mathcal{P}^0_{\mu_1,\mu_2} \cup \mathcal{P}^-_{\mu_1,\mu_2}$, we have

Lemma 5.2. *$\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$ and $\mathcal{P}_{\mu_1,\mu_2}$ is a C^1-submanifold in H with codimension 3.*

Proof. If there is $(u,v) \in \mathcal{P}^0_{\mu_1,\mu_2}$, then

$$[u]^2+[v]^2 = \gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p)+\beta(r_1+r_2)\gamma_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx+2\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,$$

and

$$2([u]^2+[v]^2) = p\gamma_p^2(\mu_1|u|_p^p+\mu_2|v|_p^p)+\beta(r_1+r_2)^2\gamma^2_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx + 4\cdot 2^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.$$

From above, we obtain

$$(2-p\gamma_p)([u]^2+[v]^2) = \beta(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx + 2(22^*_{\alpha,s}-p\gamma_p)\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.$$

Since $2-p\gamma_p<0$, $(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p>0$ and $22^*_{\alpha,s}-p\gamma_p>0$, we have $(u,v)=(0,0)$, which contradicts with $(u,v)\in S$. The remainder parts of this proof is similar with Lemma 4.2, and we omit the details here. □

Following, we show the geometry for this mass supercritical case.

Lemma 5.3. *For every $(u,v)\in S$, the function $\Psi_{\mu_1,\mu_2}(t)$ has exactly one critical point $t_{u,v}\in\mathbb{R}$ such that $t_{u,v}*(u,v)\in\mathcal{P}_{\mu_1,\mu_2}$. Moreover:*

(i) $\mathcal{P}_{\mu_1,\mu_2}=\mathcal{P}^-_{\mu_1,\mu_2}$;

(ii) $\Psi_{\mu_1,\mu_2}(t)$ *is strictly decreasing and concave on* $(t_{u,v},+\infty)$, *and* $\Psi_{\mu_1,\mu_2}(t_{u,v})=\max_{t\in\mathbb{R}}\Psi_{\mu_1,\mu_2}(t)>0$;

(iii) The map $(u,v)\mapsto t_{u,v}$ *is of class* $\mathcal{C}^1$;

(iv) If $\mathcal{P}_{\mu_1,\mu_2}(u,v)<0$, *then* $t_{u,v}<0$.

Proof. From the definition of $\Psi_{\mu_1,\mu_2}(t)$, we have

$$\begin{aligned}\Psi'_{\mu_1,\mu_2}(t) &= e^{2st}\Big[s([u]^2+[v]^2)-s\gamma_p e^{(p\gamma_p-2)st}(\mu_1|u|_p^p+\mu_2|v|_p^p)-2se^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx \\ &\quad - s\beta(r_1+r_2)\gamma_{(r_1+r_2)}e^{[(r_1+r_2)\gamma_{(r_1+r_2)}-2]st}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx\Big],\end{aligned}$$

which implies $\Psi_{\mu_1,\mu_2}(t)$ has exactly one critical point $t_{u,v}$. Since

$$\Psi_{\mu_1,\mu_2}(-\infty)=0^+, \quad \text{and} \quad \Psi_{\mu_1,\mu_2}(+\infty)=-\infty,$$

we get $t_{u,v}$ is a strict maximum point at a positive level and $t_{u,v}*(u,v)\in\mathcal{P}_{\mu_1,\mu_2}$. From $\Psi''_{\mu_1,\mu_2}(t_{u,v})\le 0$ and $\mathcal{P}^0_{\mu_1,\mu_2}=\emptyset$, we have $\Psi''_{\mu_1,\mu_2}(t_{u,v})<0$, this implies $t_{u,v}*(u,v)\in\mathcal{P}^-_{\mu_1,\mu_2}$ and $\mathcal{P}_{\mu_1,\mu_2}=\mathcal{P}^-_{\mu_1,\mu_2}$. To see (iii), we use the implicit function theorem as in Lemma 4.3. Finally, since $\Psi'_{\mu_1,\mu_2}(t)<0$ if and only if $t>t_{u,v}$, we get $P_{\mu_1,\mu_2}(u,v)=\Psi'_{\mu_1,\mu_2}(0)<0$ if and only if $t_{u,v}<0$. □

Remark 5.4. From Lemma 5.3, we see $m(a,b)=m^-(a,b)$.

Lemma 5.5. $m(a,b) = \inf_{\mathcal{P}_{\mu_1,\mu_2}} J(u,v) > 0$.

Proof. For $(u,v) \in \mathcal{P}_{\mu_1,\mu_2}$, then from (3.2), (4.1)-(4.3), we get

$$([u]^2 + [v]^2) \leq \gamma_p(C_1\mu_1 + C_2\mu_2)([u]^2 + [v]^2)^{\frac{p\gamma_p}{2}} + 2C_4([u]^2 + [v]^2)^{2^*_{\alpha,s}} + C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}([u]^2 + [v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}}.$$

Since $p\gamma_p > 2$, we obtain $\inf_{\mathcal{P}_{\mu_1,\mu_2}}([u]^2 + [v]^2) > 0$ and so

$$\begin{aligned}\inf_{\mathcal{P}_{\mu_1,\mu_2}} J(u,v) &= \inf_{\mathcal{P}_{\mu_1,\mu_2}} \left[J(u,v) - \frac{1}{2}P_{\mu_1,\mu_2}(u,v)\right] \\ &= \inf_{\mathcal{P}_{\mu_1,\mu_2}} \left[\left(\frac{\gamma_p}{2} - \frac{1}{p}\right)(\mu_1|u|_p^p + \mu_2|v|_p^p) + \beta\left(\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2} - 1\right)\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx \right.\\ &\quad \left. + \left(1 - \frac{1}{2^*_{\alpha,s}}\right)\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx\right] > 0.\end{aligned}$$

Therefore we have $m(a,b) > 0$. □

Lemma 5.6. *For any $\delta > 0$ sufficiently small, we have $0 < \sup_{\overline{B_\delta}} J(u,v) < m(a,b)$ and*

$$u \in \overline{B_\delta} \Rightarrow J(u,v),\ P_{\mu_1,\mu_2}(u,v) > 0,$$

where $B_\delta := \{(u,v) \in S : ([u]^2 + [v]^2)^{\frac{1}{2}} < \delta\}$.

Proof. Since (3.2), (4.1)–(4.3), we get

$$\begin{aligned}J(u,v) \geq{}& \frac{1}{2}([u]^2 + [v]^2) - \frac{(\mu_1C_1 + \mu_2C_2)}{p}([u]^2 + [v]^2)^{\frac{p\gamma_p}{2}} - \beta C_3([u]^2 + [v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}} \\ &- \frac{C_4}{2^*_{\alpha,s}}([u]^2 + [v]^2)^{2^*_{\alpha,s}},\end{aligned}$$

and

$$\begin{aligned}P_{\mu_1,\mu_2}(u,v) \geq{}& s([u]^2 + [v]^2) - s\gamma_p(\mu_1C_1 + \mu_2C_2)([u]^2 + [v]^2)^{\frac{p\gamma_p}{2}} - 2sC_4([u]^2 + [v]^2)^{2^*_{\alpha,s}} \\ &- s\beta(r_1 + r_2)\gamma_{(r_1+r_2)}C_3([u]^2 + [v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}}.\end{aligned}$$

Thus for $\delta > 0$ small enough, we have $J(u,v) > 0$ and $P_{\mu_1,\mu_2}(u,v) > 0$. Moreover, by Lemma 5.5, we can choose δ with smaller quantity, such that

$$J(u,v) \leq ([u]^2 + [v]^2) < m(a,b).$$ □

To use the Proposition 4.9, we need some properties about $m(a,b)$. Firstly, we get the monotonicity of $m(a,b)$. The proof is similar with *Case 2* in Lemma 3.4 and we omit this process here.

Lemma 5.7. *$m(a,b) \leq m(a_1,b_1)$ for any $0 < a_1 \leq a$, $0 < b_1 \leq b$.*

Lemma 5.8. *For $a,b > 0$ fixed, we have $\lim_{\beta\to+\infty} m(a,b) = 0^+$.*

Proof. This lemma is equivalent to prove, for any $\varepsilon > 0$, there exists $\bar{\beta} > 0$ such that

$$m(a,b) < \varepsilon \quad \text{for any } \beta \geq \bar{\beta}.$$

Firstly, from Lemma 5.6, for any $\beta > 0$, we have $m(a,b) > 0$. If we choose $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $|\varphi|_2 \leq \min\{a,b\}$, by Lemmas 5.3 and 5.7 we obtain,

$$m(a,b) \leq m(|\varphi|_2, |\varphi|_2) \leq \max_{t\in\mathbb{R}} J(t*(\varphi,\varphi)) = \max_{t\in\mathbb{R}} \left[E(t) - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t} \int_{\mathbb{R}^N} |\varphi|^{(r_1+r_2)} dx\right],$$

where

$$E(t) := e^{2ts}[\varphi]^2 - \frac{e^{p\gamma_p st}}{p}(\mu_1+\mu_2)|\varphi|_p^p - \frac{e^{22^*_{\alpha,s}st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx.$$

From $p\gamma_p - 2 > 0$ and $22^*_{\alpha,s} - 2 > 0$ we see

$$\begin{aligned} E(t) &= e^{2ts}\left([\varphi]^2 - \frac{e^{(p\gamma_p-2)st}}{p}(\mu_1+\mu_2)|\varphi|_p^p - \frac{e^{(22^*_{\alpha,s}-2)st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx\right) \\ &= e^{2ts}([\varphi]^2 + o(1)) \to 0^+, \quad \text{as } t \to -\infty. \end{aligned}$$

And there exists $\tilde{t} > 0$ such that $E(t) < \frac{\varepsilon}{4}$ for $t < -\tilde{t}$. Moreover, there exists $\bar{\beta} > 0$, such that for any $\beta \geq \bar{\beta}$,

$$\begin{aligned} &\max_{t\geq -\tilde{t}} \left[E(t) - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t}|\varphi|_{r_1+r_2}^{r_1+r_2}\right] \\ &\leq \max_{t\geq -\tilde{t}} \left[e^{2ts}[\varphi]^2 - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t}|\varphi|_{(r_1+r_2)}^{(r_1+r_2)} - \frac{e^{22^*_{\alpha,s}st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx\right] \\ &\leq \max_{t\in\mathbb{R}} \left[e^{2ts}[\varphi]^2 - \frac{e^{22^*_{\alpha,s}st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx\right] - \beta e^{-s(r_1+r_2)\gamma_{(r_1+r_2)}\tilde{t}}|\varphi|_{r_1+r_2}^{r_1+r_2} \\ &\leq \left(1 - \frac{1}{2^*_{\alpha,s}}\right)[\varphi]^{\frac{22^*_{\alpha,s}}{2^*_{\alpha,s}-1}} \left(\int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx\right)^{\frac{-1}{2^*_{\alpha,s}-1}} - \beta e^{-s(r_1+r_2)\gamma_{(r_1+r_2)}\tilde{t}}|\varphi|_{r_1+r_2}^{r_1+r_2}. \end{aligned}$$

Hence, we have $\max_{t\in\mathbb{R}} \left[E(t) - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t}|\varphi|_{r_1+r_2}^{r_1+r_2}\right] < \varepsilon$ for $\beta \geq \bar{\beta}$, and $m(a,b) < \varepsilon$. □

Thus by the above lemma, we have the following conclusion:

Lemma 5.9. *There exists $\hat{\beta}_1 > 0$, we get $m(a,b) < \frac{N+2s-\alpha}{2N-\alpha}(\frac{S^*}{2})^{\frac{2N-\alpha}{N+2s-\alpha}}$ for any $\beta > \hat{\beta}_1$.*

Lemma 5.10. *There exists $\hat{\beta}_2 > 0$ such that for any $\beta > \hat{\beta}_2$, the level satisfies*

$$m(a,b) < \min\{m(a,0), m(0,b)\}.$$

Proof. From Theorem 3.3, $m(a,0) > 0$ can be achieved by $u^* \in S_a$. Similarly, $m(0,b) > 0$ can be achieved by $v^* \in S_b$. Since

$$I_{\mu_1,a}(t*u^*) \to 0, \quad \text{and} \quad I_{\mu_2,b}(t*v^*) \to 0, \quad \text{as } t \to -\infty,$$

there is $t^* \ll -1$ which is independent of β, such that

$$\max_{t<t^*} J(u^*,v^*) < \max_{t<t^*} I_{\mu_1,a}(t*u^*) + \max_{t<t^*} I_{\mu_2,b}(t*v^*) < \min\{m(a,0), m(0,b)\}.$$

On the other hand, for $t > t^*$, firstly we have

$$\int_{\mathbb{R}^N} |t * u^*|^{r_1} |t * v^*|^{r_2} dx = e^{st(r_1+r_2)\gamma_{(r_1+r_2)}} \int_{\mathbb{R}^N} |u^*|^{r_1} |v^*|^{r_2} dx \geq C e^{st^*(r_1+r_2)\gamma_{(r_1+r_2)}},$$

for some $C > 0$. Then by Theorem 3.3, we get

$$\begin{aligned}\max_{t \geq t^*} J(t * (u^*, v^*)) &\leq \max_{t \geq t^*} I_{\mu_1,a}(t * u^*) + \max_{t \geq t^*} I_{\mu_2,b}(t * v^*) - C\beta e^{st^*(r_1+r_2)\gamma_{(r_1+r_2)}} \\ &\leq m(a,0) + m(0,b) - C\beta e^{st^*(r_1+r_2)\gamma_{(r_1+r_2)}}.\end{aligned}$$

Hence, there is $\hat{\beta}_2 > 0$, for any $\beta > \hat{\beta}_2$ such that $m(a,b) < \min\{m(a,0), m(0,b)\}$. □

To prove Theorem 1.2, we give the following minimax theorem to establish the existence of Palais–Smale sequence. At first, we show some definitions.

Definition 5.11 ([21, Definition 3.1])**.** Let Θ be a closed subset of a metric space $X \subset H$. We say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with closed boundary Θ provided

(i) every set in $\mathcal{F}$ contains Θ;

(ii) for any set $Y \in \mathcal{F}$ and any $\eta \in C([0,1] \times X, X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times X) \cup ([0,1] \times \Theta)$, we have that $\eta(\{1\} \times Y) \in \mathcal{F}$.

Definition 5.12. [33] Let M be a C^∞ m-dimensional manifold and $\widetilde{TM} = TM \setminus \{0\}$, where TM is a tangent bundle. A function $F : TM \to [0, \infty)$ is called a Finsler structure on M if F has the following properties:

(i) $F(tY) = tF(Y)$, $\forall t \in \mathbb{R}^+$;

(ii) F is C^∞ on $\widetilde{TM}$;

(iii) for every non-zero $Y \in T_xM$, the induced quadratic form g_Y is an inner product in T_xM, where

$$g_Y(U,V) := \frac{1}{2} \frac{\partial^2}{\partial_s \partial_t} (F^2(Y + sU + tV))|_{s=t=0},$$

and T_xM is the tangent space at the point x. A Finsler manifold is a C^∞-manifold M with its Finsler structure F.

Remark 5.13. From [14], we know Riemannian manifolds are special cases of Finsler manifolds. Denote $X := \mathbb{R} \times S_r$. Since $\mathbb{R}$ is a Banach space and $S_r \subset H^s(\mathbb{R}^N, R) \times H^s(\mathbb{R}^N, R)$ is a Banach manifold, similar to [24] (see (7.2) there), [26, Lemma 4.8] and [32, Theorem 6.12], we know X is a para-compact space with satisfying the requirement of locally limited refinement for each open coverage. Moreover, by [41, Section 3], we can assign X a Finsler structure and we know X is a Finsler manifold.

Proposition 5.14 ([21, Theorem 3.2])**.** *Let φ be a C^1-functional on a complete connected C^1-Finsler manifold X (without boundary) and consider a homotopy stable family $\mathcal{F}$ of compact subsets of X with a closed boundary B.*

$$c = c(\varphi, \mathcal{F}) = \inf_{Y \in \mathcal{F}} \max_{u \in Y} \varphi(u),$$

and suppose that $\sup \varphi(\Theta) < c$. *Then for any sequence of sets* $\{Y_n\}$ *in* $\mathcal{F}$ *such that* $\lim_{n\to\infty} \sup_{Y_n} \varphi = c$, *there exists a sequence* $\{u_n\}$ *in* X *such that*

$$\lim_{n\to\infty} \varphi(u_n) = c; \quad \lim_{n\to\infty} \|d\varphi(u_n)\| = 0; \quad \textit{and} \quad \lim_{n\to\infty} \operatorname{dist}(u_n, Y_n) = 0.$$

Furthermore, if $d\varphi$ *is uniformly continuous, then* u_n *can be chosen to be in* Y_n *for each* n.

Proof of Theorem 1.2. Using the strategy from [25], for $\delta > 0$ be defined by Lemma 5.6, let the function $\tilde{J} : \mathbb{R} \times H \mapsto \mathbb{R}$ as

$$\begin{aligned}\tilde{J}(t,(u,v)) := J(t*(u,v)) = \frac{e^{2st}}{2}([u]^2 + [v]^2) - \frac{e^{sp\gamma_p t}}{p}(\mu_1|u|_p^p + \mu_2|v|_p^p) \\ - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t} \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ - \frac{e^{22^*_{\alpha,s}st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx,\end{aligned}$$

then $\tilde{J} \in C^1$ and a Palais–Smale sequence for $\tilde{J}|_{\mathbb{R}\times S_r}$ is a Palais–Smale sequence for $\tilde{J}|_{\mathbb{R}\times S}$. Setting $J^c := \{(u,v) \in S, J(u,v) \le c\}$, we introduce the minimax class

$$\Gamma := \{\gamma = (\alpha,\beta) \in C([0,1], \mathbb{R} \times S_r) : \gamma(0) \in (0, \overline{B}_\delta), \gamma(1) \in (0, J^0)\},$$

with the minimax level

$$\sigma(a,b) := \inf_{\gamma\in\Gamma} \max_{(t,(u,v))\in\gamma([0,1])} \tilde{J}(t,(u,v)).$$

Let $(u,v) \in S_r$. From $[t*u]^2 + [t*v]^2 \to 0^+$ as $t \to -\infty$ and $J(t*(u,v)) \to -\infty$ as $t \to +\infty$, there is $t_0 \ll -1$ and $t_1 \gg 1$ such that

$$\gamma_{(u,v)} : \tau \in [0,1] \mapsto (0, ((1-\tau)t_0 + \tau t_1) * (u,v)) \in \mathbb{R} \times S_r, \tag{5.1}$$

which is a path in Γ and $\sigma(a,b)$ is a real number. For any $\gamma = (\alpha,\beta) \in \Gamma$, we study the function

$$\Pi_\gamma : \tau \in [0,1] \mapsto P_{\mu_1,\mu_2}(\alpha(\tau) * \beta(\tau)) \in \mathbb{R}.$$

From Lemma 5.6, we find $\Pi_\gamma(0) = P_{\mu_1,\mu_2}(\beta(0)) > 0$. Besides, from Lemma 5.3, $\Psi_{\mu_1,\mu_2}(t) > 0$ for any $t \in (-\infty, t_{u,v})$. If $(u,v) = \beta(1)$, we have $\Psi_{\mu_1,\mu_2}(0) = J(\beta(1)) \le 0$. Hence, we obtain $t_{\beta(1)} < 0$ and $\Pi_\gamma(1) = P_{\mu_1,\mu_2}(0 * \beta(1)) < 0$. Since the map $\tau \mapsto \alpha(\tau) * \beta(\tau)$ is continuous from $[0,1]$ to H, there exists $\tau_\gamma \in (0,1)$ such that $\Pi_\gamma(\tau_\gamma) = 0$, which implies $\alpha(\tau_\gamma) * \beta(\tau_\gamma) \in \mathcal{P}_{\mu_1,\mu_2} \cap S_r$ and

$$\max_{\gamma([0,1])} \tilde{J} \ge \tilde{J}(\gamma(\tau_\gamma)) = J(\alpha(\tau_\gamma) * \beta(\tau_\gamma)) \ge \inf_{\mathcal{P}_{\mu_1,\mu_2}\cap S_r} J(u,v) = m_r(a,b).$$

Therefore, $\sigma(a,b) \ge m_r(a,b)$. On the other hand, for any $(u,v) \in \mathcal{P}_{\mu_1,\mu_2} \cap S_r$, from (5.1), $\gamma_{(u,v)}$ is a path in Γ and by Lemma 5.3,

$$J(u,v) = \max_{\gamma_{(u,v)}([0,1])} \tilde{J} \ge \sigma(a,b),$$

then $m_r(a,b) \ge \sigma(a,b)$. Combining this with (5.6), we get

$$\sigma(a,b) = m_r(a,b) > \sup_{(\overline{B}_\delta \cup J^0)\cap S_r} J(u,v) = \sup_{((0,\overline{B}_\delta)\cup(0,J^0))\cap(\mathbb{R}\times S_r)} \tilde{J}.$$

From Definition 5.11, the set $\{\gamma([0,1]) : \gamma \in \Gamma\}$ is a homotopy stable family of compact subsets of $\mathbb{R} \times S_r$ with closed boundary $(0, \overline{B}_\delta) \cup (0, J^0)$. By Proposition 5.14, similar to [24,32], taking any minimizing sequence $\{\gamma_n = (\alpha_n, \beta_n)\} \subset \Gamma_n$ for $\sigma(a,b)$ with $\alpha_n \equiv 0$, and $\beta_n(\tau) \geq 0$ a.e. in $\mathbb{R}$, for every $\tau \in [0,1]$, there exists a Palais–Smale sequence $\{t_n, w_n\} \subset \mathbb{R} \times S_r$ for $\tilde{J}|_{\mathbb{R}\times S_r}$ at the level $\sigma(a,b)$, where $w_n = (u_n, v_n)$, such that,

$$\partial t\tilde{J}(t_n, w_n) \to 0, \quad \|\partial w\tilde{J}(t_n, w_n)\|_{(T_{w_n}S_r)*} \to 0, \quad \text{as } n \to \infty, \tag{5.2}$$

with an additional property

$$|t_n| + \text{dist}_H(w_n, \beta_n([0,1])) \to 0, \quad \text{as } n \to \infty. \tag{5.3}$$

From (5.3), we have t_n is bounded in both side. Besides, from the first formula of (5.2), we have $P_{\mu_1,\mu_2}(t_n * (u_n, v_n)) \to 0$, and from the second formula of (5.2) with the boundedness of t_n, for any $\varphi \in T_{w_n}S_r$,

$$dJ(t_n * w_n)[t_n * \varphi] = o_n(1)\|\varphi\| = o_n(1)\|t_n * \varphi\|, \quad \text{as } n \to \infty.$$

Following, we define $\hat{w}_n := t_n * w_n$ with $\hat{w}_n = (\hat{u}_n, \hat{v}_n)$. Therefore, $\{(\hat{u}_n, \hat{v}_n)\}$ is a Palais–Smale sequence for $J(u,v)|_{S_r}$ at the level $\sigma(a,b)$ with an additional condition $P_{\mu_1,\mu_2}(\hat{u}_n, \hat{v}_n) \to 0$. From Lemma 5.8, there exists $\hat{\beta}_1 > 0$, $m_r(a,b) \in \left(0, \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}\right)$ for any $\beta > \hat{\beta}_1$. Besides, from Lemma 5.10, there exists $\hat{\beta}_2 > 0$ such that $m_r(a,b) < \min\{m(a,0), m(0,b)\}$. Following, we may require $\beta_0 = \max\{\hat{\beta}_1, \hat{\beta}_2\}$. Then for $\beta \in (\beta_0, +\infty)$, by Proposition 4.9, we know there is $(u,v) \in H$ with $u, v > 0$ a.e. in $\mathbb{R}$, such that $(\hat{u}_n, \hat{v}_n) \to (u,v)$ in H and $J(u,v) = m_r(a,b)$. Hence, we need to show

$$\inf_{\mathcal{P}_{\mu_1,\mu_2}\cap S_r} J(u,v) = \inf_{\mathcal{P}_{\mu_1,\mu_2}} J(u,v) = m(a,b).$$

If this does not happen, there is $(\bar{u}, \bar{v}) \in \mathcal{P}_{\mu_1,\mu_2} \setminus S_r$ such that $J(\bar{u}, \bar{v}) < m_r(a,b)$. Denote $(\tilde{u}, \tilde{v}) := (|\bar{u}|^*, |\bar{v}|^*)$ as the symmetric decreasing rearrangement of $(\bar{u}, \bar{v})$ such that

$$[\tilde{u}]^2 + [\tilde{v}]^2 \leq [\hat{u}]^2 + [\hat{v}]^2, \quad J(\tilde{u}, \tilde{v}) \leq J(\bar{u}, \bar{v}), \quad \text{and} \quad P_{\mu_1,\mu_2}(\tilde{u}, \tilde{v}) \leq P_{\mu_1,\mu_2}(\bar{u}, \bar{v}) = 0.$$

If $P_{\mu_1,\mu_2}(\tilde{u}, \tilde{v}) = 0$, which $(\tilde{u}, \tilde{v}) \in \mathcal{P}_{\mu_1,\mu_2} \cap S_r$, there is a contradiction. On the other hand, if $P_{\mu_1,\mu_2}(\tilde{u}, \tilde{v}) < 0$, from Lemma 5.3, we get $t_{\tilde{u},\tilde{v}} < 0$. Therefore, by $t_{\tilde{u},\tilde{v}} * (\tilde{u}, \tilde{v}) \in \mathcal{P}_{\mu_1,\mu_2}$, we have

$$\begin{aligned}
J(\bar{u}, \bar{v}) &\leq J(t_{\tilde{u},\tilde{v}} * (\tilde{u}, \tilde{v})) - \frac{1}{2}P_{\mu_1,\mu_2}(t_{\tilde{u},\tilde{v}} * (\tilde{u}, \tilde{v})) \\
&= \left(\frac{\gamma_p}{2} - \frac{1}{p}\right)e^{p\gamma_p s t_{\tilde{u},\tilde{v}}}(\mu_1|\tilde{u}|_p^p + \mu_2|\tilde{v}|_p^p) \\
&\quad + \beta\left[\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2} - 1\right]e^{(r_1+r_2)\gamma_{(r_1+r_2)}st_{\tilde{u},\tilde{v}}}\int_{\mathbb{R}^N}|\tilde{u}|^{r_1}|\tilde{v}|^{r_2}dx \\
&\quad + \left(1 - \frac{1}{2^*_{\alpha,s}}\right)e^{22^*_{\alpha,s}st_{\tilde{u},\tilde{v}}}\int_{\mathbb{R}^N}(I_\alpha * |\tilde{u}|^{2^*_{\alpha,s}})|\tilde{v}|^{2^*_{\alpha,s}}dx \\
&< J(\bar{u}, \bar{v}),
\end{aligned}$$

which is a contradiction. Thus, we have $m_r(a,b) = m(a,b)$ and (u,v) is a ground state solution. □

6 The case: $p = r_1 + r_2 = 2_s^*$

Lemma 6.1. *Assume $s \in (0,1)$, $2s < N \le 4s$, $p = r_1 + r_2 = 2_s^*$ and $a, b, \mu_1, \mu_2, \beta > 0$. Then the following system*

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1-2} u |v|^{r_2} + (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2} u, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{p-2} v + \beta r_2 |v|^{r_2-2} v |u|^{r_1} + (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2} v, \\ \int_{\mathbb{R}^N} u^2 dx = a^2, \quad \int_{\mathbb{R}^N} v^2 dx = b^2, \quad u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{6.1}$$

has no positive solution.

Proof. Assume by contradiction that there is a positive solution (u, v) of (6.1) for some $\lambda_1, \lambda_2 \in \mathbb{R}$. On one hand, from Proposition 4.9 and [29, Lemma 2.3], we see that $\lambda_1, \lambda_2 < 0$ for $2s < N \le 4s$. On the other hand, by Proposition 2.4 we know (u, v) satisfies the Pohožaev identity such that

$$[u]^2 + [v]^2 = (\mu_1 |u|_{2_s^*}^{2_s^*} + \mu_2 |v|_{2_s^*}^{2_s^*}) + \beta 2_s^* \int_{\mathbb{R}^N} |u|^{r_1} |v|^{r_2} dx + 2 \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx. \tag{6.2}$$

Moreover since (u, v) is a weak solution to (1.1)-(1.2), it satisfies

$$\begin{aligned} [u]^2 + [v]^2 = & \int_{\mathbb{R}^N} (\lambda_1 |u|^2 + \lambda_2 |v|^2) dx + (\mu_1 |u|_{2_s^*}^{2_s^*} + \mu_2 |v|_{2_s^*}^{2_s^*}) + \beta 2_s^* \int_{\mathbb{R}^N} |u|^{r_1} |v|^{r_2} dx \\ & + 2 \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx. \end{aligned} \tag{6.3}$$

Combining (6.2)–(6.3), we show

$$\int_{\mathbb{R}^N} \lambda_1 |u|^2 + \lambda_2 |v|^2 dx = \lambda_1 a^2 + \lambda_2 b^2 = 0.$$

From which we obtain $\lambda_1 = \lambda_2 = 0$. This is clearly a contradiction with $\lambda_1, \lambda_2 < 0$. The proof is complete. □

Proof of Theorem 1.3. Theorem 1.3 follows from Lemma 6.1, then we finish the proof. □

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Declarations

We would like to thank you for following the above instructions. This will definitely speed up the publication process of your paper.

Data Availability

Date sharing is not applicable to this article as no new data were created and analyzed in this study.

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