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Existence and multiplicity of solutions for a p-Laplacian fractional system with logarithmic nonlinearity

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Abstract. This paper is concerned with the existence and multiplicity of a ground state solution for the following class of elliptic fractional type problems given by

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = \lambda h_1(x)|u|^{\theta-2}u \ln|u| + \dfrac{q}{q+r} b_1(x)|v|^r|u|^{q-2}u & \text{in } \Omega, \\ (-\Delta)_p^t v + |v|^{p-2}v = \mu h_2(x)|v|^{\theta-2}v \ln|v| + \dfrac{r}{q+r} b_2(x)|u|^r|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s,t \in (0,1)$, $N > \max\{ps, pt\}$, $\lambda, \mu > 0$, $p \le \theta$, $2 < q + r < \min\left\{\frac{pN}{N-ps}, \frac{pN}{N-pt}\right\}$, and the additional weights $h_1, h_2, b_1, b_2 \in C(\overline{\Omega})$ are such that: $b_1(x), b_2(x)$ are positive functions and $h_1(x), h_2(x)$ are sign-changing functions. The operators $(-\Delta)_p^s$ and $(-\Delta)_p^t$ represents, both, fractional p-Laplacian operator, a generalization for the fractional Laplacian $(-\Delta)^s, 0 < s < 1$ $(p = 2)$, defined in a integral way as

$$(-\Delta)^s u(x) := \frac{c(n,s)}{2} \int_{\mathbb{R}^N} \frac{2u(x) - (x+y) - u(x-y)}{|y|^{n+2s}} dy, \qquad x \in \mathbb{R}^N,$$

where $c(n,s)$ is a positive normalizing constant, and another fractional operator.

Specifically, the operators $(-\Delta)_p^s$ and $(-\Delta)_p^t$ are defined, up to a normalization constant, by the formula

$$(-\Delta)_p^\ell u(x) := \lim_{\varepsilon \to 0^+} 2 \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} dy,$$

for all $u \in C_0^\infty(\mathbb{R}^N)$, $x \in \mathbb{R}^N$, and $\ell \in \{s, t\}$.

Keywords: fractional p-Laplacian operator, logarithmic nonlinearity, Nehari manifold, mountain pass theorem.

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1 Introduction and main results

Since logarithmic nonlinearities play a significant role in depicting the mathematical and physical phenomena, they have received much attention in PDEs (see [15, 16, 19] and references therein).

In [15], the authors studied the existence and multiplicity of a class of fractional Laplacian systems with logarithmic nonlinearity in which three types of weights with certain regularity are involved.

$$\begin{cases} (-\Delta)^s u = \lambda h_1(x) u \ln(|u|) + \dfrac{p}{p+q} b(x)|v|^q|u|^{p-2}u & \text{in } \Omega, \\ (-\Delta)^t v = \mu h_2(x) v \ln(|v|) + \dfrac{r}{q+r} b(x)|u|^r|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s, t \in (0,1)$, $N > \max\{2s, 2t\}$, $\lambda, \mu > 0$, $2 < q + p < \min\left\{\frac{2N}{N-2s}, \frac{2N}{N-2t}\right\}$, and the additional weights $h_1, h_2, b \in C(\overline{\Omega})$ are such that: $b(x)$ are positive functions and $h_1(x), h_2(x)$ are sign-changing functions.

In [16], the authors studied a class of systems of equations where they showed the existence and multiplicity of solutions for a mixed local-nonlocal system with logarithmic nonlinearities

$$\begin{cases} (-\Delta)^s u_j + u_j = \lambda_j a_j(x) u_j \ln|u_j| + \displaystyle\sum_{i \neq j} \beta_{ij}|u_j|^{q-2}u_j & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s \in (0,1)$, $N > 4$, λ_j are parameters, $\beta_{ij} > 0$ for all $1 \leq i < j \leq k$, $\beta_{ij} = \beta_{ji}$ for all $i, j = 1, \ldots, k$, $a_j \in C(\overline{\Omega})$. When $1 < 2q < 2 < 2^* = \frac{2N}{N-2}$ and a_j they are functions that change sign, they obtained two different solutions using Nehari method. When $1 < q < 2 < 2^* = \frac{2N}{N-2}$ and a_j are positive constant functions, the existence of the ground state solution is obtained using the minimization method.

In [9], the authors studied a class of fractional Laplacian systems where they showed the existence of solutions using the Nehari method and multiplicity of solutions using the Lusternik–Schnirelmann category, with polynomial nonlinearity.

$$\begin{cases} (-\Delta)^s u = \lambda|u|^{p-2}u + \dfrac{2\alpha}{\alpha+\beta}|v|^\beta|u|^{\alpha-2}u & \text{in } \Omega, \\ (-\Delta)^s v = \mu|u|^{p-2}u + \dfrac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $s \in (0,1)$, $N > 4s$, $\lambda, \mu > 0$ are parameters, $\alpha + \beta = \frac{2N}{N-2s}$ is the critical Sobolev exponent.

In [6], the authors showed the existence of solutions for the fractional critical system (p, q)

Laplacian using variational method

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda Q(x)|u|^{r-2}u + \dfrac{2\alpha}{\alpha+\beta}|v|^{\beta}|u|^{\alpha-2}u & \text{in } \Omega, \\ (-\Delta)_p^{s_1} v + (-\Delta)_q^{s_2} v = \mu Q(x)|u|^{r-2}u + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < s_2 < s_1 < 1$, $1 < q < p < r < p_{s_1}^*$, $N > ps_1$, $\lambda, \mu > 0$ are parameters and $\alpha > 1$ $\beta > 1$ satisfy $\alpha + \beta = p_{s_1}^*$ with $\frac{Np}{N-ps_1}$ is the critical Sobolev exponent, and $(-\Delta)_t^s$ is the fractional t-Laplacian operator.

In [19] was concerned with the existence and asymptotic behavior of normalized ground states solutions for the following coupled Schrödinger system with logarithmic terms:

$$\begin{cases} -\Delta u_1 + \omega_1 u_1 = \mu_1 u_1 \log u_1^2 + \dfrac{p}{p+q}|u_2|^q|u_1|^{p-2}u_1, \\ -\Delta u_2 + \omega_2 u_2 = \mu_2 u_2 \log u_2^2 + \dfrac{q}{p+q}|u_1|^p|u_2|^{q-2}u_2, \\ \displaystyle\int_\Omega |u_i|^2 dx = \rho_i, \ i = 1, 2, \end{cases} \tag{1.5}$$

where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $\omega_i \in \mathbb{R}, \mu_i, \rho_i > 0, i = 1, 2$. Moreover, $p, q \geq 1, 2 \leq p + q \leq 2^*$ where $2^* := \frac{2N}{N-2}$.

In [17] the authors studied the existence of least energy solutions to the following fractional Kirchhoff problem with logarithmic nonlinearity

$$\begin{cases} M([u]_{s,t}^p)(-\Delta)_p^s u = h(x)|u|^{\theta p-2}u \ln|u| + \lambda|u|^{q-2}u, & x \in \Omega \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.6}$$

where $s \in (0, 1)$, $1 < p < Np$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $M([u]_{s,p}^p) = [u]_{s,p}^{(\theta-1)p}$, with $\theta \geq 1$ and $u, h \in C(\overline{\Omega})$ may change sign, $\lambda > 0$ and $q \in (1, p_s^*)$.

On the other hand, parabolic and hyperbolic type equations with logarithmic nonlinearity have been studied extensively in recent years. Here we only refer some results involving fractional Laplacian. [2] considered the Cauchy problem of the following Schrödinger equation

$$iu_t - (-\Delta)^s u + u \log|u|^2 = 0, \quad x \in \Omega, \ t > 0. \tag{1.7}$$

The existence of global solutions was obtained by using a compactness method. Moreover, the author obtained the existence of ground states by the Nehari manifold approach. Xiang et al. [18] studied the initial boundary value problem of the following parabolic equation involving logarithmic nonlinearity

$$u_t + [u]_s^{2(\kappa-1)}(-\Delta)^s u = |u|^{q-2}u \ln|u|^2, \quad x \in \Omega, \ t > 0. \tag{1.8}$$

where $\kappa \in (1, 2_s^*/2)$ and $2\kappa < q < 2^*$. By the potential well theory, the existence of global solutions and blow-up properties of local solutions were discussed. Particularly, using the Nehari manifold approach, the existence of ground state solutions for above stationary problem was investigated.

The main tool used in this paper is the so-called fibering method introduced by Pohozaev [11], [12] and [13], and applied to a single equation of p-Laplacian type by Drábek and Pohozaev in [8]. Bozhkov and Mitidieri [4] used this method to study the existence for multiple solution for a quasilinear system. Fibering method is very used to proof existence of multiple solution for a large class of equations. For example, Brown and Wu [5] used this method to show the existence of at least two positive solutions for the semilinear elliptic boundary-value problem. For more recent applying of fibering method we indicate [1], used to show existence of multiple solution for a class of Schrödinger equations involving indefinite weight functions, and other interesting work involving the fractional operator is [7]

In this paper motivated by [9, 15, 16, 19] we deal with the existence and multiplicity of ground state solutions for the subcritical for the following class of problems

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = \lambda h_1(x)|u|^{\theta-2}u\ln(|u|) + \dfrac{q}{q+r}b_1(x)|v|^r|u|^{q-2}u & \text{in } \Omega, \\ (-\Delta)_p^t v + |v|^{p-2}v = \mu h_2(x)|v|^{\theta-2}v\ln(|v|) + \dfrac{r}{q+r}b_2(x)|u|^r|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{P}$$

Taking $W_0^{s,j} = \{u \in W^{s,j}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$, we define the normed space W, denoted by $W := W_0^{s,p}(\Omega) \times W_0^{s,t}(\Omega)$ (see Section 3), the main results of this paper is writing as follows:

Theorem 1.1. *Problem* (P) *has a nontrivial ground state solution in W for $s,t \in (0,1)$, $2 < p+q < \min\{p_s^*, p_t^*\}$ and $h_1(x), h_2(x)$ positive functions in $C(\overline{\Omega})$.*

Theorem 1.2. *Problem* (P) *has a nontrivial ground state solution in W for $s,t \in (0,1)$, $2 < p+q < \min\{p_s^*, p_t^*\}$ and $h_1(x), h_2(x)$ two sign-changing functions in $C(\overline{\Omega})$.*

The paper is organized as follows. In Section 2 we study the variational framework. In Section 3 study Nehari manifold and fibering map analysis. In Section 4 we prove Theorem 1.1. In Section 5 we prove Theorem 1.2.

2 The variational framework

First of all, in this section we introduce the fractional Sobolev space where lies the solution for Problem (P). After defining this space, we introduce some technical results that will be used to proof the main theorems.

Let $\Omega \subset \mathbb{R}^N$ be a domain. For for $p \in [1,\infty)$ and $s \in (0,1)$, we define the fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega\times\Omega) \right\}; \tag{2.1}$$

i.e., an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_s = \|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p dx + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}$$

where

$$[u]_s = [u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}$$

is the so-called Gagliardo seminorm of u.

Moreover, we define the corresponding local fractional Sobolev spaces by

$$W^{s,p}_{\mathrm{loc}}(\Omega) := \{u \in L^p_{\mathrm{loc}}(\Omega) : \ u \in W^{s,p}(\Omega') \text{ for any } \Omega' \subset\subset \Omega\}.$$

Also, we define the space

$$W^{s,p}_0(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Now we are able to define the space $W := W^{s,p}_0(\Omega) \times W^{s,t}_0(\Omega)$ where lies the solutions of the Problem (P) in the Theorems 1.1 and 1.2. One can check that $(W, |\cdot|)$ is a normed space, with norm given by

$$\|(u,v)\| := \|(u,v)\|_W = \left([u]_s^p + |u|_p^p + [v]_t^p + |v|_p^p\right)^{1/p}.$$

Motivated by [10] we show the following lemma for system.

Lemma 2.1. *W is continuously embedding on $L^{s,p}(\Omega) \times L^{s,t}(\Omega)$.*

Proof. Let $u \in W^{s,p}(\Omega)$. Since $\Omega \subset \mathbb{R}^N$ is an extension domain for $W^{s,p}$, then there exists a constant $C_1 = C_1(N,p,s,\Omega) > 0$ such that

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)} \le C_1 \|u\|_{W^{s,p}(\Omega)} \tag{2.2}$$

with $\tilde{u}$ such that $\tilde{u}(x) = u(x)$ for x a.e. in Ω. On the other hand, by [10, Theorem 6.7], the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$, for any $r \in [p, p_s^*]$; i.e., there exists a constant $C_2 = C_2(N,p,s) > 0$ such that

$$\|\tilde{u}\|_{L^r(\mathbb{R}^N)} \le C_2 \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)}. \tag{2.3}$$

Using (2.2) and (2.3), we get

$$\|u\|_{L^r(\Omega)} = \|\tilde{u}\|_{L^r(\Omega)} \le \|\tilde{u}\|_{L^r(\mathbb{R}^N)} \le C_2 \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)} \le C_2 C_1 \|u\|_{W^{s,p}(\Omega)}. \tag{2.4}$$

Similarly we have the following inequality

$$\|v\|_{L^\eta(\Omega)} = \|\tilde{v}\|_{L^\eta(\Omega)} \le \|\tilde{v}\|_{L^\eta(\mathbb{R}^N)} \le C_3 \|\tilde{v}\|_{W^{t,p}(\mathbb{R}^N)} \le C_3 C_4 \|v\|_{W^{t,p}(\Omega)}. \tag{2.5}$$

Finally using (2.4) and (2.5) we have

$$\|(u,v)\|_{L^r(\Omega)\times L^\eta(\Omega)} \le \tilde{C} \|(u,v)\|_W \tag{2.6}$$

for some $C \in (0,\infty)$, from which follows the proposition. □

Lemma 2.2. *If $q + r < \min\{p_s^*, p_t^*\}$ then there exists a positive constant c such that*

$$\left(\int_\Omega |u|^r |v|^q dx \right)^{\frac{1}{q+r}} \le c \|(u,v)\|.$$

Proof. The proof follows from the definition

$$S_{r,q} := \inf_{(u,v)\in W\setminus\{(0,0)\}} \left\{ \frac{\|(u,v)\|^p}{\left(\int_\Omega |u|^r|v|^q dx\right)^{\frac{p}{q+r}}} \right\}. \tag{2.7}$$

and the inequality $|u|^r|v|^q \leq \frac{r}{r+q}|u|^{r+q} + \frac{q}{r+q}|v|^{r+q} \leq |u|^{r+q} + |v|^{r+q}$. □

Lemma 2.3. *Let $\xi, \eta \in \mathbb{R}^N$. Then,*

$$|\xi - \eta|^p \leq \begin{cases} 2^p\left(|\xi|^{p-2}\xi + |\eta|^{p-2}\eta\right)(\xi - \eta), & \text{for } p \geq 2, \\ \dfrac{1}{(p-1)}\dfrac{\left[(|\xi|^{p-2}\xi + |\eta|^{p-2}\eta)(\xi - \eta)\right]^{p/2}}{\left(|\xi|^p + |\eta|^p\right)^{(p-2)/p}}, & \text{for } 1 < p < 2. \end{cases} \tag{2.8}$$

The proof of this lemma can be found in [14, Lemma 6].

Definition 2.4. We say that $(u,v) \in W$ is a (weak) solution of Problem (P) if

$$\begin{aligned} &\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy + \int_\Omega |u|^{p-2}u\varphi dx \\ &\quad + \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+pt}}dxdy + \int_\Omega |v|^{p-2}v\phi dx \\ &\quad = \lambda\int_\Omega h_1(x)|u|^{\theta-2}u\ln|u|\varphi dx + \mu\int_\Omega h_2(x)|v|^{\theta-2}v\ln|v|\phi dx \\ &\quad + \frac{q}{q+r}\int_\Omega b_1(x)|v|^r|u|^{q-2}u\varphi dx + \frac{r}{q+r}\int_\Omega b_2(x)|u|^q|v|^{r-2}v\phi dx, \end{aligned}$$

for any $(\varphi, \phi) \in W$.

Now we consider the energy functional for Problem (P), $E : W \to \mathbb{R}$ defined by

$$\begin{aligned} E(u,v) = &\frac{1}{p}[u]_{s,p}^p + \frac{1}{p}\int_\Omega |u|^p dx + \frac{1}{p}[v]_{s,t}^p + \frac{1}{p}\int_\Omega |v|^p dx - \frac{\lambda}{\theta}\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ &+ \frac{\lambda}{\theta^2}\int_\Omega h_1(x)|u|^\theta dx - \frac{\mu}{\theta}\int_\Omega h_2(x)|v|^\theta \ln|v|dx + \frac{\mu}{\theta^2}\int_\Omega h_2(x)|v|^\theta dx \\ &- \frac{1}{q+r}\int_\Omega b_1(x)|u|^q|v|^r dx - \frac{1}{q+r}\int_\Omega b_2(x)|u|^q|v|^r dx. \end{aligned}$$

We also consider the functional

$$\begin{aligned} I(u,v) = &[u]_{s,p}^p + \int_\Omega |u|^p dx + [v]_{s,t}^p + \int_\Omega |v|^p dx - \lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ &- \mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx - \int_\Omega b_1(x)|u|^q|v|^r dx - \int_\Omega b_2(x)|u|^q|v|^r dx. \end{aligned}$$

As consequence of embedding on Remark 2.1 and Lemma 2.2, we obtain that the functional I

is well-defined and $E, I \in C^1(W, \mathbb{R})$. Moreover, note that, for all $(\varphi, \phi) \in W$,

$$\begin{aligned}
E'(u,v)(\varphi,\phi) = & \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy + \int_\Omega |u|^{p-2}u\varphi dx \\
& + \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+pt}}dxdy + \int_\Omega |v|^{p-2}v\phi dx \\
& - \lambda \int_\Omega h_1(x)|u|^{\theta-2}u\ln|u|\varphi dx - \mu \int_\Omega h_2(x)|v|^{\theta-2}v\ln|v|\phi dx \\
& - \frac{q}{q+r}\int_\Omega b_1(x)|v|^r|u|^{q-2}u\varphi dx - \frac{r}{q+r}\int_\Omega b_2(x)|u|^q|v|^{r-2}v\phi dx
\end{aligned}$$

and

$$\begin{aligned}
I'(u,v)(\varphi,\phi) = p & \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy + p\int_\Omega |u|^{p-2}u\varphi dx \\
& + p\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+pt}}dxdy + p\int_\Omega |v|^{p-2}v\phi dx \\
& - \lambda\theta \int_\Omega h_1(x)|u|^{\theta-2}u\varphi\ln|u|dx - \lambda\int_\Omega h_1(x)|u|^{\theta-2}u\varphi dx \\
& - \mu\theta \int_\Omega h_2(x)|v|^{\theta-2}v\phi\ln|v|dx - \mu\int_\Omega h_2(x)|v|^{\theta-2}v\phi dx \\
& - (q+r)\int_\Omega b_1(x)|v|^r|u|^{q-2}u\varphi dx - (q+r)\int_\Omega b_2(x)|u|^q|v|^{r-2}v\phi dx.
\end{aligned}$$

Some important results will be used in this paper, including those concerning Sobolev's Logarithmic Inequalities to obtain some estimates of the problem and to resolve some arisen difficult in the logarithmic term:

Lemma 2.5. *Let ρ be a positive real number. Then we have the following inequalities*

$$\ln t \leq \frac{t^\rho}{e\rho} \quad \text{for all } 1 \leq t \tag{2.9}$$

and

$$|t^\rho \ln t| \leq \frac{1}{e\rho} \quad \text{for all } 0 < t < 1. \tag{2.10}$$

where e is Euler logarithm basis.

Proof. We consider the following function $h : [1, \infty) \to \mathbb{R}$ defined by

$$h(t) = \ln t - \frac{1}{e\rho}t^\rho \quad \text{for all } t \geq 1.$$

With respect to t, just by taking a simple derivative, we deduce

$$h'(t) = \frac{1}{t} - \frac{1}{e}t^{\rho-1} \quad \text{for all } t \geq 1.$$

Then, $t_* = e^{1/\rho}$ is the unique maximum point of function h. Thus, $h(t) \leq h(t_*) = 0$ for all $t \geq 1$ and the inequality (2.9) is valid.

To prove the inequality (2.10), we consider $g : (0, 1] \to \mathbb{R}$ defined by

$$g(t) = t^\rho |\ln t| \quad \text{for all } t \in (0, 1].$$

We can verify that $t_* = e^{-1/\rho}$ is the unique minimum point of the function g. We can also check that $\lim_{t\to 0^+} g(t) = 0$. Thus,

$$g(t) \geq g(t_*) = -\frac{1}{e\rho}$$

and the proof is complete. □

Lemma 2.6. *Assume that (u_n, v_n) is bounded in W such that u_n converges to u a.e. and v_n converges to v in Ω. Then*

$$\lim_{n\to\infty} \int_\Omega h_1(x)|u_n|^\theta \ln|u_n| dx = \int_\Omega h_1(x)|u|^\theta \ln|u| dx.$$

and

$$\lim_{n\to\infty} \int_\Omega h_2(x)|v_n|^\theta \ln|v_n| dx = \int_\Omega h_2(x)|v|^\theta \ln|v| dx,$$

for $2 < \theta < q + r$.

Proof. Since that (u_n, v_n) is bounded in W, we get that $\|(u_n, v_n)\| \leq C$, for all $n \in \mathbb{N}$, we first need to discuss the two cases of $|u_n|$ as follows.

Case I. If $\frac{|u_n(x)|}{\|(u_n,v_n)\|} < 1$. By Lemma 2.5 with (2.9) with $\rho = \theta < p_s^*$, from the Sobolev Inequalities, we get

$$\begin{aligned}
\left| h_1(x)|u_n|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right) \right| &\leq C_{h_1} \left| |u|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right) \right| \\
&= C_{h_1} \|(u_n,v_n)\|^\theta \left| \frac{|u_n(x)|}{\|(u_n,v_n)\|} \right|^\theta \left| \ln\left(\frac{|u_n(x)|}{\|(u_n,v_n)\|}\right) \right| \\
&\leq \frac{C_{h_1}}{e\theta} \|(u_n,v_n)\|^\theta =: g_n^1. \qquad (2.11)
\end{aligned}$$

Since $(u_n, v_n) \rightharpoonup (0,0)$ in W, then $(u_n, v_n) \to (0,0)$ in $L^\theta(\Omega) \times L^\theta(\Omega)$ for all $\theta \in (1, \min\{p_s^*, p_t^*\})$ using the Sobolev embeddings, we have

$$\int_\Omega |g_n^1(x)| dx \leq \frac{C_{H_1}}{e\theta} \|(u_n,v_n)\|^\theta \to 0, \quad \text{as } n \to \infty. \qquad (2.12)$$

Thus, $|g_n^1| \to 0$ in $L^1(\Omega)$, which means that there exists some $g_1 \in L^1(\Omega)$ such that $|g_n^1(x)| \leq g^1(x)$ a.e. in Ω for all $n \in \mathbb{N}$.

Case II. If $\frac{|u_n(x)|}{\|(u_n,v_n)\|} > 1$. Using Lemma 2.5 with (2.9) and Hölder's inequality, we get

$$\begin{aligned}
\left| h_1(x)|u_n|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right) \right| &\leq C_{h_1} \left| |u_n|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right) \right| \\
&= C_{h_1} \|(u_n,v_n)\|^{\theta-\rho} |u_n|^\rho \left| \frac{|u_n(x)|}{\|(u_n,v_n)\|} \right|^{\theta-\rho} \left| \ln\left(\frac{|u_n(x)|}{\|(u,v)\|}\right) \right| \\
&\leq C_{h_1} \frac{\|(u_n,v_n)\|^{\theta-\rho}}{e(\theta-\rho)} |u_n|^\rho := g_2^n. \qquad (2.13)
\end{aligned}$$

Since $(u_n, v_n) \rightharpoonup (0,0)$ in W, then $(u_n, v_n) \to (0,0)$ in $L^\theta(\Omega) \times L^\theta(\Omega)$ for all $\theta \in (1, \min\{p_s^*, p_t^*\})$ using the Sobolev embeddings, we have

$$\int_\Omega |g_n^2(x)| dx \leq \frac{C_{h_1}}{e(\theta-\rho)} \|(u_n,v_n)\|^\theta \to 0, \quad \text{as } n \to \infty. \qquad (2.14)$$

Thus, $|g_n^2| \to 0$ in $L^1(\Omega)$, which means that there exists some $g_1 \in L^1(\Omega)$ such that $|g_n^2(x)| \leq g^1(x)$ a.e. in Ω for all $n \in \mathbb{N}$. Now, defining $f_n := |u_n(x)|^\theta |\varphi|$, we also have $f_n \to 0$ in $L^1(\Omega)$, which provides $f \in L^1(\Omega)$ such that $|f_n(x)| \leq f(x)$ a.e. in $\mathbb{N}$ for all $n \in \mathbb{N}$.
Combining (2.11) and (2.13) , we have the following estimate

$$\begin{aligned}\left|h_1(x)|u_n(x)|^\theta \ln\left(\frac{|u_n(x)|\cdot\|(u_n,v_n)\|}{\|(u_n,v_n)\|}\right)\right| &\leq C_{h_1}\left[\left||u_n(x)|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right)\right|\right.\\ &\qquad\left. + \ln\|(u_n,v_n)\||u_n(x)|^\theta\right]\\ &\leq g_n^1 + g_n^2 + f_n, \end{aligned} \tag{2.15}$$

where $g_n^1 + g_n^2 + f_n \in L^1(\Omega)$, for all $n \in \mathbb{N}$. For other hand, since $\{(u_n, v_n)\}$ is bounded in W, up to a subsequence, we may assume that $(u_n(x), v_n(x)) \to (u(x), v(x))$ a.e in Ω. This implies that

$$h_1(x)|u_n(x)|^\theta \ln|u_n(x)| \to h_1(x)|u(x)|^\theta \ln|u(x)|, \quad \text{a.e. in } \Omega. \tag{2.16}$$

Therefore, using (2.15) and (2.16) the Lebesgue Dominated Convergence Theorem yields that

$$\lim_{n\to+\infty}\int_\Omega h_1(x)|u_n|^\theta \ln|u_n|dx = \int_\Omega h_1(x)|u|^\theta \ln|u|dx. \tag{2.17}$$

Using the same idea of equation (2.17), we have the following convergence

$$\lim_{n\to+\infty}\int_\Omega h_2(x)|v_n|^\theta \ln|v_n|dx = \int_\Omega h_2(x)|v|^\theta \ln|v|dx. \tag{2.18}$$

The proof is concluded. □

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^N$ be a domain, $q \in [0,\infty)$ and (u_n) be a bounded sequence in $L^r(\Omega)$. If $u_n \to u$ almost everywhere on Ω as $n \to +\infty$, then for every $q \in [1, r]$.*

$$\lim_{n\to+\infty}\int_\Omega \left||u_n|^q - |u_n - u|^q - |u|^q\right|^{\frac{r}{q}} dx = 0. \tag{2.19}$$

Also recall that pointwise convergence of a bounded sequence implies weak converge (see for example [3, Proposition 4.7.12])

Lemma 2.8. *Let the sequences (u_n) and (v_n) be in $\in W_0^{s,p}(\Omega)$ such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$ and $u_n(x) \to u(x)$, $v_n(x) \to v(x)$ a.e. in Ω. Then,*

$$\lim_{n\to\infty}\int_\Omega \left[|u_n|^q|v_n|^r - |u_n - u|^q|v_n - v|^r\right]dx = \int_\Omega |u|^q|v|^r dx.$$

Proof. By direct calculation we have the following equality of integrals

$$\begin{aligned}&\int_\Omega \left(|u_n|^q|v_n|^r - |u_n - u|^q|v_n - v|^r\right)dx\\ &\quad= \int_\Omega \left(|u_n|^q - |u_n - u|^q)|v_n|^r + |u_n - u|^q(|v_n|^r - |v_n - v|^r\right)dx.\end{aligned}$$

Since that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$ and using Lemma 2.7, we get

$$\lim_{n\to+\infty}\int_\Omega \left(|u_n|^q - |u_n - u|^q - |u|^q\right)^{\frac{q+r}{q}} dx = 0. \tag{2.20}$$

which means that $|u_n|^q - |u_n - u|^q \to |u|^q$ in $L^{\frac{q+r}{q}}(\Omega)$, for other hand $|v_n|^r \rightharpoonup |v|^r$ in $L^{\frac{q+r}{r}}(\Omega)$, this implies that

$$\lim_{n\to+\infty} \int_\Omega \left(|u_n|^q - |u_n - u|^q - |u|^q\right)|v_n|^r dx = 0.$$

Also, note that $|v_n|^r - |v_n - v|^r \to |v|^q$ in $L^{\frac{q+r}{r}}(\Omega)$. As $|u_n - u|^q \rightharpoonup 0$ in $L^{\frac{q+r}{q}}(\Omega)$, this implies that

$$\lim_{n\to+\infty} \int_\Omega |u_n - u|^q \left(|v_n|^r - |v_n - v|^r\right) dx = 0. \tag{2.21}$$

This concludes the proof. □

Lemma 2.9. *Let $(u,v) \in W \setminus \{(0,0)\}$. Then*

$$\lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx$$
$$\leq (\lambda C_{h_1} + \mu C_{h_2}) L \|(u,v)\|^p + \ln\|(u,v)\| \left[\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\right]$$

where

$$L := \frac{|\Omega|}{e\theta} + \frac{1}{e(p_s^* - \theta)} p_{p_s^*}^{p_{p_s^*}} + \frac{1}{e(p_t^* - \theta)} p_{p_t^*}^{p_{p_t^*}}, \quad C_{h_1} := \max_{x\in\overline{\Omega}} |h_1(x)|, \quad C_{h_2} := \max_{x\in\overline{\Omega}} |h_2(x)|,$$

and $S_{p_s^*}^{p_{p_s^*}}, S_{p_t^*}^{p_{p_t^*}} > 0$ denote the best constants of embeddings from $W_0^{s,p}(\Omega)$, and $W_0^{t,p}(\Omega)$, respectively.

Proof. Let us consider $\Omega = \Omega_1 \cup \Omega_2$, using integration properties over Ω, where $\Omega_1 = \{x \in \Omega : |u(x)| \leq \|(u,v)\|\}$ and $\Omega_2 = \{x \in \Omega : |u(x)| > \|(u,v)\|\}$. Then

$$\int_\Omega h_1(x)|u|^\theta \ln\frac{|u(x)|}{\|(u,v)\|} dx = \int_{\Omega_1} h_1(x)|u|^\theta \ln\frac{|u(x)|}{\|(u,v)\|} dx + \int_{\Omega_2} h_1(x)|u|^\theta \ln\frac{|u(x)|}{\|(u,v)\|} dx.$$

Using (2.10) by direct calculation gives that

$$\int_{\Omega_1} h_1(x)|u|^\theta \ln\frac{|u(x)|}{\|(u,v)\|} dx \leq C_{h_1}\|(u,v)\|^\theta \int_\Omega \left|\frac{u(x)}{\|(u,v)\|}\right|^\theta \left|\ln\frac{|u(x)|}{\|(u,v)\|}\right| dx$$
$$\leq C_{h_1}\|(u,v)\|^\theta \int_\Omega \frac{1}{e\theta} dx = C_{h_1}\|(u,v)\|^\theta |\Omega| \frac{1}{e\theta}.$$

By Lemma 2.5 for $\rho = p_s^* - \theta > 0$ and direct calculation

$$\begin{aligned}
\int_{\Omega_2} h_1(x)|u|^\theta \ln\frac{|u(x)|}{\|(u,v)\|} dx &\leq C_{h_1} \int_\Omega |u|^\theta \ln\frac{|u(x)|}{\|(u,v)\|} dx \\
&\leq \frac{C_{h_1}}{e(p_s^* - \theta)} \int_\Omega |u|^\theta \left[\frac{|u(x)|}{\|(u,v)\|}\right]^{p_s^* - \theta} dx \\
&\leq \frac{C_{h_1}}{e(p_s^* - \theta)} \frac{1}{\|(u,v)\|^{p_s^* - \theta}} \int_\Omega |u|^{p_s^*} dx \\
&\leq \frac{C_{h_1}}{e(p_s^* - \theta)} \frac{1}{\|(u,v)\|^{p_s^* - \theta}} S_{p_s^*}^{p_s^*} \|u\|_{p_s^*}^{p_s^*} \\
&\leq \frac{C_{h_1}}{e(p_s^* - \theta)} \frac{1}{\|(u,v)\|^{p_s^* - \theta}} S_{p_s^*}^{p_s^*} \|(u,v)\|^{p_s^*} \\
&= \frac{C_{h_1}}{e(p_s^* - \theta)} S_{p_s^*}^{p_s^*} \|(u,v)\|^\theta.
\end{aligned}$$

Consequently we get

$$\int_\Omega |u|^\theta \ln\left[\frac{|u(x)|}{\|(u,v)\|}\right]dx \le C_{h_1}\left[\frac{|\Omega|}{e\theta} + \frac{S_{p_s^*}^{p_s^*}}{e(p_s^*-\theta)}\right]\|(u,v)\|^\theta.$$

For other hand

$$\begin{aligned}\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx &= \lambda\int_\Omega h_1(x)|u|^\theta \ln\left[\frac{|u(x)|}{\|(u,v)\|}\right]dx + \ln(\|(u,v)\|)\int_\Omega \lambda h_1(x)|u|^\theta dx \\ &\le \lambda C_{h_1}\left[\frac{|\Omega|}{e\theta} + \frac{S^{p_s^*}}{e(p_s^*-\theta)}\right]\|(u,v)\|^\theta + C_{h_1}\ln(\|(u,v)\|)\int_\Omega \lambda|u|^\theta dx.\end{aligned}$$

Similarly,

$$\mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx \le \mu C_{h_2}\left[\frac{|\Omega|}{e\theta} + \frac{S^{p_s^*}}{e(p_t^*-\theta)}\right]\|(u,v)\|^\theta + C_{h_2}\ln(\|(u,v)\|)\int_\Omega \mu|v|^\theta dx.$$

Adding the integrals for Ω_1 and Ω_2 we have

$$\begin{aligned}&\lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &\le (\lambda C_{h_1} + \mu C_{h_2})\left[\frac{|\Omega|}{e\theta} + \frac{S^{p_s^*}}{e(p_s^*-\theta)} + \frac{S^{p_s^*}}{e(p_t^*-\theta)}\right]\|(u,v)\|^\theta \\ &\quad + \ln\|(u,v)\|\int_\Omega\left[\lambda h_1(x)|u|^\theta + \mu h_2(x)|v|^\theta\right]dx.\end{aligned}$$

□

3 Nehari manifold and fibering map analysis

The main tool used in this paper is the so-called fibering method introduced by Pohozaev [11], [12] and [13]. In this section, we assume $\lambda, \mu > 0$, and the functions $h_1, h_2, b_1, b_2 \in C(\overline{\Omega})$.

We define the Nehari manifolds as

$$\mathcal{N} := \{(u,v) \in W \setminus \{(0,0)\} \mid I(u,v) = 0\}.$$

For all $(u,v) \in \mathcal{N}$, we have $(u,v) \neq 0$ and

$$\|(u,v)\|^p = \lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx + \int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx.$$

Now we define the $C^2(0,\infty)$ fibering map $\Phi_{(u,v)} : (0,\infty) \to \mathbb{R}$ as $\Phi_{(u,v)}(k) := E(k(u,v))$, for $k > 0$, it is

$$\begin{aligned}\Phi_{(u,v)}(k) &= \frac{k^p}{p}\|(u,v)\|^p - \frac{\lambda k^\theta}{\theta}\ln|k|\int_\Omega h_1(x)|u|^\theta dx - \frac{\lambda k^\theta}{\theta}\int_\Omega h_1(x)|u|^\theta\ln|u|dx \\ &+ \frac{\lambda k^\theta}{\theta^2}\int_\Omega h_1(x)|u|^\theta dx - \frac{\mu k^\theta}{\theta}\ln|k|\int_\Omega h_2(x)|v|^\theta dx - \frac{\mu k^\theta}{\theta}\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &+ \frac{\mu k^\theta}{\theta^2}\int_\Omega h_2(x)|v|^\theta dx - \frac{k^{q+r}}{q+r}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx.\end{aligned}$$

Consequentially:

$$\begin{aligned}\Phi'_{(u,v)}(k) &= k^{p-1}\|(u,v)\|^p - \lambda k^{\theta-1}\int_\Omega h_1(x)|u|^\theta\ln|ku|dx - \mu k^{\theta-1}\int_\Omega h_2(x)|v|^\theta \ln|kv|dx \\ &- k^{q+r-1}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx\end{aligned}$$

and also,

$$\begin{aligned}\Phi''_{(u,v)}(k) &= (p-1)k^{p-2}\|(u,v)\|^p - (\theta-1)k^{\theta-2}\lambda\int_\Omega h_1(x)|u|^\theta \ln|ku|dx \\ &- (\theta-1)k^{\theta-2}\mu\int_\Omega h_2(x)|v|^\theta \ln|kv|dx - \lambda k^{\theta-2}\int_\Omega h_1(x)|u|^\theta dx \\ &- \mu k^{\theta-2}\int_\Omega h_2(x)|v|^\theta dx - (q+r-1)k^{q+r-2}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx.\end{aligned}$$

Thus, one can easily verify that for all $(u,v)\in\mathcal{N}$,

$$\Phi'_{(u,v)}(1) = 0,$$

and

$$\begin{aligned}\Phi''_{(u,v)}(1) &= (p-1)\|(u,v)\|^p - \lambda(\theta-1)\int_\Omega h_1(x)|u|^\theta \ln|u|dx - \mu(\theta-1)\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &- \lambda\int_\Omega h_1(x)|u|^\theta dx - \mu\int_\Omega h_2(x)|v|^\theta dx - (q+r-1)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx,\end{aligned}$$

consequently

$$\begin{aligned}\Phi''_{(u,v)}(1) &= (p-1)\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + (p-1)\mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &+ (p-1)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx - \lambda(\theta-1)\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ &- \mu(\theta-1)\int_\Omega h_2(x)|v|^\theta \ln|v|dx - \lambda\int_\Omega h_1(x)|u|^\theta dx - \mu\int_\Omega h_2(x)|v|^\theta dx \\ &- (q+r-1)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx,\end{aligned}$$

which implies that

$$\begin{aligned}\Phi''_{(u,v)}(1) &= \lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &- \lambda\int_\Omega h_1(x)|u|^\theta dx - \mu\int_\Omega h_2(x)|v|^\theta dx - (q+r-p)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx.\end{aligned}$$

As a consequence of the previously calculus, it's make possible to rewrite the Nehari manifold $\mathcal{N}$ as

$$\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^0 \cup \mathcal{N}^-,$$

where

$$\mathcal{N}^+ = \left\{(u,v)\in\mathcal{N} \mid \Phi''_{(u,v)}(1) > 0\right\};$$
$$\mathcal{N}^0 = \left\{(u,v)\in\mathcal{N} \mid \Phi''_{(u,v)}(1) = 0\right\};$$

and

$$\mathcal{N}^- = \left\{(u,v)\in\mathcal{N} \mid \Phi''_{(u,v)}(1) < 0\right\}.$$

Lemma 3.1. *Assume $(u,v)\in W\setminus\{(0,0)\}$ and $k>0$. Then $(ku,kv)\in\mathcal{N}$ if, and only if, $\Phi'_{(u,v)}(k)=0$.*

Proof. If $k(u,v) \in \mathcal{N}$, for $k > 0$ we have $I(ku, kv) = 0$. So

$$0 = k^p \|(u,v)\|^p - \lambda k^\theta \int_\Omega h_1(x)|u|^\theta \ln |ku| dx - \mu k^\theta \int_\Omega h_2(x)|u|^\theta \ln |kv| dx \\ - k^{q+r} \int_\Omega b_1(x)|u|^q |v|^r dx - k^{q+r} \int_\Omega b_2(x)|u|^q |v|^r dx.$$

Dividing the above equation for $k > 0$, we have

$$\begin{aligned} 0 &= k^{p-1}\|(u,v)\|^p - \lambda k^{\theta-1} \int_\Omega h_1(x)|u|^\theta \ln |ku| dx - \mu k^{\theta-1} \int_\Omega h_2(x)|u|^\theta \ln |kv| dx \\ &\quad - k^{q+r-1} \int_\Omega (b_1 + b_2)(x)|u|^q |v|^r dx \\ &= \Phi'(k). \end{aligned}$$

This completes the proof. □

Lemma 3.2. *If (u,v) is a local minimizer for the functional E on $\mathcal{N}$, with $(u,v) \neq \mathcal{N}^0$, then $E'(u,v) = 0$.*

Proof. By the assumption for $u \in \mathcal{N}$, applying Lagrange's multipliers, there exists $\gamma \in \mathbb{R}$ such that

$$E'(u,v)(u,v) = \gamma I'(u,v)(u,v). \tag{3.1}$$

But because of $(u,v) \in \mathcal{N}$, we get

$$\begin{aligned} E'(u,v)(u,v) &= \|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta \ln |u| dx - \mu \int_\Omega h_2(x)|v|^\theta \ln |v| dx \\ &\quad - \int_\Omega (b_1 + b_2)(x)|u|^q |v|^r dx \\ &= 0. \end{aligned}$$

Thus, $\gamma I'(u,v)(u,v) = 0$. Now since $(u,v) \in \mathcal{N}^0$, we get

$$\begin{aligned} I'(u,v)(u,v) &= \lambda(p-\theta) \int_\Omega h_1(x)|u|^\theta \ln |u| dx + \mu(p-\theta) \int_\Omega h_2(x)|v|^\theta \ln |v| dx - \lambda \int_\Omega h_1(x)|u|^\theta dx \\ &\quad - \mu \int_\Omega h_2(x)|v|^\theta dx + (p-q-r) \int_\Omega (b_1 + b_2)(x)|u|^q |v|^r dx \\ &= \Phi''_{(u,v)}(1) \neq 0. \end{aligned}$$

Thus, $\gamma \neq 0$, and $E'(u,v) = 0$. □

Lemma 3.3. *Let $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L}$ and $2 < q + r < \min\{p_s^*, p_t^*\}$. If $b_1, b_2 \in L^\infty(\Omega)$ are non-negative functions satisfying $C_B < K$, where $C_B := \max_{x \in \overline{\Omega}} |b_1(x) + b_2(x)|$ then, for any $(u,v) \in W \setminus \{(0,0)\}$, we get*

1) If $\lambda \int_\Omega h_1(x)|u|^\theta + \mu \int_\Omega h_2(x)|v|^\theta dx \geq 0$, then there exists a unique $k_{(u,v)} > 0$ such that $\Phi'_{(u,v)}(k_{(u,v)}) = 0$ and $k_{(u,v)}(u,v) \in \mathcal{N}^-$. Moreover,

$$E(k_{(u,v)}u, k_{(u,v)}v) = \sup_{k>0} E(ku, kv).$$

2) If $\lambda \int_\Omega h_1(x)|u|^\theta + \mu \int_\Omega h_2(x)|v|^\theta dx < 0$, then there exists $k_1, k_2 > 0$ such that $0 < k_1 < k_{\max} < K_2 < \infty$ where $\Phi'_{(u,v)}(k_1) = 0 = \Phi'_{(u,v)}(k_2)$ witch $(k_1u, k_1v) \in \mathcal{N}^+$ and $(k_2u, k_2v) \in \mathcal{N}^-$. Moreover,

$$E(k_1u, k_1v) = \inf_{0<k<k_{\max}} E(ku, kv) \quad \textit{and} \quad E(k_2u, k_2v) = \sup_{k>0} E(ku, kv).$$

Proof. Define, for $k > 0$, a function $f : \mathbb{R} \to W$ given by

$$f(k) = k^{p-q-r}\|(u,v)\|^p - k^{p-q-r}\Big(\lambda \int_\Omega h_1(x)|u|^\theta \ln|ku|dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|kv|dx\Big).$$

We can rewrite $f(k)$, for $k > 0$, as

$$\begin{aligned} f(k) = k^{p-q-r}\|(u,v)\|^p &- k^{p-q-r}\ln|k|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx\Big) - k^{p-q-r}\Big(\lambda \int_\Omega h_1(x)|u|^\theta \ln|u|dx\Big) \\ &- k^{p-q-r}\ln|k|\Big(\lambda \int_\Omega h_2(x)|v|^\theta dx\Big) - k^{p-q-r}\Big(\lambda \int_\Omega h_2(x)|v|^\theta \ln|v|dx\Big). \end{aligned}$$

A direct computation shows that

$$\begin{aligned} f'(k) = k^{p-q-r-1}\Big[&(p-q-r)\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta dx - \lambda(p-q-r)\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ &- \mu \int_\Omega h_2(x)|v|^\theta dx - \mu(p-q-r)\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &- (p-q-r)\ln|k|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big)\Big]. \end{aligned}$$

Now we analyze all the possibilities:

i) If $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx > 0$, then $f \in C(0,\infty)$ and because $q + r > p$, we have

$$\lim_{k\to 0^+} f(k) = +\infty \quad \text{and} \quad \lim_{k\to\infty} f(k) = 0.$$

then there exists a unique minimum point $k_{\min} > 0$ such that $f'(k_{\min}) = 0$. Because $k_{\min} > 0$,

$$\begin{aligned} &(p-q-r)\ln|k_{\min}|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ &\quad = (p-q-r)\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta dx - \lambda(p-q-r)\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ &\qquad - \mu \int_\Omega h_2(x)|u|^\theta dx - \mu(p-q-r)\int_\Omega h_2(x)|v|^\theta \ln|v|dx. \end{aligned}$$

Thus

$$k_{\min} = \exp\left(\frac{\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta \ln|u|dx - \mu \int_\Omega h_2(x)|v|^\theta \ln|v|dx}{\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx} - \frac{1}{p-q-r}\right) \tag{3.2}$$

Obviously, f is decreasing on $(0, k_{\min})$ and increasing on $(k_{\min}, \infty)$. Then, because equation (3.2), the fact that $k_{\min} > 0$ and that $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx > 0$, we get

$$\begin{aligned} f(k_{\min}) = k_{\min}^{p-q-r}\Big[&\|(u,v)\|^p - \ln|k_{\min}|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ &- \Big(\lambda \int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v|dx\Big)\Big] \\ = k_{\min}^{p-q-r}&\Big[\frac{1}{p-q-r}\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big)\Big] \\ < 0.& \end{aligned}$$

Since we have

$$\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx > 0, \quad \text{and} \quad \lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx > 0,$$

and because

$$\Phi'_{(u,v)}(k) = k^{q+r+\theta-p-1}\left[f(k) - (k^{2p-q-r-\theta} + k^{p-q-r})\|(u,v)\|^p - k^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx\right],$$

there exists a unique $k_{(u,v)}$ such that $0 < k_{(u,v)} < k_{\min}$ such that

$$f(k_{(u,v)}) = \left(k_{(u,v)}^{2p-q-r-\theta} + k_{(u,v)}^{p-q-r}\right)\|(u,v)\|^p + k_{(u,v)}^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx,$$

and $f'(k_{(u,v)}) < 0$, we get $k_{(u,v)}(u,v) \in \mathcal{N}^-$. Moreover, it follows from $f(k) < f(k_{(u,v)})$, for all $k > k_{(u,v)}$ and $f(k) > f(k_{(u,v)})$, for all $k < k_{(u,v)}$, that

$$E(k_{(u,v)}(u,v)) = \sup_{k>0} E(ku,kv).$$

ii) If $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx = 0$, it follows from Lemma 2.9 there exists a unique $k_{(u,v)} > 0$ such that $k_{(u,v)}(u,v) \in \mathcal{N}^-$, and

$$E(k_{(u,v)}) = \sup_{k>0} E(ku,kv).$$

iii) If $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx < 0$, then $f \in C(0,\infty)$, $\lim_{k\to 0^+} f(k) = -\infty$ and $\lim_{k\to\infty} f(k) = 0$. Then f has a unique maximum point $k_{\max} > 0$ which is given by

$$k_{\max} = \exp\left(\frac{\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx}{\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx} - \frac{1}{p-q-r}\right).$$

Moreover, f is increasing on $(0, k_{\max})$ and decreasing on $(k_{\max}, \infty)$. By Lemma (2.9), we get

$$k_{\max} \geq \exp\left(\frac{(1-(\lambda C_{h_1} + \mu C_{h_2})L)\|(u,v)\|^p}{\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx} - \ln\|(u,v)\| - \frac{1}{p-q-r}\right).$$

So,

$$k_{\max}^{p-q-r} \geq \exp\left((p-q-r)\frac{(1-(\lambda C_{h_1} + \mu C_{h_2})L)\|(u,v)\|^p}{\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx} - 1\right)\|(u,v)\|^{q+r}.$$

It is known that the following inequality holds:

$$\exp(k-1) \geq k, \qquad \forall\, k \geq 0.$$

Then

$$k_{\max}^{p-q-r} \geq (p-q-r)\frac{1-(\lambda C_{h_1} + \mu C_{h_2})L}{\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx}\|(u,v)\|^{q+r}.$$

Therefore

$$\begin{aligned} f(k_{\max}) &= k_{\max}^{p-q-r} \frac{\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx}{p-q-r} \\ &\geq (1-(\lambda C_{h_1} + \mu C_{h_2})L)\|(u,v)\|^{q+r}. \end{aligned}$$

Because $\big(k_{(u,v)}^{2p-q-r-\theta} + k_{(u,v)}^{p-q-r}\big)\|(u,v)\|^p + k_{(u,v)}^{p-\theta} \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx, > 0$. By Hölder's inequality,

$$\begin{aligned} \int_\Omega (b_1+b_2)(x)|u|^p|v|^r dx &\leq C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} \left(\int_\Omega |u|^{2_s^*}\right)^{\frac{q}{p_s^*}} \left(\int_\Omega |v|^{2_t^*}\right)^{\frac{r}{p_t^*}} \\ &\leq C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r \|(u,v)\|^{q+r}, \end{aligned}$$

for all $(u,v) \in W$, where $C_B = \max_{x\in\overline{\Omega}} |b_1(x) + b_2(x)|$ and $S_{p_s^*}^q, S_{p_t^*}^r > 0$ denote the best constants of embeddings $W_0^{s,p} \hookrightarrow L^{p_s^*}$ and $W_0^{t,p} \hookrightarrow L^{p_t^*}$, respectively.

Then

$$(1-(\lambda C_{h_1} + \mu C_{h_2})L) S_{p_s^*}^q S_{p_t^*}^r |\Omega|^{\frac{q}{p_s^*}+\frac{r}{p_t^*}-1} > C_B,$$

it implies that

$$f(k_{\max}) > \left(k_{(u,v)}^{2p-q-r-\theta} + k_{(u,v)}^{p-q-r}\right)\|(u,v)\|^p + k_{(u,v)}^{p-\theta} \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx > 0 = \lim_{k\to\infty} f(k),$$

and it show us there exists k_1, k_2 in which,

$$0 < k_1 < k_{\max} < k_2 < \infty,$$

such that

$$f(k_1) = \left(k_{(u,v)}^{2p-q-r-\theta} + k_{(u,v)}^{p-q-r}\right)\|(u,v)\|^p + k_{(u,v)}^{p-\theta} \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx = f(k_2).$$

It show us that $\Phi'_{(u,v)}(k_1) = 0 = \Phi'_{(u,v)}(k_2)$. Moreover, f is increasing on $(0, k_{\max})$ and decreasing on $(k_{\max}, \infty)$. So

$$k_1(u,v) \in \mathcal{N}^+ \quad \text{and} \quad k_2(u,v) \in \mathcal{N}^-.$$

Moreover, we have

$$f(k) \geq \left(k^{2p-q-r-\theta} + k^{p-q-r}\right)\|(u,v)\|^p + k^{p-\theta} \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx, \qquad \forall k \in [k_1, k_2],$$

and

$$f(k) < \left(k^{2p-q-r-\theta} + k^{p-q-r}\right)\|(u,v)\|^p + k^{p-\theta} \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx, \qquad \forall k \in \mathbb{R}_+^* \setminus [k_1, k_2].$$

Thus

$$E(k_1(u,v)) = \inf_{0<k\leq k_{\max}} E(k(u,v)) \quad \text{and} \quad E(k_2(u,v)) = \sup_{k>0} E(k(u,v)).$$

The proof is complete. □

Remark 3.4. In what follows, we define

$$M := \frac{|\Omega|^{1-\frac{q}{p_s^*}} S_{p_s^*}^q + |\Omega|^{1-\frac{q}{p_s^*}} S_{p_t^*}^q}{q+r-p}, \quad \text{and} \quad K := \frac{1-(\lambda C_{h_1} + \mu C_{h_2})(L+M)}{|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}.$$

Lemma 3.5. *If $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$ and $C_B < K$, then $\mathcal{N}^0 = \emptyset$.*

Proof. Arguing by contradiction, let $(u,v) \in \mathcal{N}^0$. Then $I(u,v) = 0$, and

$$\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta dx - \mu \int_\Omega h_2(x)|v|^\theta dx = \int_\Omega (b_1 + b_2)(x)|u|^q|v|^r dx \tag{3.3}$$

and

$$\begin{aligned} 0 = {} & \lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ & - \lambda \int_\Omega h_1(x)|u|^\theta dx - \mu \int_\Omega h_2(x)|v|^\theta dx \\ & + (p-q-r)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx. \end{aligned} \tag{3.4}$$

From equations (3.3) and (3.4), and Lemma 2.9, we get

$$\begin{aligned} & \|(u,v)\|^p - \int_\Omega (b_1+b_2)(x)|u|^\theta dx \\ & \quad \le (\lambda C_{h_1} + \mu C_{h_2})L\|(u,v)\|^p + \ln\|(u,v)\|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big). \end{aligned}$$

Then, again by (3.4)

$$\begin{aligned} (1-(\lambda C_{h_1}+\mu C_{h_2})L)\|(u,v)\|^p & \le \ln\|(u,v)\|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ & \quad + \int_\Omega (b_1+b_2)(x)|v|^q|v|^r dx \\ & \le \ln\|(u,v)\|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ & \quad + \frac{1}{p-q-r}\Big[-\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ & \quad - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ & \quad + \lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big]. \end{aligned}$$

That is,

$$\begin{aligned} (1-(\lambda C_{h_1}+\mu C_{h_2})L)\|(u,v)\|^p & \le \ln\|(u,v)\|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ & \quad + \frac{1}{p-q-r}\Big[-\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ & \quad - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx\Big] \\ & \quad + \frac{\lambda C_{h_1}+\mu C_{h_2}}{q+r-p}\Big(|\Omega|^{1-\frac{q}{p_s^*}} S_{p_s^*}^q + |\Omega|^{1-\frac{q}{p_s^*}} S_{p_t^*}^q\Big)\|(u,v)\|^p, \end{aligned} \tag{3.5}$$

which means that

$$\begin{aligned} & \Big[1-(\lambda C_{h_1}+\mu C_{h_2})L - \frac{\lambda C_{h_1}+\mu C_{h_2}}{q+r-p}\Big(|\Omega|^{1-\frac{q}{p_s^*}} S_{p_s^*}^q + |\Omega|^{1-\frac{q}{p_s^*}} S_{p_t^*}^q\Big)\Big]\|(u,v)\|^p \\ & \quad \le \ln\|(u,v)\|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ & \qquad + \frac{1}{p-q-r}\Big(-\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx\Big). \end{aligned}$$

Because $p < q + r$, equation (3.4) show us that

$$0 > -\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx + \lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx.$$

Then, we have

$$0 > \ln\|(u,v)\|\Big(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\Big) + \frac{1}{p-q-r}\Big(-\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx\Big).$$

Thus, it follows from $\lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$ that $\|(u,v)\| \leq 1$, where, here we consider M as on Remark 3.4.

Otherwise, using (3.5), and one more time (3.3) and (3.4),

$$\begin{aligned}[1-(\lambda C_{h_1} + \mu C_{h_2})L]\|(u,v)\|^p &\leq \ln\|(u,v)\|\Big(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\Big) \\ &\quad + \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \\ &\leq (p-q-r)\ln\|(u,v)\|\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \\ &\quad + (\lambda C_{h_1} + \mu C_{h_2})M\|(u,v)\|^p. \end{aligned} \tag{3.6}$$

Now, we can note that

$$\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \leq C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*}\|(u,v)\|^{q+r}, \tag{3.7}$$

Thus

$$[1-(\lambda C_{h_1} + \mu C_{h_2})(L+M) - C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*}]\|(u,v)\|^p \leq 0.$$

Since $(\lambda C_{h_1} + \mu C_{h_2}) < \frac{1}{L+M}$, and

$$C_B < K := \frac{1-(\lambda C_{h_1} + \mu C_{h_2})(L+M)}{|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*}},$$

it follows that $\|(u,v)\| = 0$. Then $(u,v) = (0,0)$, a contradiction. Therefore $\mathcal{N}^0 = \emptyset$. □

Remark 3.6. For next result, we consider

$$\Lambda_{\lambda,\mu} := \min\left\{1, \left(\frac{1-(\lambda C_{h_1} + \mu C_{h_2})L - C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*}}{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*}}\right)^{\frac{1}{q+r-p}}\right\}.$$

Lemma 3.7. *If $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$ and $C_B < K$, then $\|(U,V)\| \geq \Lambda_{\lambda,\mu}$, for all $(U,V) \in \mathcal{N}^-$ and $\|(u,v)\| \leq 1$, for all $(u,v) \in \mathcal{N}^+$.*

Proof. Let $(U,V) \in \mathcal{N}^-$. Then

$$\|(U,V)\|^p - v\lambda \int_\Omega h_1(x)|U|^\theta \ln|U|dx - \mu \int_\Omega h_2(x)|V|^\theta \ln|V|dx = \int_\Omega (b_1+b_2)(x)|U|^q|V|^r dx, \quad (3.8)$$

and

$$\begin{aligned}(p-q-r)&\int_\Omega (b_1+b_2)(x)|U|^q|V|^r dx \\ &< \lambda \int_\Omega h_1(x)|U|^\theta dx + \mu \int_\Omega h_2(x)|V|^\theta dx \\ &\quad - (p-\theta)\Big(\lambda \int_\Omega h_1(x)|U|^\theta \ln|U|dx + \mu \int_\Omega h_2(x)|V|^\theta \ln|V|dx\Big). \end{aligned} \quad (3.9)$$

Similar to equation (3.6) in Lemma 3.5, we have

$$\begin{aligned}[1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(U,V)\|^p \le{}& \ln\|(U,V)\|\Big(\lambda \int_\Omega h_1(x)|U|^\theta dx + \mu \int_\Omega h_2(x)|V|^\theta dx\Big) \\ &+ C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*} \|(U,V)\|^{q+r}. \end{aligned} \quad (3.10)$$

We consider two cases:

i) If $\lambda \int_\Omega h_1(x)|U|^\theta dx + \mu \int_\Omega h_2(x)|V|^\theta dx > 0$, then $\|(U,V)\| \ge 1$, because otherwise, being $\|(U,V)\| < 1$, we have

$$[1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(U,V)\|^p < C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*} \|(U,V)\|^{q+r},$$

and together with $C_B < K$, we get

$$\|(U,V)\| > \left(\frac{1-(\lambda C_{h_1}+\mu C_{h_2})L}{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*}}\right)^{\frac{1}{q+r-p}} > 1, \quad (3.11)$$

a contradiction. Thus $\|(U,V)\| \ge 1$.

ii) If $\lambda \int_\Omega h_1(x)|U|^\theta dx + \mu \int_\Omega h_2(x)|V|^\theta dx < 0$, we have two more cases to analyze:

2.1) If $\|(U,V)\| > 1$, similar to case i) above, we get the same equation (3.11).

2.2) If $\|(U,V)\| < 1$, we have by equations (3.8) and (3.9),

$$\begin{aligned}&[1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(U,V)\|^p \\ &\quad< C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*} \|(U,V)\|^p \\ &\qquad + (p-\theta)\Big(\lambda \int_\Omega h_1(x)|U|^\theta \ln|U|dx + \mu \int_\Omega h_2(x)|V|^\theta \ln|V|dx\Big) \\ &\qquad + C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*} \|(U,V)\|^{q+r}, \end{aligned}$$

So, because $p \le \theta$,

$$\|(U,V)\| > \left(\frac{1-(\lambda C_{h_1}+\mu C_{h_2})L - C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*}}{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S^q_{p_s^*} S^r_{p_t^*}}\right)^{\frac{1}{q+r-p}} > 1.$$

Thus, $\|(U,V)\| \ge \Lambda_{\lambda,\mu}$, for all $(U,V) \in \mathcal{N}^-$.

Now, if $(u,v) \in \mathcal{N}^+$, a similar discussion show us that $\|(u,v)\| \leq 1$. □

Lemma 3.8. *If $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$ and $C_B < K$, then $\mathcal{N}^-$ is a closed subset o W.*

Proof. The proof follows directly from Lemmas 3.5 and 3.7. □

Lemma 3.9. *If $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$. Then the functional E is bounded from below on $\mathcal{N}$.*

Proof. Because of the relationship between the functionals E and I, for $(u,v) \in \mathcal{N}$, we have

$$\begin{aligned} E(u,v) = & \left(\frac{1}{p} - \frac{1}{\theta}\right) \|(u,v)\|^p + \frac{1}{\theta^2}\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ & - \left(\frac{1}{q+r} - \frac{1}{\theta}\right) \int_\Omega (b_1 + b_2)(x)|u|^q|v|^r dx. \end{aligned}$$

Because $(u,v) \in \mathcal{N}$, we have also

$$\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx = \|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx - \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx.$$

So

$$\begin{aligned} E(u,v) = & \left(\frac{1}{p} - \frac{1}{q+r}\right) \|(u,v)\|^p + \frac{1}{\theta^2}\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ & + \left(\frac{1}{\theta} - \frac{1}{q+r}\right) \Big(\lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx\Big). \end{aligned}$$

Now, concerning to exposed above, we consider two cases:

i) If $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx > 0$, because $0 < p < \theta < q+r$, it follows that

$$E(u,v) > \left(\frac{1}{\theta} - \frac{1}{q+r}\right) \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \geq 0.$$

ii) If $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx \leq 0$, it follows from Lemma (2.9) that

$$\begin{aligned} E(u,v) = & \left(\frac{1}{p} - \frac{1}{q+r}\right) [1 - (\lambda C_{h_1} + \mu C_{h_2})L] \|(u,v)\|^p \\ & + \left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right) \ln\|(u,v)\|\right] \Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big). \end{aligned}$$

Now, if $\frac{1}{\theta^2} - (\frac{1}{\theta} - \frac{1}{q+r}) \ln\|(u,v)\| \geq 0$, then $E(u,v) \geq 0$.
Otherwise, if $\frac{1}{\theta^2} - (\frac{1}{\theta} - \frac{1}{q+r}) \ln\|(u,v)\| < 0$, there exists a constant $C > 0$ such that

$$\begin{aligned} E(u,v) \geq & \left(\frac{1}{p} - \frac{1}{q+r}\right) [1 - (\lambda C_{h_1} + \mu C_{h_2})L] \|(u,v)\|^p \\ & + \left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right) \ln\|(u,v)\|\right] \Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ \geq & \left(\frac{1}{p} - \frac{1}{q+r}\right) [1 - (\lambda C_{h_1} + \mu C_{h_2})L] \|(u,v)\|^p \\ & - C\left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right) \ln\|(u,v)\|\right] \\ \geq & -C. \end{aligned}$$

Thus, E is bounded from below on $\mathcal{N}$. □

4 Nontrivial ground state solution for positive weight function

In this section, we establish the existence of a ground state solution for problem (P) when the weights functions $h_1, h_2 > 0$ by Mountain Pass theorem and the existence of a level $c_* \in W$, where the functional E satisfy the $(PS)_{c_*}$ condition.

Lemma 4.1. *Let $h_1, h_2 \in C(\overline{\Omega})$ and $h_1(x), h_2(x) > 0$ for all $x \in \Omega$. Then there are exist $\eta, \zeta > 0$, such that*

(i) $E(u,v) \geq \eta > 0$ for all $\|(u,v)\| = \zeta$,

(ii) There exists $(u,v) \in W$ such that $E(u,v) < 0$ if $\|(u,v)\| > \zeta$.

Proof. Because the definition of E and inequality (3.7), we have

$$E(u,v) \geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right] \|(u,v)\|^p - \frac{1}{\theta}\ln\|(u,v)\|\left(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right) - \frac{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{q+r}\|(u,v)\|^{q+r},$$

what implies that, for all $(u,v) \in W$ with $0 < \|(u,v)\| \leq 1$,we have

$$E(u,v) \geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right] \|(u,v)\|^p - \frac{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{q+r}\|(u,v)\|^{q+r}.$$

Choosing $\zeta \in (0,1]$ small enough, such that

$$\left(\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right) - \frac{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{q+r}\zeta^{q+r} > 0,$$

we have

$$E(u,v) \geq \left[\left(\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right) - \frac{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{q+r}\zeta^{q+r}\right]\zeta^p > 0,$$

for all $(u,v) \in W$, with $\|(u,v)\| = \zeta$. Thus (i) holds.

Otherwise, for all $(u,v) \in W \setminus \{0,0\}$ and $k > 0$, we have

$$\begin{aligned} &E(ku,kv) \\ &= k^{q+r}\Bigg[\frac{k^{p-q-r}}{p}\|(u,v)\|^p \\ &\quad - \frac{k^{\theta-q-r}\ln|k|}{\theta}\int_\Omega h_1(x)|u|^\theta dx - \frac{k^{\theta-q-r}}{\theta}\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \frac{k^{\theta-q-r}}{\theta^2}\lambda\int_\Omega h_1(x)|u|^\theta dx \\ &\quad - \frac{k^{\theta-q-r}\ln|k|}{\theta}\int_\Omega h_2(x)|v|^\theta dx - \frac{k^{\theta-q-r}}{\theta}\mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx + \frac{k^{\theta-q-r}}{\theta^2}\mu\int_\Omega h_2(x)|v|^\theta dx\Bigg], \end{aligned}$$

and because $2 < p < \theta < q+r$, it implies that there exists $k_0 > 0$ large enough such that $\|(k_0u, k_0v)\| > \zeta$ and $E(k_0u, k_0v) < 0$. So, taking $(u,v) = (k_0u, k_0v)$, item (ii) holds □

Lemma 4.2. *Let (u_n, v_n) be a $(PS)_{c_*}$ sequence of the functional E. Then, a sequence (u_n, v_n) is bounded in W.*

Proof. For $c_* \in \mathbb{R}$, we assume that $\{(u_n, v_n)\}_n \subset W$ with $\|(u_n, v_n)\| > 1$ is a $(PS)_{c_*}$ sequence with, it is

$$E(u_n, v_n) \to c_* \quad \text{and} \quad E'(u_n, v_n)(u_n, v_n) \to 0,$$

as $n \to \infty$. Since (u_n, v_n) is a $(PS)_{c_*}$ sequence for functional E, we have

$$\begin{aligned} c_* + o_n(1) + C\|(u_n, v_n)\| &\geq E(u_n, v_n) - \frac{1}{p}E'(u_n, v_n)(u_n, v_n) \\ &= \frac{1}{p^2}\Big(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\Big) \\ &\quad + \left(\frac{1}{p} - \frac{1}{q+r}\right)\int_\Omega (b_1 + b_2)(x)|u_n|^q|v_n|^r dx \\ &= \frac{1}{p^2}\Big(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\Big). \end{aligned} \tag{4.1}$$

We have also

$$\begin{aligned} &c_* + o_n(1) + C\|(u_n, v_n)\| \\ &\quad \geq E(u_n, v_n) - \frac{1}{q+r}E'(u_n, v_n)(u_n, v_n) \\ &\quad = \left(\frac{1}{p} - \frac{1}{q+r}\right)\|(u_n, v_n)\|^p \\ &\qquad - \left(\frac{1}{p} - \frac{1}{q+r}\right)\left(\lambda \int_\Omega h_1(x)|u_n|^\theta \ln u_n dx + \mu \int_\Omega h_2(x)|v_n|^\theta \ln v_n dx\right) \\ &\qquad + \frac{1}{p^2}\left(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\right) \\ &\quad \geq \left(\frac{1}{p} - \frac{1}{q+r}\right)\|(u_n, v_n)\|^p \\ &\qquad - \left(\frac{1}{p} - \frac{1}{q+r}\right)\left(\lambda \int_\Omega h_1(x)|u_n|^\theta \ln u_n dx + \mu \int_\Omega h_2(x)|v_n|^\theta \ln v_n dx\right). \end{aligned} \tag{4.2}$$

Combining equations (4.1) and (4.2), Lemma (2.5) and Lemma (2.9), we have

$$\begin{aligned} &\left[\left(\frac{1}{p} - \frac{1}{q+r}\right)[1 - (\lambda C_{h_1} + \mu C_{h_2})L]\right]\|(u, v)\| \\ &\quad \leq \left(\frac{1}{p} - \frac{1}{q+r}\right)\frac{p^2[(u_n, v_n)]^\sigma}{e\sigma}[c_* + C\|(u_n, v_n)\| + o(1)] \\ &\qquad + c_* + C\|(u_n, v_n)\| + o(1) \\ &\quad \leq C\left(1 + \|(u_n, v_n)\|^{1+\sigma}\right) + o(1), \end{aligned}$$

where $\sigma \in (0, p-1)$. Thus $\{(u_n, v_n)\}_n$ is bounded in W. □

Lemma 4.3. *Let (u_n, v_n) be a $(PS)_{c_*}$ sequence of the functional E. Then, functional E satisfies the $(PS)_{c_*}$ condition at any level c_*.*

Proof. By Lemmas 2.6 and 2.8, we get

$$\lim_{n\to\infty}\int_\Omega h_1(x)u|u_n|^{\theta-1}\ln|u|dx = \int_\Omega h_1(x)|u|^\theta \ln|u|dx, \tag{4.3}$$

$$\lim_{n\to\infty}\int_\Omega h_2(x)v|v_n|^{\theta-1}\ln|v|dx = \int_\Omega h_2(x)|v|^\theta \ln|v|dx, \tag{4.4}$$

$$\lim_{n\to\infty}\int_\Omega (b_1+b_2)(x)|u_n|^{q-2}u_n|v_n|^r(u_n-u)dx = 0, \tag{4.5}$$

$$\lim_{n\to\infty}\int_\Omega (b_1+b_2)(x)|v_n|^{r-2}v_n|u_n|^q(v_n-v)dx = 0. \tag{4.6}$$

Note that

$$\begin{aligned}\|(u_n-u,v_n-v)\|^p &= (E'(u_n,v_n)-E'(u,v))(u_n-u,v_n-v)\\ &\quad+\lambda\int_\Omega h_1(x)\left(|u_n|^{\theta-1}\ln|u_n|-|u|^{\theta-1}\ln|u|\right)(u_n-u)dx\\ &\quad+\mu\int_\Omega h_2(x)\left(|v_n|^{\theta-1}\ln|v_n|-|v|^{\theta-1}\ln|v|\right)(v_n-v)dx\\ &\quad+\frac{q}{q+r}\int_\Omega (b_1+b_2)(x)\left(|v_n|^r u_n^{q-2}u_n-|v|^r|u|^{q-2}u\right)(u_n-u)dx\\ &\quad+\frac{r}{q+r}\int_\Omega (b_1+b_2)(x)\left(|u_n|^q v_n^{r-2}v_n-|u|^q|v|^{r-2}v^{r-2}v\right)(v_n-v)dx.\end{aligned}$$

Since $(E'(u_n,v_n)-E'(u,v))(u_n-u,v_n-v)\to 0$ as $n\to\infty$, it follow from equations (4.3), (4.4), (4.5), (4.6) and Lemma 2.3, that $(u_n,v_n)\to(u,v)$ in W. It yields the proof. □

Now we can prove one of our main results, Theorem 1.1.

4.1 Proof of Theorem 1.1

Define $\mathfrak{N} := \{(u,v)\in W\setminus\{0,0\} \mid E'(u,v)(u,v)=0\}$. By the previous lemmas, $\mathfrak{N}$ is nonempty. Let $(u,v)\in\mathfrak{N}$, from Lemma (2.9), we have

$$\begin{aligned}0 &= \|(u,v)\|^p-(\lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx)-\int_\Omega(b_1+b_2)(x)|u|^q|v|^r dx\\ &\geq (1-(\lambda C_{h_1}+\mu C_{h_2})L)\|(u,v)\|^p-\ln\|(u,v)\|\left(\lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx\right)\\ &\quad-\int_\Omega(b_1+b_2)(x)|u|^q|v|^r dx \end{aligned}\tag{4.7}$$

If $\|(u,v)\|\leq 1$, it follows from the above equation (4.7) that

$$[1-(\lambda C_{h_1}+\mu C_{h_2})]\|(u,v)\|^p \leq \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \leq C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}\|(u,v)\|^{q+r}.$$

Then,

$$\|(u,v)\| \geq \left[\frac{1-(\lambda C_{h_1}+\mu C_{h_2})L}{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}S^q_{p_s^*}S^r_{p_t^*}}\right]^{\frac{1}{q+r-p}},$$

which implies that

$$\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \geq [1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(u,v)\|^p \geq \left[\frac{[1-(\lambda C_{h_1}+\mu C_{h_2})L]^{q+r}}{\left(C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}S^q_{p_s^*}S^r_{p_t^*}\right)^p}\right]^{\frac{1}{q+r-p}}. \tag{4.8}$$

Define

$$\tilde{C} := \left[\frac{[1-(\lambda C_{h_1}+\mu C_{h_2})L]^{q+r}}{\left(C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}S^q_{p_s^*}S^r_{p_t^*}\right)^p}\right]^{\frac{1}{q+r-p}} \quad \text{and} \quad \mathfrak{C} := \inf\{E(u,v) \mid (\tilde{U}_n,\tilde{V}_n) \in \mathfrak{N}\},$$

then $\mathfrak{C} > 0$, because otherwise, there exists $\{(\tilde{U}_n,\tilde{V}_n)\}_n \subset \mathfrak{N}$ such that $E'(\tilde{U}_n,\tilde{V}_n) \to 0$.

It follows from

$$\begin{aligned} E(u,v) = {} & \left(\frac{1}{p}-\frac{1}{\theta}\right)\|(u,v)\|^p + \frac{1}{\theta}I(u,v) \\ & + \frac{1}{\theta^2}\left(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right) \\ & - \left(\frac{1}{q+r}-\frac{1}{p}\right)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx, \end{aligned}$$

that

$$\frac{1}{\theta^2}\left(\lambda\int_\Omega h_1(x)|\tilde{U}_n|^\theta dx + \mu\int_\Omega h_2(x)|\tilde{V}_n|^\theta dx\right) - \left(\frac{1}{q+r}-\frac{1}{p}\right)\int_\Omega (b_1+b_2)(x)|\tilde{U}_n|^q|\tilde{V}_n|^r dx \to 0,$$

when $n \to \infty$, and by Lemma 4.2. But, from equation (4.8) follows that

$$\int_\Omega (b_1+b_2)(x)|\tilde{U}_n|^q|\tilde{V}_n|^r dx \geq \tilde{C} > 0,$$

and it implies that $0 \geq \tilde{C} > 0$, a contradiction. Thus, $\mathfrak{C} > 0$.

Finally, let $\{(u_n,v_n)\}_n \subset \mathfrak{N}$ be a minimizing sequence. Then $E'(u_n,v_n)(u_n,v_n) = 0$ and $\lim_{n\to\infty} E(u_n,v_n) = \mathfrak{C} > 0$. Again by Lemma 4.2, there exists $(u_0,v_0) \in W \setminus \{0,0\}$ such that

$$(u_n,v_n) \to (u_0,v_0)$$

in W. Hence $E(u_0,v_0) = \mathfrak{C}$ and $E'(u_0,v_0) = 0$, and it means that (u_0,v_0) is a nontrivial ground state solution of problem (P).

5 Nontrivial solutions for sing-changing weight functions

Lemma 5.1. *E has a nontrivial and nonnegative minimizer on $\mathcal{N}^+$.*

Proof. The proof of this lemma will be shown with two steps.

Step 1. The strong convergence of minimizing sequence. By Lemma 3.9, we get

$$c^+ = \inf_{(u,v)\in\mathcal{N}^+} E(u,v).$$

We claim that $c^+ < 0$. Indeed, for each $(u,v) \in \mathcal{N}^+$, we get

$$\|(u,v)\|^p = \lambda \int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v|dx + \int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx. \tag{5.1}$$

Substituting (5.1), in the energy functional E, we have the following expression

$$\begin{aligned} E(u,v) &= \frac{1}{p}\|(u,v)\|^p - \frac{\lambda}{\theta}\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega h_1(x)|u|^\theta dx - \frac{\mu}{\theta}\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &\quad + \frac{\mu}{\theta^2}\int_\Omega h_2(x)|v|^\theta dx - \frac{1}{q+r}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \\ &= \frac{1}{p}\left[\lambda \int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx + \int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx\right] \\ &\quad - \frac{\lambda}{\theta}\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega h_1(x)|u|^\theta dx - \frac{\mu}{\theta}\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &\quad + \frac{\mu}{\theta^2}\int_\Omega h_2(x)|v|^\theta dx - \frac{1}{q+r}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \\ &= \left(\frac{1}{p}-\frac{1}{\theta}\right)\left[\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx\right] \\ &\quad + \frac{1}{\theta^2}\left[\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right] \\ &\quad + \left(\frac{1}{p}-\frac{1}{q+r}\right)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx. \end{aligned}$$

Using the logarithmic estimates for positive weight functions in addition with $p < \theta$ and $p < q+r$, we have the following inequality

$$\begin{aligned} &(p-\theta)\left[\lambda\int_\Omega h_1(x)|u_n|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v_n|^\theta \ln|v|dx\right] \\ &\quad > (q+r-p)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx + \lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx. \end{aligned}$$

From the last inequality, we have that

$$\begin{aligned} &\lambda\int_\Omega h_1(x)|u_n|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v_n|^\theta \ln|v|dx \\ &\quad < -\frac{q+r-p}{\theta-p}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \\ &\qquad - \frac{1}{\theta-p}\left[\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right]. \end{aligned} \tag{5.2}$$

Finally, using (5.2) in the functional E, it follows that

$$\begin{aligned} E(u,v) &= \left(\frac{1}{p}-\frac{1}{\theta}\right)\left[\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx\right] \\ &\quad + \frac{1}{\theta^2}\left[\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right] \\ &\quad + \left(\frac{1}{p}-\frac{1}{q+r}\right)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \end{aligned}$$

$$< (p-q-r)\left(\frac{1}{\theta-p}-\frac{1}{p(q+r)}\right)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx$$
$$+\left(\frac{1}{\theta^2}-\frac{1}{\theta-p}\right)\left[\lambda\int_\Omega h_1(x)|u|^\theta dx+\mu\int_\Omega h_2(x)|v|^\theta dx\right] < 0.$$

Let $\{(u_n,v_n)\}\subset\mathcal{N}^+$ be a minimizing sequence. Then, we get

$$\|(u_n,v_n)\|^p = \lambda\int_\Omega h_1(x)|u_n|^\theta \ln|u|dx+\mu\int_\Omega h_2(x)|v_n|^\theta\ln|v|dx$$
$$+\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \tag{5.3}$$

and

$$\Phi''_{(u_n,v_n)}(1) = \|(u_n,v_n)\|^p-\lambda\int_\Omega h_1(x)|u_n|^\theta\ln|u_n|dx-\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx$$
$$-\int_\Omega (b_1+b_2)(x)|u_n|^q|v_n|^r dx-\lambda\int_\Omega h_1(x)|u_n|^\theta dx$$
$$-\mu\int_\Omega h_2(x)|v_n|^\theta dx+(p-q-r)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx. \tag{5.4}$$

Using $\{(u_n,v_n)\}\subset\mathcal{N}^+$ and $\Phi''_{(u_n,v_n)}(1)>0$, we conclude that

$$-\lambda\int_\Omega h_1(x)|u_n|^\theta dx-\mu\int_\Omega h_2(x)|v_n|^\theta dx+(p-q-r)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx>0. \tag{5.5}$$

Since $\{(u_n,v_n)\}$ is bounded by Lemma 3.8, up to subsequence we assume that

$$\begin{cases} (u_n,v_n)\rightharpoonup(u^+,v^+) & \text{in } W;\\ u_n\to u^+, v_n\to v^+ & \text{strongly in } L^t(\Omega),\ \text{for } 1\le t<\min\{p_s^*,p_t^*\};\\ u_n(x)\to u^+(x), v_n(x)\to v^+(x) & \text{a.e. in } \Omega. \end{cases} \tag{5.6}$$

Similar to Lemma 4.3, we obtain the following convergences:

$$\lim_{n\to\infty}\int_\Omega h_1(x)u_n|u_n|^\theta\ln|u_n|dx=\int_\Omega h_1(x)|u^+|^\theta\ln|u^+|dx, \tag{5.7}$$

and

$$\lim_{n\to\infty}\int_\Omega h_2(x)v_n|v_n|^\theta\ln|v_n|dx=\int_\Omega h_2(x)|v^+|^\theta\ln|v^+|dx, \tag{5.8}$$

Furthermore, the dominated convergence theorem of Lebesgue is valid for the product of functions $|u|^q$ and $|v|^r$, hence we have that

$$\lim_{n\to\infty}\int_\Omega b_1(x)|u_n|^q|v_n|^r dx=\int_\Omega b_2(x)|u^+|^q|v^+|^r dx, \tag{5.9}$$

and

$$\lim_{n\to\infty}\int_\Omega b_2(x)|u_n|^q|v_n|^r dx=\int_\Omega b_2(x)|u^+|^q|v^+|^r dx. \tag{5.10}$$

If $(u_n,v_n)\nrightarrow(u^+,v^+)$ in W, then

$$\|(u^+,v^+)\|^2\le\liminf_{n\to+\infty}\|(u_n,v_n)\|^2.$$

This implies that

$$\begin{aligned}
&\|(u^+,v^+)\|^p - \lambda\int_\Omega h_1(x)|u^+|^\theta \ln|u^+|dx - \mu\int_\Omega h_1(x)|v^+|^\theta \ln|v^+|dx\\
&\quad - \int_\Omega b_1(x)|u^+|^q|v^+|^r dx - \int_\Omega b_2(x)|u^+|^q|v^+|^r dx\\
&< \liminf_{n\to+\infty}\Big[\|(u_n,v_n)\|^2 - \lambda\int_\Omega h_1(x)|u_n|^\theta \ln|u_n|dx - \mu\int_\Omega h_1(x)|v_n|^\theta\ln|v_n|dx\\
&\quad - \int_\Omega b_1(x)|u_n|^q|v_n|^r dx - \int_\Omega b_2(x)|u_n|^q|v_n|^r dx\Big] = 0. \qquad (5.11)
\end{aligned}$$

Now we prove that for (u^+,v^+) there exists $0 < t_{(u^+,v^+)} \neq 1$ such that

$$t_{(u^+,v^+)}(u^+,v^+) \in \mathcal{N}^+.$$

Since $c^+ < 0$, one can show that $(u^+,v^+) \neq (0,0)$. By (5.5), we deduce that

$$\lambda\int_\Omega b_1(x)|u^+|^q dx + \mu\int_\Omega b_2(x)|v^+|^r dx < 0,$$

then by Lemma 3.3 there exists $t_{(u^+,v^+)} > 0$ such that

$$t_{(u^+,v^+)}(u^+,v^+) \in \mathcal{N}^+ \quad \text{and} \quad \Psi'_{(u^+,v^+)}(t_{(u^+,v^+)}) = 0.$$

By (5.11), $\Psi'_{(u^+,v^+)}(1)) < 0$. Thus, $t_{(u^+,v^+)} \neq 1$. Note that $t_{(u^+,v^+)}(u^+,v^+)$ is minimizer of $g(t) = E(\tau(u^+,\tau v^+)$. Thus,

$$E(t_{(u^+,v^+)}u^+, t_{(u^+,v^+)}v^+) < E(u^+,v^+)) \leq \lim_{n\to+\infty} E(u_n,v_n) = \inf_{(u,v)\in\mathcal{N}^+} E(u,v),$$

and this is absurd. Therefore, we obtain $(u_n,v_n) \to (u^+,v^+)$ in W.

Step 2. Existence of nonnegative minimizers. If $(u_n,v_n) \to (u^+,v^+)$ in W and Lemma 3.9, we get

$$-\lambda\int_\Omega h_1(x)|u^+|^\theta dx - \mu\int_\Omega h_2(x)|v^+|^\theta dx + (p-q-r)\int_\Omega (b_1+b_2)(x)|u^+|^q|v^+|^r dx > 0. \quad (5.12)$$

Thus, we obtain that $(u^+,v^+) \in \mathcal{N}^+$. This gives that (u^+,v^+) is a minimizer of E on $\mathcal{N}^+$. Proof, we can prove $(|u^+|,|v^+|)$ is also a minimizer $\mathcal{N}^+$. Since

$$E(|u^+|,|v^+|) \leq E(u^+,v^+)$$

and using (5.12) hold for $(|u^+|,|v^+|)$, it suffices to show that

$$\begin{aligned}
&\|(u^+,v^+)\|^p - \lambda\int_\Omega h_1(x)|u^+|^\theta\ln|u^+|dx - \mu\int_\Omega h_1(x)|v^+|^\theta\ln|v^+|dx\\
&= \int_\Omega b_1(x)|u^+|^q|v^+|^r dx + \int_\Omega b_2(x)|u^+|^q|v^+|^r dx. \qquad (5.13)
\end{aligned}$$

Using $[|u^+|]_s^p \leq [u^+]_s^p$, we obtain

$$\begin{aligned}
&\|(u^+,v^+)\|^p - \lambda\int_\Omega h_1(x)|u^+|^\theta\ln|u^+|dx - \mu\int_\Omega h_1(x)|v^+|^\theta\ln|v^+|dx\\
&\leq \int_\Omega b_1(x)|u^+|^q|v^+|^r dx + \int_\Omega b_2(x)|u^+|^q|v^+|^r dx. \qquad (5.14)
\end{aligned}$$

If

$$\begin{aligned}\|(u^+,v^+)\|^p - \lambda \int_\Omega h_1(x)|u^+|^\theta \ln|u^+|dx - \mu\int_\Omega h_1(x)|v^+|^\theta \ln|v^+|dx \\ < \int_\Omega b_1(x)|u^+|^q|v^+|^r dx + \int_\Omega b_2(x)|u^+|^q|v^+|^r dx,\end{aligned} \tag{5.15}$$

then $\Psi'_{(|u^+|,|v^+|)}(1)) < 0$. For $(|u^+|,|v^+|)$, by Lemma 3.3 there exist $t_{(|u^+|,|v^+|)} \in \mathcal{N}^+$ and $\Psi'_{(|u^+|,|v^+|)} t_{(|u^+|,|v^+|)} = 0$. Thus, $t_{(|u^+|,|v^+|)} \neq 1$. For other hand $t_{(|u^+|,|v^+|)}$ is a minimizer of

$$\xi(t) := E(u^+,v^+).$$

Thus,

$$E(t_{(|u^+|,|v^+|)}|u^+|, t_{(|u^+|,|v^+|)}|v^+| \le E(|u^+|,|v^+|) \le E(u^+,v^+) = \inf_{(u,v)\in\mathcal{N}^+} E(u,v).$$

This is absurd. Thus, $(|u^+|,|v^+|) \in \mathcal{N}^+$ and

$$E(|u^+|,|v^+|) = \inf_{(u,v)\in\mathcal{N}^+} E(u,v)$$

In conclusion, we get nonnegative minimizer E on $\mathcal{N}^+$. □

Lemma 5.2. *E has a nontrivial and nonnegative minimizer on $\mathcal{N}^-$.*

Proof. By Lemma 3.9, we know $c^- := \inf_{(u,v)\in\mathcal{N}^-} E(u,v)$ is attained. Let $\{(u_n,v_n)\}_n \subset \mathcal{N}^-$ be a minimizing sequence such that $E(u_n,v_n) \to c^-$. Then

$$\|(u_n,v_n)\|^p - \lambda\int_\Omega h_1(x)|u_n|^\theta dx - \mu\int_\Omega h_2(x)|v_n|^\theta dx = \int_\Omega (b_1+b_2)(x)|u_n|^q|v_n|^r dx, \tag{5.16}$$

and,

$$\begin{aligned}0 > \lambda(p-\theta)\int_\Omega h_1(x)|u_n|^\theta \ln|u_n|dx + \mu(p-\theta)\int_\Omega h_2(x)|v_n|^\theta \ln|v_n|dx \\ - \lambda(p-\theta)\int_\Omega h_1(x)|u_n|^\theta dx - \mu(p-\theta)\int_\Omega h_2(x)|v_n|^\theta dx \\ - (q+r-p)\int_\Omega (b_1+b_2)(x)|u_n|^q|v_n|^r dx.\end{aligned} \tag{5.17}$$

We claim that $\{(u_n,v_n)\}_n$ is bounded in $\mathcal{N}^-$.

Without loss of generality, we assume that $\|(u_n,v_n)\| \ge 1$. Then

$$\begin{aligned}c^- + o_n(1) = E(u_n,v_n) - \frac{1}{p}E'(u_n,v_n)(u_n,v_n) \\ \left(\frac{1}{p}-\frac{1}{\theta}\right)\left(\lambda\int_\Omega h_1(x)|u_n|^\theta \ln|u_n|dx + \mu\int_\Omega h_2(x)|v_n|^\theta \ln|v_n|dx\right) \\ + \frac{1}{\theta^2}\left(\lambda\int_\Omega h_1(x)|u_n|^\theta dx + \mu\int_\Omega h_2(x)|v_n|^\theta dx\right) \\ + \left(\frac{1}{p}-\frac{1}{q+r}\right)\int_\Omega (b_1+b_2)(x)|u_n|^q|v_n|^r dx.\end{aligned} \tag{5.18}$$

So

$$c^- + o_n(1) \geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\left(\lambda \int_\Omega h_1(x)|u_n|^\theta \ln |u_n| dx + \mu \int_\Omega h_2(x)|v_n|^\theta \ln |v_n| dx\right)$$
$$+ \frac{1}{\theta^2}\left(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\right). \tag{5.19}$$

If $\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx \geq 0$, then from equations (5.18), (5.19), we have

$$\|(u,v)\| \leq \left(\frac{1}{\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L}\right)^{\frac{1}{p-2}}.$$

If, $\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx < 0$, from equation (5.19), we have

$$\begin{aligned} c^- + o_n(1) &= E(u_n, v_n) \\ &\geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right] \|(u_n, v_n)\|^p \\ &\quad + \left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right) \ln \|(u_n, v_n)\|\right] \left(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\right). \end{aligned}$$

We have two cases to analyze:

i) If $\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right) \ln \|(u_n, v_n)\| \leq 0$, we have

$$c^- + o_n(1) = \left(\frac{1}{p} - \frac{1}{q+r}\right) [1 - (\lambda C_{h_1} + \mu C_{h_2})L] \|(u_n, v_n)\|^p,$$

which implies that

$$\|(u_n, v_n)\| \leq \left[\frac{p(q+r)(c^- + o_n(1))}{(q+r-p)[1 - (\lambda C_{h_1} + \mu C_{h_2})L]}\right]^{\frac{1}{p}}.$$

ii) If $\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right) \ln \|(u_n, v_n)\| > 0$, then $\|(u_n, v_n)\| < \exp\left(\frac{q+r-\theta}{\theta^3(q+r)}\right)$. So there exists $C > 0$ such that $\|(u_n, v_n)\| < C$.

Thus, every minimizer sequence of E on $\mathcal{N}^-$ is bounded.

Now, since $\{(u_n, v_n)\}$ is bounded in $\mathcal{N}^-$, up to a subsequence, we may assume that

$$\begin{aligned} (u_n, v_n) &\rightharpoonup (u_-, v_-) && \text{weakly in } W; \\ (u_n, v_n) &\to (u_-, v_-) && \text{strongly in } L^\nu(\Omega) \times L^\tau(\Omega); \\ (u_n, v_n) &\to (u_+, v_+) && \text{a.e in } \Omega, \end{aligned}$$

where $1 < \nu, \tau < \min\{p_s^*, p_t^*\}$. It implies that

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega h_1(x)|u_n|^\theta \ln |u_n| dx &= \int_\Omega h_1(x)|u_-|^\theta \ln |u_-| dx; \\ \lim_{n\to\infty} \int_\Omega h_2(x)|v_n|^\theta \ln |v_n| dx &= \int_\Omega h_2(x)|v_-|^\theta \ln |v_-| dx; \\ \lim_{n\to\infty} \int_\Omega h_1(x)|u_n|^\theta dx &= \int_\Omega h_1(x)|u_-|^\theta dx; \\ \lim_{n\to\infty} \int_\Omega h_1(x)|v_n|^\theta dx &= \int_\Omega h_2(x)|v_-|^\theta dx. \end{aligned}$$

An easy computation, combined with equation (5.17), shows that

$$I(u_-,v_-) \leq \liminf_{n\to\infty} I(u_n,v_n) = 0.$$

It means that $(u_-,v_-) \in \mathcal{N}^-$. Now, by Lemma 3.3(1), there exists $k_{(u,v)} > 0$ such that $I'_{(u_-,v_-)}k_{(u_-,v_-)} = 0$ and $k_{(u_-,v_-)} \neq 1$. Since $(u_n,v_n) \not\to (u_-,v_-)$, we get

$$k_{(u_-,v_-)}(u_n,v_n) \not\to k_{(u_-,v_-)}(u_-,v_-)$$

in W. Another easy computation shows that

$$E(k_{(u_-,v_-)}u_-,k_{(u_-,v_-)}v_-) \leq E(k_{(u_-,v_-)}u_n,k_{(u_-,v_-)}v_n).$$

Now, observe that the function $z(k) := E(ku_n,kv_n)$ attains its maximum at $k = k_{(u_-,v_-)}$. So

$$\begin{aligned} E(k_{(u_-,v_-)}u_-,k_{(u_-,v_-)}v_-) &< \liminf_{n\to\infty} E(k_{(u_-,v_-)}u_n,k_{(u_-,v_-)}v_n) \\ &\leq \lim_{n\to\infty} E(u_n,v_n) \\ &= \inf_{(u,v)\in\mathcal{N}^-} E(u,v), \end{aligned}$$

a contradiction. Thus $(u_n,v_n) \to (u_-,v_-)$ in W.

Because $(u_-,v_-) \in \mathcal{N}^-$, then (u_-,v_-) is a minimizer of E on $\mathcal{N}^-$. Moreover, a similar discussion as Theorem 5.1 - *step* 2- one can show that $(|u_-|,|v_-|)$ is a minimizer of E on $\mathcal{N}^-$. This yields the proof. □

Now it is possible to prove Theorem 1.2.

5.1 Proof of Theorem 1.2

From Theorems 5.1 and 5.2, E has two non-negative minimizers $(u_+,v_+) \in \mathcal{N}^+$ and $(u_-,v_-) \in \mathcal{N}^-$. Then, from Theorem 3.2, E has two non-negative critical points on W, which is non-trivial and non-negative local least energy solution of Problem (P). This two solution are distinct, because is obviously that $\mathcal{N}^- \cap \mathcal{N}^+ = \emptyset$.

We claim that (u_+,v_+) and (u_-,v_-) are not semi-trivial solution.

Supposing, otherwise, $v_t = 0$ in (u_+,v_+), we get that u_+ is a non-trivial solution of the problem

$$\begin{cases} (-\Delta)^s_p u + |u|^{p-2}u = \lambda h_1(x)|u|^{\theta-2}u\ln|u| & \text{in } \Omega, \\ (u,v) \in W^{s,p}_0(\Omega) \times W^{t,p}_0(\Omega) \end{cases} \tag{P'}$$

Then $\|(u_+,0)\|^p = [u_+]^p_s = \lambda \int_\Omega h_1(x)|u_+|^\theta \ln|u_+|dx$, and because $(u_+,0) \in \mathcal{N}^+$ and $\Phi''_{(u,v)}(1) > 0$, we get

$$\lambda(p-\theta)\int_\Omega h_1(x)|u_+|^\theta dx < \lambda\int_\Omega h_1(x)|u_+|^\theta dx.$$

Because $p < \theta$, we have $\int_\Omega h_1(x)|u_+|^\theta dx < 0$.

Now we choose $w \in W^{t,p}_0(\Omega) \setminus \{0\}$ such that $\int_\Omega h_1(x)|w|^\theta dx < 0$.

For (u_+,w), by Lemma 3.3, there exists a unique $k_1 > 0$ such that $k_1(u_+,w) \in \mathcal{N}^+$. Moreover, we have

$$k_{\max} = \exp\left(\frac{\|(u_+,w)\|^p - \lambda\int_\Omega h_1(x)|u_+|^\theta \ln|u_+|^\theta dx + \mu\int_\Omega h_2(x)|w|^\theta \ln|w|^\theta dx}{\lambda\int_\Omega h_1(x)^\theta dx + \mu\int_\Omega h_2(x)|w|^\theta dx} - \frac{1}{p-q-r}\right),$$

and

$$E(k_1 u_+, k_1 w) = \inf_{0<k<k_{\max}} E(ku_+, kw).$$

It follows that

$$c^+ \leq E(k_1 u_+, k_1 w) < E(u_+, w) < E(u_+, 0) = c^+,$$

a contradiction. Thus (u_+, v_+) is not a semi-trivial solution for problem (P).

Otherwise, (u_-, v_-) is not a semi-trivial solution for problem (P), by using the same above argument, but this time assuming $v_- = 0$. In this case $(u_-, 0)$ is a nontrivial solution for problem (P') and $\int_\Omega h_1(x)|u_-|^\theta dx > 0$ and $w \in W_0^{t,p}(\Omega) \setminus \{0\}$ is taking such that

$$\lambda \int_\Omega h_1(x)|u_-|^\theta dx + \mu \int_\Omega h_2(x)|w|^\theta dx < 0.$$

In this way we concluded what we wanted.

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References

[1] J. C. Albuquerque, K. Silva, On the extreme value of the Nehari manifold method for a class of Schrödinger equations with indefinite weight functions, *J. Differential Equations* **269**(2020), 5680–5700. https://doi.org/10.1016/j.jde.2020.04.019; MR4104938

[2] A. H. Ardila, Existence and stability of standing waves for nonlinear fractional Schrödinger equation with logarithmic nonlinearity, *Nonlinear Anal.* **155**(2017), 52–64. https://doi.org/10.1016/j.na.2017.01.006; MR3631741

[3] V. I. Bogachev, *Measure theory*, Springer-Verlag, Berlin–Heidelberg, 2007. https://doi.org/10.1007/978-3-540-34514-5; MR2267655

[4] Y. Bozhkov, E. Mitidieri, Existence of multiple solutions for quasilinear systems via fibering method, *J. Differential Equations*, **190**(2003), 239–267. https://doi.org/10.1016/S0022-0396(02)00112-2; MR1970963

[5] K. J. Brown, T. F. Wu, A fibering map approach to a semilinear elliptic boundary value problem, *Electron. J. Differential Equations* **2007**, No. 69, 9 pp. MR2308869

[6] W. Chen, Existence of solutions for critical fractional $p\&q$-Laplacian system, *Complex Var. Elliptic Equ.* **66**(2021), 626–641. https://doi.org/10.1080/17476933.2020.1734926; MR4224765

[7] J. N. Correia, G. M. Figueiredo, Existence of positive solution of the equation $(-\Delta)^s u + a(x)u = |u|^{2_s^*-2}u$, *Calc. Var. Partial Differential Equations* **58**(2019), Paper No. 63, 39 pp. https://doi.org/10.1007/s00526-019-1502-7; MR3925554

[8] P. Drábek, S. Pohozaev, Positive solutions for the p-Laplacian: application of the fibering method, *Proc. Roy. Soc. Edinburgh Sect. A* **127**(1997), 703–726. https://doi.org/10.1017/S0308210500023787; MR1465416

[9] Q. Li, Z. Yang, Multiple positive solutions for a fractional Laplacian system with critical nonlinearities. *Bull. Malays. Math. Sci. Soc.* **41**(2028), 1879–1905. https://doi.org/10.1007/s40840-016-0432-1; MR3854497

[10] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136**(2012), 521–573. https://doi.org/10.1080/17476933.2020.1734926; MR2944369

[11] S. I. Pohožaev, On an approach to nonlinear to nonlinear equations (in Russian), *Dokl. Akad. Nauk SSSR* **247**(1979), No. 6, 1327–1331. MR0550349

[12] S. I. Pohožaev, On a constructive method in calculus of variations (in Russian), *Dokl. Akad. Nauk SSSR* **298**(1988), No. 6, 1330–1333; translation in *Soviet Math. Dokl.* **37**(1988), No. 1, 274–277. MR0947797

[13] S. Pokhozhaev, On fibering method for the solutions of nonlinear boundary value problems (in Russian), *Trudy Mat. Inst. Steklov* **192**(1990), 146–163. MR1097896

[14] P. Pucci, M. Xiang, B. Zhang, Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations* **54**(2015), 2785–2806. https://doi.org/10.1007/s00526-015-0883-5; MR3412392

[15] F. Wang, H. Die, M. Xiang, Combined effects of logarithmic and superlinear nonlinearities in fractional Laplacian systems, *Anal. Math. Phys.* **11**(2021), Paper No. 9, 34 pp. https://doi.org/10.1007/s13324-020-00441-9; MR4179975

[16] Z. Wu, X. Zhang, Existence and multiplicity of solutions for a mixed local-nonlocal system with logarithmic nonlinearities, *Results Math.* **78**(2023), Paper No. 240, 25 pp. https://doi.org/10.1007/s00025-023-02019-w; MR4647962

[17] M. Xiang, D. Hu, D. Yang, Least energy solutions for fractional Kirchhoff problems with logarithmic nonlinearity, *Nonlinear Anal.* **198**(2020), 111899, 20 pp. https://doi.org/10.1016/j.na.2020.111899; MR4083633

[18] M. Xiang, D. Yang, B. Zhang, Degenerate Kirchhoff-type fractional diffusion problem with logarithmic nonlinearity. *Asymptot. Anal.* **118**(2020), 313–329. https://doi.org/10.1007/s11425-022-2172-x; MR4156523

[19] Q. Zhang, W. Zou, Existence and asymptotics of normalized solutions for the logarithmic Schrödinger system, *Sci. China Math.* **67**(2024), 2019–2048. https://doi.org/10.1007/s11425-022-2172-x; MR4784204