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Existence and exponential stability for the wave equation with nonlinear interior source and localized viscoelastic boundary feedback

Josiane Cristina de Oliveira Faria ✉

Departamento de Matemática, Universidade Estadual de Maringá, Maringá, PR, Brazil

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Abstract. In this work, we aim to investigate an integro-differential model that involves localized viscoelastic effects at the boundary of the domain under the history framework. We have established that the equation is well-posed and exhibits exponential stability when a localized admissible kernel is applied, along with the δ-condition.

Keywords: wave equation, localized boundary feedback, exponential stability.

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1 Introduction

1.1 The model and literature overview

We consider the following problem

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^\infty g(s)a(x)u_t(x,t-s)\,ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,-t) = u^0(x,-t), \quad u_t(x,0) = u_t^0(x) & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is an open bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, $u^0 : \Omega \times [0,\infty) \to \mathbb{R}$ is the prescribed past history of u. We denote by ω, ω_0, ω_1 the intersection of Ω with a neighborhood of Γ, Γ_0, Γ_1 in $\mathbb{R}^d$, respectively. In addition, $a = a(x)$, is real valued non-negative function, responsible for the localized dissipative effect, $f : \mathbb{R}^+ \to \mathbb{R}^+$ represents a source term and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative function having the form

$$g(s) = \int_s^\infty \mu(\tau)\,d\tau, \tag{1.2}$$

✉ Email: jcofaria@uem.br

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is an integrable function. Other assumptions on the functions f, g, μ and a will be precisely stated ahead.

It is worth mentioning that the study of stabilization of evolution equations subjected to boundary dissipation has been gaining more attention in the academic world over the past few years. In the absence of the viscolelastic term

$$\int_0^\infty g(s)a(x)u_t(x,t-s)ds,$$

problem (1.1) has been handled by many authors when a frictional damping term (linear or not) at the boundary is included; see for instance [7, 9, 26, 28, 40] among others. Related to viscoelastic boundary conditions, Aassila and Cavalcanti [2] studied the following problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^t k(t-s,x)u'(s)\,ds + a(x)g(u') = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u'(x) = u_1(x) & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $a : \Gamma_1 \to \mathbb{R}^+$ is such that $a(x) \geq a_0 > 0$. Under the following assumptions on functions k and g

$$k \geq 0, \qquad k' \leq 0, \qquad k'' \geq \alpha k' \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{1.4}$$

$$C_1|x|^p \leq |g(x)| \leq C_2|x|^{1/p}, \ |x| \leq 1; \qquad C_3|x| \leq |g(x)| \leq C_4|x|, \ |x| \geq 1, \tag{1.5}$$

for some positive constants α, $C_i (1 \leq i \leq 4)$, the authors obtained the energy decays exponentially if $p = 1$ and decays polynomially if $p > 1$ when $u_0 = 0$ in Γ_1, extending the work of [23] to the case of nonlinear frictional dampings at boundary. Park and collaborators, in [35], considered a similar extension to a nonlinear boundary condition of memory type with the same assumption on k but without the above assumption on u_0. They also included a nonlinear source term $|u|^\rho u$ acting on the domain Ω, which turns the problem more subtle than those previously cited. For other problems in connection with viscoelastic and dynamic boundary conditions, the reader is referred to [10], [1, 11, 19, 20, 25] and references therein.

Nowadays a question that has been extensively investigated is the role of the kernel k in a viscoelastic term of type

$$\int_0^t k(t-s,x)w(s)ds \qquad (w = u \text{ or } w = u') \tag{1.6}$$

acting on the domain and/or the boundary to provide existence, as well stability of solutions. A reasonably large class of them has been carried out for many authors. Indeed, since the highly cited article of Dafermos [15], a flurry of work has been done with increasing kernels $k \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ satisfying $k(s) > 0$ and conditions like $1 - \int_0^\infty k(s)\,ds > 0$, together with the classical conditions (1.4) and improvements of them to provided existence and stability of solutions, we quote for instance [1, 10, 21, 29, 33, 36, 38] among others. A generalization of condition (1.4) was considered by Alabau-Boussouira and Cannarsa in [3] (see also [30–32]), where the main assumption is that the kernel k solves a suitable differential inequality. Other refinements of such condition are also discussed in [24, 27, 34].

Efforts are being made to achieve a less restricted assumption on the memory kernel k. Indeed, in [12], the authors introduced a general class of kernels called admissible kernels. These kernels are allowed not being strictly decreasing and can be locally flat while still fulfilling the so-called δ-condition: for some $\delta > 0$, there exists $C \geq 1$ such that

$$k(t+s) \leq Ce^{-\delta t}k(s)$$

for every $t \geq 0$ and $s > 0$. On these conditions, some authors have explored questions related to existence and stability of solutions, see for example [6,13,14].

1.2 Contribution and article structure

As mentioned earlier, previous research on viscoelastic dissipation at boundaries has mainly focused on the standard assumptions for the kernel, k. However, we have not found any studies that explore the effects of a more general memory kernel at the system's boundary, nor in a localized framework. Therefore, this paper's main contribution is its novel approach to this topic. We consider the past history framework together with a localized admissible kernel under the δ-condition to show exponential stability without the inclusion of frictional damping, unlike some of the articles mentioned earlier. However, this approach presents certain technical difficulties that must be addressed to obtain an observability inequality, which is crucial to proving the exponential stability of the problem.

Indeed, to demonstrate the exponential stability, we draw inspiration from the works of Dehman, Gérard, Lebeau [16] and Dehman, Lebeau, and Zuazua [17]. The key step in this approach involves establishing the observability inequality:

$$E(0) \leq C\left(\int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta^t(s)|^2 \, ds \, d\Gamma \, dt\right)$$

for all $t \geq T_0$.

To prove this statement we employ a contradiction argument and seek a sequence (u_m, η_m^t) of weak solutions to the equivalent problem (2.2) such that $E_m(0) = 1$. By utilizing a boundary observability theorem by Duyckaerts, Zhang, and Zuazua [18], we aim to derive the desired contradiction by showing that $E_m(0) \to 0$ as $m \to \infty$. However, challenges arise due to the nature of μ satisfying the δ-condition, making it difficult to establish that the convergence

$$\lim_{m\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta_m^t|^2 \, ds \, d\Gamma \, dt = 0$$

implies

$$\lim_{m\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty \mu(s)a(x)|\eta_m^t|^2 \, ds \, d\Gamma \, dt = 0,$$

which is usual in this kind of problem, and is a crucial step for completing our contradiction argument.

Based on the above statements, this article is structured as follows: Section 2 discusses the well-posedness of problem (1.1) by introducing the well-known relative displacement history variable introduced by [15] to obtain an equivalent problem, as is typical in this kind of approach. In Section 3, the exponential stability of the solution is established by demonstrating an appropriate observability inequality.

2 Existence and uniqueness of solution

Through this article, we will use basic notations and results from books by [5,8,39].

In this section, we will prove the first result of this paper regarding the existence and uniqueness of solution for the system (2.2). To achieve this, we will introduce an equivalent problem that will enable us to utilize the Semigroups theory, as well the main assumptions and notations to be used throughout this paper.

As in the pioneer work of [15], and by following [22], we introduce the following new variable corresponding to the relative displacement history

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \qquad x \in \Omega,\ s > 0,\ t \geq 0, \tag{2.1}$$

in order to translate (1.1) into the autonomous problem

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^\infty \mu(s) a(x) \eta^t(x,s)\, ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \eta_s^t + \eta_t^t = u_t & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ u(x,-t) = u^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x) = u^0(x,0), \quad u_t(x,0) = u_1(x) = u_t^0(x) & \text{in } \Omega, \\ \eta^t(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \eta^0(x,s) = \eta_0(x,s) = u^0(x,0) - u^0(x,-s) & \text{on } \Omega \times (0,\infty) \end{cases} \tag{2.2}$$

in the two variables $u = u(t)$ and $\eta = \eta^t(s)$.

In the sequel, to state our main results on the well-posedness and asymptotic behavior of problem (1.1), let us consider the following assumptions and notations:

A.1. $a : \Gamma_1 \to \mathbb{R}^+ \in L^\infty(\Gamma_1) \cap C(\overline{\omega}_1)$ is such that

i. $a(x) \geq 0$ on Γ_1;

ii. $a(x) \geq a_0 > 0$ in $\omega_1 \subset\subset \Gamma_1$.

A.2. $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative function having the form

$$g(s) = \int_s^\infty \mu(\tau)\, d\tau, \tag{2.3}$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a pointwise absolutely continuous function, nonincreasing, integrable and such that

i. $l = \int_0^\infty \mu(s)\, ds \in (0,1)$;

ii. there exists a strictly increasing sequence $\{s_n\}$, with $s_0 = 0$, either finite or converging to $s_\infty \in (0,\infty]$, such that μ has jumps at $s = s_n,\ n > 0$.

A.3. $f \in C^2(\mathbb{R})$ satisfies

i. $f(0) = 0$;

ii. the primitive $F(s) = \int_0^s f(\tau)\,d\tau$ is such that

$$-\frac{\gamma|s|^2}{2} \leq F(s) \leq f(s)s + \frac{\gamma|s|^2}{2}, \tag{2.4}$$

$\gamma \in [0, \lambda_1[$, where $\lambda_1 > 0$ is the first eigenvalue corresponding to the Laplacian operator with Dirichlet boundary condition;

iii. there exists $c > 0$ such that

$$|f^{(j)}(s)| \leq c(1+|s|)^{p-j}, \quad \forall s \in \mathbb{R},\ j = 1,2, \tag{2.5}$$

where

$$p \geq 1 \ \text{ if } \ n = 2 \quad \text{and} \quad 1 \leq p < \frac{n}{n-2} \ \text{ if } \ n \geq 3. \tag{2.6}$$

Remark 2.1.

1. Notice that the function μ defined in Assumption A.2 can be unbounded in a neighborhood of zero. Moreover, μ is differentiable almost everywhere, and $\mu'(s) \leq 0$ for almost every s.

2. Observe that the growth condition on f implies that

$$|f(s)| \leq c(p)|s| + c(p)|s|^p. \tag{2.7}$$

We still note that (2.4) implies $f'(0) + \gamma \geq 0$ as well.

Now, consider $A : D(A) \subset H^1_{\Gamma_0}(\Omega) \to H^{-1}(\Omega)$ the operator $Au = -\Delta u$, with $D(A) = \{u \in H^1_{\Gamma_0}(\Omega),\ \partial_\nu u_{|\Gamma_1} = 0\}$, $h : \mathcal{M} \to L^2(\Gamma_1)$, $h(w(s)) = \int_0^\infty \mu(s)a(x)w(s)\,ds$ and $N : L^2(\Omega) \to L^2(\Omega)$ be the Neumann map

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ Ng = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial(Ng)}{\partial\nu} = g & \text{on } \Gamma_1. \end{cases}$$

Therefore, we have that

$$N^*A^*v = -v_{|\Gamma_1}, \qquad \forall v \in D(A^{\frac{1}{2}}) \tag{2.8}$$

as well as the system (2.2) is equivalent to

$$\begin{cases} u_{tt} + A(u - N[h(\eta)]) + f(u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \eta^t_s + \eta^t_t = u_t & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ u(x,-t) = u^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x) = u^0(x,0), \quad u_t(x,0) = u_1(x) = u^0_t(x) & \text{in } \Omega, \\ \eta^t(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \eta^0(x,s) = \eta_0(x,s) = u^0(x,0) - u^0(x,-s) & \text{on } \Omega \times (0,\infty). \end{cases} \tag{2.9}$$

Next, let a be a function satisfying Assumption A.1, and define the μ-weighted space with values in $L^2(\Gamma_1)$ as

$$\mathcal{M} = \left\{ \eta : \mathbb{R}^+ \to L^2(\Gamma_1);\ \int_0^\infty \mu(s) \|\sqrt{a}\eta(s)\|^2 < \infty \right\}, \tag{2.10}$$

which is a Hilbert space endowed with the inner-product

$$(\eta, \zeta)_{\mathcal{M}} = \int_0^\infty \mu(s) \int_{\Gamma_1} \sqrt{a}\eta(s)\sqrt{a}\zeta(s)\, d\Gamma\, ds.$$

Throughout this article, $\mathcal{H}$ represents the energy space

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times \mathcal{M},$$

where $H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$, and $\mathcal{H}$ is endowed with the inner product

$$\langle (u_1, v_1, \eta_1), (u_2, v_2, \eta_2) \rangle_{\mathcal{H}} = \int_\Omega (\nabla u_1 \nabla u_2 + v_1 v_2)\, dx + \int_0^\infty \mu(s) \int_{\Gamma_1} \sqrt{a}\eta_1 \sqrt{a}\eta_2\, d\Gamma\, ds.$$

Therefore, denoting $U = (u, u_t, \eta)^T$ we can write, equivalently, the system (2.9) in the form

$$\begin{cases} \frac{d}{dt} U(t) + \mathcal{S}U(t) + \mathcal{F}(U(t)) = 0, \\ U(0) = (u_0, u_1 \eta_0), \end{cases} \tag{2.11}$$

where $\mathcal{S} : D(\mathcal{S}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{S} \begin{bmatrix} u \\ v \\ \eta \end{bmatrix} = \begin{bmatrix} -v \\ A(u - N[h(\eta)]) \\ v - \eta^t_s \end{bmatrix},$$

$$D(\mathcal{S}) = \left\{ (u, v, \eta) \in \mathcal{H} :\ v \in H^1_{\Gamma_0},\ u - N[h(\eta)] \in D(A),\ \eta^t_s \in \mathcal{M},\ \eta(0) = 0 \right\},$$

which is well-defined in view of the previous explanation, and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is set by

$$\mathcal{F}(U) = (0, f(u), 0)^T,$$

being also well-defined by virtue of the growth condition on f and standard Sobolev embeddings. The Hadamard well-posedness of problem (2.11) and, consequently, of the original system (1.1), reads as follows.

Theorem 2.2. *Under the Assumptions A.1–A.3 we have:*

(i) *If $U_0 = (u_0, u_1, \eta_0) \in D(\mathcal{S})$, then there exists a unique regular solution $U = (u, u_t, \eta)$ of (2.11) such that*

$$u \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; H^1_{\Gamma_0}(\Omega)), \quad \eta \in W^{1,\infty}(0, T; \mathcal{M}),$$

with $U(t) = (u(t), u_t(t), \eta^t) \in D(\mathcal{S})$, for all $t \in [0, T]$, for a given $T > 0$.

(ii) *If $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$, then there exists a unique mild solution $U = (u, u_t, \eta)$ of (2.11) such that*

$$u \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; H^1_{\Gamma_0}(\Omega)), \quad \eta \in C([0, T], \mathcal{M}),$$

for all $T > 0$ given.

(iii) *Moreover, these solutions are continuously dependent of the initial data, in the norm of* $C([0,T],\mathcal{H})$, *for all* $T>0$.

Proof. To establish this result, firstly we shall prove that $\mathcal{S}$ is monotone and $I-\mathcal{S}$ is surjective on the space $\mathcal{H}$. Indeed, for $\eta \in D(\mathcal{S})$, define

$$\mathbb{J}[\eta] = \sum_{n\geq 1}(\mu(s_n^-) - \mu(s_n^+))\|\eta(s_n)\|_{\mathcal{M}}^2,$$

which is a nonpositive quantity in view of Assumption A.2. Following [37, Lemma 3.4], one notices that $\eta \in D(\mathcal{S})$ satisfies

$$2(\eta_s,\eta)_{\mathcal{M}} = \int_0^\infty \mu'(s)\|\eta(s)\|_{\mathcal{M}}^2\, ds + \mathbb{J}[\eta].$$

Let

$$\begin{bmatrix} u_1 \\ v_1 \\ \eta_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \\ \eta_2 \end{bmatrix} \in D(\mathcal{S}).$$

Then

$$\begin{aligned}
&\left\langle \mathcal{S}\begin{bmatrix} u_1 \\ v_1 \\ \eta_1 \end{bmatrix} - \mathcal{S}\begin{bmatrix} u_2 \\ v_2 \\ \eta_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_1 \\ \eta_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \\ \eta_2 \end{bmatrix}\right\rangle \\
&\quad = \langle -v_1+v_2, u_1-u_2\rangle_{H^1_{\Gamma_0}} + \langle Av_1 - Av_2, v_1-v_2\rangle_{L^2(\Omega)} \\
&\qquad - \langle A(N[h(\eta_1)-h(\eta_2)]), v_1-v_2\rangle_{L^2(\Omega)} + (v_1-v_2,\eta_1-\eta_2)_{L^2(\Omega)} \\
&\qquad - ((\eta_1)_s-(\eta_2)_s,\eta_1-\eta_2)_{\mathcal{M}} \\
&\quad = -\frac{1}{2}\mathbb{J}[\eta] \geq 0,
\end{aligned}$$

which shows that $\mathcal{S}$ is monotone.

Next, we will prove that $I-\mathcal{S}$ is surjective. To this end, we show the equation

$$(I-\mathcal{S})\left(\begin{bmatrix} u \\ v \\ \eta \end{bmatrix}\right) = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

has a solution

$$\begin{bmatrix} u \\ v \\ \eta \end{bmatrix} \in \mathcal{H}, \quad \text{for any} \quad h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \in \mathcal{H}.$$

The above equation is equivalent to write

$$\begin{cases} u+v = h_1, \\ v - A(u-N[h(\eta)]) = h_2 \\ \eta+\eta_s - v = h_3. \end{cases} \tag{2.12}$$

Combining the above identities we deduce in the weak space

$$\begin{cases} -A(-v-N[h(\eta)]) + v = h_2 + A(h_1) \\ \eta+\eta_s - v = h_3. \end{cases} \tag{2.13}$$

Denote

$$R(v,\eta) = (T_0 + T_1 + T_2)(v,\eta),$$
$$T_0(v,\eta) = (0, \eta + s), \quad T_1(v,\eta) = (Av + v, \eta), \quad T_2(v,\eta) = (A(N(h(\eta))), -v).$$

It is well-known that T_0 is maximal monotone in $H^1_{\Gamma_0} \times \mathcal{M}$. Also, T_1 is monotone and from the Lax–Milgram Theorem follows that it is surjective, therefore maximal monotone in $H^1_{\Gamma_0} \times \mathcal{M}$. Furthermore, T_2 is monotone and Lipschitz continuous in $H^1_{\Gamma_0} \times \mathcal{M}$. Then, using standard perturbation results in [4], we conclude that $R = (T_0 + T_1 + T_2)$ is maximal monotone and coercive, therefore the left hand term in (2.13) is surjective. Then, (2.13) possesses a unique solution $(v,\eta) \in H^1_{\Gamma_0}(\Omega) \times \mathcal{M}$. Since $u = v + h_1$ we obtain $u \in H^1_{\Gamma_0}(\Omega)$, which implies that $I - \mathcal{S}$ is surjective.

Next, to finish the proof we observe that from Assumption A.3, for a given $T > 0$, f generates a locally Lipschitz perturbation on the phase space $\mathcal{H}$ which after some standard calculations guarantees, by using the Kato's Theorem, the existence of a unique strong solution $U \in W^{1,\infty}([0,T], \mathcal{H})$ such that $U(t) \in D(S)$ for all $t \in [0,T]$. Moreover, this solution continuously depends on the initial data for any $T > 0$. □

3 Asymptotic stability result

In this section the goal is to establish the exponential stability result concerning problem (2.2).

Denoting $U = (u, u_t, \eta)$ the unique global solution to the problem (2.11) as stated in Theorem 2.2, then the couple (u,η) is the corresponding solution to the equivalent system (2.2). The associated energy functional is given by

$$E(t) = \frac{1}{2}\left[|u_t|^2 + |\nabla u|^2 + \|\eta\|^2_{\mathcal{M}} + 2\int_\Omega \int_0^u f(\tau)\, d\tau dx\right]. \tag{3.1}$$

A straightforward computation provides the identity

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_{\Gamma_1}\int_0^\infty \mu'(s)a(x)|\eta^t(s)|^2 ds\, d\Gamma,$$

which, in view on Assumption A.2, implies that $E(t)$ is a non-increasing function for all $t > 0$ and satisfies the identity

$$E(T) - E(0) = \frac{1}{2}\int_0^T \int_{\Gamma_1}\int_0^\infty \mu'(s)a(x)|\eta^t(s)|^2 ds\, d\Gamma\, dt$$

for all $T > 0$.

In order to obtain the desired stability, we need to make some complementary assumptions on the given functions, as well to make some remarks and comments which will be necessary to prove the exponential stability.

Concerning the memory kernel $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ defined in Assumption A.2 it is also assumed that

A.4. (i) there exist $\delta > 0$ and $C \geq 1$ such that

$$\mu(t+s) \leq Ce^{-\delta t}\mu(s) \tag{3.2}$$

for every $t \geq 0$ and almost every $s > 0$;

(ii) μ is not completely flat, that is, the set

$$D = \{s > 0,\ \mu'(s) < 0\}$$

has positive Lebesgue measure.

Remark 3.1.

a. A kernel μ satisfying Assumption A.4(i) is said to fulfill the δ-condition;

b. Particularly, the δ-condition implies that, for each $t \geq 0$

$$|N_t = \{s \in \mathbb{R}^+,\ Ce^{-\delta t}\mu(s) - \mu(t+s) < 0\}| = 0, \tag{3.3}$$

where $|\cdot|$ stands for the Lebesgue measure of the set.

c. If $S_\infty = \sup\{s,\ \mu(s) > 0\} < \infty$, then μ fulfills the δ-condition for every $\delta > 0$;

d. When $C = 1$, (3.2) is equivalent to the well-known condition in the literature $\mu'(s) + \delta\mu(s) \leq 0$, for almost every $s > 0$;

e. Regarding Assumption A.4(ii), it is fairly easy to show that there exists $\alpha > 0$ large enough such that the set

$$N = \{s \in \mathbb{R}^+,\ \alpha\mu'(s) + \mu(s) < 0\} \tag{3.4}$$

has positive Lebesgue measure.

In view of the aforementioned considerations, the stability result reads as follows.

Theorem 3.2. *Assume that Assumptions A.1–A.4 are in force and let $R > 0$ be a given constant. If $E(0) \leq R$, there exist $T_0 > 0$ and constants $C_0, \lambda > 0$, depending on R, verifying*

$$E(t) \leq C_0 E(0) e^{-\lambda t}, \quad \forall t > T_0. \tag{3.5}$$

As mentioned earlier, an important step to prove estimate (3.5) relies on obtaining an observability inequality through a contradiction argument. To accomplish this it is needed, among other tools, to obtain the following convergence

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty \mu(s) a(x) |\eta_n^t|^2 ds\, d\Gamma\, dt = 0,$$

for a sequence $\{(u_n, \eta_n^t)\}$ of solutions to the problem (2.2), which is not an easy task since μ satisfies the δ-condition $A.4(i)$. The proof of this convergence is stated in the following result:

Lemma 3.3. *Let $\{(u_n, \eta_n^t)\}$ be a sequence of solutions to the problem (2.2). By assuming Assumption A.4, if*

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s) a(x) |\eta_n^t|^2 ds\, d\Gamma\, dt = 0,$$

then

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty \mu(s) a(x) |\eta_n^t|^2 ds\, d\Gamma\, dt = 0.$$

Proof. First one notices that, according to Remark 4.1(e), as

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_0^\infty \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = \lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_N \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt + \lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_{\mathbb{R}^+\setminus N} \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt,$$

we have that

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_N \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt \le \lim_{n\to\infty}\alpha\int_0^T\int_{\Gamma_1}\int_0^\infty -\mu'(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = 0. \tag{3.6}$$

Next, suppose that $\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_{\mathbb{R}^+\setminus N}\mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt \neq 0$. Thus, there exists n_1 large enough such that

$$\int_{\mathbb{R}^+\setminus N}\mu(s)a(x)\|\eta_{n_1}^t\|^2_{L^2(\Gamma_1)}ds > 0,$$

for all $t \ge 0$. To not overload the notation, the index n_1 shall be omitted in the next calculations.

As in [12] consider, for $\eta_0 \in \mathcal{M}$, $U(t) = \mathcal{R}(t)(0,0,\eta_0)$. Therefore, as μ satisfies (3.2), if $\tilde{C}_1 > \max\{1,\tilde{C}\}$, one gets

$$\begin{aligned}
0 < \int_{\mathbb{R}^+\setminus N}\mu(s)a(x)\|\eta_0\|^2_{L^2(\Gamma_1)}ds &\le \int_0^\infty \mu(s)a(x)\|\eta_0\|^2_{L^2(\Gamma_1)}ds\\
&\le \int_0^\infty \mu(s-t)a(x)\|\eta_0\|^2_{L^2(\Gamma_1)}ds\\
&\le 2\|a\|\left[\int_t^\infty \mu(s)(\tilde{C}\|\eta^t(s)\|^2_{L^2(\Gamma_1)} + \|u(t)\|^2)ds\right]\\
&\le 2\|a\|\left[\tilde{C}\int_0^\infty \mu(s+t)\|\eta_0(s)\|^2_{L^2(\Gamma_1)} + \int_0^\infty \mu(s)\|u(t)\|^2 ds\right]\\
&< \tilde{C}_1(C+M)e^{-\delta t}\|\eta_0\|^2_{\mathcal{M}},
\end{aligned}$$

where $M = \|\mathcal{R}(t)\|$.

Particularly, for $t > 0$ fixed and $\eta_0(s) = \chi_{N_t}\phi(s)$, where $\|\phi\|_{L^2(\Gamma_1)} = 1$, we obtain

$$\int_0^\infty [1 - \tilde{C}_1(C+M)]e^{-\delta t}\chi_{N_t}(s)ds < 0. \tag{3.7}$$

On the other hand, for any fixed $t > 0$, define

$$\mathcal{N}_t = \{s \in \mathbb{R}^+,\ \mu(t+s) - \tilde{C}_1(C+M)e^{-\delta t}\mu(s) > 0\}.$$

Thus, from Remark 4.1(b) follows that

$$0 = \int_{\mathcal{N}_t}\mu(t+s) - \tilde{C}_1(C+M)e^{-\delta t}\mu(s)\, ds \le \tilde{C}_1(C+M)e^{-\delta t}|\mathcal{N}_t|,$$

which contradicts (3.7) and shows that

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_{\mathbb{R}^+\setminus N}\mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = 0.$$

□

Next, to the aim of obtaining the desired observability inequality which lead us to the proof of Theorem 3.2, and in order to use an appropriate boundary observability inequality in our arguments it is considered, for each $k \in \mathbb{N}$, the following approximation of problem (2.2):

$$\begin{cases} \partial_{tt}u^k - \Delta u^k + f^k(u^k) = 0 & \text{in } \Omega \times (0,\infty), \\ u^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u^k}{\partial \nu} + \displaystyle\int_0^\infty \mu(s)a(x)\eta^{t,k}(x,s)ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \partial_s \eta^{t,k} + \partial_t \eta^{t,k} = \partial_t u^k & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ u^k(x,-t) = u^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ u^k(x,0) = u_0^k(x) = u^0(x,0), \quad \partial_t u^k(x,0) = u_1^k(x) = \partial_t u^0(x) & \text{in } \Omega, \\ \eta^{t,k}(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \eta^{0,k}(x,s) = \eta_0^k(x,s) = u_0^k(x,0) - u_0^k(x,-s) & \text{on } \Omega \times (0,\infty), \end{cases} \tag{3.8}$$

where the function f^k is defined by

$$f^k(s) = \begin{cases} f(s), & |s| \leq k \\ f(k), & s \geq k \\ f(-k), & s \leq -k. \end{cases}$$

Notice that, for each k, f^k is Lipschitz continuous on $\mathbb{R}$ and the associated energy functional is given by

$$E^k(t) = \frac{1}{2}\left[|\partial_t u^k|^2 + |\nabla u^k|^2 + \|\eta^{t,k}\|^2_{\mathcal{M}} + 2\int_\Omega \int_0^{u^k} f^k(\tau)\, d\tau dx \right]. \tag{3.9}$$

An observability inequality to the truncated problem (3.8) shall be provided by the next result.

Proposition 3.4. *Let us take Assumptions A.1-A.4 and let $R > 0$ be a given constant. The solution (u^k, η^k) of (3.8) satisfies the following inequality*

$$E^k(0) \leq C\left(\int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta^{t,k}(s)|^2\, ds d\Gamma dt \right), \tag{3.10}$$

for all $T \geq T_0$ and some constant C depending only on $U_0 = (u_0, u_1, \eta_0)$, provided that $E^k(0) \leq R$.

Proof. To prove (3.10) we argue by contradiction. Indeed, if such inequality does not hold, there exist $T > T_0 > 0$, $R > 0$ and a sequence $\{(u_n^k, \eta_n^{t,k})\}$ of solutions to

$$\begin{cases} \partial_{tt}u_n^k - \Delta u_n^k + f^k(u_n^k) = 0 & \text{in } \Omega \times (0,\infty), \\ u_n^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u_n^k}{\partial \nu} + \displaystyle\int_0^\infty \mu(s)a(x)\eta_n^{t,k}(x,s)ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \partial_s \eta_n^{t,k} + \partial_t \eta_n^{t,k} = \partial_t u_n^k & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ u_n^k(x,-t) = u^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ u_n^k(x,0) = u_{0n}^k(x) = u_n^0(x,0),\ \partial_t u_n^k(x,0) = u_{1n}^k(x) = \partial_t u_n^0(x) & \text{in } \Omega, \\ \eta_n^{t,k}(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \eta_n^{0,k}(x,s) = \eta_{0n}^k(x,s) = u_{0n}^k(x,0) - u_{0n}^k(x,-s) & \text{on } \Omega \times (0,\infty), \end{cases} \tag{3.11}$$

such that $E_n^k(0) \le R$, which satisfies

$$\lim_{n\to\infty} \frac{E_n^k(0)}{\int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta_n^{t,k}(s)|^2 ds d\Gamma dt} = \infty. \tag{3.12}$$

Since $E_n^k(t) \le E_n^k(0) \le R$ for all $t \ge 0$, from (3.12) one gets

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta_n^{t,k}(s)|^2 ds d\Gamma dt = 0, \tag{3.13}$$

and also guarantees the existence of a subsequence of $\{(u_n^k, \eta_n^{t,k})\}$, still denoted by $\{(u_n^k, \eta_n^{t,k})\}$, such that

$$\begin{aligned} u_n^k &\overset{*}{\rightharpoonup} u^k \quad \text{in } L^\infty(0,T; H_0^1(\Omega)), \\ \partial_t u_n^k &\overset{*}{\rightharpoonup} \partial_t u^k \quad \text{in } L^\infty(0,T; L^2(\Omega)), \end{aligned} \tag{3.14}$$

when $n \to \infty$. By using compactness arguments we also obtain

$$u_n^k \to u^k \text{ in } L^2(0,T; L^2(\Omega)). \tag{3.15}$$

In the sequel, with respect to the limit function u^k, the proof is twofold: $u^k \neq 0$ and $u^k = 0$.

Case I: $u^k \neq 0$. Taking in mind (3.13), (3.14) and Lemma 3.3, from (3.11) one obtains, when $n \to \infty$

$$\begin{cases} \partial_{tt}u^k + \Delta u^k + f^k(u^k) = 0 & \text{in } \Omega \times (0,\infty), \\ u^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u^k}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u^k(x,-t) = u^{0,k}(x,-t) & \text{in } \Omega \times (0,\infty) \\ u^k(x,0) = u_0^k(x) = u^{0,k}(x,0), \quad \partial_t u^k(x,0) = u_1^k(x) = \partial_t u^{0,k}(x) & \text{in } \Omega, \end{cases} \tag{3.16}$$

Since f^k is globally Lipschitz, for each $k \in \mathbb{N}$ we find by the boundary observability theorem due to the Theorem 2.2 in [18] that $u^k = 0$, which presents the desired contradiction.

Case II: $u^k = 0$. Denote

$$\alpha_n = \left(E_n^k(0)\right)^{\frac{1}{2}}, \quad v_n^k = \frac{1}{\alpha_n} u_n^k, \quad \zeta_n^k = \frac{1}{\alpha_n}\eta_n^k. \tag{3.17}$$

Whereupon, $\{(v_n^k, \zeta_n^k)\}$ is solution of the normalized problem

$$\begin{cases} \partial_{tt}v_n^k - \Delta v_n^k + \dfrac{1}{\alpha_n} f^k(\alpha_n v_n^k) = 0 & \text{in } \Omega \times (0,\infty), \\ v_n^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial v_n^k}{\partial \nu} + \displaystyle\int_0^\infty \mu(s)a(x)\zeta_n^{t,k}(x,s)ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \partial_s \zeta_n^{t,k} + \partial_t \zeta_n^{t,k} = \partial_t v_n^k & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ v_n^k(x,-t) = v^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ v_n^k(x,0) = v_{0n}^k(x) = v_n^0(x,0), \ \partial_t v_n^k(x,0) = v_{1n}^k(x) = \partial_t v_n^0(x) & \text{in } \Omega, \\ \zeta_n^{t,k}(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \zeta_n^{0,k}(x,s) = \zeta_{0n}^k(x,s) = v_{0n}^k(x,0) - v_{0n}^k(x,-s) & \text{on } \Omega \times (0,\infty), \end{cases} \tag{3.18}$$

whose energy functional is defined by

$$E^{v_n^k}(t) = \frac{1}{2}\left[|\partial_t v_n^k|^2 + |\nabla v_n^k|^2 + \|\zeta_n^{t,k}\|_{\mathcal{M}}^2 + 2\int_\Omega \int_0^{v_n^k} f^k(\tau)\, d\tau dx\right]. \tag{3.19}$$

Further, as $E^{v_n^k}(t) = \frac{1}{\alpha_n^2}E_n^k(t)$ for all $t \geq 0$ we deduce

$$E^{v_n^k}(0) = \frac{1}{\alpha_n^2}E_n^k(0) = 1 \tag{3.20}$$

for all $n > 0$, and also the existence of a subsequence $\{(v_n^k, \zeta_n^k)\}$ such that such that

$$\begin{aligned} v_n^k &\overset{*}{\rightharpoonup} v^k && \text{in } L^\infty(0,T;H_0^1(\Omega)),\\ \partial_t v_n^k &\overset{*}{\rightharpoonup} \partial_t v^k && \text{in } L^\infty(0,T;L^2(\Omega)),\\ v_n^k &\to v^k && \text{in } L^2(0,T;L^2(\Omega)), \end{aligned} \tag{3.21}$$

since $E^{v_n^k}(t) \leq E^{v_n^k}(0)$ for all $t \geq 0$. Moreover, by combining (3.13) and Lemma 3.3 we get

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty \mu(s)a(x)|\zeta_n^{t_k}|^2 ds\, d\Gamma\, dt = 0. \tag{3.22}$$

If we show that $E^{v_n^k}(T)$ goes to zero uniformly for each k fixed the desired contradiction is proved, since

$$E^{v_n^k}(T) = E^{v_n^k}(0) + \int_0^T \int_{\Gamma_1} \int_0^\infty \mu'(s)a(x)\zeta_n^{t,k}\, ds d\Gamma dt.$$

Indeed, for this purpose observe that, for an eventual subsequence, $\alpha_n \to \alpha$, where $\alpha \geq 0$. Therefore we separate the proof in two subcases: $\alpha > 0$ and $\alpha = 0$.

If $\alpha > 0$, since we have $\alpha_n v^n = u_n^k \to 0$ strongly in $L^2(0,T,L^2(\Omega))$, passing to the limit in (3.18) when $n \to \infty$, and taking (3.21) and (3.22) into account, we arrive at

$$\begin{cases} \partial_{tt} v^k - \Delta v^k + \frac{1}{\alpha} f^k(0) = 0 & \text{in } \Omega \times (0,\infty),\\ v^k = 0 & \text{on } \Gamma_0 \times (0,\infty),\\ \dfrac{\partial v^k}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,\infty),\\ v^k(x,-t) = v^0(x,-t) & \text{in } \Omega \times (0,\infty)\\ v^k(x,0) = v_0^k(x) = v^0(x,0),\ \partial_t v^k(x,0) = v_1^k(x) = \partial_t v^0(x) & \text{in } \Omega \end{cases} \tag{3.23}$$

which implies, as in the Case I, that $v^k = 0$.

Now, consider $\alpha = 0$. Firstly notice that, by Taylor's formula, we have

$$\begin{aligned} \frac{1}{\alpha_n} f(\alpha_n v_n^k) &= \frac{f'(0)\alpha_n v_n^k}{\alpha_n} + \frac{R(\alpha_n v_n^k)}{\alpha_n},\\ \frac{|R(\alpha_n v_n^k)|}{\alpha_n} &\leq \frac{\alpha_n^2 |v_n^k|^2}{\alpha_n} + \frac{\alpha_n^p |v_n^k|^p}{\alpha_n}. \end{aligned} \tag{3.24}$$

Next, by defining the set $\Omega_n^t = \{x \in \Omega \text{ s.t. } |u_n^k(x,t)| > k\}$, we have, thanks to assumption $A.3$

and Sobolev's embbeding,

$$
\begin{aligned}
&\left\| \frac{1}{\alpha_n} f^k(v_n^k) - \frac{1}{\alpha_n} f(v_n^k) \right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad = \left\| \frac{1}{\alpha_n} f^k(u_n^k) - \frac{1}{\alpha_n} f(u_n^k) \right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad = \frac{1}{\alpha_n^2} \int_0^T \int_{\Omega_n^t} \left| f^k(u_n^k) - f(u_n^k) \right|^2 \, dxdt \\
&\quad \leq c \frac{1}{\alpha_n^2} \int_0^T \int_{\Omega_n^t} \left| f^k(u_n^k) \right|^2 \, dxdt + \frac{1}{\alpha_n^2} \int_0^T \int_{\Omega_n^t} \left| f(u_n^k) \right|^2 \, dxdt \\
&\quad \leq c \frac{1}{\alpha_n^2} \int_0^T \int_{\Omega_m^t} (|k|^2 + |k|^{2p}) \, dxdt + \frac{1}{\alpha_n^2} \int_0^T \int_{\Omega_n^t} (|u_n^k|^2 + |u_n^k|^{2p}) \, dxdt \\
&\quad \leq c \alpha_n^{2(p-1)} \left\| v_m^k \right\|^{2p}_{L^{2p}(0,T;L^{2p}(\Omega))} \longrightarrow 0.
\end{aligned}
\tag{3.25}
$$

Also, it is not difficult to see that, up to a subsequence,

$$
\frac{R(\alpha_n v_n^k)}{\alpha_n} \rightharpoonup 0 \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{3.26}
$$

Therefore, from (3.24) – (3.26), and since

$$
\frac{1}{\alpha_n} f^k(\alpha_n v_n^k) - f'(0) v_n^k = \frac{1}{\alpha_n} f^k(u_n^k) - f'(0) v_k = \frac{1}{\alpha_n} f^k(u_n^k) - \frac{1}{\alpha_n} f(u_n^k) + \frac{1}{\alpha_n} f(u_n^k) - f'(0) v_k,
$$

one obtain

$$
\frac{1}{\alpha_n} f^k(\alpha_n v_n^k) - (f^k)'(0) v_k \to 0 \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{3.27}
$$

By passing to the limit in (3.18) when $n \to \infty$, and taking (2.9), (3.22), (3.25) and (3.27) into account, we arrive at

$$
\begin{cases}
\partial_{tt} v^k - \Delta v^k + \frac{1}{\alpha} (f^k)'(0) v^k = 0 & \text{in } \Omega \times (0,\infty), \\
v^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\
\dfrac{\partial v^k}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,\infty), \\
v^k(x,-t) = v^0(x,-t) & \text{in } \Omega \times (0,\infty) \\
v^k(x,0) = v_0^k(x) = v^0(x,0), \quad \partial_t v^k(x,0) = v_1^k(x) = \partial_t v^0(x) & \text{in } \Omega
\end{cases}
\tag{3.28}
$$

allowing us to conclude, as before, that $v^k = 0$. Thus, convergences (3.14) and (3.15) read as

$$
\begin{aligned}
v_n^k &\overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T:H_0^1(\Omega)), \\
\partial_t v_n^k &\overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;L^2(\Omega)), \\
v_n^k &\to 0 \quad \text{in } L^2(0,T;L^2(\Omega)).
\end{aligned}
\tag{3.29}
$$

Besides that,

$$
\frac{1}{\alpha_n} f^k(\alpha_n v_n^k) \to 0 \quad \text{in } L^2(0,T,L^2(\Omega)). \tag{3.30}
$$

In light of these calculations, consider now $\phi_n^k(x,t) = \int_0^\infty \mu(s)\zeta_n^{t,k}(x,s)\,ds$ and $\theta \in C^\infty(0,T)$; $0 \le \theta < 1$; $\theta(t) = 1$ in $(\varepsilon, T-\varepsilon)$. Multiplying the first equation of (3.18) by $\psi_n = \theta\phi_n^k$ and integrating by parts, we infer

$$\begin{aligned}\mu_0 \int_0^T \int_\Omega |\partial_t v_n^k|^2 \theta dxdt = & -\int_0^T \int_\Omega \partial_t v_n^k \left(\int_0^\infty \mu(s)\partial_s v_n^k ds\right)\theta dxdt + \int_0^T \int_\Omega \nabla v_n^k \nabla \phi_n^k \theta dxdt \\ & -\int_0^T \int_\Omega \partial_t v_n^k \phi_n^k \theta_t dxdt + \int_0^T \int_{\Gamma_1} a(x)\left(\int_0^\infty \mu(s)\zeta_n^{t,k} ds\right)^2 \theta\, d\Gamma dt \\ & + \int_0^T \int_\Omega \frac{1}{\alpha_n} f^k(\alpha_n v_n^k)\phi_n^k \theta\, dxdt = I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{3.31}$$

From convergences (3.22), (3.29) and (3.30) it is not hard to conclude that

$$\lim_{n\to\infty} I_1 = \cdots = \lim_{n\to\infty} I_5 = 0.$$

Thus, $\lim_{n\to\infty} \int_\varepsilon^{T-\varepsilon} |\partial_t v_n^k|^2 dxdt = 0$, that is,

$$\lim_{n\to\infty} \int_0^T |\partial_t v_n^k|^2 dxdt = 0. \tag{3.32}$$

The next step is to show that the potential energy converges to zero. To this aim, we multiply the first equation of (3.18) by θv_n^k and integrate by parts to get

$$\begin{aligned}\int_0^T \int_\Omega |\nabla v_n^k|^2 \theta dxdt &= \int_0^T \int_\Omega |\partial_t v_n^k|^2 \theta dxdt + \int_0^T \int_\Omega \partial_t v_n^k v_n^k \theta_t dxdt \\ &\quad - \int_0^T \int_{\Gamma_1} a(x) \int_0^\infty \mu(s)\zeta_n^{t,k} v_n^k \theta ds\, d\Gamma dt \\ &\quad - \int_0^T \int_\Omega \frac{1}{\alpha_n} f^k(\alpha_n v_n^k) v_n^k \theta\, dxdt \\ &= J_1 + J_2 + J_3 + J_4,\end{aligned} \tag{3.33}$$

which, through an analysis similar to the performed previously, produces

$$\lim_{n\to\infty} \int_0^T \int_\Omega |\nabla v_n^k|^2 dxdt = 0. \tag{3.34}$$

Therefore, since $E^{v_n^k}(t)$ is non-increasing, from (3.30)–(3.34) we conclude that

$$\lim_{n\to\infty} E^{v_n^k}(T) = 0,$$

which concludes this proof. □

Proof of Theorem 3.2. Notice that since $C > 0$ in (3.10) does not depend on k, by arguing similarly to [7, Lemma 2.1 and Proposition 2.1] one can pass (3.10) to limit to obtain the observability inequality

$$E(0) \le C\left(\int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta^t(s)|^2\, dsd\Gamma dt\right) \tag{3.35}$$

and, consequently, the desired exponential stability. □

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