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On the solvability of a higher-order semilinear ODE

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Abstract. The existence of at least one or two nontrivial solutions to a general higher-order boundary value problem is established by using variational tools. Two of the results are obtained without any asymptotic behaviour at infinity of potential F of the nonlinear term f, which is a key condition in the available literature, when applying critical point theorems. Moreover, F may change sign. The last result is stated when the nonlinearity has asymptotic behaviour at both infinity and zero.

Keywords: ODE, higher-order, semilinear, Brézis–Nirenberg's linking theorem, Mountain Pass theorem, variational method.

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1 Introduction

In this paper, we investigate the existence of at least one or two solutions for the boundary value problem

$$\begin{cases} u^{(2n)} + A_{n-1}u^{(2n-2)} + \cdots + A_1u'' + A_0u + f(x,u) = 0 \text{ in } \Omega = (0, \mathrm{L}) \\ u = u'' = \cdots = u^{(2n-2)} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $A_0, A_1, \ldots, A_{n-1}$ are some given real constants, f is a continuous function on $\overline{\Omega} \times \mathbb{R}$ and $n \geq 2$.

The existence of solutions for fourth-order problems ($n = 2$), which describe the deflection of an elastic beam with supported ends, has been extensively studied in the literature (see for example [2–4, 7, 8, 11, 15, 16] and the literature cited therein).

We mention the paper [12], where (1.1) (case $n = 2$) was treated under the assumption $A_1^2 > 4A_0$ by variational tools. The authors obtained existence and multiplicity results if the potential $F(x,s) = \int_0^s f(x,t)dt$ satisfies an asymptotic behaviour at zero and for some $C > 0$ and $p > 2$

$$F(x,s) \geq C|s|^p, \qquad \forall\, x \in \Omega,\ s \in \mathbb{R}. \tag{1.2}$$

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The case $A_1^2 = 4A_0$ was treated in [15].

The existence of solutions to sixth-order equations ($n = 3$) was investigated in [8] by using Clark's theorem provided the coefficients A_0, A_1, A_2 satisfy some relations in the particular case when $f(x,s) = a(x)s^3$. Here $a(x)$ is a continuous positive and even function.

In [13], using two Brézis–Nirenberg's linking theorems, the existence of at least two or three solutions was obtained, where $F \geq 0$ has an asymptotic behaviour at zero and satisfies

$$\frac{F(x,s)}{s^2} \to +\infty, \quad \text{uniformly with respect to } x \text{ as } |s| \to \infty. \tag{1.3}$$

Note that condition (1.2) implies the weaker super-quadratic condition (1.3). Also in the new paper [1] infinitely many solutions to equation (1.1) (case $n = 3, \Omega = (0,1)$) are obtained in the case when the nonlinear term f has an oscillating behaviour and the following restriction holds

$$\max\{A_2 k, A_2 k - A_1 k^2, A_2 k - A_1 k^2 + A_0 k^3\} < 1, \tag{1.4}$$

where $k = 1/\pi^2$.

For further results on sixth-order equations we refer the reader to [5,9,14,16–18].

The existence results of this paper are obtained for a general $2n-$ order equation by variational methods and hold under different assumptions on the coefficients.

We impose here suitable conditions on the coefficients $A_0, \ldots, A_{n-1}$, allowing to define several norms equivalent to the usual norm of the working space. One of the condition we impose (relation (2.5)) represents a generalization to the higher-order case of condition $A_1^2 > 4A_0$ which plays a role in the works [16] and [12].

We see that even we restrict ourselves to the case $n = 3$ our conditions imposed to the coefficients are different from the condition (1.4) or from the results obtained in the above mentioned papers.

Moreover, we note that our first two main results are stated without any asymptotic behaviour at infinity. More precisely, we prove by using the Brézis–Nirenberg's linking theorem that an existence result holds without any behaviour at infinity if $F \geq 0$ (Theorem 3.1). By using Ekeland's variational principle we show (Theorem 3.4) that a result holds if F may change sign and if no asymptotic behaviour at infinity is required. The last existence result uses the Mountain Pass theorem and is stated when F may change sign and f satisfies an asymptotic behaviour at both zero and infinity (f behavies at $\pm\infty$ as $|s|^p, p > 1$).

2 Auxiliary results and variational settings

We consider the Hilbert space

$$H(\Omega) = \{u \in H^n(\Omega) \mid u = u'' = \cdots = u^{(2n-4)} = 0 \text{ on } \partial\Omega\}$$

endowed with the standard inner product

$$(u,v)_{H^n(\Omega)} = \int_\Omega \left(uv + u'v' + u''v'' + \cdots + u^{(n)}v^{(n)}\right) dx$$

and standard norm

$$\|u\|_{H^n(\Omega)} = (u,u)_{H^n(\Omega)}^{\frac{1}{2}}.$$

For the sake of simplicity we consider $n = 4k, k = 1, 2, 3, \ldots,$ unless otherwise stated.

We recall the meaning of a weak solution to (1.1).

Definition 2.1. A weak solution of (1.1) is a function $u \in H(\Omega)$ such that

$$\int_\Omega \Big(u^{(n)} v^{(n)} - A_{n-1} u^{(n-1)} v^{(n-1)} + \cdots - A_1 u' v' + A_0 uv + f(x,u)v \Big) dx = 0, \qquad \forall\, v \in H(\Omega).$$

A classical solution of (1.1) is a function $u \in \mathrm{C}^{2\mathrm{n}}(\overline{\Omega})$ that satisfies (1.1).

We note that since f is a continuous function on $\overline{\Omega} \times \mathbb{R}$, it follows that a weak solution of (1.1) belongs to $C^{2n}(\overline{\Omega})$ (to get the result imitate the proof in [17]).

We also recall that the set of functions

$$\left\{ \sin \frac{m\pi x}{L},\ m \in \mathbb{N}, m \geq 1 \right\}$$

is a complete orthogonal basis in $H(\Omega)$.

The symbol $P(\xi) = \xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots + A_2\xi^4 - A_1\xi^2 + A_0$ of the differential operator $L(u) = u^{(2n)} + A_{n-1}u^{(2n-2)} + \cdots + A_2 u^{(4)} + A_1 u'' + A_0 u$ plays an important role in the sequel.

Problem (1.1) has a variational structure and weak solutions in the space $H(\Omega)$ can be found as critical points of the functional

$$J : H(\Omega) \to \mathbb{R}$$

$$J(u) = \frac{1}{2} \int_\Omega \Big((u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1 (u')^2 + A_0 u^2 \Big) dx + \int_\Omega F(x,u)dx,$$

which is Fréchet differentiable and its Fréchet derivative is given by

$$\langle J'(u), v \rangle = \int_\Omega \Big(u^{(n)} v^{(n)} - A_{n-1} u^{(n-1)} v^{(n-1)} + \cdots - A_1 u' v' + A_0 uv + f(x,u)v \Big) dx,$$

for all $v \in H(\Omega)$.

Throughout the paper C denotes a universal positive constant depending on the indicated quantities, unless otherwise specified.

The next lemmas are fundamental tools in proving our existence result.

First we point out some Poincaré-type inequalities.

Lemma 2.2 ([10]). *The following relations hold true for any $u \in H(\Omega)$.*

$$\int_\Omega (u^{(k)})^2 dx \leq \left(\frac{L}{\pi} \right)^2 \int_\Omega (u^{(k+1)})^2 dx, \qquad k = 0, 1, 2, \ldots, n-1. \tag{2.1}$$

$$\int_\Omega u^2 dx \leq \left(\frac{L}{\pi} \right)^{2k} \int_\Omega (u^{(k)})^2 dx, \qquad k = 1, 2, \ldots, n. \tag{2.2}$$

In particular,

$$\int_\Omega u^2 dx \leq \left(\frac{L}{\pi} \right)^{2n} \int_\Omega (u^{(n)})^2 dx. \tag{2.3}$$

An immediate consequence of Lemma 2.2 is the inequality

$$C(L,n)\|u\|_{H^n(\Omega)} \leq \int_\Omega (u^{(n)})^2 dx \leq \|u\|_{H^n(\Omega)}, \tag{2.4}$$

which shows that the scalar product

$$(u,v)_{H(\Omega)} = \int_\Omega u^{(n)} v^{(n)} dx$$

induces a norm equivalent (denoted $\|\cdot\|_{H(\Omega)}$) to the norm $\|\cdot\|_{H^n(\Omega)}$ in the space $H(\Omega)$.

The next lemma is an extension of Lemma 8, [16] and is proved by different means.

Lemma 2.3. *Let $u \in H(\Omega)$.*

a). Suppose that $A_0, A_2, \dots, A_{n-4} \geq 0$, $A_1, A_3, \dots, A_{n-3} \leq 0$, $A_{n-2}, A_{n-1} > 0$ and

$$A_{n-1}^2 < 4A_{n-2}. \tag{2.5}$$

Then there exists a constant k such that

$$\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0 u^2 \right] dx \geq k\|u\|^2_{H^n(\Omega)}. \tag{2.6}$$

A similar estimate holds for $A_0 < 0$ but under the restriction

$$A_{n-1}^2 < 4A_{n-2}A^*, \tag{2.7}$$

where $A^* = 1 + A_0\left(\frac{L}{\pi}\right)^{2n} > 0$.

b). The same estimate (2.6) holds if for some index $j = 2, 4, \dots, n-2$

$$\frac{A_{n-j-1}^2}{A_{n-j}} < 4A_{n-j-2}, \tag{2.8}$$

where

$$A_1, A_3, \dots, A_{n-j-3}, A_{n-j+1}, \dots, A_{n-1} < 0,$$
$$A_0, A_2, \dots, A_{n-j-2}, A_{n-j+2} \dots, A_{n-2} \geq 0, \quad A_{n-j-1}, A_{n-j} > 0.$$

c). Similarly, (2.6) holds if for some index $j = 1, 3, \dots, n-1$ (2.8) is fulfilled, where

$$A_1, A_3, \dots, A_{n-j-3}, A_{n-j+1}, \dots, A_{n-1} \leq 0,$$
$$A_0, A_2, \dots, A_{n-j-2}, A_{n-j+2} \dots, A_{n-2} \geq 0, \quad A_{n-j-1} < 0, A_{n-j} > 0.$$

Remark 2.4.

1. Of course if $A_{n-1} \leq 0, A_{n-2} \geq 0, \dots, A_1 \leq 0, A_0 \geq 0$, then Lemma 2.3 is always true, i.e., there is nothing to prove.
2. We easily see that if $n = 2$ (Case a).) then we obtain exactly Lemma 8, [16] for bounded domains, i.e., our result is a direct extension to the higher-order case.
3. Note that Lemma 2.5 and Lemma 2.6 can also be seen as extensions of Lemma 8, [16] and hold for bounded domains Ω as well when $\Omega = \mathbb{R}$.

Proof. a). We see that for any real α

$$\int_\Omega \left(u^{(n)} + \alpha u^{(n-1)} \right)^2 dx = \int_\Omega \left((u^{(n)})^2 - 2\alpha(u^{(n-1)})^2 + \alpha^2(u^{(n-2)})^2 \right) dx.$$

It follows that for any α the quantity

$$Q_\alpha = \int_\Omega \left((u^{(n)})^2 - 2\alpha(u^{(n-1)})^2 + \alpha^2(u^{(n-2)})^2 \right) dx$$

is positive.

For arbitrary $\varepsilon > 0$ and by the assumptions

$$\begin{aligned}
&\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right] dx \\
&\quad \geq \int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + A_{n-2}(u^{(n-2)})^2\right] dx \\
&\quad = \Bigg\{ \varepsilon \int_\Omega \left[(u^{(n)})^2 + (u^{(n-1)})^2 + (u^{(n-2)})^2\right] dx \\
&\qquad + (1-\varepsilon)\int_\Omega \left[(u^{(n)})^2 - \frac{A_{n-1}+\varepsilon}{1-\varepsilon}(u^{(n-1)})^2 + \frac{1}{4}\left(\frac{A_{n-1}+\varepsilon}{1-\varepsilon}\right)^2 (u^{(n-1)})^2\right] dx \\
&\qquad + \left[A_{n-2} - \varepsilon - \frac{1}{4}\frac{(A_{n-1}+\varepsilon)^2}{1-\varepsilon}\right] \int_\Omega (u^{(n-2)})^2 dx \Bigg\} \\
&\quad \geq \varepsilon \int_\Omega (u^{(n)})^2 dx + (1-\varepsilon) Q_{\frac{A_{n-1}+\varepsilon}{1-\varepsilon}} + \left[A_{n-2} - \varepsilon - \frac{1}{4}\frac{(A_{n-1}+\varepsilon)^2}{1-\varepsilon}\right] \int_\Omega (u^{(n-2)})^2 dx.
\end{aligned}$$

Choosing ε sufficiently small, using that $Q_{\frac{A_{n-1}+\varepsilon}{1-\varepsilon}} \geq 0$, (2.5) and the equivalence of norms $\|\cdot\|_{H^n(\Omega)}$ and $\|\cdot\|_{H(\Omega)}$ we get the result.

b). and c). Follows from case a). □

Lemma 2.5. *Let $u \in H(\Omega)$ and $A_0 > 1$.*

Suppose that for an index i and j,

$$A_i^2 < -4A_j, \qquad \frac{A_i^2}{-4A_j} \leq A_0 - 1, \tag{2.9}$$

where $i = 2, 3, \ldots, \frac{n}{2}$, $A_i \neq A_j$, $1 \leq j \leq n-1$, $A_j < 0$, $A_i < 0$ if i is even and $A_i > 0$ if i is odd.

Then there exist the constants $k_{i,j} > 0$ such that

$$\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right] dx \geq k_{i,j}\|u\|^2_{H^n(\Omega)}. \tag{2.10}$$

Proof. a). For the sake of simplicity we consider $j = 1$ and $i = 2$, i.e.,

$$A_1, A_2 < 0, \quad A_4, \ldots, A_{n-2} \geq 0, \quad A_3, \ldots, A_{n-1} \leq 0$$

and

$$\frac{A_2^2}{-4A_1} \leq A_0 - 1, \qquad A_2^2 < -4A_1.$$

We are going to prove the required inequality for $u \in H^n(\mathbb{R})$ by using the Fourier transform.

Taking in particular $u \in H(\Omega) \cap H^n(\mathbb{R})$ we get the inequalities for bounded domains Ω.

Let $\hat{u}(\xi)$ be the Fourier transform of $u(x) \in H^n(\mathbb{R})$.

First observe that by Parseval's identity we get

$$\begin{aligned}
&\int_{\mathbb{R}} \left((u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right) dx \\
&\qquad = \int_{\mathbb{R}} \left(\xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots - A_1\xi^2 + A_0\right)\|\hat{u}(\xi)\|^2 d\xi.
\end{aligned} \tag{2.11}$$

By using elementary inequalities we get for all $\xi \in \mathbb{R}$

$$\begin{aligned} A_2\xi^4 &\le \frac{A_2^2}{-4A_1}\xi^6 + (-A_1)\xi^2 \le \frac{A_2^2}{-4A_1}\xi^{2n} + (-A_1)\xi^2 + \frac{A_2^2}{-4A_1} \\ &\le \frac{A_2^2}{-4A_1}\xi^{2n} + (-A_1)\xi^2 + A_0 - 1 \\ &\le \frac{A_2^2}{-4A_1}\xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots - A_3\xi^6 + (-A_1)\xi^2 + A_0 - 1. \end{aligned}$$

Hence

$$\begin{aligned} &\xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots + A_2\xi^4 - A_1\xi^2 + A_0 \\ &\quad \ge \left(1 - \frac{A_2^2}{-4A_1}\right)\xi^{2n} + 1 \ge \left(1 - \frac{A_2^2}{-4A_1}\right)\left(\xi^{2n} + 1\right). \end{aligned} \tag{2.12}$$

It can be easily checked that $\forall\, \xi \in \mathbb{R}$

$$\xi^{2n} + 1 \ge \frac{1}{n}\left(1 + \xi^2 + \cdots + \xi^{2n}\right). \tag{2.13}$$

From (2.12) and (2.13) we get

$$\xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots + A_2\xi^4 - A_1\xi^2 + A_0 \ge \frac{1}{n}\left(1 - \frac{A_2^2}{-4A_1}\right)\left(1 + \xi^2 + \cdots + \xi^{2n}\right). \tag{2.14}$$

Now from (2.11) and (2.14) we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \left(\left(u^{(n)}\right)^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right)dx \\ &\quad \ge \frac{1}{n}\left(1 - \frac{A_2^2}{-4A_1}\right)\int_{\mathbb{R}}\left(1 + \xi^2 + \cdots + \xi^{2n}\right)\|\hat{u}(\xi)\|^2 d\xi \\ &\quad = \frac{1}{n}\left(1 - \frac{A_2^2}{-4A_1}\right)\int_{\mathbb{R}}\left(u^2 + (u')^2 + \cdots + \left(u^{(2n)}\right)^2\right)dx \\ &\quad = k_{2,1}\|u\|^2_{H^n(\mathbb{R})}, \end{aligned}$$

which is the desired result. □

Lemma 2.6. *Let $u \in H(\Omega)$ and $A_0 > 1$.*

Suppose that for an index $i = 1, 3, \ldots, (n/2) - 1$, $A_i > 0$ and for an index $j = 2, 4, \ldots, n-2$, $A_j > 0$ the following inequality be fulfilled

$$A_i^2 < 4A_j, \qquad \frac{A_i^2}{4A_j} + A_j \le A_0 - 1, \tag{2.15}$$

where the rest of coefficients

$$A_1, A_3, \ldots, A_{i-2}, A_{i+2}, \ldots, A_{n-1} \le 0$$

and

$$A_2, A_4, \ldots, A_{j-2}, A_{j+2}, \ldots, A_{n-2} \ge 0.$$

Then there exist the constants $k_{i,j} > 0$ such that

$$\int_{\Omega}\left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right]dx \ge k_{i,j}\|u\|^2_{H^n(\Omega)}. \tag{2.16}$$

The proof is similar to the proof of Lemma 2.5 and hence is omitted.

Lemma 2.7. *Let $u \in H(\Omega)$.*
Suppose that $A_0, A_2, \dots, A_{n-2} \geq 0$, $A_1, A_3, \dots, A_{n-1} \geq 0$, and

$$1 - A_{n-1}\left(\frac{L}{\pi}\right)^2 - A_{n-3}\left(\frac{L}{\pi}\right)^6 - \dots - A_1\left(\frac{L}{\pi}\right)^{2n-2} > 0. \tag{2.17}$$

Then there exists a constant $k_1 > 0$ such that

$$\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \dots - A_1(u')^2 + A_0 u^2\right] dx \geq k_1 \|u\|^2_{H^n(\Omega)}. \tag{2.18}$$

A similar result holds if $A_0, A_2, \dots, A_{n-2} < 0$ and $A_1, A_3, \dots, A_{n-1} \geq 0$ under the assumption

$$1 - A_{n-1}\left(\frac{L}{\pi}\right)^2 + A_{n-2}\left(\frac{L}{\pi}\right)^4 - \dots - A_1\left(\frac{L}{\pi}\right)^{2n-2} + A_0\left(\frac{L}{\pi}\right)^{2n} > 0. \tag{2.19}$$

The next four lemmas gives conditions on parameters $A_i, i = 0, 1, \dots, n-1$ when the functional J is bounded below and satisfies the Palais–Smale condition. We recall here what means that J satisfies the Palais–Smale condition.

Definition 2.8. Let X be a Banach space and $J \in C^1(X, \mathbb{R})$. We say that J satisfies a Palais–Smale condition if any sequence $\{u_m\}$ in X for which $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$, has a convergent subsequence.

Lemma 2.9. *Let $u \in H(\Omega)$ and let $\alpha > 0$ be a constant. Suppose that $F \geq 0$, $A_0, A_2, \dots, A_{n-4} \geq 0$, $A_1, A_3, \dots, A_{n-3} \leq 0$, $A_{n-2}, A_{n-1} > 0$ and*

$$\frac{\alpha+1}{\alpha} A_{n-1}^2 < 4A_{n-2}. \tag{2.20}$$

Then J is bounded below and satisfies the Palais–Smale condition.
A similar statement holds for $A_0 < 0$ but under the restriction

$$\frac{\alpha+1}{\alpha} A_{n-1}^2 < 4A_{n-2}A^*, \tag{2.21}$$

where $A^* = 1 + \frac{\alpha+1}{\alpha} A_0 \left(\frac{L}{\pi}\right)^{2n} > 0$.
The same conclusion holds if we are under the hypotheses of the case b). or case c). of Lemma 2.3.

Proof. We observe that for any $\alpha > 0$ we can write $J(u)$ as a sum of

$$J(u) = \frac{1}{2}\frac{1}{\alpha+1}\int_\Omega \left(u^{(n)}\right)^2 dx + \frac{\alpha}{\alpha+1} J_1(u),$$

where

$$J_1(u) = \frac{1}{2}\int_\Omega \left[(u^{(n)})^2 - \frac{\alpha+1}{\alpha} A_{n-1}(u^{(n-1)})^2 + \dots + \frac{\alpha+1}{\alpha} A_0 u^2 + 2\frac{\alpha+1}{\alpha} F\right] dx.$$

Since (2.20) holds we can use Lemma 2.3 and the positivity of F to get that $J_1(u)$ is bounded below which implies that $J(u)$ is bounded below.

We now show that $J(u)$ satisfies the Palais–Smale condition.

Suppose that $\{u_m\}$ is a Palais–Smale sequence, i.e., there exists a constant $C > 0$ such that

$$|J(u_m)| \le C \quad \text{and} \quad J'(u_m) \to 0 \quad \text{as } m \to \infty.$$

Since $J_1(u)$ is bounded below we get that there exists a constant $C_1 > 0$ such that

$$C > \frac{1}{2}\frac{1}{\alpha+1}\int_\Omega (u_m^{(n)})^2 dx - C_1,$$

which implies that $\{u_m\}$ is a bounded sequence in $H(\Omega)$.

Since

$$J(u) = \frac{1}{2}(u,u)_{H(\Omega)} - \frac{1}{2}\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0 u^2\right]dx + \int_\Omega F dx,$$

we see that

$$J'(u) = u + K(u),$$

where

$$K : H(\Omega) \to H(\Omega)$$

is defined by

$$\langle K(u), v\rangle = -\int_\Omega \left[A_{n-1}u^{(n-1)}v^{(n-1)} + \cdots + A_1 u'v' - A_0 uv - f(x,u)v\right]dx.$$

Using the fact that the Sobolev imbedding $H(\Omega) \hookrightarrow C^{n-1}(\overline{\Omega})$ is compact we get that K is a complete continuous operator. Since $J'(u_m) \to 0$ as $m \to \infty$ it follows that

$$u_m = J'(u_m) - K(u_m)$$

is a convergent sequence and hence $J(u)$ satisfies the Palais–Smale condition. □

Using the same techniques we can prove

Lemma 2.10. *Let $u \in H(\Omega)$, $A_0 > 1$ and let $\alpha > 0$ be a constant.*

Suppose that for an index i and j,

$$\frac{\alpha+1}{\alpha}A_i^2 < -4A_j, \qquad \frac{A_i^2}{-4A_j} \le A_0 - \frac{\alpha}{\alpha+1}, \tag{2.22}$$

where $i = 2, 3, \ldots, \frac{n}{2}$, $A_i \ne A_j$, $1 \le j \le n-1$, $A_j < 0$, $A_i < 0$ if i is even and $A_i > 0$ if i is odd. Then J is bounded below and satisfies the Palais–Smale condition.

Lemma 2.11. *Let $u \in H(\Omega)$, $A_0 > 1$ and let $\alpha > 0$ be a constant.*

Suppose that for an index $i = 1, 3, \ldots, (n/2) - 1$, $A_i > 0$ and for an index $j = 2, 4, \ldots, n-2$, $A_j > 0$ the following inequality be fulfilled

$$\frac{\alpha+1}{\alpha}A_i^2 < 4A_j, \qquad \frac{A_i^2}{4A_j} + A_j \le A_0 - \frac{\alpha}{\alpha+1}, \tag{2.23}$$

where the rest of coefficients

$$A_1, A_3, \ldots, A_{i-2}, A_{i+2}, \ldots, A_{n-1} \le 0,$$

and

$$A_2, A_4, \ldots, A_{j-2}, A_{j+2}, \ldots, A_{n-2} \ge 0.$$

Then J is bounded below and satisfies the Palais–Smale condition.

Lemma 2.12. *Let $u \in H(\Omega)$ and let $\alpha > 0$ be a constant.*

Suppose that $A_0, A_2, \dots, A_{n-2} \geq 0$, $A_1, A_3, \dots, A_{n-1} \geq 0$, and

$$1 - \frac{\alpha+1}{\alpha}\left[A_{n-1}\left(\frac{L}{\pi}\right)^2 + A_{n-3}\left(\frac{L}{\pi}\right)^6 + \cdots + A_1\left(\frac{L}{\pi}\right)^{2n-2}\right] > 0. \tag{2.24}$$

Then J is bounded below and satisfies the Palais–Smale condition. A similar result holds if $A_0, A_2, \dots, A_{n-2} < 0$ and $A_1, A_3, \dots, A_{n-1} \geq 0$ under the assumption

$$1 - \frac{\alpha+1}{\alpha}\left[A_{n-1}\left(\frac{L}{\pi}\right)^2 - A_{n-2}\left(\frac{L}{\pi}\right)^4 - \cdots - A_1\left(\frac{L}{\pi}\right)^{2n-2} - A_0\left(\frac{L}{\pi}\right)^{2n}\right] > 0. \tag{2.25}$$

The main tool in our approach is the Brézis–Nirenberg's linking theorem [6].

Theorem 2.13. *Suppose that $J \in C^1(H, \mathbb{R})$ satisfies the Palais–Smale condition and has a local linking at 0. Assume that J is bounded below and $\inf_H J < 0$. Then J has at least two nontrivial critical points.*

For the sake of completeness we recall the definition of local linking.

Let the Banach space H has a direct sum decomposition $H = X \oplus Y$, where X is finite dimensional.

Definition 2.14. The functional J is said to have a local linking at 0 if for some $\rho > 0$,

$$J(x) \leq 0, \qquad \forall\, x \in X,\ \|x\| \leq \rho,$$

and

$$J(y) \geq 0, \qquad \forall\, y \in Y,\ \|y\| \leq \rho.$$

3 Main results

Our existence results read.

Theorem 3.1. *Let the function $F \geq 0$, $\forall\, x \in \Omega$, $s \in \mathbb{R}$ satisfy*

$$F(x,s) \leq K|s|^p, \qquad p > 2,\ \forall\, x \in \Omega,\ s \in \mathbb{R}, s \text{ small}, \tag{3.1}$$

where $K > 0$ is a constant. Suppose that we are under hypotheses of either Lemma 2.9, Lemma 2.10, Lemma 2.11 or Lemma 2.12. If in addition there exists a natural number $m \neq 0$ such that

$$P\left(\frac{m\pi}{L}\right) < 0, \tag{3.2}$$

then the boundary value problem (1.1) has at least two nontrivial solutions.

Proof. The proof uses the Brézis–Nirenberg's linking theorem (Theorem 2.13). Hence we have to show that J satisfies the condition imposed in Theorem 2.13.

Since we are under the hypotheses of either Lemma 2.9, Lemma 2.10, Lemma 2.11 or Lemma 2.12 it follows that J is bounded below and satisfies the Palais–Smale condition.

We now follow the proof of Lemma 8, [13] and show that $\inf_{H(\Omega)} J < 0$.

We see that $P\left(\frac{m\pi}{L}\right) \to \infty$ and since (3.2) holds we get that there exists a finite set of natural numbers $\{m_1, m_2, \dots, m_k\}$ such that $P\left(\frac{m_i\pi}{L}\right) < 0$, $i = 1, 2, \dots, k$.

Introducing the finite dimensional space

$$X = \text{span}\left\{ \sin\frac{m_1\pi x}{L}, \dots, \sin\frac{m_k\pi x}{L} \right\}$$

we see that any $\varphi \in X$ can be written

$$\varphi(x) = c_1 \sin\frac{m_1\pi x}{L} + \dots + c_k \sin\frac{m_k\pi x}{L}$$

and its norm in $L^2(\Omega)$ is given by

$$\|\varphi\|_X^2 = c_1^2 + \dots + c_k^2 = \rho^2,$$

where $c_1, \dots, c_k$ are real constants.

By (3.1) and Hölder's inequality we get for sufficiently small $\rho > 0$

$$\begin{aligned}\int_\Omega F(x, \varphi(x))dx &\le K\int_\Omega |\varphi(x)|^p dx \\ &\le K\int_\Omega \left[\left(c_1^2 + \dots + c_k^2\right)^{\frac{1}{2}} \left(\sin\frac{m_1\pi x}{L} + \dots + \sin\frac{m_k\pi x}{L}\right)^{\frac{1}{2}}\right]^p \\ &\le C(K, k, p, L)\left(c_1^2 + \dots + c_k^2\right)^{\frac{p}{2}} = C(K, k, p, L)\rho^p.\end{aligned}$$

Hence

$$\begin{aligned}J(\varphi) &\le \frac{L}{4}\sum_{i=1}^{k} P\left(\frac{m_i\pi}{L}\right)c_i^2 + C(K, k, p, L)\rho^p \\ &\le \frac{L}{4}\alpha\rho^2 + C(K, k, p, L)\rho^p = \rho^2\left(\frac{L}{4}\alpha + C(K, k, p, L)\rho^{p-2}\right) < 0,\end{aligned}$$

where $\alpha = \max\left\{P\left(\frac{m_i\pi}{L}\right), i = 1, 2, \dots, k\right\} < 0$ by hypothesis.

We now show that J has a local linking at 0.

By the above estimation, we see that for sufficiently small ρ

$$J(u) \le 0, \qquad \forall\, u \in X,\ \|u\| \le \rho.$$

Also since for any $u \in Y = X^\perp$ (bear in mind that $P\left(\frac{m_{k+1}\pi}{L}\right) \ge 0$)

$$J(u) \ge \frac{1}{2}P\left(\frac{m_{k+1}\pi}{L}\right)\|u\|_{L^2(\Omega)}^2 + \int_\Omega F(x, u)dx \ge 0,$$

we get that J has a local linking at 0 and the proof follows. □

Immediate consequences of Theorem 3.1 are the following.

Corollary 3.2. *Suppose that $P(0) > 0$ and that P takes negative values. The problem (1.1) has at least two nontrivial solutions in $\Omega = (0, L)$ provided the following relation holds true*

$$\frac{m\pi}{\xi_2} < L < \frac{m\pi}{\xi_1} \quad \textit{for some natural number } m \ne 0. \tag{3.3}$$

Here $0 < \xi_1 < \xi_2$ are the first (the smallest) two positive roots of P. Note that P may have other roots.

Corollary 3.3. *Suppose that $P(0) < 0$ and let $\xi_1 > 0$ be the smallest root of P (P may have other roots). The problem (1.1) has at least two nontrivial solutions in $\Omega = (0, L)$ provided the following relation holds true*

$$L > \frac{m\pi}{\xi_1} \quad \text{for some natural number } m \neq 0. \tag{3.4}$$

We note that the uniqueness results presented in [12, 13] as well as our Theorem 3.1 are stated under the restriction $F \geq 0$ and

$$\lim_{s\to 0} \frac{F(s)}{s^2} = 0. \tag{3.5}$$

The next result is stated when F may change sign and (3.5) is weakened.

Theorem 3.4. *Let the function F satisfy*

$$F(x,s) \geq -K_1|s|^p - K_2, \qquad \forall\, x \in \Omega,\ s \in \mathbb{R}, \tag{3.6}$$

where $0 < p < 2$, and $K_1, K_2 > 0$.

Suppose that $A_{n-1} \leq 0, A_{n-2} \geq 0, \ldots, A_1 \leq 0, A_0 \geq 0$ holds or we are under hypotheses of either Lemma 2.3, Lemma 2.5, Lemma 2.6 or Lemma 2.7. If in addition one of the following relation holds

$$\lim_{s\to 0^+} \frac{f(x,s)}{s^\alpha} = q(x) \quad \text{uniformly in } \overline{\Omega}, \tag{3.7}$$

where $q(x) \leq 0$, $\|q\|_{L^\infty(\Omega)} > 0$, $0 < \alpha < 1$

$$\lim_{s\to 0} \frac{F(x,s)}{s^2} = \beta(x) \in L^1(\Omega), \quad \text{uniformly in } \overline{\Omega}, \tag{3.8}$$

where

$$\int_\Omega \beta(x) \sin^2 \frac{\pi x}{L} dx + \frac{L}{4} P\left(\frac{\pi}{L}\right) < 0, \tag{3.9}$$

then the boundary value problem (1.1) has at least one nontrivial solution.

Proof. We choose $\rho > 0$ arbitrary but fixed and denote by

$$B_\rho = \{u \in H(\Omega) \mid \|u\|_{H(\Omega)} < \rho\}.$$

We first note that one of the relations (3.7) or (3.8) assures that

$$\mu = \inf_{\overline{B}_\rho} J(u) < 0.$$

Indeed, suppose that (3.7) holds.

We can choose the positive function $\varphi(x) = \sin \frac{\pi x}{L} \in H(\Omega)$ such that

$$\int_\Omega q(x)\varphi^{\alpha+1}(x) dx < 0.$$

Hence

$$\begin{aligned}
\lim_{s\to 0^+} \frac{J(s\varphi)}{s^{\alpha+1}} &= \frac{1}{2} \lim_{s\to 0^+} s^{1-\alpha} \int_\Omega \left((\varphi^{(n)})^2 - A_{n-1}(\varphi^{(n-1)})^2 + \cdots + A_0\varphi^2 \right) dx \\
&\quad + \lim_{s\to 0^+} \int_\Omega \frac{F(x,s\varphi)}{s^{\alpha+1}} dx \\
&= \int_\Omega \lim_{s\to 0^+} \frac{F(x,s\varphi)}{s^{\alpha+1}} dx = \int_\Omega \lim_{s\to 0^+} \frac{f(x,s\varphi)\varphi}{(\alpha+1)s^\alpha} dx \\
&= \frac{1}{\alpha+1} \int_\Omega q(x)\varphi^{\alpha+1}(x) dx < 0.
\end{aligned}$$

Similarly if (3.8) holds we see that

$$\lim_{s\to 0}\frac{J(s\varphi)}{s^2}=\frac{L}{4}P\left(\frac{\pi}{L}\right)+\int_\Omega \beta(x)\sin^2\frac{\pi x}{L}dx<0.$$

By relation (3.6), Cauchy's inequality with ε and (2.3)

$$\begin{aligned}\int_\Omega F(x,u)dx &\geq -\varepsilon\int_\Omega u^2dx-\int_\Omega\left(C(p,\varepsilon)K_1^{\frac{2}{2-p}}+K_2\right)dx \\ &\geq -\varepsilon\left(\frac{L}{\pi}\right)^{2n}\|u\|^2_{H(\Omega)}-C(p,\varepsilon,K_1,K_2,L).\end{aligned} \tag{3.10}$$

Hence if we are under hypotheses of either Lemma 2.3, Lemma 2.5, Lemma 2.6 or Lemma 2.7 we can combine (3.10) with one of relations (2.6), (2.10), (2.16) or (2.18) to get (by choosing ε sufficiently small) that $J(u)$ is bounded below on $\overline{B}_\rho$ by a negative constant.

According to the Remark, inequalities of type (2.6) are always true if $A_{n-1}\leq 0, A_{n-2}\geq 0,\ldots,A_1\leq 0, A_0\geq 0$ and hence again we obtain that $J(u)$ is bounded below.

From Ekeland's variational principle it follows that there exists a minimizing sequence $\{u_m\}\subset\overline{B}_\rho$ such that

$$J(u_m)\to\mu \quad\text{and}\quad J'(u_m)\to 0,\quad \text{as } m\to\infty.$$

Since $\{u_m\}$ is bounded we can extract (by using the Sobolev imbedding) a subsequence still denoted $\{u_m\}$ such that

$$\begin{aligned} u_m &\rightharpoonup u_0 \quad \text{weakly in } H(\Omega),\\ u_m &\to u_0 \quad \text{strongly in } C^{n-1}(\overline{\Omega}).\end{aligned}$$

Arguing as in the proof Lemma 2.9 we get that $\{u_m\}$ converges strongly to u_0 in $H(\Omega)$.

As a consequence there exists $u_0\in H(\Omega)$ such that $J'(u_0)=0$, $J(u_0)<0$ i.e., problem (1.1) has at least a nontrivial solution. □

The last existence result shows that if we impose some asymptotic assumptions to f we can allow $p>2$ in (3.6). The proof uses the Mountain Pass theorem and the following two lemmas.

The first lemma shows when $J(u)$ has a mountain pass structure

Lemma 3.5. *Suppose that we are under one of the assumptions of Lemma 2.3, Lemma 2.5 or Lemma 2.6. Let F satisfy*

$$F(x,s)\leq C|s|^t,\qquad \forall\,(x,s)\in\Omega\times\mathbb{R}, \tag{3.11}$$

where $C>0, t>2$ and relation (3.7) holds.

Then

1. *there exist two positive constants ρ and η such that*

$$J(u)_{|_{\|u\|=\rho}}\geq\eta, \tag{3.12}$$

2. *there exists $e\in H(\Omega)$ satisfying $\|u\|>\rho$ and $J(e)<0$.*

Here

$$\|u\|^2 = \int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0 u^2 \right] dx$$

is a norm since we work under the assumptions of Lemma 2.3, Lemma 2.5 or Lemma 2.6.

We also note that $J(u)$ becomes

$$J(u) = \frac{1}{2}\|u\|^2 + \int_\Omega F(x,u)dx.$$

Proof. For a proof see [10]. □

We can now apply the Mountain Pass theorem in $H(\Omega)$ to find a Cerami type sequence, i.e.,

$$\text{there exists } \{u_m\} \subset H(\Omega) \text{ such that } J(u_m) \to \lambda \text{ and } \|J'(u_m)\|_{H^*(\Omega)} \to 0. \tag{3.13}$$

The next lemma gives the boundedness of the sequence $\{u_m\}$.

Lemma 3.6. *Suppose that we are under the hypotheses of Lemma 3.5. If in addition there exist the constants $\theta \in (0,2), K_1 \in \mathbb{R}, K_2 > 0$ such that*

$$f(x,s)s \geq K_1|s|^\theta - K_2, \qquad \forall\, x \in \Omega,\ |s| > M, \tag{3.14}$$

for some $M > 0$, then the sequence $\{u_m\}$ defined by (3.13) is bounded in $H(\Omega)$.

Proof. We argue by contradiction and suppose that $\|u_m\| \to \infty$. Let $w_m = \frac{u_m}{\|u_m\|}$. Obviously $\{w_m\}$ is a bounded sequence and we can extract a subsequence, still denoted $\{w_m\}$, such that

$$w_m \to w \quad \text{strongly in } \mathrm{C}^{\mathrm{n}-1}(\overline{\Omega}).$$

For each fixed m we define

$$\Omega_m^1 = \{x \in \Omega \mid u_m(x) \leq M\} \quad \text{and} \quad \Omega_m^2 = \{x \in \Omega \mid\ u_m(x) > M\}.$$

By the continuity of f there exists a constant $C_1 > 0$ such that

$$\int_{\Omega_m^1} f(x,u_m)u_m dx \geq -C_1. \tag{3.15}$$

Since

$$\langle J'(u_m), u_m \rangle = \|u_m\|^2 + \int_\Omega f(x,u_m)u_m dx,$$

we get by combining (3.14) and (3.15) that

$$\begin{aligned} \langle J'(u_m), u_m \rangle &\geq \|u_m q|^2 - C_1 - \int_{\Omega_m^2} (K_1|u_m|^\theta - K_2)dx \\ &\geq \|u_m\|^2 - C_1 - |K_1| \int_{\Omega_m^2} |u_m|^\theta dx - K_2 \operatorname{meas}(\Omega). \end{aligned} \tag{3.16}$$

Using (3.16) and the fact that $\langle J'(u_m), u_m \rangle \to 0$, as $m \to \infty$ it follows that

$$\begin{aligned} \infty = \lim_{m\to\infty} \frac{\|u_m\|^2}{\|u_m\|^\theta} &\leq \lim_{m\to\infty} \left(\frac{\langle J'(u_m), u_m \rangle}{\|u_m\|^\theta} + |K_1| \int_\Omega |w_m|^\theta dx + \frac{C_1 + K_2 \operatorname{meas}(\Omega)}{\|u_m\|^\theta} \right) \\ &= |K_1| \int_\Omega |w|^\theta dx < \infty, \end{aligned}$$

which is a contradiction.

Hence we conclude that the sequence $\{u_m\}$ is bounded. □

The last existence result reads

Theorem 3.7. *Suppose that we are under one of the assumptions of Lemma 2.3, Lemma 2.5 or Lemma 2.6 and that relation (3.7) holds. Let $p, q, r > 1$ be such that $p \geq \sigma = \max\{q, r\}$ and $L_1, L_2, L_3 \in L^\infty(\Omega)$. If in addition*

$$\lim_{|s|\to 0} \frac{f(x,s)}{|s|^p} = L_1(x) \tag{3.17}$$

and

$$\lim_{s\to\infty} \frac{f(x,s)}{s^q} = L_2(x) > 0, \qquad \lim_{s\to-\infty} \frac{f(x,s)}{|s|^r} = L_3(x) < 0, \tag{3.18}$$

uniformly in $\overline{\Omega}$, then the boundary value problem (1.1) has at least a nontrivial solution.

Proof. Combining relations (3.17) and (3.18) we get that there exists a constant $C > 0$ such that for sufficiently large M

$$-sf(x,s) \leq C|s|^{\sigma+1}, \qquad \forall\, x \in \Omega,\ |s| > M. \tag{3.19}$$

Integrating (3.19) one has

$$-F(x,s) = -\int_0^1 f(x, us)sdu \leq \frac{C}{\sigma+1}|s|^{\sigma+1}, \qquad \forall\, x \in \Omega,\ |s| > M.$$

We can now apply Lemma 3.5 to get a sequence $\{u_m\}$ that satisfies (3.13).

On the other hand, in view of (3.18) we see that (3.14) is satisfied and hence $\{u_m\}$ is bounded. As a consequence $u_m \to u_0$ in $C^{n-1}(\overline{\Omega})$ and the proof follows. □

Finally, we give some examples as an application of our results.

Example 1. Let F satisfy (3.1) and suppose that (3.3) holds with $m = 1$. Then the boundary value problem

$$\begin{cases} u^{(2n)} + Au^{(4)} + Bu'' + Cu + f(x,u) = 0 \text{ in } \Omega = (0, L) \\ u = u'' = \cdots = u^{(2n-2)} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.20}$$

has at least two nontrivial solutions in $H(\Omega)$. Here $A < 0, B = 0, C > 0$, (2.25) holds and

$$\left(\frac{-2A}{n}\right)^{\frac{n(n-2)}{4}} + A\left(\frac{-2A}{n}\right)^{\frac{n-2}{2}} + C < 0. \tag{3.21}$$

In particular, the result holds if $n = 4, A = -2, 0 < C < 1, L = 2$.

The proof follows from Corollary 3.2. Since $P(\xi) = \xi^{2n} + A\xi^4 + C$ we study the function $\varphi(t) = t^{\frac{n}{2}} + At + C$. We can check that φ attains its minimum at $t_0 = (-2A/n)^{\frac{n-2}{2}}$. Imposing $\varphi(t_0) < 0$, i.e., (3.21) we see that P has (at least) two positive roots.

Consider $n = 3$. Then P becomes $P(\xi) = \xi^6 - A\xi^4 + B\xi^2 - C$. If

$$A > 0, \quad B < 0, \quad 0 > C > \gamma = \frac{1}{27}\left[9AB - 2A^3 - 2\left(A^2 - 3B\right)^{\frac{3}{2}}\right],$$

then P has precisely two positive roots $0 < \xi_1 < \xi_2$. As a consequence (3.20) has at least two nontrivial solutions in $H(\Omega)$ if (3.3) holds with $m = 1$.

The reader is referred to Appendix A, [13] where the authors give detailed conditions on parameters A, B, C and L which guarantee the existence of at least one or two positive solutions of $P(\xi) = \xi^6 - A\xi^4 + B\xi^2 - C$.

Example 2. Let F satisfy (3.1) and suppose that (3.4) holds (here ξ_1 is the unique solution of $P(\xi) = 0$). Consider the boundary value problem (3.20), where $C < 0$. Suppose that one of the following relations holds true

$$A, B \geq 0 \quad \text{and} \quad (2.25) \tag{3.22}$$

$$A > 0, \quad B < 0 \quad \text{and} \quad (2.25) \tag{3.23}$$

$$A < 0, \quad B > 0, \quad (3.4) \quad \text{and} \quad n\left(\frac{-2A}{n(n-1)}\right)^{\frac{n-1}{n-2}} + 2A\left(\frac{-2A}{n(n-1)}\right)^{\frac{1}{n-2}} + B > 0. \tag{3.24}$$

Then the boundary value problem (3.20) has at least two nontrivial solutions in Ω.

The proof follows from Corollary 3.3 by using the same techniques as in Example 1.

Example 3. In as similar way we can conclude that if F satisfies (3.1) and that (3.4) holds (here ξ_1 is the unique solution of $P(\xi) = 0$), then the problem

$$\begin{cases} u^{(2n)} + Au^{(2n-2)} + Bu^{(2n-4)} + Cu + f(x,u) = 0 \text{ in } \Omega = (0, \mathrm{L}) \\ u = u'' = \cdots = u^{(2n-2)} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.25}$$

has at least two nontrivial solutions in Ω. Here $A, B > 0, C < 0$ and we are under the assumptions of Lemma 2.9.

Example 4. Arguing as before, if $A, B > 0$, $A^2 > 4B$, F satisfies (3.1) and if (3.3) holds, it follows that the problem

$$\begin{cases} u^{(6)} + Au^{(4)} + Bu'' + f(x,u) = 0 \text{ in } \Omega = (0, \mathrm{L}) \\ u = u'' = u^{(4)} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.26}$$

has at least two nontrivial solutions in Ω.

Example 5. The functions $F_1(s) = \ln(1 + \ln(1 + \cdots + \ln(1 + |s|^p)))$, $p > 2$ and $F_2(s) = |s|(\arctan |s|^p + \ln(1 + |s|^p))$, $p > 1$ satisfy (3.1). Hence, under the requirements of Theorem 3.1 problem (1.1) (with f replaced by $f_1 = F_1'$ or $f_2 = F_2'$) has at least two nontrivial solutions in Ω.

It is easy to check that F_1, F_2 don't satisfy (1.3) and hence this existence result cannot be deduced from the corresponding results presented in [12] or [13] even if we restrict ourselves to the particular cases $n = 2$ or $n = 3$.

We can see that $F_3(s) = s^p - Cs^2$, where $p > 2$ is even and $C > 0$ changes sign and does not fulfill the restriction (3.5) imposed in [12, 13], but fulfills the requirements of Theorem 3.4 with $\beta = -C < 0$. Again we conclude that problem (1.1) (with f replaced by $f_3 = F_3'$) has at least a nontrivial solution if (3.9) is satisfied.

Example 6. Let $C > 0, q > 2, \alpha \in (0,1)$. Then the function f_4

$$f_4(s) = \begin{cases} -s^q - C\ln(1 + s^\alpha), & s > 0 \\ |s|^q, & s \leq 0 \end{cases}$$

satisfies the requirements of Theorem 3.7. Hence the boundary value problem (1.1) with f replaced by f_4 has at least one solution.

References

[1] G. Bonanno, R. Livrea, A sequence of positive solutions for sixth-order ordinary nonlinear differential problems, *Electron. J. Qual. Theory Differ. Equ.* **2021**, No. 20, 1–17. https://doi.org/10.14232/ejqtde.2021.1.20

[2] G. Bonanno, B. Di Bella, D. O'Regan, Non-trivial solutions for nonlinear fourth-order elastic beam equations, *Comp. Math. Appl.* **62**(2011), 1862–1869. https://doi.org/10.1016/j.camwa.2011.06.029

[3] G. Bonanno, B. Di Bella, A boundary value problem for fourth-order elastic beam equations, *J. Math. Anal. Appl.* **343**(2008), 1166–1176. https://doi.org/10.1016/j.jmaa.2008.01.049

[4] G. Bonanno, B. Di Bella, Infinitely many solutions for a fourth-order elastic beam equation, *Nonlinear Differ. Equ. Appl.* **18**(2011), 357–368. https://doi.org/10.1007/s00030-011-0099-0

[5] G. Bonanno, P. Candido, D. O'Regan, Existence of nontrivial solutions for sixth-order differential equations, *Mathematics* **9**(2021), 1852. https://doi.org/10.3390/math9161852

[6] H. Brézis, L. Nirenberg, Remarks on finding critical points, *Comm. Pure. Appl. Math.* **44**(1991), 939–963. https://doi.org/10.1002/cpa.3160440808

[7] J. Chaparova, Existence and numerical approximations of periodic solutions of semilinear fourth-order differential equations, *J. Math. Anal. Appl.* **273**(2002), 121–136. https://doi.org/10.1016/S0022-247X(02)00216-0

[8] J. Chaparova, L. A. Peletier, S. Tersian, Existence and nonexistence of nontrivial solutions of semilinear fourth- and sixth-order differential equations, *Adv. Differential Equations* **10**(2003), 1237–1258. https://doi.org/10.57262/ade/1355926160; MR2016681; Zbl 1126.34316

[9] J. Chaparova, L. A. Peletier, S. Tersian, Existence and nonexistence of nontrivial solutions of semilinear sixth-order ordinary differential equations, *Applied Math. Lett.* **17**(2004), 1207–1212. https://doi.org/10.1016/j.aml.2003.05.014

[10] C.-P. Danet, Existence and uniqueness of nontrivial solutions for a class of 2n-order ODE, submitted.

[11] G. D'Aguì, B. Di Bella, P. Winkert, Two positive solutions for nonlinear fourth-order elastic beam equations, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 37, 1–12. https://doi.org/10.14232/ejqtde.2019.1.37

[12] M. R. Grossinho, L. Sanchez, S. Tersian, On the solvability of a boundary value problem for a fourth-order ordinary differential equation, *Appl. Math. Lett.* **18**(2005), 439–444. https://doi.org/10.1016/j.aml.2004.03.011

[13] T. Gyulov, G. Morosanu, S. Tersian, Existence for a semilinear sixth-order ODE, *J. Math. Anal. Appl.* **321**(2006), 86–98. https://doi.org/10.1016/j.jmaa.2005.08.007

[14] W. Li, L. Zhang, Y. An, The existence of positive solutions for a nonlinear sixth-order boundary value problem, *ISRN Appl. Math.* **2012**, Art. ID 926952, 12 pp. https://doi.org/10.5402/2012/926952

[15] X.-L. Liu, W.-T. Li, Existence and multiplicity of solutions for fourth-order boundary value problems with parameters, *J. Math. Anal. Appl.* **327**(2007), 362–375. https://doi.org/10.1016/j.jmaa.2006.04.021

[16] S. Tersian, J. Chaparova, Periodic and homoclinic solutions of extended Fisher-Kolmogorov equations, *J. Math. Anal. Appl.* **260**(2001), 490–506. https://doi.org/10.1006/jmaa.2001.7470

[17] S. Tersian, J. Chaparova, Periodic and homoclinic solutions of some semilinear sixth-order diferential equations, *J. Math. Anal. Appl.* **272**(2002), 223–239. https://doi.org/10.1016/S0022-247X(02)00153-1

[18] L. Zhang, Y. An, Existence and multiplicity of positive solutions of a boundary value problem for sixth-order ODE with three parameters, *Bound. Value Probl.* **2010**, Art. ID 878131, 13 pp. https://doi.org/10.1155/2010/878131