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Heteroclinic solutions in singularly perturbed discontinuous differential equations: a non-generic case

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Abstract. We derive Melnikov type conditions for the persistence of heteroclinic solutions in perturbed slowly varying discontinuous differential equations. Opposite to [*J. Differential Equations* **400**(2024), 314–375] we assume that the unperturbed (frozen) equation has a parametric system of heteroclinic solutions and extend a result in [*SIAM J. Math. Anal.* **18**(1987), 612–629] and [*SIAM J. Math. Anal.* **19**(1988), 1254–1255] to higher dimensional non-Hamiltonian discontinuous singularly perturbed differential equations.

Keywords: discontinuous differential equations, heteroclinic solutions, Melnikov conditions, persistence.

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1 Introduction

Let $h(x,y), f_i(x,y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, N+1$, be C^r-functions, $r \geq 2$, bounded on $\mathbb{R}^n \times \mathbb{R}^m$ together with their derivatives, and $c_1 < c_2 < \dots < c_N < c_{N+1}$ be real numbers.

In this paper we study the problem of existence of continuous, piecewise smooth, bounded solutions of a singularly perturbed equation like

$$\begin{aligned} \dot{x} &= f(x,y), \\ \dot{y} &= \varepsilon g(x,y,\varepsilon) \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and

$$f(x,y) := \begin{cases} f_i(x,y) & \text{if } c_{i-1} < h(x,y) < c_i, \\ & i = 1, \dots, N \\ f_{N+1}(x,y) & \text{if } h(x,y) > c_N \end{cases} \tag{1.2}$$

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where we take for notational simplicity $c_0 = -\infty$. It is assumed that for all $y \in \mathbb{R}^m$, the frozen system

$$\dot{x} = f(x, y) \tag{1.3}$$

has hyperbolic fixed points $x = w_\pm(y)$ with an associated piecewise C^r, heteroclinic solution $u(t, y)$ intersecting transversally the manifolds $\mathcal{S}_i(y) = \{x \mid h(x, y) = c_i\}$. We intend to give a Melnikov like condition guaranteeing that the perturbed system (1.1) has a solution $(x(t, \varepsilon), y(t, \varepsilon))$ such that $\sup_{t\in\mathbb{R}} |x(t, \varepsilon) - u(t, y(t, \varepsilon))| \to 0$ as $\varepsilon \to 0$. This paper has been motivated by [10, 11] where the authors considered a perturbation of a smooth, Hamiltonian, three-dimensional system. The main result of our paper (Theorem 6.2) concerns higher dimension, discontinuous and not necessarily Hamiltonian systems. Moreover the approach in [10, 11] is basically geometrical, while in this paper it is based on Lyapunov–Schmidt reduction.

This paper is a continuation of series of our works [3–6] on the study of existence of bounded solutions for slowly varying discontinuous differential equations. Papers [3–5] deal with the persistence of periodic solutions in case of existence of either a single or a family of periodic solutions for the frozen system (1.3). Next, in [6] generic conditions have been given for persistence of an isolated homoclinic-heteroclinic solution for the frozen system. Thus it is a natural step to study the case when the frozen system possesses a parametric system of bounded-homoclinic-heteroclinic solutions, which is the purpose of this paper.

To prove Theorem 6.2 we use a general result in [6] concerning the characterization of bounded solutions on both the positive and the negative line for the perturbed equation, Then, in [6], this result is used, jointly with a Lyapunov–Schmidt reduction, to write down a bifurcation equation which is the scalar product of certain vectors with the difference at $t = 0$ of the value of these solutions. Now, in [6] the case is considered where this function has a simple zero at $\varepsilon = 0$, while in this paper it is identically zero at $\varepsilon = 0$. This fact makes a big difference and indeed the Melnikov functions obtained in the two cases are quite different.

We now briefly sketch the content of this paper. For the reader convenience and also for the completeness of this paper, we recall necessary results from [6] in Sections 2–5. Namely, Section 2 provides basic assumptions and defines the piecewise smooth heteroclinic solution of the unperturbed system. Section 3 recalls the definition of exponential dichotomy and extends this notion to discontinuous, piecewise linear, systems with jumps at some points; moreover some results concerning existence of bounded solutions on either $t \geq 0$ and $t \leq 0$ are extended to these systems. In Section 4 we construct families of bounded solutions and describes them in terms of some parameters. These solutions are continuous and piecewise smooth and give the bounded solutions we look for, when they assume the same value at $t = 0$. Section 5 defines the discontinuous variational equation.

Our main results are proved in Section 6 where we obtain a Melnikov-type condition assuring that the bifurcation function has a manifold of solutions. Motivated by [8], Section 7, is devoted to the construction of an example of application of the main result of this paper. Although the equation is three-dimensional and Hamiltonian, the vector field is discontinuous and then the results in [8, 10, 11] do not apply.

Finally, in Section 8 we show that the Melnikov function given here extends to the heteroclinic case with finitely many discontinuity points, the Melnikov function given in [5] for the periodic case with two discontinuity points.

In the whole paper we will use the following notation. Given a vector v or a matrix A with v^T, A^T we denote the transpose of v, A.

2 Notation and basic assumptions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $c_1 < \ldots < c_{N+1}$ be real numbers and $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a C^r-functions, $r \geq 2$, with bounded derivatives. For $\ell = 1, \ldots, N+1$, we set

$$\Omega_\ell = \{(x, y) \in \Omega \times \mathbb{R}^m \mid c_{i-1} \leq h(x, y) < c_i\},$$

where we set for simplicity, $c_0 = -\infty$. Then let $f_\ell : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ be C^r-functions, bounded together with their derivatives in $\Omega \times \mathbb{R}^m$.

First we give the definition of solutions of equation

$$\dot{x} = f_i(x, y), \quad (x, y) \in \Omega_i, \; i = 1, \ldots, N+1 \tag{2.1}$$

we are considering in this paper.

Definition 2.1. A continuous, piecewise smooth function $u(t, y)$ is a solution of equation (2.1) on $t \geq 0$ intersecting transversally the sets $\mathcal{S}_i(y) = \{x \in \Omega \mid h(x, y) = c_i\}$, $i = 1, \ldots, N$, if there exist $\eta > 0$ and C^r-functions bounded together with their derivatives $0 < t_1(y) < \ldots < t_N(y)$ such that the following conditions hold for $1 \leq i \leq N$ (note that we set $t_0(y) = 0$)

$a_1)$ $\dot{u}(t, y) = f_i(u(t, y), y)$ for $t_{i-1}(y) < t < t_i(y)$ and $\dot{u}(t, y) = f_{N+1}(u(t, y), y)$ for $t > t_N(y)$;

$a_2)$ $h(u(t_i(y), y), y) = c_i$, and $h_x(u(t_i(y), y), y)\dot{u}(t_i(y)^{\pm}, y) > 2\eta$;

$a_3)$ $c_{i-1} < h(u(t, y), y) < c_i$, for $t_{i-1}(y) < t < t_i(y)$ and $h(u(t, y), y) > c_N$, for $t > t_N(y)$.

Similarly, a continuous, piecewise smooth function $u(t, y)$ is a solution of equation (2.1) on $t \leq 0$ intersecting transversally the sets $\mathcal{S}_i(y)$, if there exist $\eta > 0$ and C^r-functions bounded together with their derivatives $t_{-N}(y) < \ldots < t_{-1}(y) < 0$ such that the following conditions hold for any $1 \leq i \leq N$:

$a_1')$ $\dot{u}(t, y) = f_i(u(t, y), y)$ for $t_{-i}(y) < t < t_{-i+1}(y)$ and $\dot{u}(t, y) = f_{N+1}(u(t, y), y)$ for $t < t_{-N}(y)$;

$a_2')$ $h(u(t_{-i}(y), y), y) = c_i$, and $h_x(u(t_{-i}(y), y), y)\dot{u}(t_{-i}(y)^{\pm}, y) < -2\eta$;

$a_3')$ $c_{i-1} < h(u(t, y), y) < c_i$, for $t_{-i}(y) < t < t_{-i+1}(y)$ and $h(u(t, y), y) > c_N$, for $t < t_{-N}(y)$.

In this paper we assume that a continuous, piecewise smooth solution $u(t, y)$ of equation (2.1) exist, for $t \in \mathbb{R}$, such that the following conditions hold.

$A_1)$ $w_0(y) := u(0, y)$ and its derivatives are bounded functions on $\mathbb{R}^m$ and $w_0(y)$ belongs to an open and bounded subset $B \subset \mathbb{R}^n$ such that $\overline{B} \times \mathbb{R}^m \subset \Omega_1$.

$A_2)$ There exist smooth and bounded functions $w_{\pm}(y)$ and $\mu_0 > 0$, such that

$$f_{N+1}(w_{\pm}(y), y) = 0,$$
$$h(w_{\pm}(y), y) - c_N > \mu_0,$$

for any $y \in \mathbb{R}^m$ and

$$\lim_{t \to \pm\infty} u(t, y) - w_{\pm}(y) = 0$$

uniformly with respect to $y \in \mathbb{R}^m$.

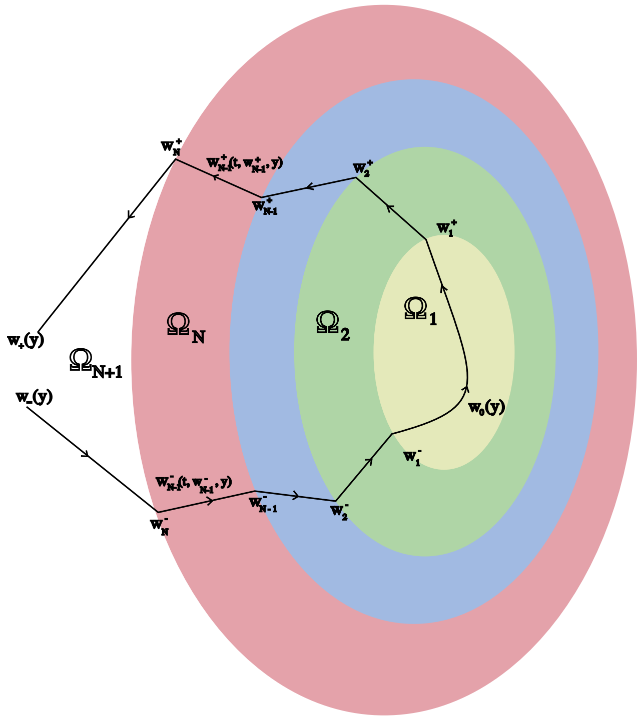

Figure 2.1: The piecewise C^1 bounded solution of (1.3). For simplicity we write $w_j^\pm$ instead of $w_j^\pm(y)$.

$A_3)$ For any $y \in \mathbb{R}^m$, $f_{N+1,x}(w_\pm(y), y)$ have k eigenvalues with negative real parts and $n-k$ eigenvalues with positive real parts, counted with multiplicities and there exists $\delta_0 > 0$ such that all these eigenvalues satisfy

$$|\operatorname{Re}\lambda(y)| > \delta_0.$$

We set

$$t_0(y) = 0, \qquad \forall y \in \mathbb{R}^m.$$

So, we are considering solutions of (2.1) which are contained in $C \times \mathbb{R}^n \subset \Omega \times \mathbb{R}^m$, where C is a compact subset of Ω. Then we may and will assume that $\Omega = \mathbb{R}^n$.

Remark 2.2. i) As in [6], all results in this paper can be easily generalised to the case where the solutions exit transversally Ω_i and enter into either Ω_{i+1} or Ω_{i-1} transversally. We can formalize all of this as follows: there exists $(j_0, \ldots, j_M)$ such that given j_i then j_{i+1} is either $j_i - 1$ or $j_i + 1$ and for $t_i(y) < t < t_{i+1}(y)$ we have

$$c_{j_i - 1} < h(u(t,y),y) < c_{j_i}.$$

Moreover

$$|h_x(u(t_i(y),y),y) f_i(u(t_i(y),y),y)| > 2\eta.$$

for any $i = 1, \ldots, N$. A similar generalization can be made for $t \leq 0$ and all other assumption will be changed accordingly.

ii) From a_2) and a_2') it follows that, for $i = 1, \dots, N$:

$$\frac{\partial}{\partial t} h(u(t,y),y)_{|t=t_i(y)} \geq 0,$$
$$\frac{\partial}{\partial t} h(u(t,y),y)_{|t=0} \geq 0$$

that is

$$h_x(u(t_i(y),y),y) f_{i-1}(u(t_i(y),y),y) \geq 0,$$
$$h_x(u(t_i(y),y),y) f_i(u(t_i(y),y),y) \geq 0.$$

Similarly

$$h_x(u(t_{-i}(y),y),y) f_i(u(t_{-i}(y),y),y) \leq 0;$$
$$h_x(u(t_{-i}(y),y),y) f_{i+1}(u(t_{-i}(y),y),y) \leq 0.$$

So a_2) and a_2') are a kind of transversality assumption on $u(t,y)$.

Let $w_0^{\pm}(y) = u(0,y)$ and set, for $i = 1, \dots, N$:

$$w_i^{\pm}(y) = u(t_{\pm i}(y), y) \in \mathcal{S}_i(y). \tag{2.2}$$

The following result has been proved in [6]

Lemma 2.3. *$w_i^{\pm}(y)$ are C^r-functions bounded together with their derivatives. Moreover $u(t,y)$ and its derivatives with respect to y are bounded uniformly with respect to y, on both $t \geq t_N^+(y)$ and $t \leq t_N^-(y)$.*

Let $i = 1, \dots, N+1$. For $t \geq 0$, let $u_i^+(t,y)$ be the solution of $\dot{x} = f_i(x,y)$ such that $u_i^+(t_{i-1}(y),y) = w_{i-1}^+(y)$. Similarly, let $u_i^-(t,y)$ be the solution of $\dot{x} = f_i(x,y)$ such that $u_i^-(t_{1-i}(y),y) = w_{i-1}^-(y)$. Note that $u_i^{\pm}(t,y)$ is defined for $t \in \mathbb{R}$ and

$$u(t,y) = \begin{cases} u_i^-(t,y) & \text{for } t_{-i}(y) \leq t \leq t_{1-i}(y),\ i = 1, \dots N+1, \\ u_i^+(t,y) & \text{for } t_{i-1}(y) \leq t \leq t_i(y),\ i = 1, \dots, N+1 \end{cases} \tag{2.3}$$

where, for simplicity, we set $t_{-N-1}(y) = -\infty$ and $t_{N+1}(y) = \infty$. Note that

$$u_i^+(t_i(y),y) = u(t_i(y),y) = w_i^+(y) = u_{i+1}^+(t_i(y),y)$$

and similarly,

$$u_i^-(t_{-i}(y),y) = u(t_{-i}(y),y) = w_i^-(y) = u_{i+1}^-(t_{-i}(y),y).$$

3 Exponential dichotomy for piecewise discontinuous systems

A basic tool in this paper is the notion of exponential dichotomy, whose definition we recall here. Let J be either $[a,\infty)$, $(-\infty,a]$, or $\mathbb{R}$ and $A(t)$, $t \in J$, be a $n \times n$ continuous matrix. We say that the linear system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n \tag{3.1}$$

has an exponential dichotomy on J if there exist a projection $P : \mathbb{R}^n \to \mathbb{R}^n$ and constants $\delta > 0$ and $K \geq 1$ such that the fundamental matrix $X(t)$ of (3.1) satisfying $X(a) = \mathbb{I}$, when $J = [a, \infty), (-\infty, a]$, or $X(0) = \mathbb{I}$ when $J = \mathbb{R}$, satisfies

$$
\begin{aligned}
|X(t)PX(s)^{-1}| &\leq Ke^{-\delta(t-s)}, \quad \text{for } s \leq t, s, t \in J, \\
|X(s)(\mathbb{I} - P)X(t)^{-1}| &\leq Ke^{-\delta(t-s)}, \quad \text{for } s \leq t, s, t \in J.
\end{aligned}
$$

K and δ are called the constant and the exponent of the exponential dichotomy.

In [6] the notion of exponential dichotomy has been extended to systems with discontinuities.

Let $t_0 < t_1 < \ldots < t_N$ be real numbers, $B_1, \ldots, B_N$ be invertible $n \times n$ matrices and $\mathcal{A}(t)$, $t \geq t_0$ be a piecewise continuous matrix with possible discontinuity jumps at $t = t_1, \ldots, t_N$, that is

$$
\mathcal{A}(t) = \begin{cases} A_i(t) & \text{if } t_{i-1} \leq t < t_i, \\ & i = 1, \ldots, N \\ A_{N+1}(t) & \text{if } t \geq t_N \end{cases} \tag{3.2}
$$

where $A_i(t)$ is continuous for $t_{i-1} \leq t \leq t_i$, $A_{N+1}(t)$ is continuous for $t \geq t_N$. Note that $\mathcal{A}(t)$ is continuous for $t \geq t_0$, $t \neq t_i$, $i \neq 1, \ldots, N$ and right-continuous at $t = t_i$, $i = 1, \ldots, N$ with possible jumps at $t = t_i$, $i = 1, \ldots, N$ given by the matrix $A_{i+1}(t_i) - A_i(t_i)$.

For $t \geq t_0$ the fundamental matrix of the linear, discontinuous, system

$$
\begin{aligned}
\dot{x} &= \mathcal{A}(t)x, \\
x(t_i^+) &= B_i x(t_i^-), \qquad i = 1, \ldots, N
\end{aligned} \tag{3.3}
$$

is defined as

$$
X_+(t) = \begin{cases} U_1(t) & \text{if } 0 \leq t < t_1, \\ U_{i+1}(t)U_{i+1}(t_i)^{-1}B_i X_+(t_i^-) & \text{if } t_i \leq t < t_{i+1}, \\ & i = 1 \ldots, N-1 \\ U_{N+1}(t)U_{N+1}(t_N)^{-1}B_N X_+(t_N^-) & \text{if } t \geq t_N, \end{cases}
$$

where $U_i(t)$ is the fundamental matrix of the linear systems

$$
\dot{x} = A_i(t)x
$$

on $\mathbb{R}$, that is $\dot{U}_i(t) = A_i(t)U_i(t)$, $t \in \mathbb{R}$, and $U_i(0) = \mathbb{I}$.

Similarly, if $t_{-N} < \ldots t_{-1} < t_0$ and

$$
\mathcal{A}(t) = \begin{cases} A_{N+1}(t) & \text{if } t \leq t_{-N}, \\ A_i(t) & \text{if } t_{-i} < t \leq t_{-i+1}, \\ & i = 1, \ldots, N \end{cases} \tag{3.4}
$$

where $A_i(t)$ is continuous for $t_{-i-1} \leq t \leq t_{-i}$ and $A_{N+1}(t)$ is continuous for $t \leq t_{-N}$, the fundamental matrix, for $t \leq t_0$, of the linear, discontinuous, system

$$
\begin{aligned}
\dot{x} &= \mathcal{A}(t)x, \\
x(t_{-i}^+) &= B_i x(t_{-i}^-)
\end{aligned} \tag{3.5}
$$

is

$$
X_-(t) = \begin{cases} U_1(t) & \text{if } t_{-1} < t \leq 0, \\ U_{i+1}(t)U_{i+1}(t_{-i})^{-1}B_i^{-1} X_-(t_{-i}^+) & \text{if } t_{-i-1} < t \leq t_{-i}, \\ & i = 1, \ldots, N-1 \\ U_{N+1}(t)U_{N+1}(t_{-N})^{-1}B_N^{-1} X_+(t_{-N}^+) & \text{if } t \leq t_{-N}. \end{cases}
$$

Note that, on $t \le t_0$, $\mathcal{A}(t)$ is continuous for $t \le t_0$, $t \ne t_{-i}$, $i \ne 1, \dots, N$ and left-continuous at $t = t_{-i}$, $i = 1, \dots, N$ with possible jumps at $t = t_{-i}$, $i = 1, \dots, N$ given by the matrix $A_i(t_{-i}) - A_{-i+1}(t_{-i})$.

Remark 3.1. As a matter of facts, for $t \ge t_0$, we will consider

$$\mathcal{A}(t) = \begin{cases} A_i(t) & \text{if } t_{i-1} \le t \le t_i, \\ & \ i = 1, \dots, N \\ A_{N+1}(t) & \text{if } t \ge t_N \end{cases}$$

and similarly for $t \le t_0$. This may cause a duplicate definition of $\mathcal{A}(t)$ at $t = t_i$, however it will be always clear which one among the functions $A_i(t)$ will be taken into account at that point.

Without loss of generality we may and will assume that $t_0 = 0$.

Note that $X_+(t)$ is continuous for $t \ne t_1, \dots, t_N$ and right-continuous at $t = t_1, \dots, t_N$ and $X_-(t)$ is continuous for $t \ne t_{-1}, \dots, t_{-N}$ and left-continuous at $t = t_{-1}, \dots, t_{-N}$.

It is clear that $\dot{X}_\pm(t) = \mathcal{A}(t) X_\pm(t)$, for any $\pm t \ge 0$, $t \ne t_{\pm 1}, \dots, t_{\pm N}$, $X_\pm(0) = \mathbb{I}$, the identity matrix, and

$$\begin{aligned} X_+(t_i^+) &= B_i X_+(t_i^-), \\ X_-(t_{-i}^+) &= B_i X_-(t_{-i}^-) \end{aligned} \tag{3.6}$$

for any $i = 1, \dots, N$. Actually we can write

$$X_+(t_i) = B_i X_+(t_i^-), \quad X_-(t_{-i}) = B_i^{-1} X_-(t_{-i}^+)$$

since $X_+(t)$ is right-continuous and $X_-(t)$ is left-continuous.

Remark 3.2. Let $\tau \ge 0$ be a fixed number. For $t \ge 0$, $x(t) = X_+(t) X_+(\tau)^{-1} \tilde{x}$ is the right-continuous solution of

$$\begin{cases} \dot{x} = \mathcal{A}(t) x, & \text{for } t \ge 0,\, t \ne t_1, \dots, t_N \\ x(t_i^+) = B_i x(t_i^-) \\ x(\tau^+) = \tilde{x}. \end{cases} \tag{3.7}$$

Indeed, it is obvious that $\dot{x}(t) = \mathcal{A}(t) x(t)$ for $t \ge 0$, $t \ne t_1, \dots, t_N$ and that $x(t_i^+) = B_i x(t_i^-)$, since $X_+(t_i^+) = B_i X_+(t_i^-)$. Moreover, for any $\tau \ge 0$ we have $x(\tau^+) = X_+(\tau^+) X_+(\tau)^{-1} \tilde{x} = X_+(\tau) X_+(\tau)^{-1} \tilde{x} = \tilde{x}$, since $X_+(t)$ is right-continuous at any $t \ge 0$.

Similarly, for $t \le 0$ and any fixed $\tau \le 0$, $x(t) = X_-(t) X_-(\tau)^{-1} \tilde{x}$ is the left-continuous solution of

$$\begin{cases} \dot{x} = \mathcal{A}(t) x, & \text{for } t \le 0,\, t \ne t_{-1}, \dots, t_{-N} \\ x(t_{-i}^-) = B_i^{-1} x(t_{-i}^+) \\ x(\tau^-) = \tilde{x}. \end{cases} \tag{3.8}$$

The following results have been proved in [6]:

Lemma 3.3. *Suppose that the linear system*

$$\dot{x} = A_{N+1}(t) x$$

has an exponential dichotomy on $t \ge t_N$ (resp. $t \le t_{-N}$) with constant K, exponent δ and projection $\mathcal{P}_+$ (resp. $\mathcal{P}_-$ when $t \le t_{-N}$). Then, the linear system (3.3) *(resp.* (3.5)*) with $\mathcal{A}(t)$ as in* (3.2) *(resp.*

(3.4)) *has an exponential dichotomy on* $\mathbb{R}_+$, *(resp.* $\mathbb{R}_-$*) with the same exponent* δ, *constant* $\tilde{K} \geq K$ *and projection*

$$\begin{aligned} \tilde{\mathcal{P}}_+ &= X_+(t_N^+)^{-1}\mathcal{P}_+X_+(t_N^+), \\ \tilde{\mathcal{P}}_- &= X_-(t_{-N}^-)^{-1}\mathcal{P}_-X_-(t_{-N}^-). \end{aligned} \tag{3.9}$$

Lemma 3.4. *Let* $\mathcal{A}(t)$ *be either as in* (3.2) *or* (3.4). *Suppose that the condition of Lemma* 3.3 *holds and let* $\tilde{\mathcal{P}}_\pm$ *be as in* (3.9). *Then* $\xi_+ \in \mathcal{R}\tilde{\mathcal{P}}_+$ *if and only if the solution of the discontinuous system* (3.3) *such that* $x(0) = \xi_+$ *is bounded for* $t \geq 0$. *Similarly,* $\xi_- \in \mathcal{N}\tilde{\mathcal{P}}_-$ *if and only if the solution of the discontinuous system* (3.5) *such that* $x(0) = \xi_-$ *is bounded for* $t \leq 0$.

Lemma 3.5. *Let* B_i, $i = 1, \ldots, N$, *be invertible* $n \times n$ *matrices and* $k(t)$ *be a bounded integrable function for* $t \geq 0$, *(resp.* $t \leq 0$*). Suppose the condition of Lemma* 3.3 *hold and set*

$$\begin{aligned} \tilde{\mathcal{P}}_+^\tau &= X_+(\tau)\tilde{\mathcal{P}}_+X_+(\tau)^{-1}, \\ \tilde{\mathcal{P}}_-^\tau &= X_-(-\tau)\tilde{\mathcal{P}}_-X_-(-\tau)^{-1} \end{aligned}$$

where $\tilde{\mathcal{P}}_\pm$ *is as in* (3.9) *and* $0 \leq \tau \in \mathbb{R}$ *is a fixed number. Then, for any* $\xi_+ \in \mathcal{R}\tilde{\mathcal{P}}_+^\tau$ *(resp.* $\xi_- \in \mathcal{N}\tilde{\mathcal{P}}_-^\tau$*) the linear inhomogeneous system*

$$\begin{aligned} \dot{x} &= \mathcal{A}(t)x + k(t), \\ x(t_i^+) &= B_i x(t_i^-), \quad i = 1, \ldots, N \\ \tilde{\mathcal{P}}_+^\tau x(\tau) &= \xi_+ \end{aligned} \tag{3.10}$$

with $t \geq 0$, *[resp.*

$$\begin{aligned} \dot{x} &= \mathcal{A}(t)x + k(t), \\ x(t_i^-) &= B_i^{-1} x(t_i^+), \\ (\mathbb{I} - \tilde{\mathcal{P}}_-^\tau)x(-\tau) &= \xi_- \end{aligned}$$

when $t \leq 0$*] has the unique right-continuous, [resp. left-continuous when* $t \leq 0$*] bounded solution*

$$\begin{aligned} x(t) = X_+(t)\tilde{\mathcal{P}}_+X_+(\tau)^{-1}\xi_+ &+ \int_\tau^t X_+(t)\tilde{\mathcal{P}}_+X_+(s)^{-1}k(s)ds \\ &- \int_t^\infty X_+(t)(\mathbb{I} - \tilde{\mathcal{P}}_+)X_+(s)^{-1}k(s)ds \end{aligned} \tag{3.11}$$

[resp.

$$\begin{aligned} x(t) = X_-(t)(\mathbb{I} - \tilde{\mathcal{P}}_-)X_-(-\tau)^{-1}\xi_- &+ \int_{-\infty}^t X_-(t)\tilde{\mathcal{P}}_-X_-(s)^{-1}k(s)ds \\ &- \int_t^{-\tau} X_-(t)(\mathbb{I} - \tilde{\mathcal{P}}_-)X_-(s)^{-1}k(s)ds \end{aligned} \tag{3.12}$$

if $t \leq 0$*]. Moreover such a solution satisfies*

$$\sup_{t \geq \tau} |x(t)| \leq K[|\xi_+| + 2\delta^{-1} \sup_{t \geq 0} |k(t)|] \tag{3.13}$$

if $t \geq 0$ *[resp.*

$$\sup_{t \leq -\tau} |x(t)| \leq K[|\xi_-| + 2\delta^{-1} \sup_{t \leq 0} |k(t)|] \tag{3.14}$$

if $t \leq 0$*].*

4 Bounded solutions on the half lines

From A_3) we know that the number of the eigenvalues of $f_{N+1,x}(w_\pm(y),y)$ with negative (and then also positive) real parts, counted with multiplicities, is independent of $y \in \mathbb{R}^m$. Moreover it also follows that all eigenvalues are bounded functions of $y \in \mathbb{R}^m$. Indeed, since $f_{N+1,x}(w_\pm(y),y)$ is bounded, the matrix $\mathbb{I} - \lambda^{-1} f_{N+1,x}(w_\pm(y),y)$ is invertible for $|\lambda| > R$, sufficiently large and independent of y. Hence all eigenvalues have to satisfy $|\lambda| \le R$.

Let δ_0 be any positive number strictly less than $\min\{|\operatorname{Re}\lambda(y)|\}$, where $\lambda(y)$ are the eigenvalues of $f_{N+1,x}(w_\pm(y),y)$. According to [7] the system $\dot{x} = f_{N+1,x}(w_\pm(y),y)x$ has an exponential dichotomy on $\mathbb{R}$ with exponent δ_0 and spectral projection (of rank k)

$$\begin{aligned} P^0_\pm(y) &= \frac{1}{2\pi i}\int_\Gamma (z\mathbb{I} - f_{N+1,x}(w_\pm(y),y))^{-1} dz \\ &= \sum_{Re\lambda(y)<0} \operatorname{Res}((z\mathbb{I} - f_{N+1,x}(w_\pm(y),y))^{-1}, z = \lambda(y)) \end{aligned}$$

where $\operatorname{Res}(F(z), z = z_0)$ is the residual of the meromorphic function $F(z)$ at z_0 and Γ is a closed curve that contains in its interior all eigenvalues of $f_{N+1,x}(w_\pm(y),y)$ with negative real parts, but none of those with positive real parts. Hence $|P^0(y)| \le M$, for any $y \in \mathbb{R}^m$ and some $M \ge 1$.

Now, recalling (2.3), from A_2) and the boundedness of $t_N(y)$, it follows immediately that

$$\lim_{t\to\pm\infty} u^\pm_{N+1}(t,y) = w_\pm(y)$$

uniformly with respect to $y \in \mathbb{R}^m$.

Let $T_+ > \sup_{y\in\mathbb{R}^m} t_N(y)$, $T_- < \inf_{y\in\mathbb{R}^m} t_{-N}(y)$ and take $0 < \delta < \delta_0$. From the roughness of exponential dichotomies (cfr. [7, Proposition 2, p. 34]) the linear systems

$$\dot{x} = f_{N+1,x}(u^+_{N+1}(t+T_+,y),y)x \tag{4.1}$$

and

$$\dot{x} = f_{N+1,x}(u^-_{N+1}(t+T_-,y),y)x \tag{4.2}$$

have an exponential dichotomy on $\mathbb{R}_+$, $\mathbb{R}_-$ resp., uniformly with respect to $y \in \mathbb{R}^m$, with projections $P_+(y)$, resp. $P_-(y)$, of rank k, constant K and exponent δ. Moreover, according to [8, Proposition 2.3], it can be assumed that, for $|y - y_0|$ sufficiently small it results: $\mathcal{N}P_+(y) = \mathcal{N}P_+(y_0)$, $\mathcal{R}P_-(y) = \mathcal{R}P_-(y_0)$ and in this case the projections are smooth with respect to y. Note that, $\mathcal{N}P_+(y) = \mathcal{N}P_+(y_0)$ and $\mathcal{R}P_-(y) = \mathcal{R}P_-(y_0)$ are equivalent to

$$\begin{aligned} &P_+(y) = P_+(y)P_+(y_0), \qquad P_+(y_0) = P_+(y_0)P_+(y) \\ &P_-(y) = P_-(y_0)P_-(y), \qquad P_-(y_0) = P_-(y)P_-(y_0). \end{aligned} \tag{4.3}$$

Let $U^\pm_i(t,y)$ be the fundamental matrix of

$$\dot{x} = f_{i,x}(u^\pm_i(t,y),y)x$$

in $\mathbb{R}_\pm$ resp., that is

$$\begin{aligned} &\dot{U}^\pm_i(t,y) = f_{i,x}(u^\pm_i(t,y),y)U^\pm_i(t,y), \quad \pm t \ge 0, \\ &U^\pm_i(0,y) = \mathbb{I}. \end{aligned}$$

As in [6] we see that

Lemma 4.1. *For any $\tau \in \mathbb{R}$ the linear system*

$$\dot{x} = f_{N+1,x}(u^+_{N+1}(t,y),y)x, \tag{4.4}$$

resp.

$$\dot{x} = f_{N+1,x}(u^-_{N+1}(t,y),y)x, \tag{4.5}$$

has an exponential dichotomy on $t \geq \tau$, resp. $t \leq \tau$, with exponent δ, constant $\tilde{K}$ independent on y and projections

$$\begin{aligned} Q_+(y) &= U^+_{N+1}(\tau,y)U^+_{N+1}(T_+,y)^{-1}P_+(y)U^+_{N+1}(T_+,y)U^+_{N+1}(\tau,y)^{-1} \\ Q_-(y) &= U^-_{N+1}(\tau,y)U^-_{N+1}(T_-,y)^{-1}P_-(y)U^-_{N+1}(T_-,y)U^-_{N+1}(\tau,y)^{-1}. \end{aligned}$$

In particular, if $\tau = T_+$, resp. $\tau = T_-$, then $Q_+(y) = P_+(y)$, resp. $Q_-(y) = P_-(y)$, and $\tilde{K} = K$.

Finally, the following result holds (see [6, Theorems 4.3, 4.5]).

Theorem 4.2. *There exist $\rho > 0$, bounded C^r-functions*

$$t^*_{-N}(\xi_-,\alpha,\varepsilon) < \ldots < t^*_{-1}(\xi_-,\alpha,\varepsilon) < t^*_0(\xi_-,\alpha,\varepsilon) = 0 < t^*_1(\xi_+,\alpha,\varepsilon) < \ldots < t^*_N(\xi_+,\alpha,\varepsilon)$$

such that, for all $i = 1,\ldots,N$,

$$\begin{aligned} \lim_{(\xi_+,\varepsilon)\to 0} |t^*_i(\xi_+,\alpha,\varepsilon) - t_i(\alpha)| &= 0, \\ \lim_{(\xi_-,\varepsilon)\to 0} |t^*_{-i}(\xi_-,\alpha,\varepsilon) - t_{-i}(\alpha)| &= 0 \end{aligned}$$

uniformly with respect to $\alpha \in \mathbb{R}^m$, and continuous, piecewise C^r, solutions of (1.1)

$$(x_\pm(t,\xi_\pm,\alpha,\varepsilon), y_\pm(t,\xi_\pm,\alpha,\varepsilon))$$

defined for $t \geq 0$ and $t \leq 0$ resp., and such that

$$\begin{aligned} c_{i-1} < h(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon)) &< c_i, && \text{for } t^*_{i-1}(\xi_+,\alpha,\varepsilon) < t < t^*_i(\xi_+,\alpha,\varepsilon), \\ h(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon)) &> c_N, && \text{for } t > t^*_N(\xi_+,\alpha,\varepsilon), \\ c_{i-1} < h(x_-(t,\xi_-,\alpha,\varepsilon),y_-(t,\xi_-,\alpha,\varepsilon)) &< c_i, && \text{for } t^*_{-i}(\xi_-,\alpha,\varepsilon) < t < t^*_{-i+1}(\xi_-,\alpha,\varepsilon), \\ h(x_-(t,\xi_-,\alpha,\varepsilon),y_-(t,\xi_-,\alpha,\varepsilon)) &> c_N, && \text{for } t < t^*_{-N}(\xi_-,\alpha,\varepsilon), \end{aligned}$$

$$\begin{aligned} h(x_+(t^*_i(\xi_+,\alpha,\varepsilon),\xi_+,\alpha,\varepsilon),y_+(t^*_i(\xi_+,\alpha,\varepsilon),\xi_+,\alpha,\varepsilon)) &= c_i, \\ h(x_-(t^*_{-i}(\xi_-,\alpha,\varepsilon),\xi_-,\alpha,\varepsilon),y_-(t^*_{-i}(\xi_-,\alpha,\varepsilon),\xi_-,\alpha,\varepsilon)) &= c_i, \\ \frac{\partial}{\partial t} h(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon))_{|t=t^*_i(\xi_+,\alpha,\varepsilon)} &> \eta, \\ \frac{\partial}{\partial t} h(x_-(t,\xi_-,\alpha,\varepsilon),y_-(t,\xi_-,\alpha,\varepsilon))_{|t=t^*_{-i}(\xi_-,\alpha,\varepsilon)} &< -\eta, \\ y_\pm(T_\pm,\xi_\pm,\alpha,\varepsilon) &= \alpha, \\ P_+(\alpha)[x(T_+) - u(T_+,\alpha)] &= \xi_+, \\ (\mathbb{I} - P_-(\alpha))[x(T_-) - u(T_-,\alpha)] &= \xi_- \end{aligned}$$

where $c_0 = -\infty$*. Moreover*

$$\begin{aligned} &\sup_{t\geq 0} |x_+(t,\xi_+,\alpha,\varepsilon) - u(t,y_+(t,\xi_+,\alpha,\varepsilon))| < \rho, \\ &\sup_{t\leq 0} |x_-(t,\xi_-,\alpha,\varepsilon) - u(t,y_-(t,\xi_-,\alpha,\varepsilon))| < \rho \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\sup_{t\geq 0} |x_+(t,\xi_+,\alpha,\varepsilon) - u(t,y_+(t,\xi_+,\alpha,\varepsilon))| \to 0 \quad \text{as } |\xi_+| + |\varepsilon| \to 0, \\ &\sup_{t\leq 0} |x_-(t,\xi_-,\alpha,\varepsilon) - u(t,y_-(t,\xi_-,\alpha,\varepsilon))| \to 0 \quad \text{as } |\xi_-| + |\varepsilon| \to 0 \end{aligned} \tag{4.7}$$

uniformly with respect to α *as well as*

$$\lim_{\varepsilon\to 0} y_\pm(0,\xi_\pm,\alpha,\varepsilon) = \alpha$$

uniformly with respect to $(\xi_\pm,\alpha)$*.*

Remark 4.3. According to Theorem 4.2 we have

$$\begin{aligned} h(x_+(t_i^*(\xi_+,\alpha,\varepsilon),\xi_+,\alpha,\varepsilon),y_+(t_i^*(\xi_+,\alpha,\varepsilon),\xi_+,\alpha,\varepsilon)) = c_i, \\ h(x_-(t_{-i}^*(\xi_-,\alpha,\varepsilon),\xi_-,\alpha,\varepsilon),y_-(t_{-i}^*(\xi_-,\alpha,\varepsilon),\xi_-,\alpha,\varepsilon)) = c_i. \end{aligned} \tag{4.8}$$

Differentiating (4.8) with respect to ξ_+, ξ_-, at $\varepsilon = 0$ we obtain a formula for the derivatives

$$\frac{\partial t_i^*}{\partial \xi_+}(\xi_+,\alpha,0), \quad \frac{\partial t_{-i}^*}{\partial \xi_-}(\xi_-,\alpha,0), \quad i = 1,\dots,N.$$

However we have to distinguish when $t \to t_i^*(\xi_+,\alpha,0)^+$ or $t \to t_i^*(\xi_+,\alpha,0)^-$ (resp. $t \to t_{-i}^*(\xi_-,\alpha,0)^+$ or $t \to t_{-i}^*(\xi_-,\alpha,0)^-$). For example if $t \to t_i^*(\xi_+,\alpha,0)^+$, $x_+(t,\xi_+,\alpha,0)$ is the solution of $\dot{x} = f_{i+1}(x,\alpha)$ and then, differentiating (4.8) with respect to ξ_+, we get, with $t_i^* = t_i^*(\xi_+,\alpha,0)$:

$$h_x(x_+(t_i^*,\xi_+,\alpha,0),\alpha)[f_{i+1}(x_+(t_i^*,\xi_+,\alpha,0),\alpha)\frac{\partial t_i^*}{\partial \xi_+}(\xi_+,\alpha,0) + x_{+,\xi_+}(t_i^{*+},\xi_+,\alpha,0)] = 0.$$

Vice versa, when $t \to t_i^*(\xi_+,\alpha,0)^-$, $x_+(t,\xi_+,\alpha,0)$ is the solution of $\dot{x} = f_i(x,\alpha)$ and then

$$h_x(x_+(t_i^*,\xi_+,\alpha,0),\alpha)[f_i(x_+(t_i^*,\xi_+,\alpha,0),\alpha)\frac{\partial t_i^*}{\partial \xi_+}(\xi_+,\alpha,0) + x_{+,\xi_+}(t_i^{*-},\xi_+,\alpha,0)] = 0.$$

Similarly we get, with $t_{-i}^* = t_{-i}^*(\xi_-,\alpha,0)$:

$$h_x(x_-(t_{-i}^*,\xi_-,\alpha,0),\alpha)[f_{i+1}(x_-(t_{-i}^*,\xi_-,\alpha,0),\alpha)\frac{\partial t_{-i}^*}{\partial \xi_-}(\xi_-,\alpha,0) + x_{-,\xi_-}(t_{-i}^{*-},\xi_-,\alpha,0)] = 0$$

and

$$h_x(x_-(t_{-i}^*,\xi_-,\alpha,0),\alpha)[f_i(x_+(t_{-i}^*,\xi_-,\alpha,0),\alpha)\frac{\partial t_{-i}^*}{\partial \xi_-}(\xi_-,\alpha,0) + x_{-,\xi_-}(t_{-i}^{*+},\xi_-,\alpha,0)] = 0.$$

We will use this remark in the next section.

5 The discontinuous variational equation

For any fixed $\alpha \in \mathbb{R}^m$ and $\ell = \pm 1, \ldots, \pm N$ we define linear operators $B_\ell(\alpha) : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$B_\ell(\alpha)x = x - \frac{h_x(u(t_\ell(\alpha),\alpha),\alpha)x}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot{u}(t_\ell(\alpha)^-,\alpha)}[\dot{u}(t_\ell(\alpha)^-,\alpha),\alpha) - \dot{u}(t_\ell(\alpha)^+,\alpha),\alpha)]. \tag{5.1}$$

The following result has been proved in [6, Propositioon 5.1, 5.2].

Proposition 5.1. *For any $\alpha \in \mathbb{R}^m$, $x \mapsto B_\ell(\alpha)x$ are invertible linear maps. Moreover $x_{+,\xi_+}(t,0,\alpha,0)$ is a solution of*

$$\begin{aligned} \dot{x} = A(t,\alpha)x &:= \begin{cases} f_{i,x}(u(t,\alpha),\alpha)x & \text{if } t_{i-1}(\alpha) \le t < t_i(\alpha), \\ & \quad i = 1,\ldots,N \\ f_{N+1,x}(u(t,\alpha),\alpha)x & \text{if } t \ge t_N(\alpha), \end{cases} \\ x(t_i(\alpha)^+) &= B_i(\alpha)x(t_i(\alpha)^-), \quad i = 1,\ldots,N \end{aligned} \tag{5.2}$$

which is C^1 for $t \neq t_i(\alpha)$, bounded for $t \ge 0$ and can be assumed to be right-continuous at $t = t_i(\alpha)$. Similarly $x_{-,\xi_-}(t,0,\alpha,0)$ is a solution of

$$\begin{aligned} \dot{x} = A(t,\alpha)x &:= \begin{cases} f_{i,x}(u(t,\alpha),\alpha)x & \text{if } t_{-i}(\alpha) < t \le t_{-i+1}(\alpha), \\ & \quad i = 1,\ldots,N \\ f_{N+1,x}(u(t,\alpha),\alpha)x & \text{if } t \le t_{-N}(\alpha) \end{cases} \\ x(t_{-i}(\alpha)^+) &= B_{-i}(\alpha)x(t_{-i}(\alpha)^-), \quad i = 1,\ldots,N \end{aligned} \tag{5.3}$$

which is C^1 for $t \neq t_{-i}(\alpha)$,bounded for $t \le 0$ and can be assumed to be left-continuous at $t = t_{-i}(\alpha)$. Finally, for $t \ge 0$, resp. $t \le 0$, the function

$$\dot{u}(t,\alpha) = \begin{cases} \dot{u}_i^+(t,\alpha) & \text{for } t_{i-1}(\alpha) \le t < t_i(\alpha), \\ & \quad i = 1,\ldots,N \\ \dot{u}_{N+1}^+(t,\alpha) & \text{for } t \ge T_N(\alpha) \end{cases}$$

resp.

$$\dot{u}(t,\alpha) = \begin{cases} \dot{u}_i^-(t,\alpha) & \text{for } t_{-i}(\alpha) < t \le t_{-i+1}(\alpha), \\ & \quad i = 1,\ldots,N \\ \dot{u}_{N+1}^-(t,\alpha) & \text{for } t \le T_{-N}(\alpha) \end{cases}$$

is a solution of (5.2) *(resp.* (5.3)*) bounded on $t \ge 0$ (resp, $t \le 0$).*

6 Main result

First we recall that $P_+(y)$ is the projections of the exponential dichotomy on $t \ge 0$, of the linear system (4.1) with constant K and exponent δ. Then, from Lemma 4.1, we see that (4.4) has an exponential dichotomy on $t \ge t_N(y)$ with exponent δ and projection

$$U_{N+1}^+(t_N(y),y)U_{N+1}^+(T_+,y)^{-1}P_+(y)U_{N+1}^+(T_+,y)U_{N+1}^+(t_N(y),y)^{-1}.$$

Similarly, the linear system (4.5) has an exponential dichotomy on $t \le t_{-N}(y)$ with exponent δ and projection

$$U_{N+1}^-(t_{-N}(y),y)U_{N+1}^-(T_-,y)^{-1}P_-(y)U_{N+1}^-(T_-,y)U_{N+1}^-(t_{-N}(y),y)^{-1}.$$

From Lemma 3.3–3.4 we obtain the following

Proposition 6.1. *For any $\alpha \in \mathbb{R}^m$, the discontinuous linear system (5.2), resp. (5.3), has an exponential dichotomy on $\mathbb{R}_+$, resp. $\mathbb{R}_-$, with projections $Q_+(\alpha)$, resp. $Q_-(\alpha)$, given by*

$$\begin{aligned} Q_+(\alpha) &= X_+(t_N(\alpha)^+,\alpha)^{-1}U^+_{N+1}(t_N(\alpha),\alpha)U^+_{N+1}(T_+,\alpha)^{-1} \\ &\quad \cdot P_+(\alpha)U^+_{N+1}(T_+,\alpha)U^+_{N+1}(t_N(\alpha),\alpha)^{-1}X_+(t_N(\alpha)^+,\alpha) \\ Q_-(\alpha) &= X_-(t_{-N}(\alpha)^-,\alpha)^{-1}U^-_{N+1}(t_{-N}(\alpha),\alpha)U^-_{N+1}(T_-,\alpha)^{-1} \\ &\quad \cdot P_-(\alpha)U^-_{N+1}(T_-,\alpha)U^-_{N+1}(t_{-N}(\alpha),\alpha)^{-1}X_-(t_{-N}(\alpha)^-,\alpha) \end{aligned}$$

where

$$\begin{aligned} X_+(t_N^+(\alpha),\alpha) &= B_N(\alpha)U^+_N(t_N(\alpha))U^+_N(t_{N-1}(\alpha),\alpha)^{-1}\dots B_1(\alpha)U^+_1(t_1(\alpha),\alpha) \\ X_-(t_{-N}(\alpha)^-,\alpha) &= \ B_{-N}(\alpha)^{-1}U^-_N(t_{-N}(\alpha))U^-_N(t_{-N+1}(\alpha),\alpha)^{-1}\dots B_{-1}(\alpha)^{-1}U^-_1(t_{-1}(\alpha),\alpha). \end{aligned}$$

Moreover $\mathcal{R}Q_+(\alpha)$ (resp. $\mathcal{N}Q_-(\alpha)$) is the space of initial conditions of solutions of (5.2), resp. (5.3), right-continuous, when $t \geq 0$ (resp. left-continuous, when $t \leq 0$) and bounded on $\mathbb{R}_+$, (resp, on $\mathbb{R}_-$).

We assume the following condition holds:

A_5) For any $\alpha \in \mathbb{R}^m$, $\dim[\mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha)] = d \leq m$.

From Proposition 5.1 we know that $\dot{u}(0,\alpha) \in \mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha)$ so

$$1 \leq \dim[\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp = d.$$

Next, from A_3) it follows that $\dim \mathcal{R}Q_+(\alpha) = k$ and $\dim \mathcal{N}Q_-(\alpha) = n-k$, hence $d \leq \min\{k, n-k\}$.

Let $\psi_1(\alpha),\dots,\psi_d(\alpha) \in \mathbb{R}^n$ be such that

$$[\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp = \operatorname{span}\{\psi_1(\alpha),\dots,\psi_d(\alpha)\}.$$

Without loss of generality we assume that $\{\psi_1(\alpha),\dots,\psi_d(\alpha)\}$ is an orthonormal set.

The purpose of this section is to prove the following

Theorem 6.2. *Suppose that A_1)–A_5) hold. Suppose further that there exists $\alpha_0 \in \mathbb{R}^m$ such that the vector function*

$$M(\alpha) := \left(\int_{-\infty}^{\infty} \psi_j(\alpha)^T G(t,\alpha) u_y(t,\alpha) g(u(t,\alpha),\alpha,0)dt \right)_{j=1,\dots,d}$$

where

$$G(t,\alpha) = \begin{cases} Q_-(\alpha)X_-(t,\alpha)^{-1} & \text{if } t \leq 0, \\ (\mathbb{I} - Q_+(\alpha))X_+(t,\alpha)^{-1} & \text{if } t \geq 0 \end{cases}$$

satisfies $M(\alpha_0) = 0$ and $\operatorname{rank} M'(\alpha_0) = d$. Then there exists $\rho > 0$ and $\varepsilon_0 > 0$ such that for $0 \leq \varepsilon \leq \varepsilon_0$ system (1.1) has a $(m-d)$-dimensional manifold of continuous, piecewise C^r solutions $(x(t), y(t))$ such that

$$\sup_{t\in\mathbb{R}} |x(t) - u(t, y(t)) < \rho,$$
$$\sup_{t\in\mathbb{R}} |x(t) - u(t, y(t))| \to 0$$

as $\varepsilon \to 0$.

Proof. Arguing as in [6, Theorem 6.2] we know that

$$
\begin{aligned}
x_+(t,\xi_+,\alpha,\varepsilon) = u(t,y_+(t,\xi_+,\alpha,\varepsilon)) &+ X_+(t,\alpha)\tilde{\xi}_+ \\
&+ \int_0^t X_+(t,\alpha)Q_+(\alpha)X_+(s,\alpha)^{-1}b_+(s)ds \\
&s - \int_t^\infty X_+(t,\alpha)(\mathbb{I}-Q_+(\alpha))X_+(s,\alpha)^{-1}b_+(s)ds
\end{aligned}
\tag{6.1}
$$

where

$$
\begin{aligned}
b_+(t) &= b_+(t,\xi_+,\alpha,\varepsilon) \\
&:= f(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon)) - f(u(t,y_+(t,\xi_+,\alpha,\varepsilon)),y_+(t,\xi_+,\alpha,\varepsilon)) \\
&\quad - \mathcal{A}(t,\alpha)[x_+(t,\xi_+,\alpha,\varepsilon) - u(t,y_+(t,\xi_+,\alpha,\varepsilon))] \\
&\quad - \varepsilon u_y(t,y_+(t,\xi_+,\alpha,\varepsilon))g(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon),\varepsilon)
\end{aligned}
$$

and

$$\tilde{\xi}_+ = Q_+(\alpha)[x_+(0,\xi_+,\alpha,\varepsilon) - u(0,y_+(0,\xi_+,\alpha,\varepsilon))] \in \mathcal{R}Q_+(\alpha).$$

Moreover, for ε sufficiently small, the map $(\xi_+,\alpha) \mapsto (\tilde{\xi}_+, y_+(0,\xi_+,\alpha,\varepsilon))$ from $\mathcal{R}P_+(\alpha) \times \mathbb{R}^m$ into $\mathcal{R}Q_+(\alpha) \times \mathbb{R}^m$ is linearly invertible.

Similarly, for $|\alpha_- - y_0|$ sufficiently small we have

$$
\begin{aligned}
x_-(t,\xi_-,\alpha_-,\varepsilon) = u(t,y_-(t,\xi_-,\alpha_-,\varepsilon)) &+ X_-(t,\alpha_-)\tilde{\xi}_- \\
&+ \int_{-\infty}^t X_-(t,\alpha_-)Q_-(\alpha_-)X_-(s,\alpha_-)^{-1}b_-(s)ds \\
&- \int_t^0 X_-(t,\alpha_-)(\mathbb{I}-Q_-(\alpha_-))X_-(s,\alpha_-)^{-1}b_-(s)ds
\end{aligned}
\tag{6.2}
$$

where

$$
\begin{aligned}
b_-(t) &= b_-(t,\xi_-,\alpha_-,\varepsilon) \\
&:= f(x_-(t,\xi_-,\alpha_-,\varepsilon),y_-(t,\xi_-,\alpha_-,\varepsilon)) - f(u(t,y_-(t,\xi_-,\alpha_-,\varepsilon)),y_-(t,\xi_-,\alpha_-,\varepsilon)) \\
&\quad - \mathcal{A}(t,\alpha_-)[x_-(t,\xi_-,\alpha_-,\varepsilon) - u(t,y_-(t,\xi_-,\alpha_-,\varepsilon))] \\
&\quad - \varepsilon u_y(t,y_-(t,\xi_-,\alpha_-,\varepsilon))g(x_-(t,\xi_-,\alpha_-,\varepsilon),y_-(t,\xi_-,\alpha_-,\varepsilon),\varepsilon)
\end{aligned}
$$

and

$$\tilde{\xi}_- = [\mathbb{I} - Q_-(\alpha_-)][x_-(0,\xi_-,\alpha_-,\varepsilon) - u(0,y_-(0,\xi_+,\alpha_-,\varepsilon))] \in \mathcal{N}Q_-(\alpha_-).$$

Moreover, for ε sufficiently small, the map $(\xi_-,\alpha_-) \mapsto (\tilde{\xi}_-, y_-(0,\xi_-,\alpha_-,\varepsilon))$ from $\mathcal{N}P_-(\alpha_-) \times \mathbb{R}^m$ into $\mathcal{N}Q_-(\alpha_-) \times \mathbb{R}^m$ is linearly invertible.

From (6.1)-(6.2) we get, for $|\alpha - y_0| + |\alpha_- - y_0|$ sufficiently small

$$
\begin{aligned}
&x_+(0,\xi_+,\alpha,\varepsilon) - x_-(0,\xi_-,\alpha_-,\varepsilon) \\
&\quad = u(0,y_+(0,\xi_+,\alpha,\varepsilon)) - u(0,y_-(0,\xi_-,\alpha_-,\varepsilon)) + \tilde{\xi}_+ - \tilde{\xi}_- \\
&\qquad - \int_0^\infty (\mathbb{I}-Q_+(\alpha))X_+(s,\alpha)^{-1}b_+(s)ds - \int_{-\infty}^0 Q_-(\alpha_-)X_-(s,\alpha_-)^{-1}b_-(s)ds.
\end{aligned}
\tag{6.3}
$$

Then the system

$$
\begin{cases}
x_+(0,\xi_+,\alpha,\varepsilon) = x_-(0,\xi_-,\alpha_-,\varepsilon), \\
y_+(0,\xi_+,\alpha,\varepsilon) = y_-(0,\xi_-,\alpha_-,\varepsilon)
\end{cases}
$$

is equivalent to

$$\begin{cases} \tilde{\xi}_+ - \tilde{\xi}_- = k(\xi_+, \xi_-, \alpha, \alpha_-, \varepsilon), \\ y_-(0, \xi_-, \alpha_-, \varepsilon) - y_+(0, \xi_+, \alpha, \varepsilon) = 0 \end{cases} \tag{6.4}$$

where

$$k(\xi_+, \xi_-, \alpha, \alpha_-, \varepsilon) = \int_0^\infty (\mathbb{I} - Q_+(\alpha))X_+(s,\alpha)^{-1} b_+(s)ds + \int_{-\infty}^0 Q_-(\alpha_-)X_-(s,\alpha_-)^{-1} b_-(s)ds.$$

Differentiating $b_+(t) = b_+(t, \xi_+, \alpha, \varepsilon)$ with respect to ξ_+ at $\xi_+ = 0$, $\varepsilon = 0$ and also using $x_+(t, 0, \alpha, 0) = u(t, \alpha)$, $y_+(t, 0, \alpha, 0) = \alpha$, we see that, for $t_{i-1}(\alpha) < t < t_i(\alpha)$, we have

$$\tfrac{\partial b_+}{\partial \xi_+}(t, 0, \alpha, 0) = [f_{i,x}(u(t,\alpha), \alpha) - A(t,\alpha)]x_{+,\xi_+}(t, 0, \alpha, 0) = 0$$

and for $t > t_N(\alpha)$:

$$\tfrac{\partial b_+}{\partial \xi_+}(t, 0, \alpha, 0) = [f_{N+1,x}(u(t,\alpha), \alpha) - A(t,\alpha)]x_{+,\xi_+}(t, 0, \alpha, 0) = 0.$$

Then

$$\frac{\partial}{\partial \xi_+} \left[\int_0^\infty (\mathbb{I} - Q_+(\alpha))X_+(s,\alpha)^{-1} b_+(s)ds \right]_{\xi_+=0, \varepsilon=0} = 0.$$

Similarly we get, for $|\alpha_- - y_0|$ sufficiently small,

$$\frac{\partial}{\partial \xi_-} \left[\int_{-\infty}^0 Q_-(\alpha_-)X_-(s,\alpha_-)^{-1} b_-(s)ds \right]_{\xi_-=0, \varepsilon=0} = 0.$$

As a consequence (6.4) reads:

$$\begin{aligned} \tilde{\xi}_+ - \tilde{\xi}_- &= R_1(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon), \\ \alpha_- - \alpha &= R_2(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon) \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} R_1(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon) &= k(\xi_+, \xi_-, \alpha, \alpha_-, \varepsilon) \\ \tilde{\xi}_+ &= Q_+(\alpha)[x_+(0, \xi_+, \alpha, \varepsilon) - u(0, y_+(0, \xi_+, \alpha, \varepsilon))] \\ \tilde{\xi}_- &= [\mathbb{I} - Q_-(\alpha_-)][x_-(0, \xi_-, \alpha_-, \varepsilon) - u(0, y_-(0, \xi_-, \alpha_-, \varepsilon))]. \end{aligned}$$

Note that, being $(\xi_+, \xi_-) \mapsto (\tilde{\xi}_+, \tilde{\xi}_-)$ linearly invertible, we derive: $R_1(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\tilde{\xi}_+|^2 + |\tilde{\xi}_-|^2 + |\varepsilon|)$ and $R_2(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\varepsilon|)$, uniformly with respect to (α, α_-).

Now, as $\tilde{\xi}_- \in \mathcal{N}Q_-(\alpha_-)$, we have

$$(\mathbb{I} - Q_-(\alpha))\tilde{\xi}_- = \tilde{\xi}_- - (Q_-(\alpha) - Q_-(\alpha_-))\tilde{\xi}_-$$

and hence

$$\frac{1}{2}|\tilde{\xi}_-| \le |(\mathbb{I} - Q_-(\alpha))\tilde{\xi}_-| \le 2|\tilde{\xi}_-|$$

provided $|\alpha_- - y_0|$ and $|\alpha - y_0|$ are sufficiently small. Hence the map $\tilde{\xi}_- \mapsto (\mathbb{I} - Q_-(\alpha))\tilde{\xi}_-$ from $\mathcal{N}Q_-(\alpha_-)$ into $\mathcal{N}Q_-(\alpha)$ is linearly invertible. Then, setting

$$\bar{\xi}_+ = \tilde{\xi}_+, \quad \bar{\xi}_- = (\mathbb{I} - Q_-(\alpha))\tilde{\xi}_-, \tag{6.6}$$

(6.5) can be written as

$$\begin{aligned} \bar{\xi}_+ - \bar{\xi}_- &= \bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon), \\ \alpha_- - \alpha &= \bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) \end{aligned} \tag{6.7}$$

with

$$\begin{aligned}(\bar{\xi}_+, \bar{\xi}_-) &\in \mathcal{R}Q_+(\alpha) \times \mathcal{N}Q_-(\alpha),\\ \bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) &= R_1(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon),\\ \bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha_+, \alpha_-, \varepsilon) &= R_2(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon)\end{aligned}$$

where $(\tilde{\xi}_+, \tilde{\xi}_-)$ is the point corresponding to $(\bar{\xi}_+, \bar{\xi}_-)$ through (6.6). Note that it holds $\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\bar{\xi}_+|^2 + |\bar{\xi}_-|^2 + |\varepsilon|)$, $\bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\varepsilon|)$ uniformly with respect to (α, α_-).

Let α, α_- be such that $|\alpha - y_0|$ and $|\alpha_- - y_0|$ are sufficiently small. The map $(\bar{\xi}_+, \bar{\xi}_-) \mapsto \bar{\xi}_+ - \bar{\xi}_-$ is a linear map from $\mathcal{R}Q_+(\alpha) \times \mathcal{N}Q_-(\alpha)$ into $\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)$ whose kernel is $\mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha)$ which, by assumption A_5), is d-dimensional.

Let $W(\alpha) \subset \mathcal{R}Q_+(\alpha)$ be a complement of $\mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha)$ in $\mathcal{R}Q_+(\alpha)$, so that

$$\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha) = W(\alpha) \oplus \mathcal{N}Q_-(\alpha).$$

Note that $\dim W(\alpha) = k - d$ and

$$\mathbb{R}^n = [\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)] \oplus \operatorname{span}\{\psi_1(\alpha), \dots, \psi_d(\alpha)\}.$$

Next, let $Q(\alpha) : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection such that $\mathcal{R}Q(\alpha) = \mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)$ and $\mathcal{N}Q(\alpha) = \operatorname{span}\{\psi_1(\alpha), \dots, \psi_d(\alpha)\}$. Since

$$(\mathbb{I} - Q(\alpha))x \in \mathcal{N}Q(\alpha) = \operatorname{span}\{\psi_1(\alpha), \dots, \psi_d(\alpha)\}$$

and $(\psi_1(\alpha), \dots, \psi_d(\alpha))$ is orthonormal we get

$$\begin{aligned}(\mathbb{I} - Q(\alpha))x &= \sum_{j-1}^{d} \langle \psi_j(\alpha), (\mathbb{I} - Q(\alpha))x \rangle \psi_j(\alpha)\\ &= \sum_{j-1}^{d} \langle (\mathbb{I} - Q(\alpha))\psi_j(\alpha), x \rangle \psi_j(\alpha) = \sum_{j-1}^{d} (\psi_j(\alpha)^T x)\psi_j(\alpha).\end{aligned}$$

Hence we replace (6.7) with

$$\begin{aligned}\bar{\xi}_+ - \bar{\xi}_- &= Q(\alpha)\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon),\\ \alpha - \alpha_- &= \bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon),\\ \psi_j^T(\alpha)\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) &= 0.\end{aligned} \tag{6.8}$$

We solve (6.8) for $(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-) \in W(\alpha) \times \mathcal{N}Q_-(\alpha) \times \mathbb{R}^m \times \mathbb{R}^m$ in terms of ε.

Since $\dim[\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)] = n - d$, we see that for any fixed ε

$$\begin{aligned}\bar{\xi}_+ - \bar{\xi}_- &= Q(\alpha)\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon)],\\ \alpha - \alpha_- &= \bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon)\end{aligned} \tag{6.9}$$

is essentially a system of $n - d + m$ equations in the $n - d + 2m$ variables $(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-)$ such that, when $\varepsilon = 0$, has the solution

$$(\bar{\xi}_+, \bar{\xi}_-) = (0, 0), \qquad \alpha_- = \alpha.$$

The Jacobian matrix at this point is

$$J = \begin{pmatrix} L & 0 & 0 \\ 0 & \mathbb{I}_{\mathbb{R}^m} & -\mathbb{I}_{\mathbb{R}^m} \end{pmatrix}$$

where $L : W \times \mathcal{N}Q_-(\alpha) \to W \oplus \mathcal{N}Q_-(\alpha)$ is the invertible linear map given by $L(\bar{\xi}_+, \bar{\xi}_-) = \bar{\xi}_+ - \bar{\xi}_-$. We have

$$\operatorname{rank} J = n - d + m$$

hence, for $\varepsilon \neq 0$ and sufficiently small (6.9) has a m-dimensional C^r-manifold of solutions

$$\bar{\xi}_+ = \bar{\xi}_+(\alpha, \varepsilon), \quad \bar{\xi}_- = \bar{\xi}_-(\alpha, \varepsilon), \quad \alpha_- = \alpha_-(\alpha, \varepsilon)$$

where

$$\begin{aligned} |\bar{\xi}_\pm(\alpha, \varepsilon)| &= O(|\varepsilon|), \\ |\alpha_-(\alpha, \varepsilon) - \alpha| &= O(|\varepsilon|) \end{aligned} \tag{6.10}$$

uniformly with respect to α. Plugging this solution in the third equation in (6.8) we obtain the system of equations

$$\psi_j^T(\alpha)\bar{R}_1(\bar{\xi}_+(\alpha, \varepsilon), \bar{\xi}_-(\alpha, \varepsilon), \alpha, \alpha_-(\alpha, \varepsilon), \varepsilon) = 0, \quad j = 1, \ldots, d.$$

As $\bar{R}_1(0, 0, \alpha, \alpha, 0) = 0$ we see that this equation is identically satisfied when $\varepsilon = 0$, so we replace it with

$$\mathcal{M}(\alpha, \varepsilon) = 0$$

where $\mathcal{M}(\alpha, \varepsilon)$ is the C^{r-1}-function:

$$\mathcal{M}(\alpha, \varepsilon) = \begin{cases} \varepsilon^{-1} \left(\psi_j^T(\alpha)[\bar{R}_1(\bar{\xi}_+(\alpha, \varepsilon), \bar{\xi}_-(\alpha, \varepsilon), \alpha, \alpha_-(\alpha, \varepsilon), \varepsilon) \right)_{j=1,\ldots,d} & \text{for } \varepsilon \neq 0, \\ \left[\left(\frac{\partial}{\partial \varepsilon} \psi_j^T(\alpha)\bar{R}_1(\bar{\xi}_+(\alpha, \varepsilon), \bar{\xi}_-(\alpha, \varepsilon), \alpha, \alpha_-(\alpha, \varepsilon), \varepsilon) \right)_{j=1,\ldots,d} \right]_{|\varepsilon=0} & \text{for } \varepsilon = 0. \end{cases}$$

We have already observed that $\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\bar{\xi}_+|^2 + |\bar{\xi}_-|^2 + |\varepsilon|)$ uniformly with respect to (α, α_-), then

$$\mathcal{M}(\alpha, 0) = \left(\psi_j^T(\alpha)\bar{R}_{1,\varepsilon}(0, 0, \alpha, \alpha, 0)] \right)_{j=1,\ldots,d}.$$

We now compute $\bar{R}_{1,\varepsilon}(0, 0, \alpha, \alpha, 0)$. Since the map $(\tilde{\xi}_+, \tilde{\xi}_-) \mapsto (\bar{\xi}_+, \bar{\xi}_-)$ where $\bar{\xi}_+ = \tilde{\xi}_+$ and $\bar{\xi}_- = (\mathbb{I} - Q(\alpha))\tilde{\xi}_-$ is a linear isomorphism we see that

$$\bar{R}_1(0, 0, \alpha, \alpha, \varepsilon) = k(0, 0, \alpha, \alpha, \varepsilon)$$

and hence

$$\begin{aligned} \bar{R}_{1\varepsilon}(0, 0, \alpha, \alpha, 0) &= k_\varepsilon(0, 0, \alpha, \alpha, 0) \\ &= \int_{-\infty}^0 Q_-(\alpha)X_-(t, \alpha)^{-1}\frac{\partial b_-}{\partial \varepsilon}(t, 0, 0, \alpha, \alpha, 0)dt \\ &\quad + \int_0^\infty (\mathbb{I} - Q_+(\alpha))X_+(t, \alpha)^{-1}\frac{\partial b_+}{\partial \varepsilon}(t, 0, 0, \alpha, \alpha, 0)dt \end{aligned}$$

that is

$$\mathcal{M}(\alpha,0) = \left(\int_{-\infty}^{\infty} \psi_j^T(t,\alpha) b_\varepsilon(t,0,0,\alpha,\alpha,0)dt\right)_{j=1,\dots,d}$$

where

$$\psi_j^T(t,\alpha) = \begin{cases} \psi_j^T(\alpha)Q_-(\alpha)X_-(t,\alpha)^{-1} & \text{if } t<0, \\ \psi_j^T(\alpha)(\mathbb{I}-Q_+(\alpha))X_+(t,\alpha)^{-1} & \text{if } t \geq 0 \end{cases} \tag{6.11}$$

and

$$b_\varepsilon(t,0,0,\alpha,\alpha,0) = \begin{cases} \frac{\partial b_-}{\partial \varepsilon}(t,0,0,\alpha,\alpha,0) & \text{if } t<0, \\ \frac{\partial b_+}{\partial \varepsilon}(t,0,0,\alpha,\alpha,0) & \text{if } t \geq 0. \end{cases}$$

Now, it is easy to check that

$$\begin{cases} \frac{\partial b_-}{\partial \varepsilon}(t,0,0,\alpha,\alpha,0) & \text{if } t<0, \\ \frac{\partial b_+}{\partial \varepsilon}(t,0,0,\alpha,\alpha,0) & \text{if } t \geq 0 \end{cases} = -u_y(t,\alpha)g(u(t,\alpha),\alpha,0).$$

Hence

$$\mathcal{M}(\alpha,0) = -\left(\int_{-\infty}^{\infty} \psi_j^T(t,\alpha)u_y(t,\alpha)g(u(t,\alpha),\alpha,0)dt\right)_{j=1,\dots,d} = -M(\alpha). \tag{6.12}$$

The thesis follows now from the Implicit Function Theorem. □

Remark 6.3. i) The orthonormal basis $(\psi_1(\alpha),\dots,\psi_d(\alpha))$ of $[\mathcal{R}Q_+(\alpha)+\mathcal{N}Q_-(\alpha)]^\perp$ can be replaced by any independent set $(\tilde\psi_1(\alpha),\dots,\tilde\psi_d(\alpha))$ such that

$$\mathbb{R}^n = [\mathcal{R}Q_+(\alpha)+\mathcal{N}Q_-(\alpha)] \oplus \operatorname{span}\{\tilde\psi_1(\alpha),\dots,\tilde\psi_d(\alpha)\}.$$

Indeed, let $\langle\cdot,\cdot\rangle$ be a scalar product on $\mathbb{R}^n$ such that

$$[\mathcal{R}Q_+(\alpha)+\mathcal{N}Q_-(\alpha)]^\perp = \operatorname{span}\{\tilde\psi_1(\alpha),\dots,\tilde\psi_d(\alpha)\}$$

and let $(\psi_1(\alpha),\dots,\psi_d(\alpha))$ be an orthonormal basis of $\operatorname{span}\{\tilde\psi_1(\alpha),\dots,\tilde\psi_d(\alpha)\}$. Then a smooth, invertible $d\times d$ matrix $N(\alpha)$ exists such that

$$(\tilde\psi_1(\alpha)\dots\tilde\psi_d(\alpha)) = (\psi_1(\alpha)\dots\psi_d(\alpha))N(\alpha).$$

Set

$$\tilde M(\alpha) = \left[\int_{-\infty}^{\infty} \tilde\psi_j^T(t,\alpha)u_y(t,\alpha)g(u(t,\alpha),\alpha,0)dt\right]_{j=1,\dots,d}.$$

We have

$$\begin{aligned} &[\tilde\psi_j(\alpha)^T u_y(t,\alpha)g(u(t,\alpha),\alpha,0)]_{j=1,\dots,d} \\ &\quad = (\tilde\psi_1(\alpha)\dots\tilde\psi_d(\alpha))^T[u_y(t,\alpha)g(u(t,\alpha),\alpha,0)] \\ &\quad = N(\alpha)^T(\psi_1(\alpha)\dots\psi_d(\alpha))^T u_y(t,\alpha)g(u(t,\alpha),\alpha,0) \\ &\quad = N(\alpha)^T[\psi_j(\alpha)^T[u_y(t,\alpha)g(u(t,\alpha),\alpha,0)]_{j=1,\dots,d} \end{aligned}$$

that is

$$\tilde M(\alpha) = N(\alpha)^T M(\alpha).$$

Now, assuming that $M(\alpha_0)=0$ and $\operatorname{rank} M'(\alpha_0) = d$, we see that $\tilde M(\alpha_0) = N(\alpha_0)^T M(\alpha_0) = 0$ and

$$\tilde M'(\alpha_0) = N(\alpha_0)^T M'(\alpha_0).$$

So $\tilde{M}(\alpha_0) = 0$ and $\operatorname{rank} \tilde{M}'(\alpha_0) = d$. The vice versa is proved in the same way using the equality

$$M(\alpha) = [N(\alpha)^T]^{-1} \tilde{M}(\alpha).$$

ii) The adjoint system to (5.2) and (5.3) is given by [1]

$$\begin{aligned} \dot{w} &= -A^T(t,\alpha)w \quad \text{if } t \geq 0, \\ w(t_i(\alpha)^+) &= (B_i^*(\alpha)^T)^{-1} w(t_i(\alpha)^-), \\ w(t_{-i}(\alpha)^+) &= (B_{*,i}(\alpha)^T)^{-1} w(t_{-i}(\alpha)^-). \end{aligned} \tag{6.13}$$

It is easy to check that, if $\psi(\alpha) \in [\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp$, the function $\psi(t,\alpha)$ defined in (6.11) is a bounded solution of (6.13). We prove that if

$$\operatorname{span}\{\psi_1(\alpha), \dots, \psi_d(\alpha)\} = [\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp$$

then $\{\psi_1(t,\alpha), \dots, \psi_d(t,\alpha)\}$ is a basis for the space of the bounded solutions of (6.13). Indeed, the fundamental matrix of (6.13) on $t \geq 0$ is $[X_+(t,\alpha))^T]^{-1}$, and the fundamental matrix of (6.13) on $t \leq 0$ is $[X_-(t,\alpha))^T]^{-1}$. As a consequence (6.13) has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections $(\mathbb{I} - Q_+^T)$ and $(\mathbb{I} - Q_-^T)$ respectively. So, the space of bounded solutions of (6.13), C^1 for $t \neq t_{\pm i}(\alpha)$, are those whose initial condition belongs to

$$\mathcal{R}(\mathbb{I} - Q_+^T(\alpha)) \cap \mathcal{N}(\mathbb{I} - Q_-^T(\alpha)) = (\mathcal{R}Q_+ v)^\perp \cap (\mathcal{N}Q_-(\alpha))^\perp = (\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha))^\perp.$$

Then the dimension of the space of solutions of (6.13), bounded on $\mathbb{R}$, is d and vectors $\{\psi_1(t,\alpha), \dots, \psi_d(t,\alpha)\}$ span this space.

As in [5,9] we see that if $x(t,\alpha)$ and $\psi(t,\alpha)$ are bounded solutions on $\mathbb{R}$ of (5.2)–(5.3) and (6.13) resp., both continuous for $t \neq t_{\pm i}(\alpha)$ then $\psi(t,\alpha)^T x(t,\alpha)$ is constant on $\mathbb{R}$.

7 An example

The simplest case of application of Theorem 6.2 is when $d = 1$ that is when

$$\mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha) = \operatorname{span}\{\dot{u}(0,\alpha)\}.$$

This condition is trivially satisfied when $n = 2$ since in this case $k = n - k = 1$. Moreover, when $n = 2$, we also have $\dim \mathcal{R}Q_+(\alpha) = \dim \mathcal{N}Q_-(\alpha) = 1$ and hence

$$\mathcal{R}Q_+(\alpha) = \mathcal{N}Q_-(\alpha) = \operatorname{span}\{\dot{u}(0,\alpha)\}. \tag{7.1}$$

In this section we consider examples of applications of Theorem 6.2 with $n = 2$, $m = 1$ and $d = 1$. Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The following result that has been proved in [6]:

Proposition 7.1. *Consider the system in* $\mathbb{R}^3$*:*

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, y), \\ \dot{x}_2 &= F_2(x_1, x_2, y), \\ \dot{y} &= \varepsilon g(x_1, x_2, y). \end{aligned} \tag{7.2}$$

and suppose that for any $\alpha \in \mathbb{R}$, the unperturbed equation

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, \alpha), \\ \dot{x}_2 &= F_2(x_1, x_2, \alpha) \end{aligned} \tag{7.3}$$

has a solution $u(t,\alpha)$ satisfying assumptions $A_1) - A_5)$. Let

$$A(t,\alpha) = [a_{jk}(t,\alpha)]_{1\le j,k\le 2} := [F_{j,x_k}(u_1(t), u_2(t), \alpha)]_{1\le j,k\le 2},$$

$B_i(\alpha)$ as in (5.1) and

$$v(t,\alpha) := e^{-\int_0^t a_{11}(s,\alpha)+a_{22}(s,\alpha)ds} J\dot{u}(t,y_0) = e^{-\int_0^t a_{11}(s,\alpha)+a_{22}(s,\alpha)ds} \begin{pmatrix} -\dot{u}_2(t,\alpha) \\ \dot{u}_1(t,\alpha) \end{pmatrix}. \tag{7.4}$$

Then the space of bounded solution of the adjoint variational system are of the form

$$\psi(t,\alpha) = \begin{cases} \mu_{-N}(\alpha)v(t,\alpha) & \text{for } t \le t_{-N}(\alpha), \\ \mu_{-i}(\alpha)v(t,\alpha), & \text{for } t_{-i-1}(\alpha) < t \le t_{-i}(\alpha), \\ \mu_i(\alpha)v(t,\alpha), & \text{for } t_i(\alpha) \le t < t_{i+1}(\alpha), \\ \mu_N(\alpha)v(t,\alpha) & \text{for } t \ge t_N(\alpha) \end{cases}$$

where $\mu_{-N}(\alpha) \neq 0$ is arbitrary and, for any $i = 1,\dots,N$,

$$\begin{aligned} \mu_{-i+1}(\alpha)v(t_{-i}(\alpha)^+,\alpha) &= \mu_{-i}(\alpha)[B_{-i}(\alpha)^T]^{-1}v(t_{-i}(\alpha)^-,\alpha), \\ \mu_i(\alpha)v(t_i(\alpha)^+,\alpha) &= \mu_{i-1}(\alpha)[B_i(\alpha)^T]^{-1}v(t_i(\alpha)^-,\alpha). \end{aligned} \tag{7.5}$$

Remark 7.2. i) From (7.5) we have

$$\begin{aligned} \mu_{-i+1}(\alpha)J\dot{u}(t_{-i}(\alpha)^+,\alpha)) &= \mu_{-i}(\alpha)[B_{-i}^T(\alpha)]^{-1}J\dot{u}(t_{-i}(\alpha)^-,\alpha)), \\ \mu_i(\alpha)J\dot{u}(t_i(\alpha)^+,\alpha)) &= \mu_{i-1}(\alpha)[B_i^T(\alpha)]^{-1}J\dot{u}(t_i(\alpha)^-,\alpha)) \end{aligned}$$

and then

$$\begin{aligned} \mu_i(\alpha)\|\dot{u}(t_i(\alpha)^+,\alpha))\|^2 &= \mu_i(\alpha)\langle J\dot{u}(t_i(\alpha)^+,\alpha)), J\dot{u}(t_i(\alpha)^+,\alpha))\rangle, \\ &= \mu_{i-1}(\alpha)\langle [B_i^T(\alpha)]^{-1}J\dot{u}(t_i(\alpha)^-,\alpha)), J\dot{u}(t_i(\alpha)^+,\alpha))\rangle \end{aligned}$$

and similarly

$$\begin{aligned} \mu_{-i+1}(\alpha)\|\dot{u}(t_{-i}(\alpha)^+,\alpha))\|^2 &= \mu_{-i}(\alpha)\langle J\dot{u}(t_{-i}(\alpha)^+,\alpha)), J\dot{u}(t_{-i}(\alpha)^+,\alpha))\rangle, \\ &= \mu_{-i}(\alpha)\langle [B_{-i}^T(\alpha)]^{-1}J\dot{u}(t_{-i}(\alpha)^-,\alpha)), J\dot{u}(t_{-i}(\alpha)^+,\alpha))\rangle. \end{aligned}$$

Hence all $\mu_i(\alpha)$'s can be computed in terms of $\dot{u}(t_i(\alpha)^\pm,\alpha))$.

ii) Since $\mu_{-N}(\alpha) \neq 0$ and all $B_i(\alpha)$, $B_{-i}(\alpha)$ are invertible, we see that $\mu_{\pm i}(\alpha) \neq 0$ for all $i = 0,\dots,N$.

The next Proposition, proved in [6], states that in some circumstances all $\mu_i(\alpha)$'s are equal. This case is of particular interest, since then we can take $\psi(t,\alpha) = v(t,\alpha)$ and the Melnikov condition reads

$$\Delta(\alpha_0) = 0, \quad \operatorname{rank}\Delta'(\alpha_0) = d$$

where

$$\Delta(\alpha) := \int_{-\infty}^{\infty} e^{-\int_0^t a_{11}(s,\alpha)+a_{22}(s,\alpha)ds} \begin{pmatrix} -\dot{u}_2(t,\alpha) \\ \dot{u}_1(t,\alpha) \end{pmatrix}^T \begin{pmatrix} u_{1,\alpha}(t,\alpha) \\ u_{2,\alpha}(t,\alpha) \end{pmatrix} g(u(t,\alpha),\alpha,0)dt.$$

If, moreover, $a_{11}(t,\alpha) + a_{22}(t,\alpha) = 0$ we have

$$\Delta(\alpha) = \int_{-\infty}^{\infty} [\dot{u}_1(t,\alpha)u_{2,\alpha}(t,\alpha) - \dot{u}_2(t,\alpha)u_{1,\alpha}(t,\alpha)]g(u(t,\alpha),\alpha,0)dt. \tag{7.6}$$

Proposition 7.3. *Equations (7.5) are satisfied with $\mu_i = 1$, $i = -N, \dots, N$, if and only if there exist $\nu_i^{\pm}(\alpha)$, $i = 1, \dots, N$, such that*

$$J[f_{i+1}(u(t_{\pm i}(\alpha),\alpha),\alpha) - f_i(u(t_{\pm i}(\alpha),\alpha),\alpha)] = \nu_i(\alpha)^{\pm} h_x(u(t_{\pm i}(\alpha),\alpha),\alpha)^T. \tag{7.7}$$

For example, suppose

$$h(x,y) = x_k$$

where either $k = 1$ or $k = 2$. Recalling that

$$f_i(x,y) = \begin{pmatrix} F_1^i(x_1,x_2,y) \\ F_2^i(x_1,x_2,y) \end{pmatrix}$$

we get, omitting arguments (that can be either $(u(t_{-i}), y_0)$ or $(u(t_i), y_0)$) for simplicity:

$$J[f_{i+1} - f_i] = \begin{pmatrix} F_2^i - F_2^{i+1} \\ F_1^{i+1} - F_1^i \end{pmatrix}$$

and then (7.7) holds if and only if

$$F_k^i(u(t_{\pm i}(\alpha)),\alpha) = F_k^{i+1}(u(t_{\pm i}(\alpha)),\alpha) \tag{7.8}$$

for all $i = 1, \dots, N$.

As a concrete example we consider the following two dimensional equation (see [8]):

$$\ddot{x} = \lambda p(t)\Phi(x)$$

where $\lambda \gg 1$ is a large parameter, $p(t) > 0$ is a positive, C^2, periodic function, and $\Phi(x)$ is a piecewise C^2 function such that

$$\Phi(x) = \begin{cases} \Phi_-(x) & \text{if } x < \frac{1}{2}, \\ \Phi_+(x) & \text{if } x > \frac{1}{2}. \end{cases}$$

Then $h(x_1,x_2) = x_1$ and the discontinuity manifold is $\mathcal{S} = \{x_1 = \frac{1}{2}\}$.

Taking $\lambda = \varepsilon^{-2}$ and changing the time scale $t \mapsto \varepsilon^{-1} t$, the equation reads

$$\begin{aligned} \ddot{x} &= p(y)\Phi(x), \\ \dot{y} &= \varepsilon \end{aligned} \tag{7.9}$$

or

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= p(y)\Phi(x_1), \\ \dot{y} &= \varepsilon. \end{aligned}$$

We assume that $x = 0$, $\dot{x} = 0$ is a hyperbolic fixed point of equation $\ddot{x} = \Phi(x)$ with an associated solution $(u(t), \dot{u}(t))$, homoclinic to $(0,0)$ and such that

$$\begin{aligned} 0 < u(t) &< \frac{1}{2} \quad \text{for } t < t_- \text{ or } t > t_+, \\ u(t) &> \frac{1}{2} \quad \text{for } t_- < t < t_+, \\ u(t_+) = u(t_-) &= \frac{1}{2}, \\ \dot{u}(t_\pm) &\neq 0. \end{aligned} \tag{7.10}$$

Then (7.9) has the family of homoclinic solutions

$$(u(t,\alpha),\dot u(t,\alpha)) = (u(t\sqrt{p(\alpha)}), \sqrt{p(\alpha)}\dot u(t\sqrt{p(\alpha)}))$$

that satisfy assumptions A_1)–A_4). Note that, according to assumption we have

$$t_\pm(\alpha) = \frac{t_\pm}{\sqrt{p(\alpha)}}$$

and

$$\left|\begin{pmatrix} u(t,\alpha) \\ \dot u(t,\alpha)\end{pmatrix}\right| \le \sqrt{1+p(\alpha)}e^{-\delta t\sqrt{p(\alpha)}} \le \sqrt{1+p_{\max}}e^{-\delta t\sqrt{p_{\min}}}$$

where $0 < p_{\min} := \min\{p(\alpha)\} < \max\{p(\alpha)\} := p_{\max}$.

As $F_1(x_1,x_2,y) = x_2$ is continuous and $h(x_1,x_2,y) = x_1$ we see that (7.8) is certainly satisfied. Then

$$\begin{aligned}\Delta(\alpha) &= \int_{-\infty}^{\infty}\sqrt{p(\alpha)}\dot u(t\sqrt{p(\alpha)})\left[\frac{p'(\alpha)}{2\sqrt{p(\alpha)}}\dot u(t\sqrt{p(\alpha)}) + \sqrt{p(\alpha)}\ddot u(t\sqrt{p(\alpha)})\frac{tp'(\alpha)}{2\sqrt{p(\alpha)})}\right] \\ &\quad - p(\alpha)\ddot u(t\sqrt{p(\alpha)}\dot u(t\sqrt{p(\alpha)})\frac{tp'(\alpha)}{2\sqrt{p(\alpha)})}dt = \frac{1}{2}p'(\alpha)\int_{-\infty}^{\infty}\dot u(t\sqrt{p(\alpha)})^2dt \\ &= \frac{p'(\alpha)}{2\sqrt{p(\alpha)}}\int_{-\infty}^{\infty}\dot u(t)^2dt.\end{aligned}$$

As a consequence $\Delta(\alpha)$ has a simple zero at $\alpha = \alpha_0$ if and only if $p(\alpha)$ has a non degenerate critical point at $\alpha = \alpha_0$. From Theorem 6.2 we conclude with the following

Proposition 7.4. *Let $\Phi_\pm(x)$ be C^2-functions and suppose that*

$$\ddot x = \Phi(x) := \begin{cases} \Phi_-(x) & \text{if } 0 < x < \frac{1}{2}, \\ \Phi_+(x) & \text{if } x > \frac{1}{2}\end{cases}$$

has the hyperbolic fixed point $(u,\dot u) = (0,0)$ together with a homoclinic orbit such that (7.10) holds. Then, if $p(\alpha)$ is a periodic C^2-functions having a non-degenerate maximum (or minimum) at $\alpha = \alpha_0$ then there exists $\lambda_0 \gg 1$ and a unique, C^1, $\alpha(\lambda)$ such that $\lim_{\lambda\to\infty}\alpha(\lambda) = \alpha_0$ and for $\lambda > \lambda_0$ the perturbed equation

$$\ddot x = \lambda p(t)\Phi(x)$$

has a solution $(x(t,\lambda),\dot x(t,\lambda))$ such that

$$\sup_{t\in\mathbb{R}}|x(t,\lambda) - u(t,t\lambda^{-\frac{1}{2}} + \alpha(\lambda))| \to 0,$$
$$\sup_{t\in\mathbb{R}}|\dot x(t,\lambda) - \dot u(t,t\lambda^{-\frac{1}{2}} + \alpha(\lambda)))| \to 0$$

as $\lambda \to \infty$.

As a concrete example we take

$$\Phi_\pm(x) = \mp x, \quad p(y) = 2 + \sin y$$

so that system (7.9) reads

$$\begin{aligned} \ddot{x} &= (2+\sin(y))x, & x < \frac{1}{2}, \\ \ddot{x} &= -(2+\sin(y))x, & x > \frac{1}{2}, \\ \dot{y} &= \varepsilon. \end{aligned} \tag{7.11}$$

The homoclinic solution of the frozen system ($\varepsilon = 0$) is

$$u(t) = \begin{cases} \frac{e^{\frac{\pi}{4}}}{2} e^t & \text{if } t \le -\frac{\pi}{4}, \\ \frac{1}{\sqrt{2}} \cos t & \text{if } -\frac{\pi}{4} \le t \le \frac{\pi}{4}, \\ \frac{e^{\frac{\pi}{4}}}{2} e^{-t} & \text{if } t \ge \frac{\pi}{4}. \end{cases}$$

Solving $p'(\alpha) = \cos\alpha = 0$, we get $\alpha_1 = \frac{\pi}{2}$ and $\alpha_2 = \frac{3\pi}{2}$ with $p''(\alpha_{1,2}) = -\sin\alpha_{1,2} = \mp 1 \neq 0$. Now we add some numerical figures of solutions of (7.11) near

$$v(t) = u\left(t\sqrt{2+\sin(\alpha_1+\varepsilon t)}\right)$$

and

$$w(t) = u\left(t\sqrt{2+\sin(\alpha_2+\varepsilon t)}\right)$$

for ε small, say $\varepsilon = 0.1$ and for

$$y(0) \sim \alpha_{1,2}. \tag{7.12}$$

Note

$$v(0) = w(0) = u(0) = \frac{1}{\sqrt{2}}, \qquad \dot{v}(0) = \dot{w}(0) = \dot{u}(0) = 0.$$

Here we draw some pictures of the solutions of equation (7.11) where we take $y(0) = \frac{\pi}{2} \pm 0.05$. In all these pictures we take $\varepsilon = 0.1$. Figures 7.2–7.5 in the paper show the curves of $(t, x(t))$

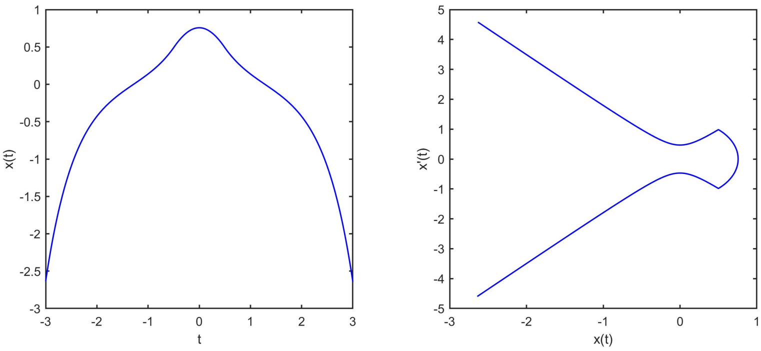

Figure 7.1: The plot of $(v(t), \dot{v}(t))$ for $t \in (-10, 10)$.

and $(x(t), x'(t))$ corresponding to $\varepsilon = 0.1$.

In the case of keeping initial conditions unchanged but taking $\varepsilon = 0.01$, we have the following Figures 7.6–7.9.

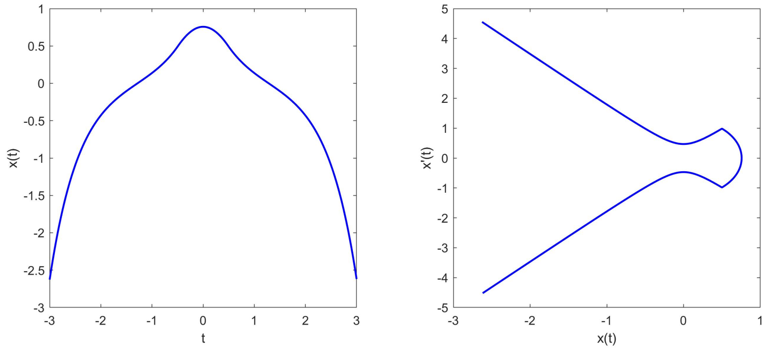

Figure 7.2: The plot of $(t, x(t))$ and $(x(t), \dot{x}(t))$ for $t = [-3, 3]$ with $y(0) = \frac{\pi}{2} + 0.05$, $x(0) = \frac{1}{\sqrt{2}} + 0.05$, $\dot{x}(0) = 0$. Note that the solution escapes very quickly from a neighbourhood of the fixed point $x = \dot{x} = 0$ as $t \to \pm\infty$.

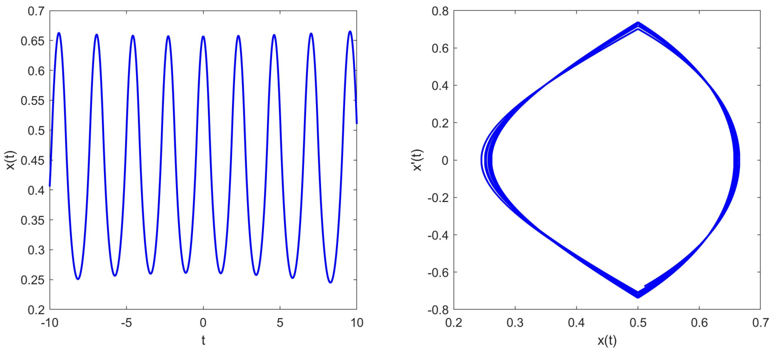

Figure 7.3: The plot of $(t, x(t))$ and $(x(t), \dot{x}(t))$ for $t = [-10, 10]$ with $y(0) = \frac{\pi}{2} + 0.05$, $x(0) = \frac{1}{\sqrt{2}} - 0.05$, $\dot{x}(0) = 0$. Here the solution looks like a periodic solution in the bounded domain $0.2 \leq x \leq 0.7$, $-0.8 \leq \dot{x} \leq 0.8$.

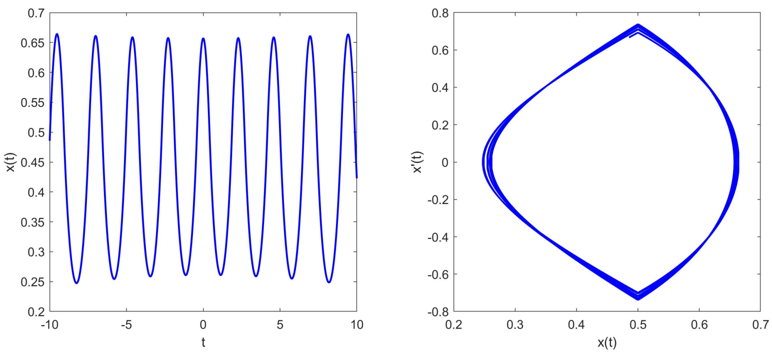

Figure 7.4: The plot of $(t, x(t))$ and $(x(t), \dot{x}(t))$ for $t = [-10, 10]$ with $y(0) = \frac{\pi}{2} - 0.05$, $x(0) = \frac{1}{\sqrt{2}} - 0.05$, $\dot{x}(0) = 0$. Also in this case the solution looks like a periodic solution in the bounded domain $0.2 \leq x \leq 0.7$, $-0.8 \leq \dot{x} \leq 0.8$.

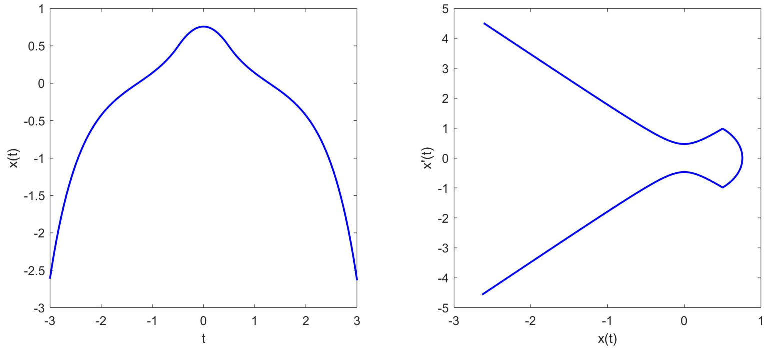

Figure 7.5: The plot of $(t, x(t))$ and $(x(t), \dot{x}(t))$ for $t = [-3, 3]$ with $y(0) = \frac{\pi}{2} - 0.05$, $x(0) = \frac{1}{\sqrt{2}} + 0.05$, $\dot{x}(0) = 0$. For these initial values the solution escapes very quickly from a neighbourhood of the fixed point $x = \dot{x} = 0$ as $t \to \pm\infty$.

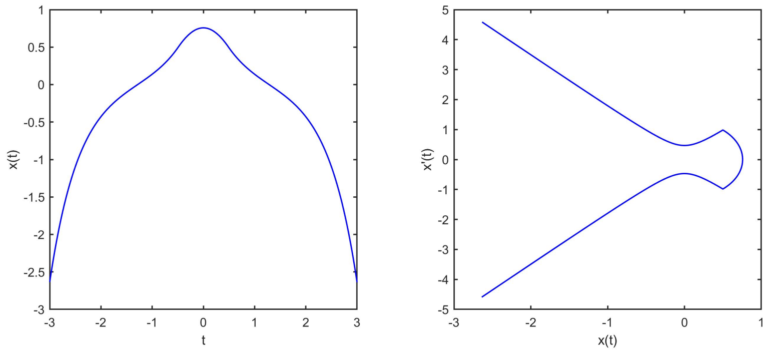

Figure 7.6: Corresponding to the case of $\varepsilon = 0.01$ in Figure 7.2.

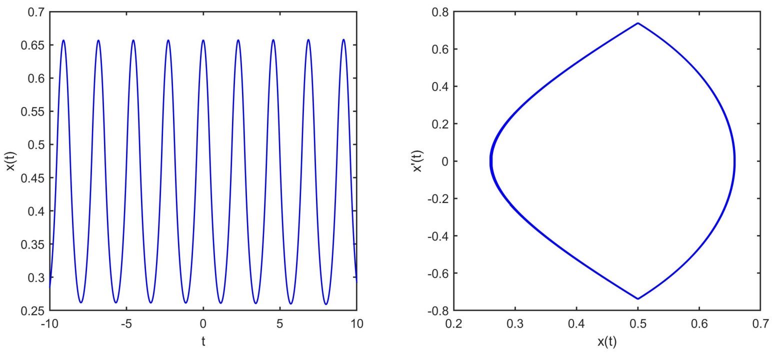

Figure 7.7: Corresponding to the case of $\varepsilon = 0.01$ in Figure 7.3.

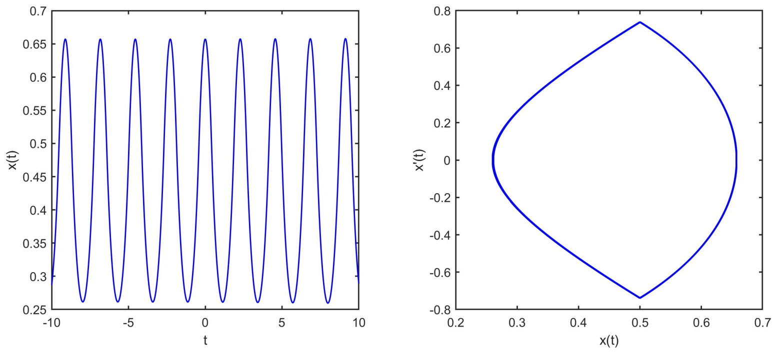

Figure 7.8: Corresponding to the case of $\varepsilon = 0.01$ in Figure 7.4.

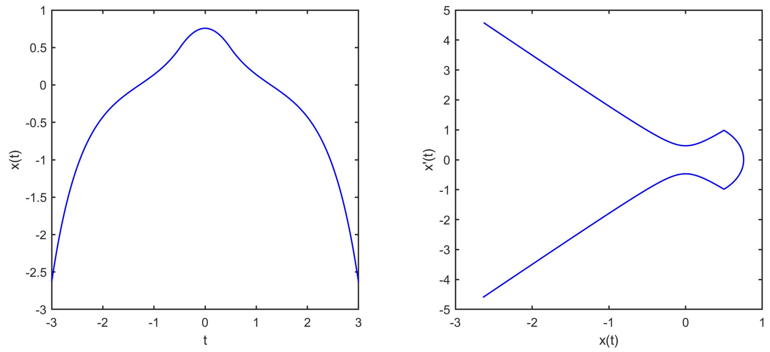

Figure 7.9: Corresponding to the case of $\varepsilon = 0.01$ in Figure 7.5.

8 Concluding remark

According to the results in [4], with the correction given in [5], the Melnikov function in the periodic case, with two discontinuity points and a family of periodic solutions $u(t,\alpha)$ of the unperturbed equation, is

$$\int_{-T(\alpha)/2}^{T(\alpha)/2} \psi_j(t,\alpha)^T f_y(u(t,\alpha),\alpha) y_\varepsilon(t,0,\alpha,0)\, dt \\ + \psi_j(t_*(\alpha)^+,\alpha)^T \frac{h_{y,*} y_{\varepsilon,*}}{h_{x,*} f_{+,*}} (f_{-,*} - f_{+,*}) + \psi_j(t^*(\alpha)^+,\alpha)^T \frac{h_y^* y_\varepsilon^*}{h_x^* f_-^*} (f_+^* - f_-^*).$$

In the following we prove that $-M(\alpha)$ extends the above expression to the heteroclinic case (i.e. with ∞ replacing $T(\alpha)$) with several discontinuity points.

Differentiating $\dot{u}(t,y) = f(u(t,y),y)$ with respect to y we see that, for $t \neq t_{\pm i}(\alpha)$, $i = 1,\ldots,N$:

$$\dot{u}_y(t,\alpha) = A(t,\alpha) u_y(t,\alpha) + f_y(u(t,\alpha),\alpha)$$

and then

$$\frac{d}{dt}(u_y(t,\alpha) y_\varepsilon(t,0,\alpha,0)) = \dot{u}_y(t,\alpha) y_\varepsilon(t,0,\alpha,0) + u_y(t,\alpha) \dot{y}_\varepsilon(t,0,\alpha,0) \\ = A(t,\alpha) u_y(t,\alpha) y_\varepsilon(t,0,\alpha,0) + f_y(u(t,\alpha),\alpha) y_\varepsilon(t,0,\alpha,0) + u_y(t,\alpha) g(u(t,\alpha),\alpha).$$

So

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \psi_j^T(t,\alpha)u_y(t,\alpha)g(u(t,\alpha),\alpha,0)dt \\
&\quad = \int_{-\infty}^{\infty} \psi_j^T(t,\alpha)\left[\frac{d}{dt}(u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)) - A(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)\right. \\
&\qquad \left. - f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)\right]dt \\
&\quad = \int_{-\infty}^{\infty} \psi_j^T(t,\alpha)\frac{d}{dt}(u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)) + \dot{\psi}_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0) \\
&\qquad - \psi_j^T(t,\alpha)f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)dt \\
&\quad = \int_{-\infty}^{\infty} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)] - \psi_j^T(t,\alpha)f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)dt.
\end{aligned}
$$

Then the j-th component of $-M(\alpha)$, say $-M_j(\alpha)$, is

$$
\begin{aligned}
-M_j(\alpha) = &\int_{-\infty}^{\infty} \psi_j^T(t,\alpha)f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)dt \\
&- \int_{-\infty}^{\infty} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt.
\end{aligned}
$$

Using the continuity of $y_\varepsilon(t,\xi,\alpha,\eta)$ we get:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt \\
&\quad = \int_{-\infty}^{t_{-N}(\alpha)} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt \\
&\qquad + \sum_{i=-N}^{N-1} \int_{t_i(\alpha)}^{t_{i+1}(\alpha)} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt \\
&\qquad + \int_{t_N(\alpha)}^{\infty} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt \\
&\quad = \psi_j^T(t_{-N}(\alpha)^-,\alpha)u_y(t_{-N}(\alpha)^-,\alpha)y_\varepsilon(t_{-N}(\alpha),0,\alpha,0) \\
&\qquad + \sum_{i=-N}^{N-1}\Big[\psi_j^T(t_{i+1}(\alpha)^-,\alpha)u_y(t_{i+1}(\alpha)^-,\alpha)y_\varepsilon(t_{i+1}(\alpha),0,\alpha,0) \\
&\qquad - \psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)y_\varepsilon(t_i(\alpha),0,\alpha,0)\Big] \\
&\qquad - \psi_j^T(t_N(\alpha)^+,\alpha)u_y(t_N(\alpha)^+,\alpha)y_\varepsilon(t_N(\alpha),0,\alpha,0) \\
&\quad = \psi_j^T(t_{-N}(\alpha)^-,\alpha)u_y(t_{-N}(\alpha)^-,\alpha)y_\varepsilon(t_{-N}(\alpha),0,\alpha,0) \\
&\qquad + \sum_{i=-N+1}^{N} \psi_j^T(t_i(\alpha)^-,\alpha)u_y(t_i(\alpha)^-,\alpha)y_\varepsilon(t_i(\alpha),0,\alpha,0) \\
&\qquad - \sum_{i=-N}^{N-1} \psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)y_\varepsilon(t_i(\alpha),0,\alpha,0) \\
&\qquad - \psi_j^T(t_N(\alpha)^+,\alpha)u_y(t_N(\alpha)^+,\alpha)y_\varepsilon(t_N(\alpha),0,\alpha,0)
\end{aligned}
$$

$$
\begin{aligned}
&= \psi_j^T(t_{-N}(\alpha)^-,\alpha)u_y(t_{-N}(\alpha)^-,\alpha)y_\varepsilon(t_{-N}(\alpha),0,\alpha,0)\\
&\quad+\sum_{i=-N+1}^{N-1}[\psi_j^T(t_i(\alpha)^-,\alpha)u_y(t_i(\alpha)^-,\alpha)-\psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)]y_\varepsilon(t_i(\alpha),0,\alpha,0)\\
&\quad+\psi_j^T(t_N(\alpha)^-,\alpha)u_y(t_N(\alpha)^-,\alpha)y_\varepsilon(t_N(\alpha),0,\alpha,0)\\
&\quad-\psi_j^T(t_{-N}(\alpha)^+,\alpha)u_y(t_{-N}(\alpha)^+,\alpha)y_\varepsilon(t_{-N}(\alpha),0,\alpha,0)\\
&\quad-\psi_j^T(t_N(\alpha)^+,\alpha)u_y(t_N(\alpha)^+,\alpha)y_\varepsilon(t_N(\alpha),0,\alpha,0)\\
&=\sum_{i=-N}^{N}[\psi_j^T(t_i(\alpha)^-,\alpha)u_y(t_i(\alpha)^-,\alpha)-\psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)]y_\varepsilon(t_i(\alpha),0,\alpha,0)\\
&=\sum_{i=-N,i\neq 0}^{N}[\psi_j^T(t_i(\alpha)^-,\alpha)u_y(t_i(\alpha)^-,\alpha)-\psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)]y_\varepsilon(t_i(\alpha),0,\alpha,0).
\end{aligned}
$$

The last equality follows from the fact that $\psi_j(t,\alpha)$ and $u_y(t,\alpha)$ are continuous at $t=t_0(\alpha)=0$.

Next, from (3.6)–(6.11) we see that, for any $\ell=\pm1,\dots,\pm N$, we have

$$\psi_j^T(t_\ell(\alpha)^-,\alpha)=\psi_j^T(t_\ell(\alpha)^+,\alpha)B_\ell(\alpha)$$

Hence:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt\\
&\qquad\sum_{\ell=-N,\ell\neq 0}^{N}\psi_j^T(t_\ell(\alpha)^+,\alpha)[B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha)-u_y(t_\ell(\alpha)^+,\alpha)]y_\varepsilon(t_\ell(\alpha),0,\alpha,0).
\end{aligned}
$$

From (5.1) we obtain

$$
\begin{aligned}
&B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha)-u_y(t_\ell(\alpha)^-,\alpha)\\
&\qquad=-\frac{h_x(u(t_\ell(\alpha),\alpha),\alpha)u_y(t_\ell(\alpha)^-,\alpha)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,\alpha),\alpha)}(\dot u(t_\ell(\alpha)^-,\alpha)-\dot u(t_\ell(\alpha)^+,\alpha)).
\end{aligned}
$$

Differentiating $h(u(t_\ell(\alpha),\alpha),\alpha)=c_{|\ell|}$ with respect to α we get

$$
\begin{aligned}
&h_x(u(t_\ell(\alpha),\alpha),\alpha)u_y(t_\ell(\alpha)^-,\alpha)\\
&\qquad=-h_x(u(t_\ell(\alpha),\alpha),y)\dot u(t_\ell(\alpha)^-,\alpha)t_\ell'(\alpha)-h_y(u(t_\ell(\alpha),\alpha),\alpha)
\end{aligned}
$$

and then

$$
\begin{aligned}
&B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha)-u_y(t_\ell(\alpha)^-,\alpha)\\
&\quad=\frac{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,y)t_\ell'(\alpha)+h_y(u(t_\ell(\alpha),\alpha),\alpha)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,\alpha)}(\dot u(t_\ell(\alpha)^-,\alpha)-\dot u(t_\ell(\alpha)^+,\alpha))\\
&\quad=\left[t_\ell'(\alpha)+\frac{h_y(u(t_\ell(\alpha),\alpha),\alpha)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,\alpha)}\right](\dot u(t_\ell(\alpha)^-,\alpha)-\dot u(t_\ell(\alpha)^+,\alpha)).
\end{aligned}
$$

So

$$
\begin{aligned}
&B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha)-u_y(t_\ell(\alpha)^+,\alpha)\\
&\qquad=u_y(t_\ell(\alpha)^-,\alpha)-u_y(t_\ell(\alpha)^+,\alpha)+t_\ell'(\alpha)[\dot u(t_\ell(\alpha)^-,\alpha)-\dot u(t_\ell(\alpha)^+,\alpha)]\\
&\qquad\quad+\frac{h_y(u(t_\ell(\alpha),\alpha),\alpha)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,\alpha)}(\dot u(t_\ell(\alpha)^-,\alpha)-\dot u(t_\ell(\alpha)^+,\alpha)).
\end{aligned}
\tag{8.1}
$$

Next, for $t_\ell(\alpha) < t < t_{\ell+1}(\alpha)$ we have:

$$u(t,\alpha) = u(t_\ell(\alpha)^-,\alpha) + \int_{t_\ell(\alpha)}^t \dot{u}(s,\alpha)ds.$$

Hence

$$u_y(t_\ell(\alpha)^+,\alpha) = \dot{u}(t_\ell(\alpha)^-,\alpha)t'_\ell(\alpha) + u_y(t_\ell(\alpha)^-,\alpha) - \dot{u}(t_\ell(\alpha)^+,\alpha)t'_\ell(\alpha)$$

and then

$$u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^+,\alpha) = [\dot{u}(t_\ell(\alpha)^+,\alpha) - \dot{u}(t_\ell(\alpha)^-,\alpha)]t'_\ell(\alpha). \tag{8.2}$$

Plugging (8.2) into (8.1) we finally obtain:

$$\begin{aligned}&[B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^+,\alpha)]y_\varepsilon(t_\ell(\alpha),0,\alpha,0)\\ &\quad = \frac{h_y(u(t_\ell(\alpha),\alpha),\alpha)y_\varepsilon(t_\ell(\alpha),0,\alpha,0)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot{u}(t_\ell(\alpha)^-,\alpha)}(\dot{u}(t_\ell(\alpha)^-,\alpha) - \dot{u}(t_\ell(\alpha)^+,\alpha))\end{aligned}$$

Putting everything together we finally get:

$$\begin{aligned}-M_j(\alpha) = &\int_{-\infty}^{\infty} \psi_j^T(t,\alpha)f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)dt\\ &+\sum_{i=1}^N \psi_j^T(t_{-i}(\alpha)^+,\alpha)\frac{h_y(u(t_{-i}(\alpha),\alpha),\alpha)y_\varepsilon(t_{-i}(\alpha),0,\alpha,0)}{h_x(u(t_{-i}(\alpha),\alpha),\alpha)f_{i+1}(u(t_{-i}(\alpha),\alpha),\alpha)}\\ &\cdot(f_i(u(t_{-i}(\alpha),\alpha),\alpha) - f_{i+1}(u(t_{-i}(\alpha),\alpha),\alpha))\\ &+\sum_{i=1}^N \psi_j^T(t_i(\alpha)^+,\alpha)\frac{h_y(u(t_i(\alpha),\alpha),\alpha)y_\varepsilon(t_i(\alpha),0,\alpha,0)}{h_x(u(t_i(\alpha),\alpha),\alpha)f_i(u(t_i(\alpha),\alpha),\alpha)}\\ &\cdot(f_{i+1}(u(t_i(\alpha),\alpha),\alpha) - f_i(u(t_i(\alpha),\alpha),\alpha)).\end{aligned} \tag{8.3}$$

This completes the proof that $-M(\alpha)$ extends the Melnikov function for the periodic case with two discontinuity points to the heteroclinic case with a finite number of discontinuity points.

Acknowledgements

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