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Tightening Poincaré–Bendixson theory after counting separately the fixed points on the boundary and interior of a planar region

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Abstract. This paper tightens the classical Poincaré–Bendixson theory for a positively invariant, simply-connected compact set $\mathcal{M}$ in a continuously differentiable planar vector field by further characterizing for any point $p \in \mathcal{M}$, the composition of the limit sets $\omega(p)$ and $\alpha(p)$ after counting separately the fixed points on $\mathcal{M}$'s boundary and interior. In particular, when $\mathcal{M}$ contains finitely many boundary but no interior fixed points, $\omega(p)$ contains only a single fixed point, and when $\mathcal{M}$ may have infinitely many boundary but no interior fixed points, $\omega(p)$ can, in addition, be a continuum of fixed points. When $\mathcal{M}$ contains only one interior and finitely many boundary fixed points, $\omega(p)$ or $\alpha(p)$ contains exclusively a fixed point, a closed orbit or the union of the interior fixed point and homoclinic orbits joining it to itself. When $\mathcal{M}$ contains in general a finite number of fixed points and neither $\omega(p)$ nor $\alpha(p)$ is a closed orbit or contains just a fixed point, at least one of $\omega(p)$ and $\alpha(p)$ excludes all boundary fixed points and consists only of a number of the interior fixed points and orbits connecting them.

Keywords: Poincaré–Bendixson theory, planar vector field, limit set.

2020 Mathematics Subject Classification: 34C05, 37C10, 34C60.

1 Introduction

Determining the asymptotic behavior of general continuous vector fields, even qualitatively, is still a daunting task. In the nineteenth century, Poincaré studied this problem for planar systems by focusing on the global behavior of the systems' trajectories without integrating the corresponding differential equations [7, 13]. The analysis was later completed by Bendixson [2]. The related classical results are commonly referred to as the Poincaré–Bendixson theorem [2, 7, 9–11, 14–17]. Consider the vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^2 \tag{1.1}$$

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where f is $\mathbf{C}^1$ on an open set $\mathcal{U}$ in $\mathbb{R}^2$. A point $x^* \in \mathbb{R}^2$ is a "fixed point" of the vector field if $f(x^*) = 0$. Denote the omega and alpha limit sets of a point p by $\omega(p)$ and $\alpha(p)$, respectively.

Theorem 1.1 (Poincaré–Bendixson theorem [23, Theorem 9.0.6], [12, Theorem 1.8]). *For the vector field* (1.1)*, let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant complex for the vector field containing a finite number of fixed points. For any $p \in \mathcal{M}$, one of the following holds:*

1. *$\omega(p)$ is a fixed point;*
2. *$\omega(p)$ is a closed orbit;*
3. *$\omega(p)$ consists of a finite number of fixed points $p_1, \dots, p_n$ and orbits γ with $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$, where p_i and p_j are not necessarily different. Moreover, for two distinct fixed points p_i and p_j, there exists at most one orbit γ such that $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$.*

From this theorem, although possibilities such as strange attractors and chaotic orbits can be easily ruled out, the third case in the theorem still gives rise to sometimes a large number of possible limiting behaviors. For example, when $\mathcal{M}$ contains just four fixed points on its boundary, there can be more than 300 different compositions of $\omega(p)$ even under the simplifying assumption that there is at most one homoclinic orbit at each fixed point. Some existing results have tried to reduce the possible scenarios; in [1, Theorem 68], [18, Theorem 3] the third case has been stated more precisely by stipulating that the trajectories γ must be the continuations of one another, and in [19, Section 3.7, Theorem 3] the number of homoclinic orbits at each fixed point is limited by one when the vector field is "relatively prime analytic". However, then for the example just mentioned, $\omega(p)$ can still have more than 50 different compositions. This example shows that if one is interested in categorizing all possible asymptotic behaviors of a planar system qualitatively, a greatly needed task in fields such as mathematical biology [4], one may still encounter difficulty even with the help of the existing most tightened form of Poincaré–Bendixson theorem.

The aim of this paper is to reduce the number of possible compositions of the limit sets of a vector field when knowing the number of fixed points on the boundary and in the interior of a given positively invariant, simply-connected compact set $\mathcal{M}$.

Notations: Let $\phi(t, x)$ denote the flow generated by the vector field (1.1), which is the solution of (1.1) passing through x at time t. For a point $p \in \mathbb{R}^2$, let $\mathcal{O}(p)$ denote the *orbit* of p defined by $\mathcal{O}(p) = \{x \in \mathbb{R}^2 \mid x = \phi(t, p), t \in \mathbb{R}\}$, and $\mathcal{O}_+(p)$ denote the *positive semi-orbit* of p, defined by $\mathcal{O}_+(p) = \{x \in \mathbb{R}^2 \mid x = \phi(t, p), t \geq 0\}$ [23]. Correspondingly, for $p_1, p_2 \in \mathcal{O}_+(p)$, define the segment semi-orbit *$\mathcal{O}_+(p)$ from p_1 to p_2* as $\mathcal{O}_+(p_2) - \mathcal{O}_+(p_1)$. A *homoclinic orbit* is a trajectory that joins a fixed point to itself. For a set $\mathcal{M}$, denote its *interior* by Int $\mathcal{M}$, its *boundary* by $\partial\mathcal{M}$, and its closure by $\overline{\mathcal{M}}$.

2 Main results

We first review some basic relevant results. The following lemma is applicable to higher dimensional spaces, but we restrict it here to the plane.

Lemma 2.1 ([23, Proposition 8.1.3], [3, Theorem 3-3.6]). *For the vector field* (1.1)*, let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set. Then for any point $p \in \mathcal{M}$, it holds that $\omega(p)$ is nonempty, connected, and compact.*

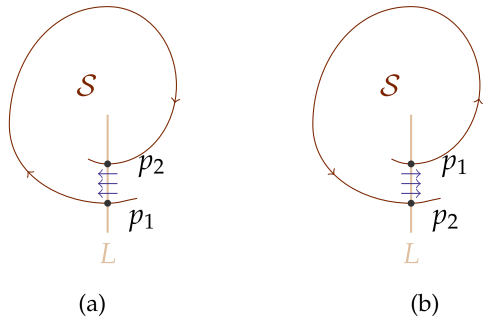

Figure 2.1: The two possible cases for the positive semi-orbit $\mathcal{O}_+(p)$ in the proof of Theorem 2.4.

A continuous connected arc in the plane is said to be *transverse to the vector field*, if the vector field has no fixed points on the arc and nowhere becomes tangent to the arc [11]. By a *transversal* we refer to a closed line segment $\mathcal{L}$ that is transverse to the vector field. Due to the continuity of the vector field, clearly one can construct a transversal through any non-fixed point. The following lemma illustrates how the flow through a point p approaches a transversal through a non-fixed omega limit point $q \in \omega(p)$ when it exists.

Lemma 2.2 ([8, reformulation of Lemma 1.26])**.** *For the vector field* (1.1)*, consider a point $p \in \mathcal{U}$ such that $\mathcal{O}(p) \subset \mathcal{U}$. Let $q \in \omega(p)$ be a non-fixed point of* (1.1) *and let $\mathcal{L}$ be a transversal through q. Then there exists a sequence $\{t_i\} \to \infty$, such that $\{\phi(t_i, p)\} \in \mathcal{L}$ and $\{\phi(t_i, p)\} \to q$.*

The following result guarantees the existence of a fixed point inside a closed orbit [3, 6, 9, 23]:

Lemma 2.3 ([23, Corollary 6.0.2])**.** *Enclosed by any closed orbit of* (1.1) *in $\mathcal{U}$, there must be at least one fixed point.*

Now we are ready to present the main results of the paper.

2.1 $\mathcal{M}$ has no interior fixed point

Theorem 2.4 (No interior fixed points, positively invariant vector field)**.** *For the vector field* (1.1)*, consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains a finite number of fixed points, all on $\partial\mathcal{M}$. Then for any $p \in \mathcal{M}$, $\omega(p)$ is a fixed point on $\partial\mathcal{M}$.*

Proof. From Theorem 1.1, it suffices to prove that $\omega(p)$ contains only fixed points since then only situation 1 is possible and the corresponding fixed point can only be on $\partial\mathcal{M}$ as Int $\mathcal{M}$ contains no fixed points. We prove this by contradiction, so assume on the contrary that there is a non-fixed point $q \in \omega(p)$. Then one can construct a transversal $\mathcal{L}$ through q, and from Lemma 2.2, we know that $\mathcal{O}_+(p)$ intersects $\mathcal{L}$ for infinitely many times and such intersection points are in $\mathcal{M}$ since $\mathcal{O}_+(p) \subset \mathcal{M}$. So one can pick two consecutive intersection points p_1 and p_2 such that the line segment p_1p_2 lies in $\mathcal{M}$. Should p_1 and p_2 coincide, $\omega(p)$ would be a closed orbit, lying in $\mathcal{M}$, but encircling no fixed point as all the fixed points are on $\partial\mathcal{M}$. This cannot happen in view of Lemma 2.3, and thus, p_1 and p_2 must be distinct.

As illustrated by Fig. 2.1, we construct the simply-connected compact set $\mathcal{S}$ whose boundary is formed by the segment semi-orbit $\mathcal{O}_+(p)$ from p_1 to p_2 and the line segment p_1p_2. Since

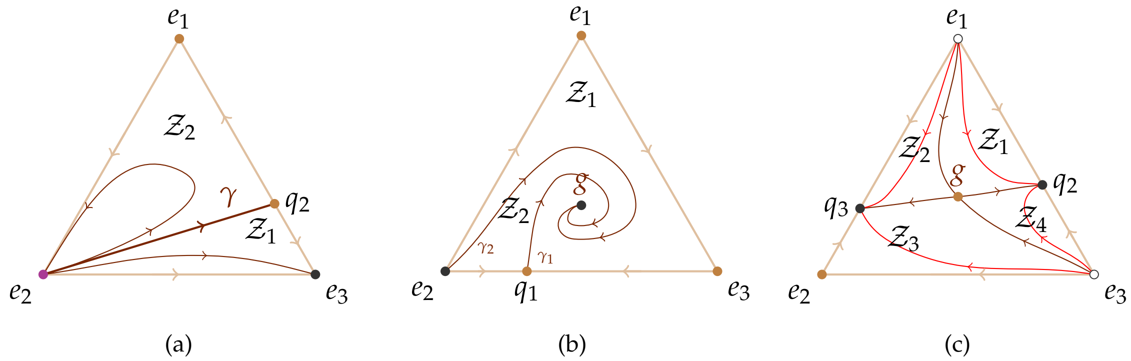

Figure 2.2: Phase portrait examples for an invariant compact set Δ. **(a)** e_1 and q_2 are hyperbolic saddle, e_3 is a hyperbolic stable and e_2 is a center fixed point. The stable invariant manifold of q_2 divides Δ into $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Theorem 2.5 and the local stability results imply that for each $z \in \text{Int } \mathcal{Z}_1$, $\alpha(z) = e_2$ and $\omega(z) = e_3$, and for each $z \in \text{Int } \mathcal{Z}_2$, $\alpha(z) = \omega(z) = e_2$. **(b)** e_1, e_3 and q_1 are hyperbolic saddle, e_2 is a hyperbolic unstable, and g is a hyperbolic stable fixed point. Because of Theorem 2.7, the local stability results and the fact that no limit cycle exists, $\omega(p) = \{g\}$ for all $p \in \text{int}(\Delta)$. Hence, the unique out-going trajectory from q_1, denoted by γ_1, converges to g. The rest of the orbits in $\text{int}(\Delta)$ start from e_2 and end at g. This is because any out-going trajectory from e_2, e.g., γ_2, together with γ_1 divide the simplex into the zones $\mathcal{Z}_1$ and $\mathcal{Z}_2$, each of which satisfy the condition of $\mathcal{M}$ in Theorem 2.7. Hence, every trajectory in $\text{Int } \mathcal{Z}_i, i = 1, 2$, starts from e_2 and end at g. **(c)** e_2 and g are hyperbolic saddle, e_1 and e_3 are hyperbolic unstable and q_3 and q_2 are hyperbolic stable fixed points. The trajectories γ_1 and γ_2 lie on the unstable invariant manifold of g. Because of Theorem 2.7 and the local stability results, the unstable invariant manifold of g is confined to q_2 and q_3 and the stable invariant manifold of g is confined to e_1 and e_3. This results in the four zones $\mathcal{Z}_1, \ldots, \mathcal{Z}_4$. In view of Theorem 2.4, $\forall z \in \text{Int } \mathcal{Z}_1$, $\alpha(z) = e_1$ and $\omega(z) = q_2$, $\forall z \in \text{Int } \mathcal{Z}_2$, $\alpha(z) = e_1$ and $\omega(z) = q_3$, $\forall z \in \text{Int } \mathcal{Z}_3$, $\alpha(z) = e_3$ and $\omega(z) = q_3$, and $\forall z \in \text{Int } \mathcal{Z}_4$, $\alpha(z) = e_3$ and $\omega(z) = q_2$.

$\mathcal{O}_+(p)$ always intersects $\mathcal{L}$ from the same side to the other, the orientation of the p_1-to-p_2 semi-orbit with respect to the line segment p_1p_2 must be one of the two cases shown in Fig. 2.1. From the definition of $\mathcal{L}$, the vector field at any point on p_1p_2 intersects p_1p_2 from the same side of the line, and thus $\mathcal{S}$ is either positively invariant as shown in Fig. 2.1.(a) or negatively invariant as shown in Fig. 2.1.(b).

Since the boundary p_1-to-p_2 semi-orbit and p_1p_2 both lie in $\mathcal{M}$, we know that $\mathcal{S} \subseteq \mathcal{M}$. Hence, $\text{Int } \mathcal{S} \subseteq \text{Int } \mathcal{M}$ and contains no fixed point. Moreover, neither $\mathcal{O}_+(p)$ nor $\mathcal{L}$ contains any fixed point, so $\partial\mathcal{S}$ does not contain any fixed point. Therefore, $\mathcal{S}$ contains no fixed point. Consequently, if $\mathcal{S}$ is positively invariant, applying Theorem 1.1, we know that for any point $s \in \mathcal{S}$, $\omega(s)$ can only be a closed orbit confined in $\mathcal{S}$. But this contradicts Lemma 2.3. If on the other hand, $\mathcal{S}$ is negatively invariant, we apply the same argument after inverting the direction of the vector field and again reach the same contradiction. So the proof is complete. □

In term of the example given in the introduction, Theorem 2.4 implies that $\omega(p)$ in the example can only be one of the fixed points, so at most four possibilities. If in addition to being positively invariant, $\mathcal{M}$ is also negatively invariant, i.e., $\mathcal{M}$ is invariant, then Theorem 2.4 can get even more strengthened.

Theorem 2.5 (No interior fixed points, invariant vector field)**.** *For the vector field* (1.1)*, consider an invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains a finite number of fixed points, all on $\partial\mathcal{M}$. Then for any $p \in \mathcal{M}$, both $\omega(p)$ and $\alpha(p)$ are fixed points, not necessarily different, on $\partial\mathcal{M}$.*

Proof. Theorem 2.4 implies that for any $p \in \mathcal{M}$, $\omega(p)$ contains only a single fixed point on $\partial\mathcal{M}$. The same holds for $\alpha(p)$ after reversing the direction of the vector field since $\mathcal{M}$ is also negatively invariant. This completes the proof. □

Fig. 2.2 demonstrates an example from planar *replicator dynamics* [20–22], where the triangle $e_1e_2e_3$, known as a *face*, is invariant. Part (a) corresponds to Theorem 2.5. The reader may refer to [4,5] for all 49 possible qualitatively different phase portraits of the dynamics.

2.2 $\mathcal{M}$ has no interior, but infinitely many boundary fixed points

We obtain the following theorem that is the counterpart of Theorem 2.4 when the vector field may have infinitely many fixed points on $\partial\mathcal{M}$.

Theorem 2.6. *For the vector field* (1.1)*, consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that has no interior fixed point, but may contain an infinite number of fixed points on $\partial\mathcal{M}$. Then for any $p \in \mathcal{M}$, one of the following two holds:*

1. *$\omega(p)$ is a fixed point on $\partial\mathcal{M}$;*
2. *$\omega(p)$ is a continuum of fixed points on $\partial\mathcal{M}$.*

Proof. Following similar steps as those in the proof for Theorem 2.4, one can construct the simply-connected compact set $\mathcal{S}$ as illustrated in Fig. 2.1. Using similar arguments for $\mathcal{S}$ as those in the proof for Theorem 2.4, after applying Theorem 6.1 in [7], which is the extension of Poincaré–Bendixson theorem to the case when there are infinitely many fixed points, one knows that $\omega(p)$ does not contain any fixed point. On the other hand, $\omega(p)$ has to be connected in view of Lemma 2.1, so it can only be a connected subset of the fixed points in $\mathcal{M}$, which is either a fixed point or a continuum of fixed points on $\partial\mathcal{M}$. □

2.3 $\mathcal{M}$ has exactly one interior fixed point

Now we present the counterpart of Theorem 2.4 discussing the case when $\mathcal{M}$ contains exactly one interior and finitely many boundary fixed points.

Theorem 2.7 (One interior fixed point)**.** *For the vector field* (1.1)*, consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains exactly one interior fixed point x^* and a finite number of fixed points on its boundary. Then for any $p \in \mathcal{M}$, at least one of the following holds:*

1. *$\omega(p)$ is a fixed point, a closed orbit encircling x^* or the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself;*
2. *$\alpha(p)$ is $\{x^*\}$, a closed orbit encircling x^* or the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself.*

Proof. We investigate all possibilities for $\omega(p)$ and show that each results in one of the cases of the theorem. Should $\omega(p)$ be a singleton fixed point or a closed orbit that has to encircle g according to Lemma 2.3, we arrive at Part 1. of the theorem. So consider the situation when $\omega(p)$ is neither. It then follows Theorem 1.1 that $\omega(p)$ contains non-fixed points; we pick one such point q and construct a transversal $\mathcal{L}$ through q. From Lemma 2.2, we know that $\mathcal{O}_+(p)$ intersects $\mathcal{L}$ for infinitely many times. Consider two consecutive intersections p_1 and p_2 which have to be distinctive since $\omega(p)$ is not a closed orbit. We construct the simply-connected compact set $\mathcal{S}$ whose boundary is formed by the semi-orbit $\mathcal{O}_+(p)$ from p_1 to p_2 and the line segment p_1p_2. Similar to the proof of Theorem 2.4, one can show that:

(i) $\mathcal{S}$ is in the form of one of the two cases shown in Fig. 2.1,

(ii) $\mathcal{S}$ is positively invariant in Case (a) of the figure and negatively invariant in Case (b), and

(iii) $x^* \in \operatorname{Int} \mathcal{S}$ is the only fixed point in $\mathcal{S}$.

If $\mathcal{S}$ is positively invariant, $\mathcal{O}_+(p) \cap \operatorname{Int} \mathcal{S} \neq \emptyset$, implying the existence of some $s_p \in \mathcal{O}_+(p) \cap \operatorname{Int} \mathcal{S}$. Consequently, $\omega(s_p) = \omega(p)$. Then, applying Theorem 1.1, we know that $\omega(s_p)$ consists of a number of fixed points in $\mathcal{S}$ and the orbits connecting them. However, since x^* is the only fixed point in $\operatorname{Int} \mathcal{M}$, such orbits can only connect x^* to itself. So $\omega(s_p)$ is the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself, so is $\omega(p)$. So in this case Part 1 of the theorem holds.

Otherwise, if $\mathcal{S}$ is negatively invariant, then there exists a point $s_p \in \mathcal{O}_-(p) \cap \operatorname{Int} \mathcal{S}$ where $\mathcal{O}_-(p)$ is the same as $\mathcal{O}_+(p)$, but when time is reversed. Consequently, after reversing the direction of the vector field, one can check the three cases in Theorem 1.1 as $\omega(s_p)$ lead to the three cases in Part 2 of the theorem respectively. □

Theorem 2.7 is indeed restricting the third case of Theorem 1.1, for at least one of the ω or α limit sets. Note that if, in addition, x^* is hyperbolic and the vector field contains no closed orbits, then for any point $p \in \mathcal{M}$, either $\omega(p)$ is a fixed point or $\alpha(p) = \{x^*\}$. See Fig. 2.2.(b) and (c) for two examples. We highlight that the first case in Theorem 2.7 may not cover all possibilities for $\omega(p)$ (see Fig. 2.3); however, then the second case of the Theorem will be in force, determining the structure of $\alpha(p)$.

It is also worth mentioning that some cases in Part 1 and Part 2 of Theorem 2.7 never take place at the same time. For example, it is impossible to have both $\omega(p)$ and $\alpha(p)$ being the union of $\{x^*\}$ and a homoclinic orbit joining x^* to itself. We exclude such cases for general positively invariant compact regions as follows. A point is *periodic* if it is on a closed orbit.

Proposition 2.8. *Let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set under the vector field* (1.1)*. For any non-periodic point $p \in \mathcal{M}$, if $\omega(p) = \alpha(p)$, then the limit sets contain only fixed points.*

Either Lemma 9.0.2 in [23] or the results on the characterization of non-periodic orbits in [6] can be used for the proof, which we skip here. In case $\mathcal{M}$ contains finitely many fixed points, we can sharpen the result of Proposition 2.8 by using Proposition 8.1.3 in [23].

Corollary 2.9. *For the vector field* (1.1)*, let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set containing a finite number of fixed points. Then for any non-periodic point $p \in \mathcal{M}$, if $\omega(p) = \alpha(p)$, then the limit sets exclusively contain a single fixed point.*

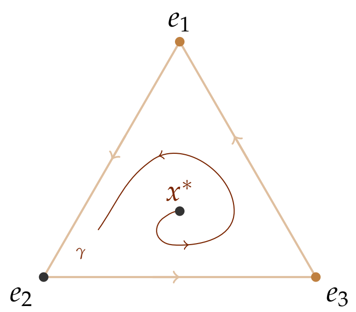

Figure 2.3: Phase portrait example for an invariant compact set $\mathcal{M}$ defined by the triangle $e_1e_2e_3$, where e_1, e_2 and e_3 are fixed points. There is exactly one interior fixed point, g. For every point p in the interior of $\mathcal{M}$, the ω-limit set of p equals $\partial\mathcal{M}$, that is the union of the fixed points e_1, e_2 and e_3 and the heteroclinic orbits connecting them to each other. This is not covered by the first case of Theorem 2.7. However, $\alpha(p) = \{x^*\}$, which is satisfied by the second case of Theorem 2.7.

2.4 $\mathcal{M}$ has finitely many interior fixed points

Following the previous subsection of having one interior fixed point in the positively invariant compact set $\mathcal{M}$, we now extend the result to the more general case of having finitely many interior fixed points in $\mathcal{M}$.

Theorem 2.10 (Finitely many interior fixed points)**.** *For the vector field* (1.1), *consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ containing a finite number of fixed points. Then for any point $p \in \mathcal{M}$, at least one of the following holds:*

1. *$\omega(p)$ is a fixed point, a closed orbit encircling at least one interior fixed point or the union of some interior fixed points together with the orbits connecting them;*
2. *$\alpha(p)$ is an interior fixed point, a closed orbit encircling at least one interior fixed point or the union of some interior fixed points together with the orbits connecting them.*

Proof. The proof is similar to that for Theorem 2.7 and we omit it here. □

Compared to the classical form of Poincaré–Bendixson Theorem 1.1, what Theorem 2.10 has further clarified is the role of the interior fixed points of $\mathcal{M}$ play to influence the topological structure of the limit sets. For example, as an immediate result of Theorem 2.10, if the third case of Theorem 1.1 takes place for p, then $\omega(p)$ and $\alpha(p)$ cannot be free of interior fixed points at the same time; in other words, unless $\omega(p)$ is simply a fixed point or a closed orbit, some interior fixed points must be in either $\omega(p)$ or $\alpha(p)$. Another implication of Theorem 2.10 is the exclusion of the boundary fixed points from one of $\omega(p)$ and $\alpha(p)$. From Theorem 2.10, if $\omega(p)$ is not simply a fixed point, then at least one of $\omega(p)$ or $\alpha(p)$ does not contain any boundary fixed point. In a sense, this implies that the interior fixed points are more important for determining the structures of the limit sets. Finally, we note that Corollary 2.9 can also be utilized here to rule out some trivial cases when $\omega(p)$ and $\alpha(p)$ are the same.

At the end of this section, we present the following version of Theorem 2.10 without requiring $\mathcal{M}$ to be simply connected.

Theorem 2.11. *For the vector field (1.1), consider a positively invariant, compact set $\mathcal{M} \subset \mathcal{U}$ that contains a finite number of fixed points. Then for any $p \in \mathcal{M}$, at least one of the following holds:*

1. *$\omega(p)$ is a fixed point, a closed orbit or the union of some interior fixed points with the orbits connecting them;*

2. *$\alpha(p)$ is one of the interior fixed points, a closed orbit or the union of some interior fixed points with the orbits connecting them.*

Proof. The proof is similar to that of Theorem 2.7. The difference is that if $\omega(p)$ or $\alpha(p)$ is a closed orbit, it may encircle areas that do not belong to $\mathcal{M}$. □

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