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Existence and uniqueness of solutions of a fourth-order boundary value problem with non-homogeneous boundary conditions

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Abstract. Let $m \geq 2$ and $a, b, c > 0$. We consider the existence and uniqueness of solutions for the fourth order iterative boundary value problem,

$$x^{(4)}(t) = -f(t, x(t), x^{[2]}(t), \dots, x^{[m]}(t)), \qquad -a \leq t \leq a$$

where $x^{[2]}(t) = x(x(t))$ and for $j = 3, \dots, m$, $x^{[j]}(t) = x(x^{[j-1]}(t))$, with solutions satisfying one of the following sets of conjugate boundary conditions:

$$x(-a) = -a, \qquad x'(-a) = b, \qquad x''(-a) = c, \qquad x(a) = a,$$
$$x(-a) = -a, \qquad x(a) = a, \qquad x'(a) = b, \qquad x''(a) = c.$$

The main tool used is the Schauder fixed point theorem.

Keywords: differential equations, iterative differential equations, boundary value problems.

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1 Introduction

In this paper we consider existence and uniqueness of solutions for the fourth-order iterative boundary value problem,

$$x^{(4)}(t) = -f(t, x(t), x^{[2]}(t), \dots, x^{[m]}(t)), \qquad -a \leq t \leq a \tag{1.1}$$

where $x^{[2]}(t) = x(x(t))$, and for $j = 3, \dots, m, x^{[j]}(t) = x(x^{[j-1]})(t)$, with solutions satisfying one of the boundary conditions:

$$x(-a) = -a, \qquad x'(-a) = b, \qquad x''(-a) = c, \qquad x(a) = a, \tag{1.2}$$
$$x(-a) = -a, \qquad x(a) = a, \qquad x'(a) = b, \qquad x''(a) = c. \tag{1.3}$$

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We assume throughout that $f : [-a, a] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Iterative differential equations are a special case of state-dependent differential equations. They have applications in a wide variety of fields, including climate change [12], economics [6], electrodynamics [4], infectious diseases [8], mechanical models [9], neural networks [2], and population dynamics [13].

One of the earliest works in iterative differential equations was by Petuhov [14] who, in 1965, studied existence and uniqueness of solutions of $x''(t) = \lambda x(x(t))$ with the condition x maps the interval $[-T, T]$ into itself, and that $x(0) = x(T) = \alpha$. Eder [5] then studied the existence, uniqueness, and analyticity of solutions of $x'(t) = x(x(t))$, proving that every solution is either monotonic or vanishes. In 1990, Wang [16] obtained a solution to $x'(t) = f(x(x(t))), x(a) = a$ using Schauder's fixed point theorem, and in 1993 Fečkan [7] used the Contraction Mapping Principle to show existence of local solutions of $x'(t) = f(x(x(t)), x(0) = 0$.

More recently, Kaufmann [10] established existence and uniqueness results for the second-order boundary value problem $x''(t) = f(t, x(t), x^{[m]}(t)), x(a) = a,\ x(b) = b$ using Schauder's fixed point theorem and the Contraction Mapping Principle. In 2020, Cheraiet, Bouakkaz, and Khemis [3] studied the third-order equation $x''(t) + f(t, x(t), x^{[2]}(t) \ldots, x^{[n]}(t)) = 0$ with conditions $x(0) = x''(0) = 0, \alpha \int_0^\eta x(t)\,dt = x(T)$ with $\eta \in (0, T)$. Meanwhile, in 2022, Kaufmann [11] considered the fourth-order equation $x^{(4)}(t) = f(t, x(t), x^{[2]}(t), \ldots, x^{[m]}(t))$ subject to the Lidstone conditions $x(a) = x(-a) = x''(a) = x''(-a) = 0$, establishing conditions for existence and uniqueness of solutions. The main goal of this paper is to further the results of [11].

In Section 2, we will rewrite (1.1), (1.2) as an integral equation, and state conditions under which the solution of the integral equation will be a solution of the boundary value problem. We will also state properties of the Green's function and of the norm of the difference of two iterative functions. In Section 3, we will state and prove results concerning the existence and uniqueness of solutions of (1.1), (1.2). In Section 4, we present the equivalent inversion of (1.1), (1.3) and state, without proof, the analogous existence and uniqueness results. Examples will be included to illustrate results.

2 Preliminaries

Our main goal of Section 2 is to invert (1.1), (1.2) into an integral equation. We will accomplish this by first inverting the non-homogeneous equation with homogeneous boundary conditions, and then solving the homogeneous equation with non-homogeneous boundary conditions. The inversion of (1.1), (1.2) will be the sum of the two expressions. We will end the section with a lemma on the Green's function and the norm of the difference of iterations, and then a statement of Schauder's fixed point theorem.

We will begin the inversion by considering

$$x^{(4)}(t) = -g(t), \quad -a \le t \le a, \tag{2.1}$$

$$x(-a) = x'(-a) = x''(-a) = x(a) = 0. \tag{2.2}$$

Integrate $x^{(4)}(t) = -g(t))$ from $-a$ to t twice and apply the boundary condition $x''(-a) = 0$.

$$x''(t) = x'''(-a)(t+a) - \int_{-a}^{t} (t-s)g(s)\,ds. \tag{2.3}$$

Integrating (2.3) and applying the condition $x'(-a) = 0$ yields,

$$x'(t) = x'''(-a)\frac{(t+a)^2}{2} - \int_{-a}^{t} \frac{(t-s)^2}{2} g(s)\, ds.$$

When we integrate once more and apply the condition $x(-a) = 0$, we obtain,

$$x(t) = x'''(-a)\frac{(t+a)^3}{6} - \int_{-a}^{t} \frac{(t-s)^3}{6} g(s)\, ds. \tag{2.4}$$

The constant $x'''(-a)$ is found by applying the condition $x(a) = 0$,

$$x'''(-a) = \int_{-a}^{a} \frac{(a-s)^3}{8a^3} g(s)\, ds. \tag{2.5}$$

When we plug (2.5) into (2.4) we get,

$$x(t) = \int_{-a}^{a} \frac{(a-s)^3(t+a)^3}{48a^3} g(s)\, ds - \int_{-a}^{t} \frac{(t-s)^3}{6} g(s)\, ds.$$

Finally, we can split the first integral and combine it with the second to obtain

$$x(t) = \int_{-a}^{t} \frac{(a-s)^3(t+a)^3 - 8a^3(t-s)^3}{48a^3} g(s)\, ds + \int_{t}^{a} \frac{(a-s)^3(t+a)^3}{48a^3} g(s)\, ds.$$

Thus, we have shown that if x is a solution to (2.1), (2.2), then x satisfies the integral equation

$$x(t) = \int_{-a}^{a} G(t,s) g(s)\, ds \tag{2.6}$$

where

$$G(t,s) = \frac{1}{48a^3} \begin{cases} (a-s)^3(t+a)^3 - 8a^3(t-s)^3, & -a \le s \le t \le a, \\ (a-s)^3(t+a)^3, & -a \le t \le s \le a. \end{cases} \tag{2.7}$$

It is easy to show that if x is a solution of

$$\begin{aligned} &x^{(4)}(t) = 0, \\ &x(-a) = -a, \quad x'(-a) = b, \quad x''(-a) = c, \quad x(a) = a, \end{aligned}$$

then x is given by

$$x(t) = -a + b(t+a) + \frac{c}{2}(t+a)^2 + \frac{1-b-ac}{4a^2}(t+a)^3. \tag{2.8}$$

Consequently, if x is a solution of (1.1), (1.2), then x will then be the sum of (2.7) and (2.8). That is, x is a solution of the integral equation

$$\begin{aligned} x(t) = &\int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s))\, ds \\ &- a + b(t+a) + \frac{c}{2}(t+a)^2 + \frac{1-b-ac}{4a^2}(t+a)^3, \end{aligned}$$

where $G(t,s)$ is given in (2.7).

In order for solutions to be well-defined, we also require the image of x be in the interval $[-a,a]$; that is, in order for $x(x^{[m]})(t)$ to be defined, we need $-a \le x(t) \le a$ for all $t \in [-a,a]$. Knowing this, we can show that if $x \in C[-a,a]$, satisfies $-a \le x(t) \le a$ for all t, and satisfies the integral equation (**??**), then it satisfies (1.1), (1.2). This gives us the following lemma.

Lemma 2.1. *The function $x \in C^4[-a,a]$ is a solution of* (1.1), (1.2) *if and only if $x \in C[-a,a]$ satisfies $-a \le x(t) \le a$, and the integral equation*

$$x(t) = \int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s))\, ds \\ - a + b(t+a) + \frac{c}{2}(t+a)^2 + \frac{1-b-ac}{4a^2}(t+a)^3,$$

where $G(t,s)$ is defined in (2.7).

In order to prove the existence and uniqueness of solutions of (1.1), (1.2), we will need to have a bound for our Green's functions. For that, we will use the following lemma.

Lemma 2.2. *The Green's function given in* (2.7) *satisfies the following inequality:*

$$0 \le G(t,s) \le \frac{4a^3}{3}.$$

Proof. First note that $(a-s)^3(t+a)^3$ is an increasing function of t, so $(a-s)^3(t+a)^3 \le (a-s)^3(s+a)^3$. Since $\max_{s\in[-a,a]}(a-s)^3(s+a)^3$ occurs when $s = 0$, and equals a^6, then $\frac{(a-s)^3(t+a)^3}{48a^3} \le \frac{a^3}{48}$.

Now consider the function $(a-s)^3(t+a)^3 - 8a^3(t-s)^3, s \le t$. Since $(t-s)^3$ is positive when $s \le t$, then $(a-s)^3(t+a)^3 - 8a^3(t-s)^3 \le (a-s)^3(t+a)^3$. Now, $(a-s)^3(t+a)^3$ is an increasing function of t, so $(a-s)^3(t+a)^3 \le 8a^3(a-s)^3$. But, $8a^3(a-s)^3$ is a decreasing function of s for $-a \le s$, so $8a^3(a-s)^3 \le 64a^6$. That is, $\frac{(a-s)^3(t+a)^3-8a^3(t-s)^3}{48a^3} \le \frac{64a^6}{48a^3} = \frac{4a^3}{3}$. Finally, since $\frac{a^3}{48} < \frac{4a^3}{3}$, we obtain that the upper bound on our Green's function is $\frac{4a^3}{3}$.

Similar procedures can be used to obtain the lower bound on our Green's function, that $0 \le G(t,s)$. □

We will use the Banach space $\Phi = (C[-a,a], \|\cdot\|)$ with the norm $\|x\| = \max_{t\in[-a,a]} |x(t)|$. Define the operator $T : C[-a,a] \to C[-a,a]$ by

$$(Tx)(t) = \int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s))\, ds \\ - a + b(t+a) + \frac{c}{2}(t+a)^2 + \frac{1-b-ac}{4a^2}(t+a)^3 \qquad (2.9)$$

where $G(t,s)$ is defined in (2.7).

We will also need the subspace

$$\Phi(J,M) = \{x \in \Phi : \|x\| \le J, |x(t_2) - x(t_1)| \le M|t_1 - t_2|, t_1, t_2 \in [-a,a]\}.$$

as well as the following lemma, which is proved in [17], [15].

Lemma 2.3. *If $x, y \in \Phi(J,M)$, then*

$$|x^{[m]}(t_1) - x^{[m]}(t_2)| \le M^m|t_1 - t_2|, \qquad m = 0,1,2,\ldots,$$

for all $t_1, t_2 \in [-a,a]$ and

$$\|x^{[m]}(t_1) - x^{[m]}(t_2)\| \le \sum_{k=0}^{m-1} M^k \|x - y\|.$$

We end this section by stating Schauder's fixed point theorem [1].

Theorem 2.4 (Schauder)**.** *Let A be a nonempty compact convex subset of a Banach space and let $T : A \to A$ be continuous. Then T has a fixed point in A.*

3 Existence and uniqueness results for (1.1), (1.2)

In this section, we will state and prove our existence and uniqueness results for (1.1), (1.2). Let $T : C[-a,a] \to C[-a,a]$ be defined as (2.9). Throughout the section we will assume the following conditions hold.

(H1) There exists an $\alpha_\ell \in L[-a,a], \ell = 1,2,\ldots,m+1$, such that

$$|f(t,x_1,\ldots,x_{m+1}) - f(t,y_1,\ldots,y_{m+1})| \leq \sum_{\ell=1}^{m+1} \alpha_\ell(t)\|x_\ell - y_\ell\|$$

for all $t \in [-a,a]$ and $x_i, y_i \in \mathbb{R}, i = 1,2,\ldots,m+1$.

(H2) There exists a $K \in \mathbb{R}$ such that $0 < K < \frac{3(1-b-ac)}{a^3}$ and

$$-K \leq f\big(t, x(t), x^{[2]}(t), \ldots, x^{[m]}(t)\big) < 0$$

for all $t \in [-a,a]$.

Notice that $(H2)$ puts further conditions on b and c, namely that $1 > b + ac > 0$.

Theorem 3.1. *Suppose that condition (H1) and (H2) holds. Then there exists a solution to* (1.1), (1.2).

Proof. Consider the convex, compact nonempty set $\Phi(a,M)$, where

$$M = |3 - 2b - ac| + K\left(6a^3 + \frac{1}{18}\right).$$

To use the Schauder fixed point theorem, we need for $T : \Phi(a,M) \to \Phi(a,M)$. We first show that $-a \leq (Tx)(t) \leq a$ for all $t \in [-a,a]$.

$$\begin{aligned}
(Tx)'(t) &= \frac{1}{16a^3}\int_{-a}^{a}(a-s)^3(t+a)^2 f(s)\,ds - \frac{1}{2}\int_{-a}^{a}(t-s)^2 f(s)\,ds \\
&\quad + \frac{3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b \\
&\geq \frac{-K}{16a^3}\int_{-a}^{a}(a-s)^3(t+a)^2\,ds + \frac{3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b \\
&\geq \frac{-Ka}{4}(t+a)^2 + \frac{3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b \\
&= \frac{-Ka^3 + 3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b.
\end{aligned}$$

Since $(H2)$ holds, then

$$\frac{-Ka^3 + 3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b > 0$$

for all $t \in [-a,a]$. That is, $(Tx)'(t) > 0$ and hence $(Tx)(t)$ is a strictly increasing function of t. Since $(Tx)(\pm a) = \pm a$, then $-a \leq (Tx)(t) \leq a$ and furthermore $\|Tx\| \leq a$.

We need to show that for given $t_1, t_2 \in [-a, a]$, $\big|(Tx)(t_2) - (Tx)(t_1)\big| \le M|t_2 - t_1|$, where M is defined as above. We may assume, without loss of generality, that $t_2 \le t_1$. To this end we first note that

$$
\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| = {} & \int_{-a}^{a} |G(t_2,s) - G(t_1,s)||f(s, x(s), x^{[2]}(s), \dots, x^{[m]}(s)|\,ds \\
& + b\big((t_2+a) - (t_1+a)\big) + \frac{c}{2}\big((t_2+a)^2 - (t_1+a)^2\big) \\
& + \frac{1-b-ac}{4a^2}\big((t_2+a)^3 - (t_1+a)^3\big) \\
\le {} & K\int_{-a}^{a} \big|G(t_2,s) - G(t_1,s)\big|\,ds \\
& + b|t_2 - t_1| + 2ac|t_2 - t_1| + \frac{12a^2(1-b-ac)}{4a^2}|t_2 - t_1| \\
\le {} & K\int_{-a}^{a} |G(t_2,s) - G(t_1,s)|\,ds + (3 - 2b - ac)|t_2 - t_1|.
\end{aligned}
$$

Now consider $\int_{-a}^{a} |G(t_2,s) - G(t_1,s)|\,ds$. Since $t_2 \le t_1$, we can rewrite the integral as

$$
\begin{aligned}
\int_{-a}^{a} \big|G(t_2,s) - G(t_1,s)\big|\,ds \le {} & \int_{-a}^{t_1} \big|G(t_2,s) - G(t_1,s)\big|\,ds + \int_{t_1}^{t_2} \big|G(t_2,s) - G(t_1,s)\big|\,ds \\
& + \int_{t_2}^{a} \big|G(t_2,s) - G(t_1,s)\big|\,ds.
\end{aligned}
$$

Given that $t_1 \le t_2$, the first term on the right satisfies

$$
\begin{aligned}
& \int_{-a}^{t_1} |G(t_2,s) - G(t_1,s)|\,ds \\
& \quad \le \frac{1}{48a^3}\int_{-a}^{t_1} \big|(a-s)^3\big((t_2+a)^3 - (t_1+a)^3\big)\big| + 8a^3\big|(t_2-s)^3 - (t_1-s)^3\big|\,ds \\
& \quad \le \frac{1}{48a^3}\left(\left(4a^4 - \frac{(a-t_1)^4}{4}\right)(12a^2)\right)|t_2 - t_1| + \frac{1}{48a^3}(192a^6)|t_2 - t_1| \\
& \quad \le 5a^3|t_2 - t_1|.
\end{aligned}
$$

Also, due to the bound on our Green's function,

$$
\begin{aligned}
\int_{t_1}^{t_2} \big|G(t_2,s) - G(t_1,s)\big|\,ds \le & \frac{1}{48a^3}\left(\frac{8a^3}{3}\right)|t_2 - t_1| \\
\le & \frac{1}{18}|t_2 - t_1|.
\end{aligned}
$$

And finally,

$$
\begin{aligned}
\int_{t_2}^{a} |G(t_2,s) - G(t_1,s)|\,ds & \le \frac{1}{48a^3}\int_{t_2}^{a} \big|(a-s)^3\big((t_2+a)^3 - (t_1+a)^3\big)\big|\,ds \\
& \le \frac{1}{48a^3}\left(\frac{(a-t_2)^4}{4}(12a^2)\right)|t_2 - t_1| \\
& \le a^3|t_2 - t_1|.
\end{aligned}
$$

That is,

$$
\int_{-a}^{a} |G(t_2,s) - G(t_1,s)|\,ds \le \left(6a^3 + \frac{1}{18}\right)|t_2 - t_1|.
$$

Consequently,

$$|(Tx)(t_2) - (Tx)(t_1)| \le \left(|3-2b-ac| + K\left(6a^3 + \frac{1}{18}\right)\right)|t_2 - t_1|$$
$$= M|t_2 - t_1|.$$

Therefore, $T : \Phi(a, M) \to \Phi(a, M)$.

Lastly, it can be shown through standard arguments that T is continuous. Hence, by Schauder's fixed point theorem, there is a fixed point x of T, $(Tx)(t) = x(t)$, which by Lemma 2.1 is a solution of (1.1), (1.2). □

Example 3.2. *Consider the following boundary value problem with parameter k.*

$$x^{(4)}(t) = kt^2 \cos(x^{[2]}(t)) \tag{3.1}$$
$$x\left(-\frac{\pi}{3}\right) = -\frac{\pi}{3}, \quad x\left(\frac{\pi}{3}\right) = \frac{\pi}{3}, \tag{3.2}$$
$$x'\left(-\frac{\pi}{3}\right) = \frac{2}{3}, \quad x''\left(-\frac{\pi}{3}\right) = \frac{1}{\pi^2}. \tag{3.3}$$

Here, $m = 2$ and $f(t, x, x^{[2]}) = -kt^2\cos(x^{[2]})$. Let $\alpha(t) = kt^2$. Then,

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le \alpha(t)|x_2 - y_2| \tag{3.4}$$

for all $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. Also, $-k \le f(t, x, x^{[2]}) \le 0$. So, for all $0 < k < \frac{3(1-b-ac)}{a^3} = \frac{27(\pi-1)}{\pi^4} \approx$ 0.5936099, there exists a solution to (3.1), (3.2), (3.3), according to Theorem 3.1.

We are now ready for our uniqueness result.

Theorem 3.3. *Suppose that (H1) and (H2) hold and that*

$$\frac{4a^3}{3}\sum_{\ell=1}^{m+1}\int_{-a}^{a}\alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}M^k < 1. \tag{3.5}$$

Then, there exists a unique solution to (1.1), (1.2).

Proof. By Theorem 3.1 and Lemma 2.1, there exists a solution of (1.1), (1.2), which is a fixed point of T. Assume x and y are two distinct fixed points of T. Then,

$$\begin{aligned}
\|x-y\| &= |(Tx)(t) - (Ty)(t)| \\
&\le \frac{4a^3}{3}\int_{-a}^{a}\sum_{\ell=1}^{m+1}\alpha_\ell(s)\|x^{[\ell]} - y^{[\ell]}\|\,ds \\
&\le \left(\frac{4a^3}{3}\int_{-a}^{a}\sum_{\ell=1}^{m+1}\alpha_\ell(s)\sum_{k=0}^{l-1}M^k\,ds\right)\|x-y\| \\
&< \|x-y\|
\end{aligned}$$

This contradiction implies $x = y$, and our fixed point is unique. □

It should be noted that the results in Theorem 3.3 can also be obtained using the Banach fixed point theorem.

Example 3.4. To illustrate our uniqueness result, again consider the boundary value problem (3.1), (3.2), (3.3). Again, note that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le kt^2|x_2 - y_2|$$

for all $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. So, $\alpha_0(t) = \alpha_1(t) = 0$ and $\alpha_2(t) = kt^2$. The left side of (3.5) becomes

$$\begin{aligned} \frac{4a^3}{3}\sum_{\ell=1}^{m+1}\int_{-a}^{a}\alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}M^k &= \frac{4(\frac{\pi}{3})^3}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} ks^2\,ds\,\left(1+M+M^2\right) \\ &= \frac{4\pi^3}{81}\frac{2k\pi^3}{81}\left(1+M+M^2\right) \\ &= \frac{8\pi^6}{6561}\left(1+M+M^2\right)k. \end{aligned}$$

In our case, $M < 5.8958889$. So, whenever $k < \frac{6561}{8\pi^6(1+M+M^2)} \approx 0.020478$, there exists a unique solution to (3.1), (3.2), (3.3) according to Theorem 3.3.

4 Other results

In this section, we give the corresponding results from Section 3 for (1.1), (1.3). The proof of the results in this section are similar to those found in Section 3. As such, we only point out the main differences. We begin by considering the boundary value problem (1.1), (1.3).

As in Section 2, we can show that if x is a solution of (1.1), (1.3), then $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) = &\int_{-a}^{a} G(t,s)f(s,x(s),x^{[2]}(s),\dots,x^{[m]}(s))\,ds \\ &+ a - b(a-t) + \frac{c}{2}(a-t)^2 - \frac{1-b+ac}{4a^2}(a-t)^3, \end{aligned} \tag{4.1}$$

where

$$G(t,s) = \frac{1}{48a^3}\begin{cases} (a-t)^3(s+a)^3, & -a \le s \le t \le a, \\ (a-t)^3(s+a)^3 - 8a^3(s-t)^3, & -a \le t \le s \le a. \end{cases} \tag{4.2}$$

The Green's function $G(t,s)$ in (4.2) satisfies Lemma 2.2.

In addition to $(H1)$, we will need the following condition.

(H3) There exists an $L \in \mathbb{R}$ such that $0 < L < \frac{3(1-b+ac)}{a^3}$ and

$$0 < f(t,x(t),x^{[2]}(t),\dots,x^{[m]}(t)) < L$$

for all $t \in [-a,a]$.

Notice that $(H3)$ puts further conditions on b and c, namely that $1 + ac > b > 0$.

Theorem 4.1. *Suppose that conditions $(H1)$ and $(H3)$ hold. Then there exists a solution to (1.1), (1.3).*

Proof. For this proof, the space $\Phi(a,M)$ where $M = |b| + L\left(\frac{1}{18}\right)$ is needed. The rest of the proof follows the sames steps as Theorem 3.1. □

Example 4.2. *Consider the following boundary value problem with parameter k.*

$$x^{(4)}(t) = -kt^2 \cos(x^{[2]}(t)) \tag{4.3}$$

$$x\left(-\frac{\pi}{3}\right) = -\frac{\pi}{3}, \quad x\left(\frac{\pi}{3}\right) = \frac{\pi}{3}, \tag{4.4}$$

$$x'\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad x''\left(\frac{\pi}{3}\right) = \frac{1}{\pi^2}. \tag{4.5}$$

Here, $m = 2$ *and* $f(t, x, x^{[2]}) = kt^2 \cos(x^{[2]})$. *Let* $\alpha(t) = kt^2$. *Then,*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le \alpha(t)|x_2 - y_2|$$

for all $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. *Also,* $0 \le f(t, x, x^{[2]}) \le L$. *So, for all* $0 < k < \frac{3(1-b+ac)}{a^3} = \frac{243\pi+162}{6\pi^4} \approx 1.583369$, *there exists a solution to* (4.3), (4.4), (4.5), *according to Theorem* 4.1.

Theorem 4.3. *Suppose that* (H1) *and* (H3) *hold and that*

$$\frac{4a^3}{3} \sum_{\ell=1}^{m+1} \int_{-a}^{a} \alpha_\ell(s)\, ds \sum_{k=0}^{l-1} M^k < 1. \tag{4.6}$$

Then, there exists a unique solution to (1.1), (1.3).

Example 4.4. *To illustrate our uniqueness result, again consider the boundary value problem* (4.3), (4.4), (4.5). *Again, note that*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le kt^2|x_2 - y_2|$$

for all $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. *So,* $\alpha_0(t) = \alpha_1(t) = 0$ *and* $\alpha_2(t) = kt^2$. *The left side of* (4.6) *becomes*

$$\begin{aligned}
\frac{4a^3}{3} \sum_{\ell=1}^{m+1} \int_{-a}^{a} \alpha_\ell(s)\, ds \sum_{k=0}^{\ell-1} M^k &= \frac{4(\frac{\pi}{3})^3}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} ks^2\, ds \left(1 + M + M^2\right) \\
&= \frac{4\pi^3}{81} \frac{2k\pi^3}{81} \left(1 + M + M^2\right) \\
&= \frac{8\pi^6}{6561} \left(1 + M + M^2\right) k.
\end{aligned}$$

In this example, $M < .5879649$. *So, whenever* $k < \frac{6561}{8\pi^6(1+M+M^2)} \approx 0.4411629$, *there exists a unique solution to* (4.3), (4.4), (4.5) *according to Theorem* 4.3.

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