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Exact solution of the Susceptible–Exposed–Infectious–Recovered–Deceased (SEIRD) epidemic model

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Abstract. An exact solution of an initial value problem for the Susceptible–Exposed–Infectious–Recovered–Deceased (SEIRD) epidemic model is derived, and various properties of the exact solution are obtained. It is shown that the parametric form of the exact solution satisfies some linear differential system including a positive solution of an Abel differential equation of the second kind. In this paper Abel differential equations play an important role in establishing the exact solution of the SEIRD differential system, in particular the number of infected individuals can be represented in a simple form by using a positive solution of an initial value problem for an Abel differential equation. Uniqueness of positive solutions of an initial value problem to SEIRD differential system is also investigated, and it is shown that the exact solution is a unique solution in the class of positive solutions.

Keywords: exact solution, SEIRD epidemic model, initial value problem, linear differential system, Abel differential equation, uniqueness of positive solutions.

2020 Mathematics Subject Classification: 34A34.

1 Introduction

Recently there is an increasing interest in mathematical approach to the epidemic models. Since the pioneering work of Bernoulli [2], a vast literature and research papers has been published so far (cf. [4, 5, 9]), and studies of epidemic models have become one of the important areas in mathematical biology. In particular we mention Kermack and McKendrick [11] in which the Susceptible-Infectious-Recovered (SIR) epidemic model was proposed. Exact solutions of epidemic models have been investigated in recent years. We refer to Bohner, Streipert and Torres [3], Harko, Lobo and Mak [10] and Shabbir, Khan and Sadiq [16] and Yoshida [19] for SIR epidemic models, to Yoshida [18] for Susceptible–Infectious–Recovered–Deceased (SIRD) epidemic models, and to Yoshida [20] for Susceptible-Exposed-Infectious-Recovered (SEIR) epidemic models.

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The Susceptible–Exposed–Infectious–Recovered–Deceased (SEIRD) epidemic models have been an important and interesting subject to study (cf. [6, 12–15, 17]). However, there appears to be no known results about exact solutions of SEIRD epidemic models. The objective of this paper is to establish an exact solution of an initial value problem for SEIRD epidemic model. Our method is an adaptation of that used in Yoshida [20], and is based on the existence of unique positive solution of an initial value problem for Abel differential equations of the second kind. We refer the reader to Abel [1] and Davis [8] for Abel differential equations. Uniqueness of positive solutions of an initial value problem to SEIRD differential system is also studied, and we find that the exact solution is a unique solution in the class of positive solutions.

We study the Susceptible–Expose–Infectious–Recovered–Deceased (SEIRD) epidemic model

$$\frac{dS(t)}{dt} = -\beta S(t)I(t), \tag{1.1}$$

$$\frac{dE(t)}{dt} = \beta S(t)I(t) - \delta E(t), \tag{1.2}$$

$$\frac{dI(t)}{dt} = \delta E(t) - \gamma I(t) - \mu I(t), \tag{1.3}$$

$$\frac{dR(t)}{dt} = \gamma I(t), \tag{1.4}$$

$$\frac{dD(t)}{dt} = \mu I(t) \tag{1.5}$$

for $t > 0$, where β, γ, δ and μ are positive constants. The initial condition to be considered is the following:

$$S(0) = \tilde{S}, \quad E(0) = \tilde{E}, \quad I(0) = \tilde{I}, \quad R(0) = \tilde{R},\ D(0) = \tilde{D}. \tag{1.6}$$

It is assumed throughout this paper that:

(A_1) $\tilde{I} > 0$;

(A_2) $\tilde{S} > \frac{\delta \tilde{E}}{\beta \tilde{I}}$;

(A_3) $\tilde{E} > \frac{\gamma+\mu}{\delta}\tilde{I}$;

(A_4) $\tilde{R} \geq 0$;

(A_5) $\tilde{D} \geq 0$ and $\tilde{D}$ satisfies

$$N - \tilde{R} > \tilde{S}e^{(\beta/\mu)\tilde{D}} + \tilde{D};$$

(A_6) $\tilde{S} + \tilde{E} + \tilde{I} + \tilde{R} + \tilde{D} = N$ (positive constant).

In Section 2 we obtain a parametric solution of an initial value problem for SEIRD differential system, and in Section 3 we derive an exact solution of an initial value problem for SEIRD differential system. Section 4 is devoted to various properties of the exact solution of SEIRD differential system. In Section 5 we show that there exists one, and only one, solution of an initial value problem for SEIRD differential system in the class of positive solutions.

2 Parametric solution of an initial value problem for SEIRD differential system

In this section we show that a positive solution of the initial value problem (1.1)–(1.6) can be represented in a parametric form.

Since

$$\frac{d}{dt}\left(S(t)+E(t)+I(t)+R(t)+D(t)\right)=\frac{dS(t)}{dt}+\frac{dE(t)}{dt}+\frac{dI(t)}{dt}+\frac{dR(t)}{dt}+\frac{dD(t)}{dt}=0$$

by (1.1)–(1.5), we obtain

$$S(t)+E(t)+I(t)+R(t)+D(t)=k \qquad (t\geq 0)$$

for some constant k. The hypothesis (A_6) implies

$$k=S(0)+E(0)+I(0)+R(0)+D(0)=\tilde{S}+\tilde{E}+\tilde{I}+\tilde{R}+\tilde{D}=N,$$

and therefore

$$S(t)+E(t)+I(t)+R(t)+D(t)=N \qquad (t\geq 0).$$

We state the following important lemma.

Lemma 2.1. *If $(S(t),E(t),I(t),R(t),D(t))$ is a solution of the SEIRD differential system (1.1)–(1.5) such that $S(t)>0$ for $t>0$, then*

$$D''(t)+(\delta+\gamma+\mu)D'(t)=\delta\mu\left(N-\tilde{R}+\frac{\gamma}{\mu}\tilde{D}-\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)}-\left(1+\frac{\gamma}{\mu}\right)D(t)\right) \tag{2.1}$$

for $t>0$.

Proof. We see from (1.1) and (1.5) that

$$D'(t)=\mu I(t)=\mu\left(\frac{S'(t)}{-\beta S(t)}\right)=-\frac{\mu}{\beta}(\log S(t))',$$

and integrating the above on $[0,t]$ gives

$$D(t)-\tilde{D}=-\frac{\mu}{\beta}(\log S(t)-\log\tilde{S}).$$

Hence we obtain

$$\log S(t)=-\frac{\beta}{\mu}(D(t)-\tilde{D})+\log\tilde{S}$$

and therefore

$$S(t)=\exp\left(\log\tilde{S}-\frac{\beta}{\mu}D(t)+\frac{\beta}{\mu}\tilde{D}\right)=\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)}. \tag{2.2}$$

It follows from (1.5) that $I(t)=D'(t)/\mu$, and hence $I'(t)=D''(t)/\mu$. Therefore, (1.3) implies that

$$\begin{aligned}E(t)&=\frac{1}{\delta}\left(I'(t)+(\gamma+\mu)I(t)\right)\\&=\frac{1}{\delta}\left(\frac{D''(t)}{\mu}+(\gamma+\mu)\frac{D'(t)}{\mu}\right)\\&=\frac{1}{\delta\mu}\left(D''(t)+(\gamma+\mu)D'(t)\right).\end{aligned} \tag{2.3}$$

It is obvious that

$$R'(t) = \gamma I(t) = \gamma \frac{D'(t)}{\mu} = \frac{\gamma}{\mu} D'(t),$$

and hence

$$R(t) = \frac{\gamma}{\mu} D(t) + k$$

for some constant k. Letting $t = 0$ yields

$$k = \tilde{R} - \frac{\gamma}{\mu} \tilde{D},$$

and therefore

$$R(t) = \frac{\gamma}{\mu} D(t) + \tilde{R} - \frac{\gamma}{\mu} \tilde{D}. \tag{2.4}$$

We observe, using (1.5), (2.2)–(2.4), that

$$\begin{aligned} \frac{D'(t)}{\mu} &= I(t) \\ &= N - S(t) - E(t) - R(t) - D(t) \\ &= N - \tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu)D(t)} - \frac{1}{\delta\mu} \left(D''(t) + (\gamma + \mu) D'(t) \right) - \frac{\gamma}{\mu} D(t) - \tilde{R} + \frac{\gamma}{\mu} \tilde{D} - D(t) \end{aligned}$$

which implies

$$\frac{1}{\delta\mu} D''(t) + \left(\frac{1}{\mu} + \frac{\gamma + \mu}{\delta\mu} \right) D'(t) = N - \tilde{R} + \frac{\gamma}{\mu} \tilde{D} - \tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu)D(t)} - \left(1 + \frac{\gamma}{\mu} \right) D(t).$$

Multiplying the above by $\delta\mu$ yields the desired identity (2.1). □

By a *solution* of the SEIRD differential system (1.1)–(1.5) we mean a vector-valued function $(S(t), E(t), I(t), R(t), D(t))$ of class $C^1(0, \infty) \cap C[0, \infty)$ which satisfies (1.1)–(1.5). Associated with every continuous function $f(t)$ on $[0, \infty)$, we define

$$f(\infty) := \lim_{t \to \infty} f(t).$$

Lemma 2.2. *Let $(S(t), E(t), I(t), R(t), D(t))$ be a solution of the SEIRD differential system (1.1)–(1.5) such that $S(t) > 0$, $E(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for $t > 0$. Then there exists the limit $D(\infty)$.*

Proof. Since $I(t) > 0$ for $t > 0$, it follows from (1.5) that $D'(t) = \mu I(t) > 0$ for $t > 0$, and therefore $D(t)$ is increasing on $[0, \infty)$. It is easy to see that $D(t)$ is bounded from above in light of

$$D(t) = N - S(t) - E(t) - I(t) - R(t) < N \ (t > 0).$$

Hence there exists the limit $D(\infty)$. □

Theorem 2.3. *Let $(S(t), E(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.6) such that $S(t) > 0$, $E(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for $t > 0$. Then the solution*

$(S(t), E(t), I(t), R(t), D(t))$ can be represented in the following parametric form:

$$S(\varphi(u)) = \tilde{S}e^{(\beta/\mu)\tilde{D}}u, \tag{2.5}$$

$$E(\varphi(u)) = \tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv, \tag{2.6}$$

$$\begin{aligned} I(\varphi(u)) = N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\gamma+\mu}{\beta}\log u - \tilde{E}e^{-\delta\varphi(u)} \\ - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv, \end{aligned} \tag{2.7}$$

$$R(\varphi(u)) = -\frac{\gamma}{\beta}\log u + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}, \tag{2.8}$$

$$D(\varphi(u)) = -\frac{\mu}{\beta}\log u \tag{2.9}$$

for $e^{-(\beta/\mu)D(\infty)} < u \leq e^{-(\beta/\mu)\tilde{D}}$, where

$$t = \varphi(u) = \int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)}, \tag{2.10}$$

with $\psi(u)$ satisfying the Abel differential equation of the second kind

$$\psi'\psi - \frac{\delta+\gamma+\mu}{u}\psi = -\delta\,\frac{\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u}{u} \tag{2.11}$$

for $e^{-(\beta/\mu)D(\infty)} < u < e^{-(\beta/\mu)\tilde{D}}$, and the following conditions

$$\psi\big(e^{-(\beta/\mu)\tilde{D}}\big) = \beta\tilde{I},$$
$$\lim_{u\to e^{-(\beta/\mu)D(\infty)}+0} \psi(u) = 0,$$
$$\psi(u) > 0 \quad \text{in } (e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}}].$$

Proof. Since $D'(t) = \mu I(t) > 0$ for $t > 0$ in view of (1.5), we find that $D(t)$ is increasing on $[0,\infty)$. Then there exists the limit $D(\infty)$ by Lemma 2.2. It is easy to check that $u = u(t) = e^{-(\beta/\mu)D(t)}$ is decreasing on $[0,\infty)$, $e^{-(\beta/\mu)D(\infty)} < u \leq e^{-(\beta/\mu)\tilde{D}}$ and $\lim_{t\to\infty} u(t) = e^{-(\beta/\mu)D(\infty)}$. Hence there exists the inverse function $\varphi(u) \in C^1(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$ of $u = u(t)$ such that

$$t = \varphi(u) \qquad \left(e^{-(\beta/\mu)D(\infty)} < u \leq e^{-(\beta/\mu)\tilde{D}}\right),$$

$\varphi(u)$ is decreasing in $(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}}]$, $\lim_{u\to e^{-(\beta/\mu)D(\infty)}+0} \varphi(u) = \infty$, and $\varphi\big(e^{-(\beta/\mu)\tilde{D}}\big) = 0$. Substituting $t = \varphi(u)$ into (2.1) in Lemma 2.1 yields

$$\begin{aligned} &D''(\varphi(u)) + (\delta+\gamma+\mu)D'(\varphi(u)) \\ &\quad = \delta\mu\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\varphi(u))} - \left(1+\frac{\gamma}{\mu}\right)D(\varphi(u))\right) \end{aligned} \tag{2.12}$$

for $e^{-(\beta/\mu)D(\infty)} < u < e^{-(\beta/\mu)\tilde{D}}$. Differentiating both sides of $u = e^{-(\beta/\mu)D(\varphi(u))}$ with respect to u yields

$$\begin{aligned} 1 &= -\frac{\beta}{\mu}D'(\varphi(u))\varphi'(u)e^{-(\beta/\mu)D(\varphi(u))} \\ &= -\frac{\beta}{\mu}D'(\varphi(u))\varphi'(u)u, \end{aligned}$$

and therefore

$$D'(\varphi(u)) = -\frac{\mu}{\beta}\frac{1}{\varphi'(u)u}. \tag{2.13}$$

Since $D'(t) \in C^1(0,\infty)$ by means of (1.5) and $\varphi(u) \in C^1(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$, we see that $D'(\varphi(u)) \in C^1(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$, and consequently $1/(\varphi'(u)u) \in C^1(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$. We differentiate (2.13) with respect to u to obtain

$$D''(\varphi(u))\varphi'(u) = -\frac{\mu}{\beta}\left(\frac{1}{\varphi'(u)u}\right)',$$

and hence

$$D''(\varphi(u)) = -\frac{\mu}{\beta}\left(\frac{1}{\varphi'(u)u}\right)'\frac{1}{\varphi'(u)}. \tag{2.14}$$

It is obvious that

$$D(\varphi(u)) = -\frac{\mu}{\beta}\log u \tag{2.15}$$

in light of $u = e^{-(\beta/\mu)D(\varphi(u))}$. Combining (2.12)–(2.15), we get

$$\begin{aligned} -\frac{\mu}{\beta}\left(\frac{1}{\varphi'(u)u}\right)'\frac{1}{\varphi'(u)} + (\delta+\gamma+\mu)\left(-\frac{\mu}{\beta}\frac{1}{\varphi'(u)u}\right) \\ = \delta\mu\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u - \left(1+\frac{\gamma}{\mu}\right)\left(-\frac{\mu}{\beta}\log u\right)\right) \end{aligned}$$

or

$$\begin{aligned} \frac{\mu}{\beta}\left(-\frac{1}{\varphi'(u)u}\right)'\left(-\frac{1}{\varphi'(u)u}\right) - \frac{\mu}{\beta}\frac{\delta+\gamma+\mu}{u}\left(-\frac{1}{\varphi'(u)u}\right) \\ = -\delta\mu\frac{1}{u}\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\mu}{\beta}\left(1+\frac{\gamma}{\mu}\right)\log u\right). \end{aligned} \tag{2.16}$$

Letting

$$\psi(u) := -\frac{1}{\varphi'(u)u}, \tag{2.17}$$

we observe that $\psi(u)$ satisfies (2.11). Since $t = \varphi(u) > 0$ for $e^{-(\beta/\mu)D(\infty)} < u < e^{-(\beta/\mu)\tilde{D}}$, we see from (1.5), (2.13) and (2.17) that

$$\psi(u) = \frac{\beta}{\mu}D'(\varphi(u)) = \beta I(\varphi(u)) > 0$$

in $(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$. If we define

$$\begin{aligned} \psi(e^{-(\beta/\mu)\tilde{D}}) &:= \lim_{u\to e^{-(\beta/\mu)\tilde{D}}-0}\psi(u) = \frac{\beta}{\mu}\lim_{u\to e^{-(\beta/\mu)\tilde{D}}-0}D'(\varphi(u)) \\ &= \frac{\beta}{\mu}\lim_{t\to+0}D'(t) = \frac{\beta}{\mu}\mu I(0) = \beta\tilde{I} > 0, \end{aligned}$$

then $\psi(u)$ is a positive continuous function in $(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}}]$. It follows from (2.17) that

$$t = \varphi(u) = \int_{e^{-(\beta/\mu)\tilde{D}}}^{u}\varphi'(\xi)d\xi = \int_{u}^{e^{-(\beta/\mu)\tilde{D}}}\frac{d\xi}{\xi\psi(\xi)},$$

and therefore (2.10) holds. Since $\lim_{u\to e^{-(\beta/\mu)D(\infty)}+0} \varphi(u) = \infty$, it is necessary that $\lim_{u\to e^{-(\beta/\mu)D(\infty)}+0} \psi(u) = 0$.

Now we establish the representation formulae (2.5)–(2.9). We see from (2.2) and (2.15) that

$$S(\varphi(u)) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\varphi(u))} = \tilde{S}e^{(\beta/\mu)\tilde{D}}u,$$
$$D(\varphi(u)) = -\frac{\mu}{\beta}\log u,$$

which are the desired representations (2.5) and (2.9). Combining (1.1) with (1.2) yields the first order linear differential equation

$$E'(t) + \delta E(t) = -S'(t)$$

which implies

$$E(t) = \tilde{E}e^{-\delta t} - e^{-\delta t}\int_0^t e^{\delta\xi}S'(\xi)d\xi. \tag{2.18}$$

Differentiating (2.2), we obtain

$$S'(t) = -\frac{\beta}{\mu}\tilde{S}e^{(\beta/\mu)\tilde{D}}D'(t)e^{-(\beta/\mu)D(t)}. \tag{2.19}$$

Substitution of (2.19) into (2.18) gives

$$E(t) = \tilde{E}e^{-\delta t} + \frac{\beta}{\mu}\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_0^t e^{\delta\xi}D'(\xi)e^{-(\beta/\mu)D(\xi)}d\xi. \tag{2.20}$$

By changing the variables $D(\xi) = s$, we obtain

$$\begin{aligned} J := \int_0^t e^{\delta\xi}D'(\xi)e^{-(\beta/\mu)D(\xi)}d\xi &= \int_{\tilde{D}}^{D(t)} e^{\delta D^{-1}(s)}e^{-(\beta/\mu)s}ds \\ &= \int_{\tilde{D}}^{D(t)} e^{\delta\varphi(e^{-(\beta/\mu)s})}e^{-(\beta/\mu)s}ds \\ &= \frac{\mu}{\beta}\int_{D(t)}^{\tilde{D}} e^{\delta\varphi(e^{-(\beta/\mu)s})}\left(e^{-(\beta/\mu)s}\right)' ds \end{aligned}$$

in view of $D^{-1}(s) = \varphi(e^{-(\beta/\mu)s})$. Letting $v = e^{-(\beta/\mu)s}$ yields

$$J = \frac{\mu}{\beta}\int_{e^{-(\beta/\mu)D(t)}}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv. \tag{2.21}$$

Combining (2.20) with (2.21), we are led to

$$E(t) = \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{e^{-(\beta/\mu)D(t)}}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv. \tag{2.22}$$

Substituting $t = \varphi(u)$ into (2.22), we arrive at (2.6). We observe, using (2.4), that

$$R(\varphi(u)) = \frac{\gamma}{\mu}D(\varphi(u)) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = -\frac{\gamma}{\beta}\log u + \tilde{R} - \frac{\gamma}{\mu}\tilde{D},$$

which is equal to (2.8). Since $I(\varphi(u)) = N - S(\varphi(u)) - R(\varphi(u)) - D(\varphi(u)) - E(\varphi(u))$, (2.7) follows from (2.5), (2.6), (2.8) and (2.9). □

Corollary 2.4. *Let $(S(t), E(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.6) such that $S(t) > 0$, $E(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for $t > 0$. Then we obtain the following relations:*

$$S(t) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)}, \tag{2.23}$$

$$E(t) = \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{e^{-(\beta/\mu)D(t)}}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta D^{-1}(-(\mu/\beta)\log v)}dv, \tag{2.24}$$

$$\begin{aligned} I(t) = {} & N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)} - \frac{\gamma+\mu}{\mu}D(t) - \tilde{E}e^{-\delta t} \\ & - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{e^{-(\beta/\mu)D(t)}}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta D^{-1}(-(\mu/\beta)\log v)}dv, \end{aligned} \tag{2.25}$$

$$R(t) = D(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} \tag{2.26}$$

for $t \geq 0$.

Proof. It is easy to see that

$$u = \varphi^{-1}(t) = e^{-(\beta/\mu)D(t)}, \tag{2.27}$$

$$\varphi(v) = D^{-1}(-(\mu/\beta)\log v) \tag{2.28}$$

in the proof of Theorem 2.3. Combining (2.5)–(2.8), (2.27) and (2.28), we are led to (2.23)–(2.26). □

3 Exact solution of an initial value problem for SEIRD differential system

In this section we establish an exact solution of an initial value problem for SEIRD differential system (1.1)–(1.5) by utilizing Theorem 2.3 in Section 2.

The following lemma follows from a result of Yoshida [18, Lemma 3] by replacing $\tilde{R}, \tilde{D}, \gamma, \mu$ by $\tilde{D}, \tilde{R}, \mu, \gamma$, respectively.

Lemma 3.1. *Under the hypothesis* (A_5), *the transcendental equation*

$$x = \frac{\mu}{\gamma+\mu}N - \frac{\mu}{\gamma+\mu}\tilde{R} + \frac{\gamma}{\gamma+\mu}\tilde{D} - \frac{\mu}{\gamma+\mu}\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x}$$

has a unique solution $x = \alpha$ such that

$$\tilde{D} < \alpha < N$$

(*cf.* Figure 3.1).

We assume that the following hypothesis

(A_7) $\tilde{S} < \frac{\gamma+\mu}{\beta}e^{(\beta/\mu)(\alpha-\tilde{D})}$

holds in the rest of this paper. We note that (A_7) is equivalent to the following

(A_7') $\frac{\gamma+\mu}{\beta} > N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \frac{\gamma+\mu}{\mu}\alpha$

in view of $\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} = N - \tilde{R} + (\gamma/\mu)\tilde{D} - ((\gamma+\mu)/\mu)\alpha$.

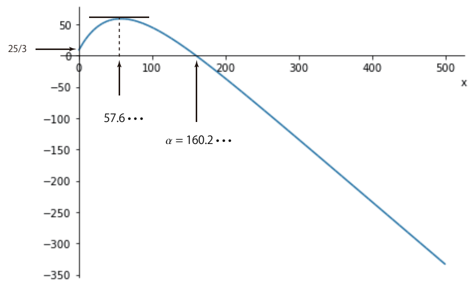

Figure 3.1: Variation of $(\mu/(\gamma+\mu))N - (\mu/(\gamma+\mu))\tilde{R} + (\gamma/(\gamma+\mu))\tilde{D} - (\mu/(\gamma+\mu))\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x} - x$ for $N = 1000, \tilde{S} = 950, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.2/1000$, $\gamma = 0.05$ and $\mu = 0.01$. In this case we find that $(\mu/(\gamma+\mu))(N-\tilde{S}) = 25/3$ and $0 < \alpha = 160.2\ldots < 1000$.

Remark 3.2. Combining (A_2) with (A_3), we have

$$\tilde{S} > \frac{\delta\tilde{E}}{\beta\tilde{I}} > \frac{\gamma+\mu}{\beta}.$$

Lemma 3.3. *There exists a unique positive solution $w(x)$ of the initial value problem for the Abel differential equation*

$$w'w + \frac{\beta(\delta+\gamma+\mu)}{\mu}w = \frac{\beta\delta}{\mu}\left(\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x} - \frac{\beta(\gamma+\mu)}{\mu}x\right) \qquad (\tilde{D} < x < \alpha), \tag{3.1}$$

subject to the initial condition

$$w(\tilde{D}) = \beta\tilde{I}. \tag{3.2}$$

Proof. Let

$$f(x) := N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x} - \frac{\gamma+\mu}{\mu}x.$$

Since $f'(x) = 0$ for

$$x = \tilde{x} = \frac{\mu}{\beta}\log\left(\frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\mu)\tilde{D}}\right),$$

we see that $\tilde{D} < \tilde{x} < \alpha$ by means of (A_7) and Remark 3.2, and that $f'(x) > 0$ for $\tilde{D} < x < \tilde{x}$ and $f'(x) < 0$ for $\tilde{x} < x < \alpha$. Hence, $f(x)$ is increasing in $[\tilde{D}, \tilde{x})$ and decreasing in $(\tilde{x}, \alpha)$. Since $f(\tilde{D}) = N - \tilde{R} - \tilde{S} - \tilde{D} = \tilde{E} + \tilde{I} > 0$ and $\lim_{x\to\alpha-0} f(x) = 0$ by Lemma 3.1, it follows that $f(x) \in C[\tilde{D}, \alpha)$, $f(x) > 0$ in $[\tilde{D}, \alpha)$ and $\lim_{x\to\alpha-0} f(x) = 0$. Therefore there exists a unique positive solution $w(x)$ of the initial value problem (3.1), (3.2) by a result of Yoshida [20, Theorem 3] (*cf.* Figure 3.2). □

Lemma 3.4. *There exists a unique positive solution $\psi(u)$ of the initial value problem for the Abel differential equation*

$$\psi'\psi - \frac{\delta+\gamma+\mu}{u}\psi = -\delta\frac{\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u}{u} \tag{3.3}$$

$$(e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}})$$

with the initial condition

$$\psi\left(e^{-(\beta/\mu)\tilde{D}}\right) = \beta\tilde{I} \tag{3.4}$$

(*cf*. Figure 3.3).

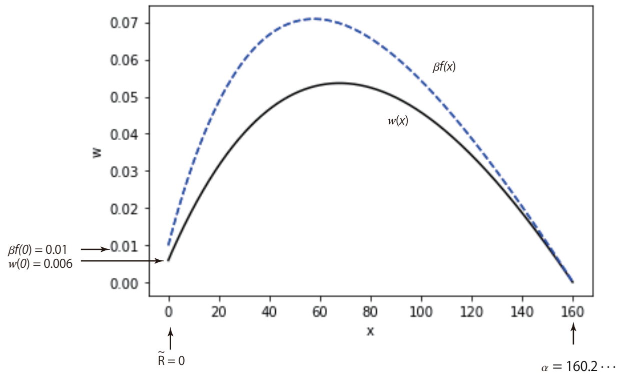

Figure 3.2: Variations of $\beta f(x)$ (dashed curve), and $w(x)$ (solid curve) obtained by the numerical integration of the initial value problem (3.1), (3.2), for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05, \delta = 0.2, \mu = 0.01$ and $\alpha = 160.2\ldots$ In this case we obtain $\beta f(0) = \beta N - \beta\tilde{S} = 0.01$ and $w(0) = \beta\tilde{I} = 0.006$.

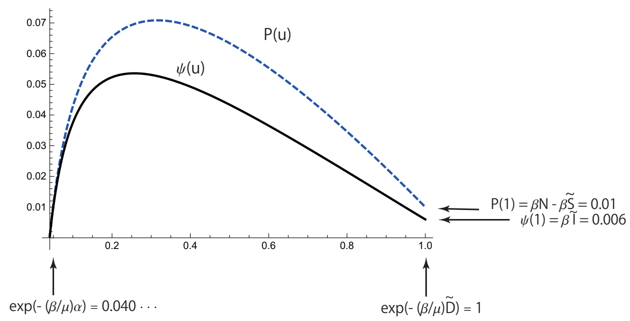

Figure 3.3: Variations of $P(u) := \beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma + \mu)\log u$ (dashed curve) and $\psi(u)$ (solid curve) obtained by the numerical integration of the initial value problem (3.3), (3.4) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05, \delta = 0.2, \mu = 0.01$ and $\alpha = 160.2\ldots$ In this case we get $e^{-(\beta/\mu)\alpha} = 0.040\ldots$, $e^{-(\beta/\mu)\tilde{D}} = 1$, $P(1) = \beta N - \beta\tilde{S} = 0.01$ and $\psi(1) = \beta\tilde{I} = 0.006$.

Proof. Let $w(x)$ be a unique positive solution of the initial value problem (3.1), (3.2). We define $\psi(u)$ by

$$\psi(u) := w\left(-\frac{\mu}{\beta}\log u\right)$$

and find that

$$\psi'(u) = w'\left(-\frac{\mu}{\beta}\log u\right)\left(-\frac{\mu}{\beta}\frac{1}{u}\right),$$

and hence

$$\begin{aligned}
\psi'(u)\psi(u) &= -\frac{\mu}{\beta}\frac{1}{u}w'\left(-\frac{\mu}{\beta}\log u\right)w\left(-\frac{\mu}{\beta}\log u\right)\\
&= -\frac{\mu}{\beta}\frac{1}{u}\left[-\frac{\beta(\delta+\gamma+\mu)}{\mu}w\left(-\frac{\mu}{\beta}\log u\right)\right.\\
&\quad\left.+\frac{\beta\delta}{\mu}\left(\beta N-\beta\tilde{R}+((\beta\gamma)/\mu)\tilde{D}-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u+(\gamma+\mu)\log u\right)\right]\\
&= \frac{\delta+\gamma+\mu}{u}\psi(u)-\delta\,\frac{\beta N-\beta\tilde{R}+((\beta\gamma)/\mu)\tilde{D}-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u+(\gamma+\mu)\log u}{u}
\end{aligned}$$

for $e^{-(\beta/\mu)\alpha}<u<e^{-(\beta/\mu)\tilde{D}}$ by means of (3.1). Hence $\psi(u)$ satisfies (3.3). It is easily seen from (3.2) that

$$\psi(e^{-(\beta/\mu)\tilde{D}})=w(\tilde{D})=\beta\tilde{I}$$

and therefore (3.4) is satisfied. The uniqueness of $\psi(u)$ follows from that of $w(x)$. It can be shown that

$$\psi(u)>0 \text{ in } (e^{-(\beta/\mu)\alpha},e^{-(\beta/\mu)\tilde{D}}] \tag{3.5}$$

since $\psi(u)=w(-(\mu/\beta)\log u)$ and $w(x)>0$ in $[\tilde{D},\alpha)$. □

Lemma 3.5. *The unique positive solution $\psi(u)$ of the initial value problem (3.3), (3.4) satisfies the following relation*

$$\begin{aligned}
\psi(u) &= \beta N-\beta\tilde{R}+\frac{\beta\gamma}{\mu}\tilde{D}-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u+(\gamma+\mu)\log u\\
&\quad-\beta\left(\tilde{E}e^{-\delta\varphi(u)}+\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}}e^{\delta\varphi(v)}dv\right)
\end{aligned}\tag{3.6}$$

for $e^{-(\beta/\mu)\alpha}<u\le e^{-(\beta/\mu)\tilde{D}}$, where

$$\varphi(u):=\int_u^{e^{-(\beta/\mu)\tilde{D}}}\frac{d\xi}{\xi\psi(\xi)}. \tag{3.7}$$

Conversely, the function $\psi(u)$ satisfying (3.5), (3.6) is a solution of the initial value problem (3.3), (3.4).

Proof. We note that (3.6) is some kind of integral equation of $\psi(u)$, in light of (3.7). Let $\psi(u)$ be the unique positive solution of the problem (3.3), (3.4), and define $z(u)$ by

$$z(u):=\psi(u)-P(u), \tag{3.8}$$

where

$$P(u)=\beta N-\beta\tilde{R}+\frac{\beta\gamma}{\mu}\tilde{D}-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u+(\gamma+\mu)\log u. \tag{3.9}$$

Dividing (3.3) by $\psi(u)$, we obtain

$$\begin{aligned}
\psi'(u) &= \frac{\delta+\gamma+\mu}{u}-\delta\,\frac{P(u)}{u\psi(u)}=\frac{\gamma+\mu}{u}-\delta\,\frac{P(u)-\psi(u)}{u\psi(u)}\\
&= \frac{\gamma+\mu}{u}+\delta\,\frac{z(u)}{u\psi(u)}.
\end{aligned}\tag{3.10}$$

On the other hand, differentiating (3.8) yields

$$\psi'(u) = -\beta\tilde{S}e^{(\beta/\mu)\tilde{D}} + \frac{\gamma+\mu}{u} + z'(u). \tag{3.11}$$

Combining (3.10) with (3.11), we get

$$z'(u) - \frac{\delta}{u\psi(u)}z(u) = \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}$$

or

$$z'(u) + \delta\varphi'(u)z(u) = \beta\tilde{S}e^{(\beta/\mu)\tilde{D}} \tag{3.12}$$

which is a linear differential equation of first order. It is clear that

$$\begin{aligned} z\big(e^{-(\beta/\mu)\tilde{D}}\big) &= \psi\big(e^{-(\beta/\mu)\tilde{D}}\big) - \left(\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S} - \frac{\beta(\gamma+\mu)}{\mu}\tilde{D}\right) \\ &= \beta\tilde{I} - \beta\big(N - \tilde{R} - \tilde{S} - \tilde{D}\big) \\ &= -\beta\tilde{E}. \end{aligned} \tag{3.13}$$

Now we solve the initial value problem (3.12), (3.13). Multiplying (3.12) by $e^{\delta\varphi(u)}$ yields

$$\big(e^{\delta\varphi(u)}z(u)\big)' = \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{\delta\varphi(u)}$$

and then integrating the above on $[u, e^{-(\beta/\mu)\tilde{D}}]$ gives

$$z\big(e^{-(\beta/\mu)\tilde{D}}\big) - e^{\delta\varphi(u)}z(u) = \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv.$$

Taking account of (3.13), we obtain

$$z(u) = -\beta\left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right). \tag{3.14}$$

Combining (3.8) with (3.14), we observe that $\psi(u)$ satisfies (3.6) for $e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}}$. If $u = e^{-(\beta/\mu)\tilde{D}}$, then $\psi\big(e^{-(\beta/\mu)\tilde{D}}\big) = \beta\tilde{I}$ by (3.4) and the right hand side of (3.6) with $u = e^{-(\beta/\mu)\tilde{D}}$ is equal to $\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S} - (\beta(\gamma+\mu)/\mu)\tilde{D} - \beta\tilde{E} = \beta\tilde{I}$. Therefore (3.6) holds for $u = e^{-(\beta/\mu)\tilde{D}}$.

Conversely we suppose that the function $\psi(u)$ satisfies (3.5), (3.6), and let $e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}}$. Differentiating (3.6) with respect to u yields

$$\begin{aligned} \psi'(u) = &-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}} + \frac{\gamma+\mu}{u} - \beta\tilde{E}e^{-\delta\varphi(u)}\big(-\delta\varphi'(u)\big) \\ &-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\left(e^{-\delta\varphi(u)}\big(-\delta\varphi'(u)\big)\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv - 1\right) \\ = &\frac{\gamma+\mu}{u} - \beta\delta\tilde{E}e^{-\delta\varphi(u)}\frac{1}{u\psi(u)} \\ &-\beta\delta\tilde{S}e^{(\beta/\mu)\tilde{D}}\frac{1}{u\psi(u)}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv. \end{aligned} \tag{3.15}$$

It follows from (3.6) that

$$-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv = \psi(u) - P(u) + \beta\tilde{E}e^{-\delta\varphi(u)}. \tag{3.16}$$

We combine (3.15) with (3.16) to obtain

$$\begin{aligned}\psi'(u) &= \frac{\gamma+\mu}{u} - \beta\delta\tilde{E}e^{-\delta\varphi(u)}\frac{1}{u\psi(u)} + \delta\,\frac{\psi(u) - P(u) + \beta\tilde{E}e^{-\delta\varphi(u)}}{u\psi(u)} \\ &= \frac{\gamma+\mu}{u} - \delta\,\frac{P(u)-\psi(u)}{u\psi(u)} \\ &= \frac{\delta+\gamma+\mu}{u} - \delta\,\frac{P(u)}{u\psi(u)}\end{aligned}$$

and consequently, $\psi(u)$ satisfies (3.3) for $e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}}$. It is easy to see from (3.6) that

$$\begin{aligned}\psi(e^{-(\beta/\mu)\tilde{D}}) &= \beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S} - (\beta(\gamma+\mu)/\mu)\tilde{D} - \beta\tilde{E} \\ &= \beta N - \beta\tilde{R} - \beta\tilde{D} - \beta\tilde{S} - \beta\tilde{E} \\ &= \beta(N - \tilde{R} - \tilde{D} - \tilde{S} - \tilde{E}) = \beta\tilde{I}\end{aligned}$$

in view of $\varphi(e^{-(\beta/\mu)\tilde{D}}) = 0$, and therefore (3.4) is satisfied. □

Proposition 3.6. *Let $\psi(u)$ be the unique positive solution of the initial value problem* (3.3), (3.4)*, then we obtain the following inequalities*:

$$\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u > \psi(u) > 0, \tag{3.17}$$

$$\begin{aligned}&\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u \\ &\quad > \beta\left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right) > 0\end{aligned} \tag{3.18}$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$.

Proof. Since $\psi(u) > 0$ in $(e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}}]$, the relation (3.6) in Lemma 3.5 means

$$\begin{aligned}&\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u \\ &\quad > \beta\left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi(v)}dv\right)\end{aligned}$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$. It is clear that

$$\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv > 0 \tag{3.19}$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, and therefore (3.18) follows. Since (3.19) holds, the relation (3.6) implies that

$$\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u > \psi(u) > 0$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, which is the desired inequality (3.17). □

Proposition 3.7. *Let $\psi(u)$ be the unique positive solution of the initial value problem* (3.3), (3.4)*, then we see that*

$$\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \psi(u) = 0, \tag{3.20}$$

$$\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)} \int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)} dv \right) = 0 \tag{3.21}$$

(*cf*. Figure 3.4).

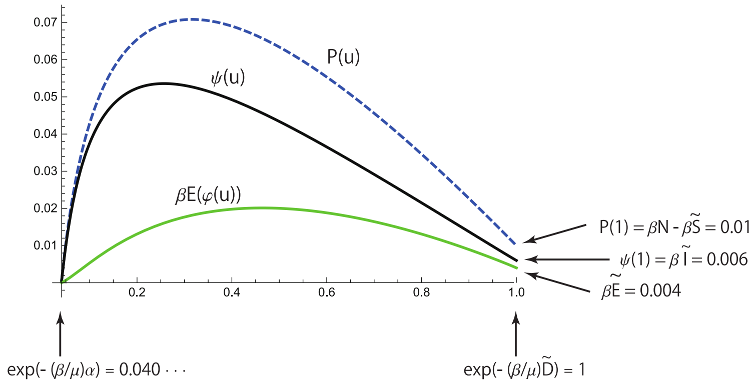

Figure 3.4: Variations of $P(u) = \beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma + \mu)\log u$ (dashed curve), $\beta E(\varphi(u))$ (green curve), and $\psi(u)$ (solid curve) obtained by the numerical integration of the initial value problem (3.3), (3.4) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05, \delta = 0.2, \mu = 0.01$ and $\alpha = 160.2\ldots$ In this case we have $e^{-(\beta/\mu)\alpha} = 0.040\ldots$, $e^{-(\beta/\mu)\tilde{D}} = 1$, $P(1) = \beta N - \beta\tilde{S} = 0.01$, $\psi(1) = \beta\tilde{I} = 0.006$ and $\beta\tilde{E} = 0.004$. Moreover, $\lim_{u \to e^{-(\beta/\mu)\alpha}+0} P(u) = 0$, $\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \psi(u) = 0$, and $\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \beta E(\varphi(u)) = \lim_{u \to e^{-(\beta/\mu)\alpha}+0} (P(u) - \psi(u)) = 0$.

Proof. Since

$$\begin{aligned} &\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \left(\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u \right) \\ &\quad = \lim_{x \to \alpha-0} \beta \left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x} - \frac{\gamma+\mu}{\mu}x \right) = 0 \end{aligned}$$

by Lemma 3.1, Proposition 3.6 implies that (3.20) and (3.21) hold by taking the limit as $u \to e^{-(\beta/\mu)\alpha} + 0$ in (3.17) and (3.18). □

Lemma 3.8. *Let $\psi(u)$ be the unique positive solution of the initial value problem* (3.3), (3.4)*. Then there exists the inverse function $\varphi^{-1}(t) \in C^1(0, \infty)$ of the function*

$$t = \varphi(u) = \int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)} \tag{3.22}$$

for $e^{-(\beta/\mu)\alpha} < u \leq e^{-(\beta/\mu)\tilde{D}}$, such that $\varphi^{-1}(t)$ is decreasing on $[0, \infty)$, $\varphi^{-1}(0) = e^{-(\beta/\mu)\tilde{D}}$ and $\lim_{t\to\infty} \varphi^{-1}(t) = e^{-(\beta/\mu)\alpha}$.

Proof. We easily see that $\varphi(u) \in C^1(e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}})$, $\varphi(u)$ is decreasing in $(e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}}]$ and $\varphi(e^{-(\beta/\mu)\tilde{D}}) = 0$. We divide (3.3) by $(\delta+\gamma+\mu)\psi(u)^2$ to obtain

$$\frac{1}{u\psi(u)} = \frac{\delta}{\delta+\gamma+\mu}\frac{P(u)}{u\psi(u)^2} + \frac{1}{\delta+\gamma+\mu}\frac{\psi'(u)}{\psi(u)}, \tag{3.23}$$

and therefore

$$\begin{aligned}\varphi(u) &= \int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)} \\ &= \frac{\delta}{\delta+\gamma+\mu}\int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{P(\xi)}{\xi\psi(\xi)^2}d\xi + \frac{1}{\delta+\gamma+\mu}\int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{\psi'(\xi)}{\psi(\xi)}\,d\xi \\ &\geq \frac{1}{\delta+\gamma+\mu}\big(\log(\beta\tilde{I}) - \log\psi(u)\big),\end{aligned} \tag{3.24}$$

where $P(u)$ is defined by (3.8). We see that $\lim_{u\to e^{-(\beta/\mu)\alpha}+0}\log\psi(u) = -\infty$ in view of (3.20), and that $\lim_{u\to e^{-(\beta/\mu)\alpha}+0}\varphi(u) = \infty$ by taking the limit as $u \to e^{-(\beta/\mu)\alpha}+0$ in (3.24). Hence there exists the inverse function $\varphi^{-1}(t)$ which has the desired properties. □

The following is our main theorem.

Theorem 3.9. *The function $(S(t), E(t), I(t), R(t), D(t))$ defined by*

$$S(t) = \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t), \tag{3.25}$$

$$E(t) = \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv, \tag{3.26}$$

$$\begin{aligned}I(t) = {}& N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t) - \tilde{E}e^{-\delta t} \\ & - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv,\end{aligned} \tag{3.27}$$

$$R(t) = -\frac{\gamma}{\beta}\log\varphi^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}, \tag{3.28}$$

$$D(t) = -\frac{\mu}{\beta}\log\varphi^{-1}(t) \tag{3.29}$$

is a solution of the initial value problem (1.1)–(1.6)*, where $\varphi(u)$ and $\varphi^{-1}(t)$ are given in Lemma* 3.8.

Proof. First note that

$$\begin{aligned}(\varphi^{-1}(t))' &= \frac{1}{\varphi'(u)}\Big|_{u=\varphi^{-1}(t)} = -u\psi(u)\Big|_{u=\varphi^{-1}(t)} \\ &= -\varphi^{-1}(t)\psi(\varphi^{-1}(t)) = -\beta\varphi^{-1}(t)I(t)\end{aligned} \tag{3.30}$$

by taking account of (3.6) and (3.27). We see from (3.25) and (3.30) that

$$\begin{aligned}S'(t) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}(\varphi^{-1}(t))' \\ &= -\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)I(t) \\ &= -\beta S(t)I(t)\end{aligned} \tag{3.31}$$

and therefore (1.1) follows. A direct calculation yields

$$\begin{aligned} E'(t) =& -\delta\tilde{E}e^{-\delta t} \\ &+ \tilde{S}e^{(\beta/\mu)\tilde{D}}\left(-\delta e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv + e^{-\delta t}\left(-e^{\delta t}(\varphi^{-1}(t))'\right)\right) \\ =& -\delta\tilde{E}e^{-\delta t} - \delta\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv - \tilde{S}e^{(\beta/\mu)\tilde{D}}(\varphi^{-1}(t))' \\ =& -\delta E(t) + \beta S(t)I(t) \end{aligned} \tag{3.32}$$

in view of (3.26) and (3.31), and hence (1.2) is satisfied. An easy computation shows that

$$\begin{aligned} I'(t) =& -\tilde{S}e^{(\beta/\mu)\tilde{D}}(\varphi^{-1}(t))' + \frac{\gamma+\mu}{\beta}\frac{(\varphi^{-1}(t))'}{\varphi^{-1}(t)} \\ &- \left(\tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right)' \\ =& \beta S(t)I(t) + \frac{\gamma+\mu}{\beta}(-\beta I(t)) - E'(t) \\ =& \beta S(t)I(t) - (\gamma+\mu)I(t) - \left(-\delta E(t) + \beta S(t)I(t)\right) \\ =& \delta E(t) - \gamma I(t) - \mu I(t) \end{aligned} \tag{3.33}$$

in view of (3.30)–(3.32). Thus, (1.3) holds. It is easily seen from (3.30) that

$$R'(t) = -\frac{\gamma}{\beta}\frac{(\varphi^{-1}(t))'}{\varphi^{-1}(t)} = -\frac{\gamma}{\beta}(-\beta I(t)) = \gamma I(t)$$

which is the equation (1.4). Similarly we obtain

$$D'(t) = -\frac{\mu}{\beta}\frac{(\varphi^{-1}(t))'}{\varphi^{-1}(t)} = -\frac{\mu}{\beta}(-\beta I(t)) = \mu I(t)$$

which is the desired equation (1.5). It is easy to see that

$$\begin{aligned} S(0) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(0) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\tilde{D}} = \tilde{S}, \\ E(0) &= \tilde{E} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\int_{\varphi^{-1}(0)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv = \tilde{E}, \\ I(0) &= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S} + \frac{\gamma+\mu}{\beta}\left(-\frac{\beta}{\mu}\tilde{D}\right) - \tilde{E} \\ &= N - \tilde{R} - \tilde{S} - \tilde{D} - \tilde{E} = \tilde{I}, \\ R(0) &= -\frac{\gamma}{\beta}\left(-\frac{\beta}{\mu}\tilde{D}\right) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = \tilde{R}, \\ D(0) &= -\frac{\mu}{\beta}\left(-\frac{\beta}{\mu}\tilde{D}\right) = \tilde{D} \end{aligned}$$

in light of $\varphi^{-1}(0) = e^{-(\beta/\mu)\tilde{D}}$. Therefore, (1.6) is satisfied. □

Theorem 3.10. *Let* $(S(t), E(t), I(t), R(t), D(t))$ *be the exact solution* (3.25)–(3.29) *of the initial value problem* (1.1)–(1.6)*. Then,* $\big(\hat{S}(u), \hat{E}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u)\big)$ *defined by*

$$\big(\hat{S}(u), \hat{E}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u)\big) := \big(S(\varphi(u)), E(\varphi(u)), I(\varphi(u)), R(\varphi(u)), D(\varphi(u))\big)$$

satisfies the linear differential system

$$\frac{d\hat{S}(u)}{du} = \frac{\hat{S}(u)}{u}, \tag{3.34}$$

$$\frac{d\hat{E}(u)}{du} - \frac{\delta}{u\psi(u)}\hat{E}(u) = -\frac{\hat{S}(u)}{u}, \tag{3.35}$$

$$\frac{d\hat{I}(u)}{du} - \frac{\gamma+\mu}{\beta}\frac{1}{u} = -\frac{\delta}{u\psi(u)}\hat{E}(u), \tag{3.36}$$

$$\frac{d\hat{R}(u)}{du} = -\frac{\gamma}{\beta}\frac{1}{u}, \tag{3.37}$$

$$\frac{d\hat{D}(u)}{du} = -\frac{\mu}{\beta}\frac{1}{u} \tag{3.38}$$

for $u \in (e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}})$*, and the initial condition*

$$\hat{S}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{S}, \tag{3.39}$$

$$\hat{E}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{E}, \tag{3.40}$$

$$\hat{I}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{I}, \tag{3.41}$$

$$\hat{R}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{R}. \tag{3.42}$$

$$\hat{D}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{D}. \tag{3.43}$$

Proof. It follows from (3.30) that

$$\hat{I}(u) = I(\varphi(u)) = \frac{1}{\beta}\psi(u). \tag{3.44}$$

Since $S(t)$ satisfies (1.1), we obtain

$$S'(\varphi(u)) = -\beta S(\varphi(u))I(\varphi(u)) = -\beta\hat{S}(u)\hat{I}(u).$$

Therefore we arrive at

$$\begin{aligned}\frac{d\hat{S}(u)}{du} &= \frac{dS(t)}{dt}\Big|_{t=\varphi(u)}\varphi'(u) = S'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right)\\ &= \left(-\beta\hat{S}(u)\hat{I}(u)\right)\left(-\frac{1}{u\psi(u)}\right)\\ &= \frac{\hat{S}(u)}{u}\end{aligned}$$

in light of (3.44), and hence (3.34) holds. Using (1.2) and (3.44), we get

$$\begin{aligned}\frac{d\hat{E}(u)}{du} &= \frac{dE(t)}{dt}\Big|_{t=\varphi(u)}\varphi'(u) = E'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right)\\ &= \left(\beta\hat{S}(u)\hat{I}(u) - \delta\hat{E}(u)\right)\left(-\frac{1}{u\psi(u)}\right)\\ &= -\frac{\hat{S}(u)}{u} + \frac{\delta}{u\psi(u)}\hat{E}(u),\end{aligned}$$

which is equal to (3.35). We observe, using (1.3), that

$$\begin{aligned}
\frac{d\hat{I}(u)}{du} &= \frac{dI(t)}{dt}\Big|_{t=\varphi(u)} \varphi'(u) \\
&= \left(\delta\hat{E}(u) - \gamma\hat{I}(u) - \mu\hat{I}(u)\right)\left(-\frac{1}{u\psi(u)}\right) \\
&= -\delta\frac{\hat{E}(u)}{u\psi(u)} + (\gamma+\mu)\frac{\hat{I}(u)}{u\psi(u)} \\
&= -\frac{\delta}{u\psi(u)}\hat{E}(u) + \frac{\gamma+\mu}{\beta}\frac{1}{u},
\end{aligned}$$

and therefore (3.36) follows. We are led to

$$\begin{aligned}
\frac{d\hat{R}(u)}{du} &= \frac{dR(t)}{dt}\Big|_{t=\varphi(u)} \varphi'(u) = R'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= \gamma\hat{I}(u)\left(-\frac{1}{u\psi(u)}\right) \\
&= -\frac{\gamma}{\beta}\frac{1}{u}
\end{aligned}$$

by use of (1.4) and (3.44). Thus (3.37) is obtained. Similarly we have

$$\begin{aligned}
\frac{d\hat{D}(u)}{du} &= \frac{dD(t)}{dt}\Big|_{t=\varphi(u)} \varphi'(u) = D'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= \mu\hat{I}(u)\left(-\frac{1}{u\psi(u)}\right) \\
&= -\frac{\mu}{\beta}\frac{1}{u},
\end{aligned}$$

which is the desired equation (3.38). It is easily seen that

$$\begin{aligned}
\hat{S}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= S\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = S(0) = \tilde{S}, \\
\hat{E}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= E\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = E(0) = \tilde{E}, \\
\hat{I}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= I\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = I(0) = \tilde{I}, \\
\hat{R}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= R\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = R(0) = \tilde{R}, \\
\hat{D}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= D\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = D(0) = \tilde{D}.
\end{aligned}$$

Hence, (3.39)–(3.43) are satisfied. □

Theorem 3.11. *Solving the initial value problem* (3.34)–(3.43)*, we obtain the parametric solution* (2.5)–(2.9) *for* $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$.

Proof. Since (3.34) is equivalent to

$$\frac{d}{du}\left(\frac{1}{u}\hat{S}(u)\right) = 0,$$

we have

$$\hat{S}(u) = ku$$

for some constant k. We see from (3.39) that

$$\hat{S}\left(e^{-(\beta/\mu)\tilde{D}}\right) = ke^{-(\beta/\mu)\tilde{D}} = \tilde{S}$$

which implies

$$k = \tilde{S}e^{(\beta/\mu)\tilde{D}}.$$

Therefore we obtain

$$\hat{S}(u) = \tilde{S}e^{(\beta/\mu)\tilde{D}}u. \tag{3.45}$$

It follows from (3.45) that

$$-\frac{\hat{S}(u)}{u} = -\tilde{S}e^{(\beta/\mu)\tilde{D}}$$

and hence (3.35) reduces to

$$\frac{d\hat{E}(u)}{du} - \frac{\delta}{u\psi(u)}\hat{E}(u) = -\tilde{S}e^{(\beta/\mu)\tilde{D}} \tag{3.46}$$

which can be rewritten as

$$\frac{d\hat{E}(u)}{du} + \delta\varphi'(u)\hat{E}(u) = -\tilde{S}e^{(\beta/\mu)\tilde{D}}. \tag{3.47}$$

Multiplying (3.47) by $e^{\delta\varphi(u)}$ gives

$$\frac{d}{du}\left(e^{\delta\varphi(u)}\hat{E}(u)\right) = -\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{\delta\varphi(u)},$$

and an integration of the above on $[u, e^{-(\beta/\gamma)\tilde{R}}]$ yields

$$\hat{E}(u) = e^{-\delta\varphi(u)}\left(\tilde{E} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right),$$

which is equal to (2.6). Multiplying (3.36) by β, we have

$$\frac{d(\beta\hat{I}(u))}{du} - \frac{\gamma+\mu}{u} = -\frac{\beta\delta}{u\psi(u)}\hat{E}(u). \tag{3.48}$$

Define $z(u)$ by

$$z(u) := \beta\hat{I}(u) - \left(\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u\right),$$

then we obtain

$$\frac{dz(u)}{du} = \frac{d(\beta\hat{I}(u))}{du} + \beta\tilde{S}e^{(\beta/\mu)\tilde{D}} - \frac{\gamma+\mu}{u}. \tag{3.49}$$

Combining (3.48) with (3.49), we get

$$\begin{aligned}\frac{dz(u)}{du} &= \beta\tilde{S}e^{(\beta/\mu)\tilde{D}} - \frac{\beta\delta}{u\psi(u)}\hat{E}(u)\\ &= -\beta\left(-\tilde{S}e^{(\beta/\mu)\tilde{D}} + \frac{\delta}{u\psi(u)}\hat{E}(u)\right).\end{aligned} \tag{3.50}$$

It follows from (3.46) and (3.50) that

$$\frac{dz(u)}{du} = -\beta \frac{d\hat{E}(u)}{du},$$

and therefore

$$z(u) = -\beta \hat{E}(u) + k$$

for some constant k. Since

$$\begin{aligned} z\big(e^{-(\beta/\mu)\tilde{D}}\big) &= \beta \hat{I}\big(e^{-(\beta/\mu)\tilde{D}}\big) - \big(\beta N - \beta \tilde{R} - \beta \tilde{S} - \beta \tilde{D}\big) \\ &= \beta \tilde{I} - \big(\beta N - \beta \tilde{R} - \beta \tilde{S} - \beta \tilde{D}\big) = -\beta \tilde{E} \end{aligned}$$

and $-\beta \hat{E}\big(e^{-(\beta/\mu)\tilde{D}}\big) = -\beta \tilde{E}$, we see that $k = 0$, and therefore $z(u) = -\beta \hat{E}(u)$, i.e.,

$$\beta \hat{I}(u) = \big(\beta N - \beta \tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta \tilde{S} e^{(\beta/\mu)\tilde{D}} u + (\gamma + \mu) \log u\big) - \beta \hat{E}(u),$$

which is equivalent to (2.7). Solving (3.37) yields

$$\hat{R}(u) = -\frac{\gamma}{\beta} \log u + k$$

for some constant k. The initial condition (3.42) implies

$$\hat{R}\left(e^{-(\beta/\mu)\tilde{D}}\right) = -\frac{\gamma}{\beta} \log e^{-(\beta/\mu)\tilde{D}} + k = \frac{\gamma}{\mu}\tilde{D} + k = \tilde{R}$$

and hence $k = \tilde{R} - (\gamma/\mu)\tilde{D}$. Hence we obtain

$$\hat{R}(u) = -\frac{\gamma}{\beta} \log u + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}.$$

Similarly we find that

$$\hat{D}(u) = -\frac{\mu}{\beta} \log u. \qquad \square$$

Remark 3.12. Let $I(t)$ be given by (3.27). Then $I(t)$ can be represented in the simple form

$$I(t) = \frac{1}{\beta}\psi\big(\varphi^{-1}(t)\big)$$

by taking account of (3.6) and (3.27).

4 Various properties of solution

This section is devoted to various properties of solution by investigating the exact solution of the initial value problem (1.1)–(1.6).

Theorem 4.1. *Let $D(t)$ be given by (3.29). Then we find that $D(\infty) = \alpha$,*

$$D(\infty) = N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu)D(\infty)} - \frac{\gamma}{\mu} D(\infty), \tag{4.1}$$

and that $D(t)$ is an increasing function on $[0, \infty)$ such that

$$\tilde{D} \le D(t) < \alpha = D(\infty).$$

Proof. We easily see that

$$\begin{aligned} D(\infty) = \lim_{t\to\infty} D(t) &= \lim_{t\to\infty} -\frac{\mu}{\beta}\log\,\varphi^{-1}(t) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0} -\frac{\mu}{\beta}\log\,u \\ &= \alpha. \end{aligned}$$

Since $\alpha = D(\infty)$, the identity (4.1) follows from the definition of α (see Lemma 3.1). In light of $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$, we obtain

$$-\frac{\mu}{\beta}\log\,e^{-(\beta/\mu)\tilde{D}} \le D(t) < -\frac{\mu}{\beta}\log\,e^{-(\beta/\mu)\alpha}$$

or

$$\tilde{D} \le D(t) < \alpha = D(\infty).$$

It is easy to check that $D(t)$ is increasing on $[0,\infty)$ in view of the fact that $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$. □

Theorem 4.2. *Let $S(t)$ be given by (3.25). Then we deduce that*

$$S(\infty) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\infty)}, \tag{4.2}$$

and that $S(t)$ is a decreasing function on $[0,\infty)$ such that

$$\tilde{S} \ge S(t) > \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} = S(\infty).$$

Proof. The identity (4.2) follows from

$$\begin{aligned} S(\infty) = \lim_{t\to\infty} S(t) &= \lim_{t\to\infty} \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0} \tilde{S}e^{(\beta/\mu)\tilde{D}}u \\ &= \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} \\ &= \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\infty)}. \end{aligned}$$

Since $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$, we have

$$\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} < \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) \le \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\tilde{D}}.$$

Therefore we get

$$\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} < S(t) \le \tilde{S}.$$

Since $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, we observe that $S(t)$ is also decreasing on $[0,\infty)$. □

Theorem 4.3. *Let $R(t)$ be given by (3.28). Then we conclude that*

$$R(\infty) = \frac{\gamma}{\mu}D(\infty) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}, \tag{4.3}$$

and that $R(t)$ is an increasing function on $[0,\infty)$ such that

$$\tilde{R} \le R(t) < R(\infty).$$

Proof. We obtain

$$\begin{aligned} R(\infty) = \lim_{t\to\infty} R(t) &= \lim_{t\to\infty}\left(-\frac{\gamma}{\beta}\log\varphi^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}\right) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0}\left(-\frac{\gamma}{\beta}\log u + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}\right) \\ &= \frac{\gamma}{\mu}\alpha + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} \\ &= \frac{\gamma}{\mu}D(\infty) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}. \end{aligned}$$

Since $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$, we get

$$\frac{\gamma}{\mu}\tilde{D} + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} \le R(t) < \frac{\gamma}{\mu}\alpha + \tilde{R} - \frac{\gamma}{\mu}\tilde{D},$$

or

$$\tilde{R} \le R(t) < \frac{\gamma}{\mu}D(\infty) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = R(\infty).$$ □

Theorem 4.4. *Let $E(t)$ be given by (3.26). Then we find that*

$$E(\infty) = 0, \quad E(t) > 0 \quad \text{on } [0,\infty),$$

and $E(t)$ has the maximum $\max_{t\ge 0} E(t)$ at some $t = T_1 \in \{T;\, E'(T) = 0\}$, where

$$\begin{aligned} E'(T) = &\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(T)\right)\psi(\varphi^{-1}(T)) \\ &- \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(T) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(T)\right). \end{aligned}$$

Proof. We easily check that

$$\begin{aligned} E(\infty) &= \lim_{t\to\infty} E(t) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0}\left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right) \\ &= 0 \end{aligned}$$

in light of of (3.20) in Proposition 3.7. Since $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$ $(t \ge 0)$ and $\hat{E}(u) > 0$ for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$ (cf. (3.19)), it is easily seen that $E(t) = \hat{E}(\varphi^{-1}(t)) > 0$ on $[0,\infty)$. The hypothesis (A_2) implies that the right differential derivative $E'_+(0)$ is positive because

$$E'_+(0) = \lim_{t\to+0} E'(t) = \lim_{t\to+0}\big(\beta S(t)I(t) - \delta E(t)\big) = \beta\tilde{S}\tilde{I} - \delta\tilde{E} > 0.$$

Since the definition of $E'_+(0)$ implies

$$0 < E'_+(0) = \lim_{t\to+0}\frac{E(t) - E(0)}{t} = \lim_{t\to+0}\frac{E(t) - \tilde{E}}{t},$$

we see that for $\varepsilon = (1/2)E'_+(0) > 0$ there exists a number $\delta_\varepsilon > 0$ such that

$$\left|\frac{E(t) - \tilde{E}}{t} - E'_+(0)\right| < \frac{1}{2}E'_+(0)$$

holds for $0 < t < \delta_\varepsilon$, and hence

$$\frac{1}{2}E'_+(0) < \frac{E(t) - \tilde{E}}{t}$$

or

$$E(t) > \tilde{E} + \frac{1}{2}E'_+(0)t > \tilde{E}$$

holds for $0 < t < \delta_\varepsilon$. Since $E(\infty) = 0$, there exists a number $\tilde{T}$ such that $E(\tilde{T}) = \tilde{E}$ and $E(t) \le \tilde{E}$ for $t \ge \tilde{T}$. Therefore there exists $\max_{0\le t\le \tilde{T}} E(t) = E(T_1)$ $(> \tilde{E})$ at some $t = T_1$ $(< \tilde{T})$. Since $E(t) \le \tilde{E}$ for $t \ge \tilde{T}$, we observe that $\max_{t\ge 0} E(t) = \max_{0\le t\le \tilde{T}} E(t) = E(T_1)$. It is obvious that $E'(T_1) = 0$. It can be shown from (3.25)–(3.27) and (3.44) that

$$\begin{aligned} E'(t) = &-\delta E(t) + \beta S(t)I(t) \\ = &-\delta E(t) + \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)\psi(\varphi^{-1}(t)) \\ = &-\delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t) - I(t)\right) \\ &+ \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)\psi(\varphi^{-1}(t)) \\ = &\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)\right)\psi(\varphi^{-1}(t)) \\ &-\delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t)\right). \end{aligned} \tag{4.4}$$

□

Remark 4.5. If u_1 is a unique solution of the equation

$$\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}u\right)\psi(u) = \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\gamma+\mu}{\beta}\log u\right),$$

then we get

$$T_1 = \varphi(u_1) = \int_{u_1}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)}$$

in view of (3.22) (*cf.* Figure 4.1).

In case $E'(T_1) = 0$, we obtain $\beta S(T_1)I(T_1) = \delta E(T_1)$ by (1.2), and therefore $E(T_1) = (\beta/\delta)S(T_1)I(T_1)$. Hence, in Theorem 4.4 we see that

$$\max_{t\ge 0} E(t) = E(T_1) = \frac{\beta}{\delta}S(T_1)I(T_1).$$

Letting

$$\Psi(u) := \left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}u\right)\psi(u),$$

we observe that $\Psi(u)$ is a solution of the initial value problem for the Abel differential equation

$$\begin{aligned} &\Psi'(u)\Psi(u) - \frac{\tilde{S}e^{(\beta/\mu)\tilde{D}}}{(\delta/\beta) + \tilde{S}e^{(\beta/\mu)\tilde{D}}u}\Psi(u)^2 - \frac{\delta+\gamma+\mu}{u}\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}u\right)\Psi(u) \\ &= -\delta\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}u\right)^2 \\ &\quad\times\frac{\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u}{u} \end{aligned} \tag{4.5}$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, with the initial condition

$$\Psi\left(e^{-(\beta/\mu)\tilde{D}}\right) = \beta\left(\frac{\delta}{\beta} + \tilde{S}\right)\tilde{I}. \tag{4.6}$$

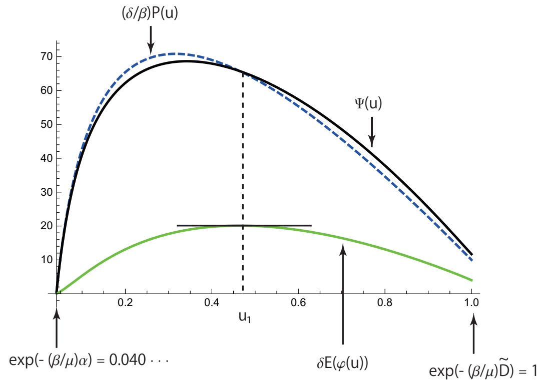

Figure 4.1: Variations of $(\delta/\beta)P(u) = \delta(N - \tilde{R} + (\gamma/\mu)\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + ((\gamma+\mu)/\beta)\log u)$ (dashed curve), $\delta E(\varphi(u))$ (green curve) and $\Psi(u)$ (solid curve) obtained by the numerical integration of the initial value problem (4.5), (4.6) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05$, $\delta = 0.2$ and $\mu = 0.01$. In this case we see that there exists a unique u_1 such that $(\delta/\beta)P(u_1) = \Psi(u_1)$, and that T_1 is calculated by $T_1 = \varphi(u_1) = \int_{u_1}^{1} \frac{d\xi}{\xi\psi(\xi)}$, where $\psi(u)$ is a unique positive solution of the initial value problem $\psi'\psi - \frac{0.26}{u}\psi = -0.2\,\frac{0.2-0.19u+0.06\log u}{u}$ $(0.040\ldots < u < 1)$, $\psi(1) = 0.006$.

Theorem 4.6. *Let $I(t)$ be given by (3.27). Then we see that*

$$I(\infty) = 0,$$
$$I(t) > 0 \quad \text{on } [0,\infty),$$

and $I(t)$ has the maximum $\max_{t\ge 0} I(t)$ at some $t = T_2 \in \{T;\ I'(T) = 0\}$, where

$$I'(T) = -\frac{\delta+\gamma+\mu}{\beta}\psi(\varphi^{-1}(T)) + \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(T) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(T)\right).$$

Proof. It follows from (3.20) and (3.30) that

$$\begin{aligned} I(\infty) &= \lim_{t\to\infty} I(t) = \lim_{t\to\infty}\frac{1}{\beta}\psi(\varphi^{-1}(t)) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0}\frac{1}{\beta}\psi(u) \\ &= 0. \end{aligned}$$

Since $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$ $(t \ge 0)$ and $\psi(u) > 0$ for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, we find that $I(t) = (1/\beta)\psi(\varphi^{-1}(t)) > 0$ on $[0,\infty)$. The hypothesis (A_3) implies that the right differential derivative $I'_+(0)$ is positive because

$$I'_+(0) = \lim_{t\to+0} I'(t) = \lim_{t\to+0}\left(\delta E(t) - \gamma I(t) - \mu I(t)\right) = \delta\tilde{E} - (\gamma+\mu)\tilde{I} > 0,$$

and therefore there exists a number $\delta_1 > 0$ such that $I(t) > \tilde{I}$ in $(0, \delta_1)$ as in the proof of Theorem 4.3. Since $I(\infty) = 0$, we can use the same arguments as in the proof of Theorem 4.3 to conclude that there exists the maximum $\max_{t \geq 0} I(t) = I(T_2)$ for some T_2. Then $I'(T_2) = 0$, and we obtain

$$\begin{aligned} I'(t) &= \tilde{S}e^{(\beta/\mu)\tilde{D}} \varphi^{-1}(t)\psi(\varphi^{-1}(t)) - \frac{\gamma+\mu}{\beta}\psi(\varphi^{-1}(t)) - E'(t) \\ &= \tilde{S}e^{(\beta/\mu)\tilde{D}} \varphi^{-1}(t)\psi(\varphi^{-1}(t)) - \frac{\gamma+\mu}{\beta}\psi(\varphi^{-1}(t)) \\ &\quad - \left[\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) \right) \psi(\varphi^{-1}(t)) \right. \\ &\quad \left. -\delta \left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t) \right) \right] \\ &= -\frac{\delta+\gamma+\mu}{\beta}\psi(\varphi^{-1}(t)) \\ &\quad + \delta \left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t) \right) \end{aligned}$$

in light of (3.30), (3.33), (3.44) and (4.4). □

Remark 4.7. In case u_2 is a unique solution of the equation

$$\frac{\delta+\gamma+\mu}{\beta}\psi(u) = \delta \left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\gamma+\mu}{\beta}\log u \right),$$

then we get

$$T_2 = \varphi(u_2) = \int_{u_2}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)}$$

(*cf*. Figure 4.2). If $I'(T_2) = 0$, (1.3) implies that $\delta E(T_2) = (\gamma+\mu)I(T_2)$, and in Theorem 4.6 we see that

$$\max_{t\geq 0} I(t) = I(T_2) = \frac{\delta}{\gamma+\mu}E(T_2).$$

Theorem 4.8. *The function $E(t) + I(t)$ has the maximum*

$$\max_{t\geq 0} (E(t)+I(t)) = \tilde{S} + \tilde{E} + \tilde{I} - \frac{\gamma+\mu}{\beta}\left(1 + \log\tilde{S} - \log\frac{\gamma+\mu}{\beta}\right)$$

at

$$t = T_3 := \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}\right) = \int_{(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\mu)\tilde{D}})}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)} = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

Moreover, $E(t) + I(t)$ is increasing in $[0, T_3)$ and is decreasing in (T_3, ∞).

Proof. We see from (3.26) and (3.27) that

$$E(t) + I(t) = N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t). \tag{4.7}$$

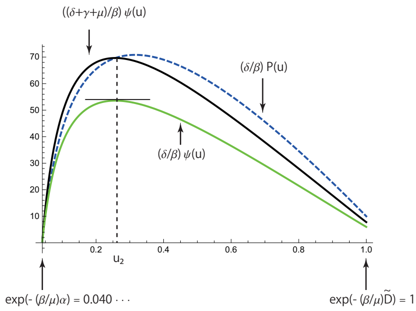

Figure 4.2: Variations of $(\delta/\beta)P(u) = \delta(N - \tilde{R} + (\gamma/\mu)\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + ((\gamma + \mu)/\beta)\log u)$ (dashed curve), $((\delta + \gamma + \mu)/\beta)\psi(u)$ (solid curve), and $(\delta/\beta)\psi(u)$ (green curve) obtained by the numerical integration of the initial value problem (4.5), (4.6) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05$, $\delta = 0.2$ and $\mu = 0.01$. In this case we find that there exists a unique u_2 such that $(\delta/\beta)P(u_2) = ((\delta + \gamma + \mu)/\beta)\psi(u_2)$, and that T_2 is calculated by $T_2 = \varphi(u_2) = \int_{u_2}^{1} \frac{d\xi}{\xi\psi(\xi)}$, where $\psi(u)$ is the unique positive solution of the same initial value problem as in Figure 4.1.

Differentiating (4.7) with respect to t gives

$$\begin{aligned} E'(t) + I'(t) &= -\tilde{S}e^{(\beta/\mu)\tilde{D}}\left(\varphi^{-1}(t)\right)' + \frac{\gamma+\mu}{\beta}\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} \\ &= \left(-\tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\right)\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} \\ &= \left(-S(t) + \frac{\gamma+\mu}{\beta}\right)\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)}. \end{aligned}$$

Since

$$\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} = -\psi\left(\varphi^{-1}(t)\right) < 0$$

by (3.30), we observe that $E'(t) + I'(t) = 0$ for

$$t = T_3 = \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}\right) = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

Note that

$$e^{-(\beta/\mu)\alpha} < \frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}} = \frac{\gamma+\mu}{\beta\tilde{S}}e^{-(\beta/\mu)\tilde{D}} < e^{-(\beta/\mu)\tilde{D}}$$

in view of (A_7) and Remark 3.2. In light of (3.22) we obtain

$$T_3 = \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}\right) = \int_{(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\mu)\tilde{D}})}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)} = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

It is easy to check that $E'(t) + I'(t) > 0$ [resp. < 0] if and only if $t < T_3$ [resp. $> T_3$], because $\varphi^{-1}(t)$ is decreasing on $[0, \infty)$. Therefore we conclude that $E(t) + I(t)$ is increasing in $[0, T_3)$

and is decreasing in (T_3, ∞). It can be shown that

$$\begin{aligned}\max_{t\geq 0}\left(E(t)+I(t)\right) &= N-\tilde{R}+\frac{\gamma}{\mu}\tilde{D}-\tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(T_3)+\frac{\gamma+\mu}{\beta}\log\varphi^{-1}(T_3)\\ &= N-\tilde{R}+\frac{\gamma}{\mu}\tilde{D}-\tilde{S}e^{(\beta/\mu)\tilde{D}}\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}+\frac{\gamma+\mu}{\beta}\log\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}\right)\\ &= N-\tilde{R}+\frac{\gamma}{\mu}\tilde{D}-\frac{\gamma+\mu}{\beta}+\frac{\gamma+\mu}{\beta}\left(\log\frac{\gamma+\mu}{\beta}-\log\tilde{S}-\frac{\beta}{\mu}\tilde{D}\right)\\ &= \tilde{S}+\tilde{E}+\tilde{I}-\frac{\gamma+\mu}{\beta}\left(1+\log\tilde{S}-\log\frac{\gamma+\mu}{\beta}\right). \qquad\square\end{aligned}$$

Remark 4.9. Since $u_3 = (\gamma+\mu)/\left(\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\right) = ((\gamma+\mu)/(\beta\tilde{S}))e^{-(\beta/\mu)\tilde{D}}$ is a unique solution of the equation

$$\left(N-\tilde{R}+\frac{\gamma}{\mu}\tilde{D}-\tilde{S}e^{(\beta/\mu)\tilde{D}}u+\frac{\gamma+\mu}{\beta}\log u\right)'=0,$$

we obtain

$$T_3=\varphi(u_3)=\int_{((\gamma+\mu)/(\beta\tilde{S}))e^{-(\beta/\mu)\tilde{D}}}^{e^{-(\beta/\mu)\tilde{D}}}\frac{d\xi}{\xi\psi(\xi)}$$

(*cf.* Figure 4.3).

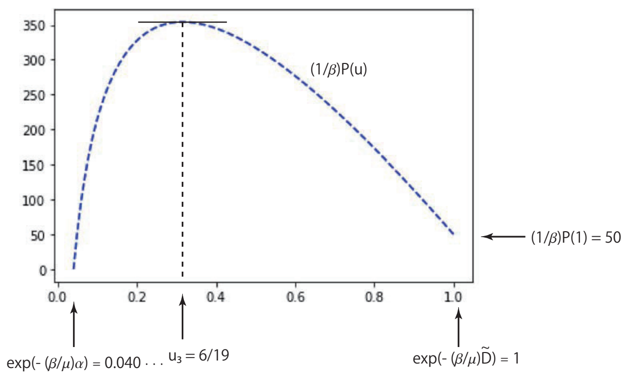

Figure 4.3: Variation of $(1/\beta)P(u) = N-\tilde{R}+(\gamma/\mu)\tilde{D}-\tilde{S}e^{(\beta/\mu)\tilde{D}}u+((\gamma+\mu)/\beta)\log u$ (dashed curve) for $N=1000, \tilde{S}=950, \tilde{I}=30, \tilde{E}=20, \tilde{R}=0, \beta=0.3/1000, \gamma=0.1$ and $\delta=0.2$. In this case we observe that there exists a unique $u_3=6/19$ such that $(1/\beta)P'(u_3)=0$, and that T_3 is calculated by $T_3=\varphi(u_3)=\int_{6/19}^{1}\frac{d\xi}{\xi\psi(\xi)}=41.9\ldots$, where $\psi(u)$ is the unique positive solution of the same initial value problem as in Figure 4.1.

Theorem 4.10. *The following relation holds:*

$$S(\infty)=\tilde{S}+\tilde{E}+\tilde{I}+\frac{\gamma+\mu}{\beta}\log\frac{S(\infty)}{\tilde{S}}.$$

Proof. Since $E(\infty) = I(\infty) = 0$, we obtain

$$\begin{aligned} S(\infty) &= N - R(\infty) - D(\infty) \\ &= N - \left(\frac{\gamma}{\mu}D(\infty) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}\right) - D(\infty) \\ &= N - \tilde{R} - \tilde{D} + \frac{\gamma+\mu}{\beta}\left(\frac{\beta}{\mu}\tilde{D} - \frac{\beta}{\mu}D(\infty)\right) \\ &= \tilde{S} + \tilde{E} + \tilde{I} + \frac{\gamma+\mu}{\beta}\log\left(e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\infty)}\right) \\ &= \tilde{S} + \tilde{E} + \tilde{I} + \frac{\gamma+\mu}{\beta}\log\frac{S(\infty)}{\tilde{S}} \end{aligned}$$

by use of (4.2) and (4.3). □

Theorem 4.11. *We find that*

$$S'(\infty) = E'(\infty) = I'(\infty) = R'(\infty) = D'(\infty) = 0.$$

Proof. Since $E(\infty) = I(\infty) = 0$, we see from (1.1)–(1.5) that

$$\begin{aligned} S'(\infty) &= -\beta S(\infty)I(\infty) = 0, \\ E'(\infty) &= \beta S(\infty)I(\infty) - \delta E(\infty) = 0, \\ I'(\infty) &= \delta E(\infty) - \gamma I(\infty) - \mu I(\infty) = 0, \\ R'(\infty) &= \gamma I(\infty) = 0, \\ D'(\infty) &= \mu I(\infty) = 0. \end{aligned}$$

□

Remark 4.12. The hypothesis (A_5) is satisfied if $\tilde{D} = 0$, since $N > \tilde{S} + \tilde{R}$.

Remark 4.13. It follows from Theorems 4.1–4.6 that $S(t) > 0, E(t) > 0, I(t) > 0$ for $t \geq 0$ and $R(t) > 0, D(t) > 0$ for $t > 0$ (*cf.* Figure 4.4).

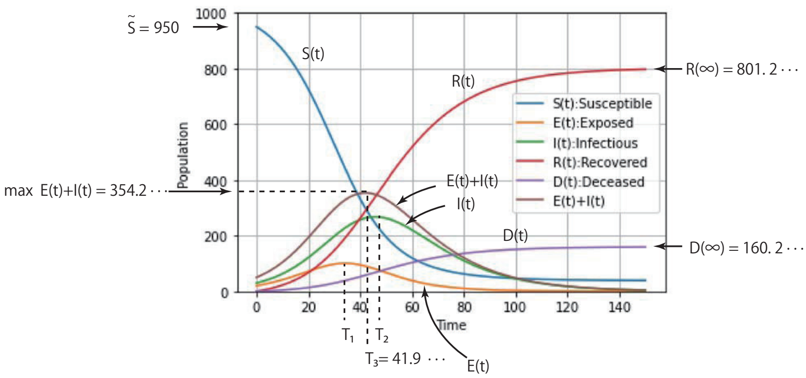

Figure 4.4: Variations of $S(t)$, $E(t)$, $I(t)$, $R(t)$, $D(t)$ and $E(t)+I(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.6) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05$, $\delta = 0.2$ and $\mu = 0.01$.

Remark 4.14. We note that

$$E'_+(0) + I'_+(0) = \beta\tilde{S}\tilde{I} - (\gamma+\mu)\tilde{I},$$

and that $E'_+(0) + I'_+(0) \le 0$ is equivalent to $\tilde{S} \le (\gamma+\mu)/\beta$. Let $E'_+(0) + I'_+(0) \le 0$ be satisfied, and let $P(u)$ be given by (3.9). Then we see that

$$\frac{1}{\beta}P'(u) = -\tilde{S}e^{(\beta/\mu)\tilde{D}} + \frac{\gamma+\mu}{\beta}\frac{1}{u} = 0$$

at $u = ((\gamma+\mu)/(\beta\tilde{S}))e^{-(\beta/\mu)\tilde{D}}$ $(\ge e^{-(\beta/\mu)\tilde{D}})$ and that $(1/\beta)P(u)$ is increasing in $\big(e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}}\big]$, $\lim_{u\to e^{-(\beta/\mu)\alpha}+0}(1/\beta)P(u) = 0$ and $(1/\beta)P\big(e^{-(\beta/\mu)\tilde{D}}\big) = \tilde{E}+\tilde{I} > 0$. Since $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, $\varphi^{-1}(0) = e^{-(\beta/\mu)\tilde{D}}$ and $\lim_{t\to\infty}\varphi^{-1}(t) = e^{-(\beta/\mu)\alpha}$, we conclude that $E(t)+I(t) = (1/\beta)P\big(\varphi^{-1}(t)\big)$ is decreasing on $[0,\infty)$, $E(0)+I(0) = \tilde{E}+\tilde{I}$, and $E(\infty)+I(\infty) = 0$ (*cf.* Figure 4.5).

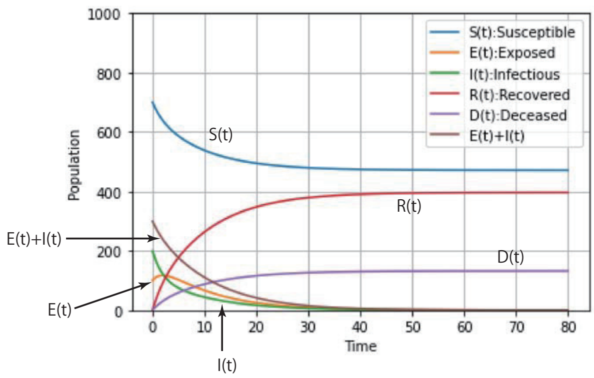

Figure 4.5: Variations of $S(t)$, $E(t)$, $I(t)$, $R(t)$, $D(t)$ and $E(t)+I(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.6) for $N = 1000, \tilde{S} = 700, \tilde{E} = 100, \tilde{I} = 200, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.3/1000, \gamma = 0.3$, $\delta = 0.2$ and $\mu = 0.1$. In this case we find that $E'_+(0) = 22 > 0$, $I'_+(0) = -60 < 0$ and $E'_+(0) + I'_+(0) = -38 < 0$.

Remark 4.15. The function $D(t)$ given by (3.29) is a positive and increasing solution of the initial value problem for (2.1) with the initial conditions $D(0) = \tilde{D}$ and $D'_+(0) = \mu\tilde{I}$. In fact, it follows from Theorem 4.1 that $D(t)$ is an increasing function such that $D(t) > 0$ for $t > 0$. Since

$$\begin{aligned} D'(t) &= -\frac{\mu}{\beta}\frac{\big(\varphi^{-1}(t)\big)'}{\varphi^{-1}(t)} = -\frac{\mu}{\beta}\big(-\psi(\varphi^{-1}(t))\big) = \frac{\mu}{\beta}\psi(\varphi^{-1}(t)), \\ D''(t) &= -\frac{\mu}{\beta}\varphi^{-1}(t)\psi'(\varphi^{-1}(t))\psi(\varphi^{-1}(t)) \end{aligned}$$

in light of (3.30), we arrive at

$$\begin{aligned}
&D''(t) + (\delta+\gamma+\mu)D'(t)\\
&\quad = -\frac{\mu}{\beta}\varphi^{-1}(t)\left(\psi'(\varphi^{-1}(t))\psi(\varphi^{-1}(t)) - (\delta+\gamma+\mu)\frac{\psi(\varphi^{-1}(t))}{\varphi^{-1}(t)}\right)\\
&\quad = -\frac{\mu}{\beta}\varphi^{-1}(t)\left(-\delta\frac{\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + (\gamma+\mu)\log\varphi^{-1}(t)}{\varphi^{-1}(t)}\right)\\
&\quad = \delta\mu\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t)\right)\\
&\quad = \delta\mu\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)} - \frac{\gamma+\mu}{\mu}D(t)\right)
\end{aligned}$$

in view of (3.3). It is easy to check that $D(0) = -(\mu/\beta)\log\varphi^{-1}(0) = -(\mu/\beta)\log e^{-(\beta/\mu)\tilde{D}} = \tilde{D}$ and

$$\begin{aligned}
D'_+(0) &= \lim_{\varepsilon\to+0} D'(\varepsilon) = \lim_{\varepsilon\to+0}\frac{\mu}{\beta}\psi(\varphi^{-1}(\varepsilon))\\
&= \frac{\mu}{\beta}\psi(\varphi^{-1}(0)) = \frac{\mu}{\beta}\psi(e^{-(\beta/\mu)\tilde{D}}) = \frac{\mu}{\beta}\beta\tilde{I} = \mu\tilde{I}.
\end{aligned}$$

Remark 4.16. Let $D(t)$ be given by (3.29). Then the functions $S(t), E(t), I(t)$ and $R(t)$ given by (2.23)–(2.26) reduce to (3.25)–(3.28), respectively, since

$$e^{-(\beta/\mu)D(t)} = \varphi^{-1}(t), \qquad t = D^{-1}(-(\mu/\beta)\log\varphi^{-1}(t)) \quad \text{and} \quad \varphi(v) = D^{-1}(-(\mu/\beta)\log v).$$

Remark 4.17. If we suppose the hypothesis

(A'_4) $\tilde{R} \geq 0$ and $\tilde{R}$ satisfies

$$N - \tilde{D} > \tilde{S}e^{(\beta/\gamma)\tilde{R}} + \tilde{R},$$

then the transcendental equation

$$y = \frac{\gamma}{\gamma+\mu}N - \frac{\gamma}{\gamma+\mu}\tilde{D} + \frac{\mu}{\gamma+\mu}\tilde{R} - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)y} \tag{4.8}$$

has a unique solution $y = \alpha_*$ such that

$$\tilde{R} < \alpha_* < N$$

(see Yoshida [18, Lemma 3]). Since the equation (4.8) reduces to the transcendental equation in Lemma 3.1 by the transformation $y = \tilde{R} - (\gamma/\mu)(\tilde{D} - x)$, we find that $\alpha_* = \tilde{R} - (\gamma/\mu)(\tilde{D} - \alpha)$. We define

$$\varphi_*(w) := \int_w^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi\psi_*(\xi)}$$

for $e^{-(\beta/\gamma)\alpha_*} < w \leq e^{-(\beta/\gamma)\tilde{R}}$, where $\psi_*(\xi)$ is a unique positive solution of the initial value problem

$$\begin{aligned}
&\psi'_*(\xi)\psi_*(\xi) - \frac{\delta+\mu+\gamma}{\xi}\psi_*(\xi)\\
&\qquad = -\delta\frac{\beta N - \beta\tilde{D} + ((\beta\mu)/\gamma)\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\xi + (\mu+\gamma)\log\xi}{\xi}\\
&\qquad\quad (e^{-(\beta/\gamma)\alpha_*} < \xi < e^{-(\beta/\gamma)\tilde{R}}),\\
&\psi_*(e^{-(\beta/\gamma)\tilde{R}}) = \beta\tilde{I}.
\end{aligned}$$

It follows from the transformation

$$\xi = e^{-(\beta/\gamma)\tilde{R}} e^{(\beta/\mu)\tilde{D}} u$$

that

$$\varphi_*(w) = \int_{e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\mu)\tilde{D}} w}^{e^{-(\beta/\mu)\tilde{D}}} \frac{du}{u\psi_*\big(e^{-(\beta/\gamma)\tilde{R}} e^{(\beta/\mu)\tilde{D}} u\big)},$$

where $e^{-(\beta/\mu)\alpha} < e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\mu)\tilde{D}} w \leq e^{-(\beta/\mu)\tilde{D}}$. It is easy to check that $\psi_*(e^{-(\beta/\gamma)\tilde{R}} e^{(\beta/\mu)\tilde{D}} u)$ is a solution of the initial value problem (3.3), (3.4), and therefore

$$\psi_*\big(e^{-(\beta/\gamma)\tilde{R}} e^{(\beta/\mu)\tilde{D}} u\big) = \psi(u)$$

by the uniqueness of solutions of the initial value problem (3.3), (3.4). Hence we obtain

$$\varphi_*(w) = \varphi\big(e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\mu)\tilde{D}} w\big). \tag{4.9}$$

Let $\varphi_*^{-1}(t)$ and $\varphi^{-1}(t)$ be the inverse functions of

$$t = \varphi_*(w), \quad t = \varphi\big(e^{(\beta/\gamma)\tilde{R}} e^{-(\beta/\mu)\tilde{D}} w\big),$$

respectively, then we see that

$$\varphi_*^{-1}(t) = e^{-(\beta/\gamma)\tilde{R}} e^{(\beta/\mu)\tilde{D}} \varphi^{-1}(t) \quad (0 \leq t < \infty). \tag{4.10}$$

It is easy to see that the hypothesis (A_7) is equivalent to

(A_8) $\tilde{S} < \dfrac{\mu+\gamma}{\beta} e^{(\beta/\gamma)(\alpha_* - \tilde{R})}$.

Let $\big(S_*(t), E_*(t), I_*(t), R_*(t), D_*(t)\big)$ be the exact solution of the initial value problem (1.1)–(1.6) by starting our arguments utilizing (1.4) instead of (1.5). Then we get

$$\begin{aligned}
S_*(t) &= \tilde{S} e^{(\beta/\gamma)\tilde{R}} \varphi_*^{-1}(t), \\
E_*(t) &= \tilde{E} e^{-\delta t} + \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-\delta t} \int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta \varphi_*(v)} dv, \\
I_*(t) &= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} \varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta} \log \varphi_*^{-1}(t) \\
&\qquad - \tilde{E} e^{-\delta t} - \tilde{S} e^{(\beta/\gamma)\tilde{R}} e^{-\delta t} \int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta \varphi_*(v)} dv, \\
R_*(t) &= -\frac{\gamma}{\beta} \log \varphi_*^{-1}(t), \\
D_*(t) &= -\frac{\mu}{\beta} \log \varphi_*^{-1}(t) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}.
\end{aligned}$$

We observe, using (4.10), that

$$\begin{aligned}
S_*(t) &= \tilde{S} e^{(\beta/\gamma)\tilde{R}} \varphi_*^{-1}(t) = \tilde{S} e^{(\beta/\gamma)\tilde{R}} \big(e^{-(\beta/\gamma)\tilde{R}} e^{(\beta/\mu)\tilde{D}} \varphi^{-1}(t)\big) \\
&= \tilde{S} e^{(\beta/\mu)\tilde{D}} \varphi^{-1}(t) = S(t).
\end{aligned}$$

It follows from (4.9) and (4.10) that

$$\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv = \int_{e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} \exp\left(\delta\varphi\left(e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\mu)\tilde{D}}v\right)\right)dv$$
$$= e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}} \int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(w)}dw$$

and hence

$$\begin{aligned} E_*(t) &= \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta t}\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv \\ &= \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(w)}dw = E(t). \end{aligned} \tag{4.11}$$

Since

$$\begin{aligned} &N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\log\varphi_*^{-1}(t) \\ &= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t), \end{aligned}$$

we deduce that $I_*(t) = I(t)$ in view of (4.11). It is easy to see that

$$\begin{aligned} R_*(t) &= -\frac{\gamma}{\beta}\log\varphi_*^{-1}(t) \\ &= -\frac{\gamma}{\beta}\left(-\frac{\beta}{\gamma}\tilde{R} + \frac{\beta}{\mu}\tilde{D} + \log\varphi^{-1}(t)\right) \\ &= -\frac{\gamma}{\beta}\log\varphi^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = R(t), \end{aligned}$$

and that

$$\begin{aligned} D_*(t) &= -\frac{\mu}{\beta}\log\varphi_*^{-1}(t) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\ &= -\frac{\mu}{\beta}\left(-\frac{\beta}{\gamma}\tilde{R} + \frac{\beta}{\mu}\tilde{D} + \log\varphi^{-1}(t)\right) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\ &= -\frac{\mu}{\beta}\log\varphi^{-1}(t) = D(t). \end{aligned}$$

Consequently we conclude that

$$\big(S_*(t), E_*(t), I_*(t), R_*(t), D_*(t)\big) \equiv \big(S(t), E(t), I(t), R(t), D(t)\big) \quad \text{on } [0,\infty).$$

Remark 4.18. The hypotheses (A_4') and (A_5) are equivalent to

(A_4'') $0 \le \tilde{R} < \dfrac{\gamma}{\beta}\log\left(1 + (\tilde{E}/\tilde{S}) + (\tilde{I}/\tilde{S})\right)$;

(A_5') $0 \le \tilde{D} < \dfrac{\mu}{\beta}\log\left(1 + (\tilde{E}/\tilde{S}) + (\tilde{I}/\tilde{S})\right)$,

respectively.

5 Uniqueness of positive solutions

This section is devoted to the uniqueness of positive solutions of the initial value problem (1.1)–(1.6). As a consequence we conclude that the exact solution (3.25)–(3.29) is the unique solution in the class of positive solutions.

A solution $(S(t), E(t), I(t), R(t), D(t))$ of the SEIRD differential system (1.1)–(1.5) is said to be *positive* if $S(t) > 0, E(t) > 0, I(t) > 0, R(t) > 0$ and $D(t) > 0$ for $t > 0$.

Theorem 5.1. *Let $(S_i(t), E_i(t), I_i(t), R_i(t), D_i(t))$ $(i = 1, 2)$ be solutions of the initial value problem (1.1)–(1.6) such that $S_i(t) > 0, E_i(t) > 0, I_i(t) > 0, R_i(t) > 0$ for $t > 0$. Then we find that*

$$(S_1(t), E_1(t), I_1(t), R_1(t), D_1(t)) \equiv (S_2(t), E_2(t), I_2(t), R_2(t), D_2(t)) \quad \text{on } [0, \infty). \tag{5.1}$$

Proof. First we note that $D_i(t) > 0$ for $t > 0$ $(i = 1, 2)$ since $D_i'(t) = \mu I_i(t) > 0$ for $t > 0$ and $D_i(0) = \tilde{D} \geq 0$. It follows from Lemma 2.1 that $D_i(t)$ $(i = 1, 2)$ satisfies (2.1) and the initial condition

$$D_i(0) = \tilde{D}, \qquad \lim_{\varepsilon \to +0} D_i'(\varepsilon) = \mu \tilde{I}$$

in view of (1.5) and (1.6). It is easy to see that

$$\boldsymbol{z}_i(t) := \big(D_i(t), D_i'(t)\big) \qquad (i = 1, 2)$$

are positive solutions of the initial value problem

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t)), \qquad t > 0,$$
$$\boldsymbol{y}_+(0) = \lim_{\varepsilon \to +0} \boldsymbol{y}(\varepsilon) = (\tilde{D}, \mu \tilde{I}),$$

where $\boldsymbol{f}(\boldsymbol{y})$ is a function defined by

$$\boldsymbol{f}(\boldsymbol{y}) = \left(y_2, -(\delta + \gamma + \mu) y_2 + +\delta\mu \left(N - \tilde{R} + \frac{\gamma}{\mu} \tilde{D} - \tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu) y_1} - \left(1 + \frac{\gamma}{\mu}\right) y_1 \right) \right)$$

for $\boldsymbol{y} = (y_1, y_2)$ such that $y_1 > 0$ and $y_2 > 0$. Since

$$\frac{\partial \boldsymbol{f}}{\partial y_1}(\boldsymbol{y}) = \left(0, \beta\delta\tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu) y_1} - \delta(\gamma + \mu)\right),$$
$$\frac{\partial \boldsymbol{f}}{\partial y_2}(\boldsymbol{y}) = (1, -(\delta + \gamma + \mu)),$$

we obtain

$$\left| \frac{\partial \boldsymbol{f}}{\partial y_k}(\boldsymbol{y}) \right| \leq \max\left\{ \beta\delta\tilde{S} e^{(\beta/\mu)\tilde{D}} + \delta(\gamma + \mu), 1 + (\delta + \gamma + \mu) \right\} \ (\equiv K) \ (k = 1, 2)$$

for $\boldsymbol{y} = (y_1, y_2)$ such that $y_1 > 0$ and $y_2 > 0$, where the magnitude of a vector $\boldsymbol{y}$, denoted by $|\boldsymbol{y}|$, is defined by

$$|\boldsymbol{y}| = |y_1| + |y_2| \quad \text{for } \boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2.$$

Therefore, $\boldsymbol{f}(\boldsymbol{y})$ satisfies a Lipschitz condition on $(0, \infty) \times (0, \infty)$ with Lipschitz constant K (see Coddington [7, p.248, Theorem 1]). Since

$$\boldsymbol{z}_i'(t) = \boldsymbol{f}(\boldsymbol{z}_i(t)), \qquad t > 0 \qquad (i = 1, 2),$$

integrating the above on $[\varepsilon, t]$ $(\varepsilon > 0)$ and then taking the limit as $\varepsilon \to +0$ yield

$$\boldsymbol{z}_i(t) - \lim_{\varepsilon \to +0} \boldsymbol{z}_i(\varepsilon) = \int_0^t \boldsymbol{f}(\boldsymbol{z}_i(s))ds, \qquad t > 0,$$

and we observe, using $\lim_{\varepsilon \to +0} \boldsymbol{z}_i(\varepsilon) = (\tilde{D}, \mu\tilde{I})$, that

$$\boldsymbol{z}_i(t) = (\tilde{D}, \mu\tilde{I}) + \int_0^t \boldsymbol{f}(\boldsymbol{z}_i(s))ds, \qquad t > 0.$$

Therefore we obtain

$$\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t) = \int_0^t \left(\boldsymbol{f}(\boldsymbol{z}_1(s)) - \boldsymbol{f}(\boldsymbol{z}_2(s))\right) ds, \qquad t > 0$$

and hence

$$|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)| \leq K \int_0^t |\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)| \, ds, \qquad t > 0$$

since $\boldsymbol{f}(\boldsymbol{y})$ satisfies a Lipschitz condition with Lipschitz constant K. Defining

$$W(t) := \int_0^t |\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)| \, ds,$$

we obtain

$$W'(t) - KW(t) \leq 0, \qquad t > 0,$$

or

$$\left(e^{-Kt}W(t)\right)' \leq 0, \qquad t > 0.$$

Since $e^{-Kt}W(t) \leq W(0) = 0$ $(t \geq 0)$, we see that $W(t) \leq 0$ $(t \geq 0)$. Hence

$$|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)| \leq KW(t) \leq 0, \qquad t > 0,$$

which yields

$$\boldsymbol{z}_1(t) = \boldsymbol{z}_2(t), \qquad t > 0.$$

Therefore we conclude that

$$D_1(t) \equiv D_2(t) \quad \text{on } (0, \infty).$$

Since $D_1(0) = D_2(0) = \tilde{D}$, we observe that

$$D_1(t) \equiv D_2(t) \quad \text{on } [0, \infty).$$

It follows from Corollary 2.4 that $S_i(t)$ $(i = 1, 2)$ can be represented by

$$S_i(t) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D_i(t)}$$

for $t \geq 0$. Since $D_1(t) = D_2(t)$ for $t \geq 0$, we deduce that $S_1(t) = S_2(t)$ for $t \geq 0$. Similarly we find that $E_1(t) = E_2(t)$ $(t \geq 0)$, $I_1(t) = I_2(t)$ $(t \geq 0)$ and $R_1(t) = R_2(t)$ $(t \geq 0)$. Consequently we conclude that (5.1) holds. □

Theorem 5.2. *Assume that the hypotheses* (A_1)–(A_7), (A_4') *hold. The function* $(S(t), E(t), I(t), R(t), D(t))$ *given by*

$$\begin{aligned}
S(t) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t),\\
E(t) &= \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\\
&= \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta t}\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv,\\
I(t) &= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t)\\
&\quad - \tilde{E}e^{-\delta t} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi_*(v)}dv\\
&= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\log\varphi_*^{-1}(t)\\
&\quad - \tilde{E}e^{-\delta t} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta t}\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv,\\
R(t) &= -\frac{\gamma}{\beta}\log\varphi^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = -\frac{\gamma}{\beta}\log\varphi_*^{-1}(t),\\
D(t) &= -\frac{\mu}{\beta}\log\varphi^{-1}(t) = -\frac{\mu}{\beta}\log\varphi_*^{-1}(t) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}
\end{aligned}$$

is a positive solution of the initial value problem (1.1)–(1.6)*, and is unique in the class of positive solutions.*

Proof. Combining Theorem 3.9, Remarks 4.13 and 4.17, we see that $(S(t), E(t), I(t), R(t), D(t))$ given above is a positive solution of the initial value problem (1.1)–(1.6). Uniqueness of positive solutions follows from Theorem 5.1. □

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