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Existence of nodal solutions to some nonlinear boundary value problems for ordinary differential equations of fourth order

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Abstract. In this paper, we study the existence of nodal solutions of some nonlinear boundary value problems for ordinary differential equations of fourth order with a spectral parameter in the boundary condition. To do this, we first study the global bifurcation of solutions from zero and infinity of the corresponding nonlinear eigenvalue problems in classes with a fixed oscillation count. Then, using these global bifurcation results, we prove the existence of solutions of the considered nonlinear boundary value problems with a fixed number of nodes.

Keywords: nonlinear problem, eigenvalue parameter, bifurcation point, nodal solution, component

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1 Introduction

In this paper, we consider the existence of nodal solutions to the following nonlinear boundary value problem for ordinary differential equations of fourth order

$$\ell(y) \equiv (p(x)y''(x))'' - (q(x)y'(x))' = \chi r(x)h(y(x)),\ x \in (0,l), \tag{1.1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{1.2}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{1.3}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{1.4}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{1.5}$$

where $Ty \equiv (py'')' - qy'$, p is a positive twice continuously differentiable function on $[0,l]$, q is a non-negative continuously differentiable function on $[0,l]$, χ is a positive number, $r(x)$ is

✉Corresponding author. Email: z_aliyev@mail.ru a positive continuous function on $[0,l]$, α, β, γ, a,b,c,d are real constants such that $\alpha,\beta,\gamma \in [0,\pi/2]$ and $\sigma = bc - ad > 0$. The nonlinear term h has the form $f+g$, where f and g are real-valued continuous on $\mathbb{R}$ functions that satisfy the following conditions:

$$\underline{f}_0,\ \overline{f}_0,\ \underline{f}_\infty,\ \overline{f}_\infty \in \mathbb{R} \quad \text{with} \quad \underline{f}_0 \neq \overline{f}_0,\ \underline{f}_\infty \neq \overline{f}_\infty, \tag{1.6}$$

where

$$\underline{f}_0 = \liminf_{|s|\to 0} \frac{f(s)}{s}, \qquad \overline{f}_0 = \limsup_{|s|\to 0} \frac{f(s)}{s}, \tag{1.7}$$

$$\underline{f}_\infty = \liminf_{|s|\to +\infty} \frac{f(s)}{s}, \qquad \overline{f}_\infty = \limsup_{|s|\to +\infty} \frac{f(s)}{s}; \tag{1.8}$$

$$sg(s) > 0 \quad \text{for } s \in \mathbb{R}\setminus\{0\}; \tag{1.9}$$

there exist positive constants g_0, $g_\infty \in (0,+\infty)$ such that

$$g_0 = \lim_{|s|\to 0} \frac{g(s)}{s}, \qquad g_\infty = \lim_{|s|\to +\infty} \frac{g(s)}{s}. \tag{1.10}$$

The subject of this paper is to determine the interval of χ, in which there are solutions to problem (1.1)–(1.5) that have a fixed number of simple zeros in $(0,l)$.

It is well known that boundary value problems for ordinary differential equations arise in the study of many different processes of natural science, see [9,10,12,14,17] and the references therein. For example, problem (1.1)–(1.5) arises when studying of bending (of deformation) of a homogeneous rod, in the cross sections of which a longitudinal force acts and at the right end of which the mass is concentrated or on this end a tracking force acts.

Problems similar to (1.1)–(1.5) for ordinary differential equations of second and fourth orders have been considered before in, for example, [8, 11, 13, 16, 18–22, 26–28]. In [8, 11, 18–21, 26], the authors using the global bifurcation results of [1, 2, 7, 8, 11, 18, 23–25] show that there are nontrivial solutions of the considered nonlinear problems, which have the usual nodal properties (unfortunately, there are gaps in the proofs of the main assertions in [11, Theorems 2.2 and 3.1] and [18, Theorem 3.1]). Similar results were obtained in the paper [22] by analytical methods involving the Prüfer angular functions. Should be noted that in [13, 26, 27], problems with local and nonlocal boundary conditions are considered and the existence of positive solutions of these problems is established.

In the present paper, using the global bifurcation results from [1–4, 6] and removing the above gaps (see the proof of Steps 1–3 of Theorem 3.1), we prove the existence of two different solutions to problem (1.1)–(1.5) with a fixed number of nodal points.

The rest of this article is organized as follows. Section 2 provides, which we need in the future, known facts about the unilateral global bifurcation of solutions from zero and infinity of nonlinear eigenvalue problems for fourth-order ordinary differential equations. In Section 3, we determine an interval for a parameter χ, in which there are nodal solutions to problem (1.1)–(1.5). In this case, the proof of the main theorem, i.e. Theorem 3.1 consists of 4 steps. In Step 1, using (1.6), (1.7) and the first condition from (1.10), we find bifurcation intervals from zero and prove the existence of two families of unbounded components of the solution set of problem (1.1)–(1.5) bifurcating from these intervals and contained in classes with a fixed number of nodes. In Step 2, using (1.6), (1.8) and the second condition from (1.10), we find bifurcation intervals from infinity and prove the existence of two families of unbounded components of the set of solutions bifurcating from these intervals and contained in classes with a fixed number of nodes in the neighborhood of these intervals, which either intersect another bifurcation interval, or intersect the line of trivial solutions, or have unbounded projections onto the line of trivial solutions. In Step 3, it is established that the global components of solutions to problem (1.1)–(1.5) bifurcating from intervals infinity are also contained in the corresponding classes with a fixed number of nodes and coincide with the corresponding components of solutions bifurcating from intervals of the line of trivial solutions.

2 Preliminary

We consider the following linear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda r(x) y(x),\ x \in (0,l), \\ y \in (b.c.)_\lambda, \end{cases} \qquad . \tag{2.1}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $(b.c.)_\lambda$ is a set of functions satisfying the boundary conditions (1.2)–(1.5).

The spectral properties of (2.1) were studied in [15], where, in particular, it was shown that the spectrum of this problem is discrete and consists of an infinitely increasing sequence $\{\lambda_k\}_{k=1}^\infty$ of real and simple eigenvalues. Moreover, if $c = 0$, then eigenfunction $y_k(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue λ_k has exactly $k-1$ simple zeros in $(0, 1)$; if $c \neq 0$, then there exists $N \in \mathbb{N}$ such that the eigenfunction $y_k(x)$ corresponding to the eigenvalue λ_k has for $k \leq N$ exactly $k-1$ and for $k > N$ exactly $k-2$ simple zeros in $(0, l)$.

Remark 2.1. Throughout what follows we will assume that the coefficients of boundary conditions are chosen such that the first eigenvalue of problem (2.1) is positive.

Let E be a Banach space $C^3[0,l] \cap BC_0$ with the norm $\|y\|_3 = \sum_{s=0}^{3} \| y^{(s)} \|_\infty$, where $\|y\|_\infty = \max_{x\in[0,l]} |y\,(x)|$ and BC_0 is a set of functions which satisfy the boundary conditions (1.2)–(1.4).

From now on ν will denote an element of $\{+,-\}$ that is, either $\nu = +$ or $\nu = -$.

In a recent paper [4, §2, pp. 4–5], using the Prüfer type transformation for each $k \in \mathbb{N}$ and each ν, the authors constructed sets $\mathcal{S}_k^\nu$ of functions $y \in E$, which have the oscillatory properties of eigenfunctions of the linear problem (2.1) and their derivatives. Note that the sets $\mathcal{S}_k^+$, $\mathcal{S}_k^-$ and $\mathcal{S}_k = \mathcal{S}_k^+ \cup \mathcal{S}_k^-$ are pairwise disjoint open subsets of E. Moreover, it was shown in [1, Lemma 2.2] that if $y \in \partial\mathcal{S}_k^\nu\ (\partial\mathcal{S}_k)$, then y has at least one zero of multiplicity 4 in $(0, l)$.

To study the existence of solutions to problem (1.1)–(1.5) with a fixed number of nodes, consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y) = \lambda r(x) y + \tilde{h}(x, y, y', y'', y''', \lambda),\ x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{2.2}$$

Here $\tilde{h}$ has a representation $\tilde{f} + \tilde{g}$, where $\tilde{f}$ and $\tilde{g}$ are real-valued continuous functions on $[0,l] \times \mathbb{R}^5$ that satisfy the following conditions: there exist constants $\tilde{M} > 0$ and sufficiently small $\tau_0 > 0$ such that

$$\left| \frac{\tilde{f}(x,y,s,v,w,\lambda)}{y} \right| \leq \tilde{M},\ \ (x,y,s,v,w) \in [0,l] \times \mathbb{R}^4,\ 0 < |y| + |s| + |v| + |w|\ \leq \tau_0, \\ y \neq 0,\ \ \lambda \in \mathbb{R}; \tag{2.3}$$

$$\tilde{g}(x,y,s,v,w,\lambda) = o\left(|y|+|s|+|v|+|w|\right) \text{ as } |y|+|s|+|v|+|w| \to 0, \tag{2.4}$$

uniformly in $x \in [0,l]$ and $\lambda \in \Lambda$ for each bounded interval $\Lambda \subset \mathbb{R}$, or there exist constants $\tilde{\tilde{M}} > 0$ and sufficiently large $\varkappa_0 > 0$ such that

$$\begin{array}{l} \left|\frac{\tilde{f}(x,y,s,v,w,\lambda)}{y}\right| \le \tilde{\tilde{M}},\ (x,y,s,v,w) \in [0,l]\times\mathbb{R}^4,\ |y|+|s|+|v|+|w| \ge \varkappa_0, \\ y \ne 0,\ \lambda \in \mathbb{R}; \end{array} \tag{2.5}$$

$$\tilde{g}(x,\, y,\, s,\, v,\, w,\, \lambda) = o\left(|y|+|s|+|v|+|w|\right) \text{ as } |y|+|s|+|v|+|w| \to \infty, \tag{2.6}$$

uniformly in $x \in [0,l]$ and $\lambda \in \Lambda$.

If conditions (2.3) and (2.4) are satisfied, then the bifurcation of nontrivial solutions of problem (2.2) from the line of trivial solutions $\mathbb{R}\times\{0\} = \{(\lambda,0) : \lambda \in \mathbb{R}\}$ is considered. In this case, the global bifurcation of nontrivial solutions of problem (2.2) is studied in [4], where the following results are obtained.

Lemma 2.2 ([4, Lemmas 3 and 4]). *Let conditions (2.3) and (2.4) be satisfied. Then for each $k \in \mathbb{N}$ and each ν the set of bifurcation points of (2.2) with respect to the set $\mathbb{R}\times\mathcal{S}_k^\nu$ is nonempty and lies in $\tilde{I}_k \times \{0\}$, where $\tilde{I}_k = \left[\lambda_k - \frac{\tilde{M}}{r_0}, \lambda_k + \frac{\tilde{M}}{r_0}\right]$, $r_0 = \min_{x\in[0,l]} r(x)$.*

For each $k \in \mathbb{N}$ and each ν let $\tilde{D}_k^\nu$ be the union of all the components of the set of nontrivial solutions to problem (2.2) bifurcating from the points of the interval $\tilde{I}_k \times \{0\}$ with respect to $\mathbb{R}\times\mathcal{S}_k^\nu$. Moreover, let $D_k^\nu = \tilde{D}_k^\nu \cup (\tilde{I}_k \times \{0\})$. Note that D_k^ν is connected, but $\tilde{D}_k^\nu$ may not be connected in $\mathbb{R}\times E$.

Theorem 2.3 ([4, Theorem 3]). *Let conditions (2.3) and (2.4) be satisfied. Then for each $k \in \mathbb{N}$ and each ν the set $\tilde{D}_k^\nu$ is nonempty, lies in $\mathbb{R}\times S_k^\nu$ and is unbounded in $\mathbb{R}\times E$.*

In the case when conditions (2.5) and (2.6) are satisfied, then we consider the bifurcation of nontrivial solutions to problem (2.2) from infinity, or rather from the line $\mathbb{R}\times\{\infty\} = \{(\lambda,\infty) : \lambda \in \mathbb{R}\}$. Global bifurcation of nontrivial solutions of problem (2.2) from infinity with respect to the set $\mathbb{R}\times S_k^\nu$ was considered in [3] in the case of $\tilde{f} \equiv 0$. Using the results of [1,3] and [4] following the corresponding arguments in [6], we can obtain the following results.

Lemma 2.4. *Let conditions (2.5) and (2.6) be satisfied. Then for each $k \in \mathbb{N}$ and each ν the set of asymptotic bifurcation points of problem (2.2) with respect to the set $\mathbb{R}\times S_k^\nu$ is nonempty and lies in $\tilde{\tilde{I}}_k \times \{\infty\}$, where $\tilde{\tilde{I}}_k = \left[\lambda_k - \frac{\tilde{\tilde{M}}}{r_0}, \lambda_k + \frac{\tilde{\tilde{M}}}{r_0}\right]$.*

For each $k \in \mathbb{N}$ and each ν let $\tilde{\tilde{D}}_k^\nu$ be the union of all the components of the set of nontrivial solutions to problem (2.2) bifurcating from the points of the interval $\tilde{\tilde{I}}_k \times \{\infty\}$ with respect to the set $\mathbb{R}\times S_k^\nu$. Moreover, let $D_k^{\nu,*} = \tilde{\tilde{D}}_k^\nu \cup \left(\tilde{\tilde{I}}_k \times \{\infty\}\right)$ (in this case we add the points $\{(\lambda,\infty) : \lambda \in \mathbb{R}\}$ to our space $\mathbb{R}\times E$ and define an appropriate topology on the resulting set). Note that $D_k^{\nu,*}$ is connected.

Theorem 2.5. *For each $k \in \mathbb{N}$ and each ν the set $\tilde{\tilde{D}}_k^\nu$ is nonempty and for this set at least one of the following statements holds:*

(i) the set $\tilde{\tilde{D}}_k^\nu$ meets $\tilde{\tilde{I}}_{k'} \times \{\infty\}$ with respect to $\mathbb{R}\times\mathcal{S}_{k'}^{\nu'}$ for some $(k',\nu') \ne (k,\nu)$;

(ii) the set $\tilde{\tilde{D}}_k^\nu$ meets $\mathbb{R}\times\{0\}$ for some $\lambda \in \mathbb{R}$;

(iii) the projection of $\tilde{\tilde{D}}_k^\nu$ on $\mathbb{R}\times\{0\}$ is unbounded.

In addition, if cases (ii) and (iii) are not satisfied for the union $\tilde{\tilde{D}}_k = \tilde{\tilde{D}}_k^+ \cup \tilde{\tilde{D}}_k^-$, then case (i) is satisfied for it with $k' \ne k$.

3 Existence of solutions to problem (1.1)–(1.5) with fixed oscillation count

In this section we will determine the interval of χ, in which there exist nodal solutions of problem (1.1)–(1.5).

Theorem 3.1. *Let $g_0 > -\underline{f}_0$, $g_\infty > -\underline{f}_\infty$, and for some $k \in \mathbb{N}$ one of the following conditions is satisfied:*

$$\frac{\lambda_k}{g_0 + \underline{f}_0} < \chi < \frac{\lambda_k}{g_\infty + \overline{f}_\infty}; \qquad \frac{\lambda_k}{g_\infty + \underline{f}_\infty} < \chi < \frac{\lambda_k}{g_0 + \overline{f}_0}.$$

Then there are two solutions $\tilde{y}_k^+$ and $\tilde{y}_k^-$ of problem (1.1)–(1.5) such that $\tilde{y}_k^+ \in \mathcal{S}_k^+$ and $\tilde{y}_k^- \in \mathcal{S}_k^-$, i.e., $\tilde{y}_k^+$ has either $k-1$ or $k-2$ simple zeros in $(0,l)$ and is positive near $x=0$, and $\tilde{y}_k^-$ has either $k-1$ or $k-2$ simple zeros in $(0,l)$ and is negative near $x=0$.

Proof. To prove the theorem, consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda \chi r(x) g(y(x)) + \chi r(x) f(y(x)), \ x \in (0,l), \\ y \in (b.c.)_\lambda, \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue parameter.

Step 1. It follows from the first condition of (1.10) that the function $g(s)$, $s \in \mathbb{R}$, can be represented in the following form

$$g(s) = s g_0 + \rho(s), \tag{3.2}$$

where $\rho(s)$ is a real-valued continuous functions on $\mathbb{R}$ that satisfies the condition

$$\lim_{|s| \to 0} \frac{\rho(s)}{s} = 0. \tag{3.3}$$

Let $\zeta(u) = \max_{|s| \in [0,u]} |\rho(s)|$. It is obvious that the function $\zeta(u)$ is nondecreasing on $[0, +\infty)$.

It follows from (3.3) that for any sufficiently small $\varepsilon > 0$ one can find a sufficiently small $\delta_\varepsilon > 0$ such that for any $s \in \mathbb{R}$ satisfying condition $|s| < \delta_\varepsilon$ we have $|\rho(s)| < \varepsilon |s|$. Then we have

$$\frac{\zeta(u)}{u} < \varepsilon \quad \text{for any } u \in (0, \delta_\varepsilon). \tag{3.4}$$

Since the function $\zeta(u)$ is nondecreasing on $[0, +\infty)$ for any $x \in [0,l]$ we get

$$\frac{|\rho(y(x))|}{\|y\|_3} \leq \frac{\zeta(\|y\|_\infty)}{\|y\|_3} \leq \frac{\zeta(\|y\|_3)}{\|y\|_3}. \tag{3.5}$$

Let $y \in E$ such that $\|y\|_3 < \delta_\varepsilon$. Then by (3.4) we have

$$\frac{\zeta(\|y\|_3)}{\|y\|_3} < \varepsilon,$$

and consequently, for any $x \in [0,l]$ we get

$$\frac{|\rho(y(x))|}{\|y\|_3} < \varepsilon \quad \text{for any } x \in [0,l], \tag{3.6}$$

in view of (3.5). Therefore, it follows from (3.6) that

$$\|\rho(y)\|_\infty = o(\|y\|_3) \quad \text{as } \|y\|_3 \to 0. \tag{3.7}$$

Considering (3.2), the problem (3.1) can be written in the following equivalent form

$$\begin{cases} \ell(y)(x) = \lambda\chi r(x) g_0 y(x) + \chi r(x) f(y(x)) + \lambda\chi r(x)\rho(y(x)),\ x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.8}$$

Let $\delta_0 > 0$ be a sufficiently small number. Then it follows from (1.6) and (1.7) that there exists sufficiently small $\sigma_0 \in (0,\tau_0)$ such that

$$\underline{f}_0 - \frac{g_0\delta_0}{2} < \frac{f(s)}{s} < \overline{f}_0 + \frac{g_0\delta_0}{2} \quad \text{for any } s \in \mathbb{R},\ 0 < |s| < \sigma_0. \tag{3.9}$$

Relation (3.9) implies that

$$\left|\frac{f(s)}{s}\right| \le \tilde{M}_0 \quad \text{for any } s \in \mathbb{R},\ \ 0 < |s| < \sigma_0, \tag{3.10}$$

where $\tilde{M}_0 = \max\left\{\left|\underline{f}_0 - \frac{g_0\delta_0}{2}\right|, \left|\overline{f}_0 + \frac{g_0\delta_0}{2}\right|\right\} > 0$. Then by (3.7) (see also (3.6)) and (3.10) it follows from Lemma 2.2 that for each $k \in \mathbb{N}$ and each ν the set of bifurcation points of (3.8) (or (3.1)) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$ is nonempty. If $(\lambda^*, 0)$ is a bifurcation point of problem (3.8) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$, then there exists a sequence $\{(\lambda_n^*, y_n^*)\}_{n=1}^\infty \subset \mathbb{R} \times \mathcal{S}_k^\nu$ such that

$$\begin{cases} \ell(y_n^*)(x) = \lambda_n^*\chi r(x) g_0 y_n^*(x) + \chi r(x) f(y_n^*(x)) + \lambda_n^*\chi r(x)\rho(y_n^*(x)),\ x \in (0,l), \\ y_n^* \in (b.c.)_{\lambda_n^*}, \end{cases} \tag{3.11}$$

and

$$(\lambda_n^*, y_n^*) \to (\lambda^*, 0) \quad \text{in } \mathbb{R} \times E \text{ as } n \to \infty. \tag{3.12}$$

Let

$$\varphi_n^*(x) = \begin{cases} -\frac{f(\tilde{y}_n^*(x))}{\tilde{y}_n^*(x)} & \text{if } \tilde{y}_n^*(x) \ne 0, \\ 0 & \text{if } \tilde{y}_n^*(x) = 0. \end{cases} \tag{3.13}$$

Then by (3.13) it follows from (3.11) that for each $n \in \mathbb{N}$ the pair (λ_n^*, y_n^*) is a solution of the following linearizable problem

$$\begin{cases} \frac{1}{\chi r(x) g_0}\,\ell(y)(x) + \frac{1}{g_0}\,\varphi_n^*(x) y(x) = \lambda y(x) + \frac{1}{g_0}\,\lambda\rho(y(x)),\ x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.14}$$

In view of (3.12) we can choose $n \in \mathbb{N}$ so large that

$$-\left(\frac{\overline{f}_0}{g_0} + \frac{\delta_0}{2}\right) < \frac{1}{g_0}\varphi_n^*(x) < -\left(\frac{\underline{f}_0}{g_0} - \frac{\delta_0}{2}\right) \quad \text{for any } x \in [0,l], \tag{3.15}$$

in view of (3.9) and (3.13).

It follows from [5, Remark 4.2 and Theorem 4.3] that for each fixed $n \in \mathbb{N}$ the eigenvalues of the linear eigenvalue problem

$$\begin{cases} \frac{1}{\chi r(x) g_0}\,\ell(y)(x) + \frac{1}{g_0}\,\varphi_n^*(x) y(x) = \lambda y(x),\ x \in (0,l), \\ y \in (b.c.)_\lambda, \end{cases} \tag{3.16}$$

are real and simple, and form an infinitely increasing sequence $\{\lambda_{k,n}^*\}_{k=1}^{\infty}$; moreover, the eigenfunction $y_{k,n}^*(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue $\lambda_{k,n}^*$ lies in $\mathcal{S}_k$.

In view of relation (3.15), by following the arguments in Lemmas 5.1 and 5.3 of [1] we get

$$\tilde{\lambda}_k - \frac{\overline{f}_0}{g_0} - \frac{\delta_0}{2} \leq \lambda_{k,n}^* \leq \tilde{\lambda}_k - \frac{\underline{f}_0}{g_0} + \frac{\delta_0}{2}, \tag{3.17}$$

where $\tilde{\lambda}_k$, $k \in \mathbb{N}$, is a kth eigenvalue of the linear eigenvalue problem

$$\begin{cases} \frac{1}{\chi r(x) g_0}\, \ell(y)(x) = \lambda y(x), \; x \in (0, l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.18}$$

Since $(\lambda_{k,n}^*, 0)$ is a unique bifurcation point of problem (3.14) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$ by (3.12) we can again choose $n \in \mathbb{N}$ so large that

$$\lambda_{k,n}^* - \frac{\delta_0}{2} < \lambda_n^* < \lambda_{k,n}^* + \frac{\delta_0}{2}\,. \tag{3.19}$$

Then it follows from (3.17) and (3.19) that

$$\tilde{\lambda}_k - \frac{\overline{f}_0}{g_0} - \delta_0 < \lambda_n^* < \tilde{\lambda}_k - \frac{\underline{f}_0}{g_0} + \delta_0\,, \tag{3.20}$$

whence, with regard to (3.12), we obtain

$$\tilde{\lambda}_k - \frac{\overline{f}_0}{g_0} - \delta_0 \leq \lambda^* \leq \tilde{\lambda}_k - \frac{\underline{f}_0}{g_0} + \delta_0\,. \tag{3.21}$$

As can be seen from (3.18) that $\lambda_k = \chi g_0 \tilde{\lambda}_k$ for each $k \in \mathbb{N}$. Consequently, it follows from (3.21) that

$$\frac{\lambda_k}{\chi g_0} - \frac{\overline{f}_0}{g_0} - \delta_0 \leq \lambda^* \leq \frac{\lambda_k}{\chi g_0} - \frac{\underline{f}_0}{g_0} + \delta_0\,. \tag{3.22}$$

Since δ_0 is arbitrary small enough, it follows from (3.22) that

$$\frac{\lambda_k}{\chi g_0} - \frac{\overline{f}_0}{g_0} \leq \lambda^* \leq \frac{\lambda_k}{\chi g_0} - \frac{\underline{f}_0}{g_0}\,. \tag{3.23}$$

Thus, (3.23) shows that the bifurcation points of problem (3.1) (or (3.8)) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$ are contained in the interval $I_k^0 \times \{0\}$, where

$$I_k^0 = \left[\frac{\lambda_k}{\chi g_0} - \frac{\overline{f}_0}{g_0}, \frac{\lambda_k}{\chi g_0} - \frac{\underline{f}_0}{g_0} \right].$$

Then, by Theorem 2.3, for each $k \in \mathbb{N}$ and each ν there exists a component $D_{k,0}^\nu$ of the set of solutions of problem (3.1), which contains $I_k^0 \times \{0\}$, lies in $(\mathbb{R} \times \mathcal{S}_k^\nu) \cup (I_k^0 \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.

Step 2. By the second condition in (1.10) we can represent the function $g(s)$, $s \in \mathbb{R}$, as follows:

$$g(s) = s g_\infty + \varrho(s), \tag{3.24}$$

where

$$\lim_{|s|\to+\infty} \frac{\varrho(s)}{s} = 0. \tag{3.25}$$

Let $\varsigma(u) = \max_{|s|\in[0,u]} |\varrho(s)|$. It is obvious that the function $\varsigma(u)$ is nondecreasing on $[0,+\infty)$.

In view of (3.25), for any sufficiently small $\varepsilon > 0$ there exists a sufficiently large $\Delta_\varepsilon > 0$ such that

$$|\varrho(s)| < \frac{1}{2}\,\varepsilon|s| \quad \text{for any } s \in \mathbb{R},\ |s| > \Delta_\varepsilon. \tag{3.26}$$

Let $u \in [\Delta_\varepsilon, \infty)$ be arbitrary. Then we have

$$\varsigma(u) = \max\left\{\max_{|s|\in[0,\Delta_\varepsilon]} |\varrho(s)|,\ \max_{|s|\in[\Delta_\varepsilon,u]} |\varrho(s)|\right\}. \tag{3.27}$$

Let $K_\varepsilon = \max_{|s|\in[0,\Delta_\varepsilon]} |\varrho(s)|$. We will choose $\Delta_{1,\varepsilon} > \Delta_\varepsilon$ so large that $\frac{K_\varepsilon}{\Delta_{1,\varepsilon}} < \frac{1}{2}\,\varepsilon$. Now let $u > \Delta_{1,\varepsilon}$. Then by (3.26) it follows from (3.27) that

$$\begin{aligned}\frac{\varsigma(u)}{u} &= \frac{\max\{K_\varepsilon, \max_{|s|\in[\Delta_\varepsilon,u]} |\varrho(s)|\}}{u} \le \frac{\max\{K_\varepsilon, \frac{1}{2}\,\varepsilon\, u\}}{u} \\ &= \max\left\{\frac{K_\varepsilon}{u}, \frac{1}{2}\,\varepsilon\right\} \le \max\left\{\frac{K_\varepsilon}{\Delta_{1,\varepsilon}}, \frac{1}{2}\,\varepsilon\right\} \le \frac{1}{2}\,\varepsilon < \varepsilon.\end{aligned} \tag{3.28}$$

Since the function $\varsigma(u)$ is nondecreasing on $[0,+\infty)$ for any $x \in [0,l]$ we have

$$\frac{|\varrho(y(x))|}{\|y\|_3} \le \frac{\varsigma(\|y\|_\infty)}{\|y\|_3} \le \frac{\varsigma(\|y\|_3)}{\|y\|_3}. \tag{3.29}$$

If $\|y\|_3 > \Delta_{1,\varepsilon}$, then by (3.28) it follows from (3.29) that

$$\frac{|\varrho(y(x))|}{\|y\|_3} < \varepsilon \quad \text{for any } x \in [0,l],$$

which shows that

$$\|\varrho(y)\|_\infty = o(\|y\|_3) \quad \text{as } \|y\|_3 \to \infty. \tag{3.30}$$

Taking into account (3.24), we can rewrite the problem (3.1) in the following equivalent form

$$\begin{cases}\ell(y)(x) = \lambda\chi r(x) g_\infty y(x) + \chi r(x) f(y(x)) + \lambda\chi r(x)\varrho(y(x)),\ x \in (0,l),\\ y \in (b.c.)_\lambda.\end{cases} \tag{3.31}$$

Using [1, Lemma 5.1], Lemma 2.4, relations (1.6), (1.8), (3.30) and following the above arguments, we can show that if $(\tilde\lambda^*, \infty)$ is an asymptotic bifurcation point of problem (3.1) (or (3.31)), then

$$\tilde\lambda^* \in I_k^\infty = \left[\frac{\lambda_k}{\chi g_\infty} - \frac{\overline{f}_\infty}{g_\infty}, \frac{\lambda_k}{\chi g_\infty} - \frac{\underline{f}_\infty}{g_\infty}\right].$$

Hence it follows from Theorem 2.5 that for each $k \in \mathbb{N}$ and each ν there exists a component $D_{k,\infty}^\nu$ of the set of solutions to problem (3.1) containing $I_k^\infty \times \{\infty\}$ and for which at least one of the following statements holds:

(i) the set $D_{k,\infty}^\nu$ meets $I_{k'}^\infty \times \{\infty\}$ with respect to $\mathbb{R} \times \mathcal{S}_{k'}^{\nu'}$ for some $(k',\nu') \neq (k,\nu)$;

(ii) the set $D^{\nu}_{k,\infty}$ meets $\mathbb{R}\times\{0\}$ for some $\lambda\in\mathbb{R}$;

(iii) the projection of $D^{\nu}_{k,\infty}$ on $\mathbb{R}\times\{0\}$ is unbounded.

Step 3. By following the arguments in Theorem 3.3 of [25] we can show that for each $k\in\mathbb{N}$ and each ν, $D^{\nu}_{k,\infty}\backslash(I^{\infty}_k\times\{\infty\})\subset\mathbb{R}\times\mathcal{S}^{\nu}_k$, and consequently, alternative (i) above for $D^{\nu}_{k,\infty}$ cannot hold. Moreover, if $D^{\nu}_{k,\infty}$ meets $\mathbb{R}\times\{0\}$ for some $\lambda\in\mathbb{R}$, then $\lambda\in I^0_k$. Similarly, if $D^{\nu}_{k,0}$ meets $\mathbb{R}\times\{\infty\}$ for some $\lambda\in\mathbb{R}$, then $\lambda\in I^{\infty}_k$. Hence we conclude that if $D^{\nu}_{k,\infty}$ has a bounded projection on $\mathbb{R}\times\{0\}$, then $D^+_{k,0}=D^+_{k,\infty}$ and $D^-_{k,0}=D^-_{k,\infty}$.

Now we show that for each $k\in\mathbb{N}$ and each ν the set $D^{\nu}_{k,\infty}$ has a bounded projection on $\mathbb{R}\times\{0\}$. Indeed, otherwise there exists a sequence $\{(\overline{\lambda}_n,\overline{y}_n)\}_{n=1}^{\infty}\subset(D^{\nu}_{k,\infty}\backslash\mathcal{Q}_{k,\infty})\subset(\mathbb{R}\times\mathcal{S}^{\nu}_k)$ such that

$$\lim_{n\to\infty}\overline{\lambda}_n=\pm\infty, \tag{3.32}$$

where $\mathcal{Q}_{k,\infty}$ is a some neighborhood of $I^{\infty}_k\times\{\infty\}$.

By (1.6)–(1.10) there exists a positive constants κ_0, κ_1 and κ_2 such that

$$\kappa_0\leq\frac{g(s)}{s}\leq\kappa_1\quad\text{and}\quad\left|\frac{f(s)}{s}\right|\leq\kappa_2\quad\text{for any } s\in\mathbb{R},\ s\neq 0. \tag{3.33}$$

We define the functions $\overline{\varphi}_n(x)$ and $\overline{\phi}_n(x)$, $x\in[0,l]$, as follows:

$$\overline{\varphi}_n(x)=\begin{cases}\frac{g(\overline{y}_n(x))}{\overline{y}_n(x)} & \text{if } \overline{y}_n(x)\neq 0,\\ 0 & \text{if } \overline{y}_n(x)=0,\end{cases}\qquad \overline{\phi}_n(x)=\begin{cases}-\frac{f(\overline{y}_n(x))}{\overline{y}_n(x)} & \text{if } \overline{y}_n(x)\neq 0,\\ 0 & \text{if } \overline{y}_n(x)=0.\end{cases} \tag{3.34}$$

Since $\overline{y}_n\in\mathcal{S}^{\nu}_k$ by (3.34) it follows from (3.1) that $\overline{\lambda}_n$ for each $n\in\mathbb{N}$ is kth eigenvalue of the following linear eigenvalue problem

$$\begin{cases}\ell(y)(x)+\chi r(x)\overline{\phi}_n(x)\,y(x)=\lambda\chi r(x)\overline{\varphi}_n(x)\,y(x)),\ x\in(0,l),\\ y\in(b.c.)_{\lambda}.\end{cases} \tag{3.35}$$

By (3.33) from (3.34) we get

$$\kappa_0\leq\overline{\varphi}_n(x)\leq\kappa_1\quad\text{and}\quad|\overline{\phi}_n(x)|\leq\kappa_2\quad\text{for any } x\in[0,l]. \tag{3.36}$$

It is known (see [1,4]) that problem (3.35) reduces to the spectral problem for the self-adjoint operator in the Hilbert space $H=L_2(0,l)\oplus\mathbb{C}$ with corresponding scalar product. In view of (3.36), by the maximum-minimum property of eigenvalues (see [1,2]) we obtain that the eigenvalues of problem (3.35) are uniformly bounded from below with respect to $n\in\mathbb{N}$. Consequently, the relation

$$\lim_{n\to\infty}\overline{\lambda}_n=-\infty$$

is not possible. Should be noted that the relation

$$\lim_{n\to\infty}\overline{\lambda}_n=+\infty,$$

is also impossible, since for a sufficiently large n, by [5, Theorem 4.3], the number of zeros of the function $\overline{y}_n$ will be large enough, which contradicts the condition $\overline{y}_n\in\mathcal{S}^{\nu}_k$.

Therefore, for any $k\in\mathbb{N}$ we have

$$D^+_{k,0}=D^+_{k,\infty}\quad\text{and}\quad D^-_{k,0}=D^-_{k,\infty}. \tag{3.37}$$

Step 4. It is obvious that any solution to problem (3.1) of the form $(1, y)$ gives a solution y to problem (1.1)–(1.5). In order for problem (1.1)–(1.5) to have a solution y which is contained in $\mathcal{S}_k^\nu$ for some $k \in \mathbb{N}$, by (3.37) it is sufficient that on the real axis $\mathbb{R}$ the interval I_k^0 lies to the left of 1 and the interval I_k^∞ lies to the right of 1, or the interval I_k^0 lies to the right of 1, and the interval I_k^∞ lies to the left of 1.

Let the conditions $g_0 > -\underline{f}_0$ and $g_\infty > -\underline{f}_\infty$ be satisfied. Hence we have $g_\infty > -\overline{f}_\infty$. If the condition $\frac{\lambda_k}{g_0+\underline{f}_0} < \chi < \frac{\lambda_k}{g_\infty+\overline{f}_\infty}$ is satisfied, then we get

$$\frac{\lambda_k}{\chi g_0} - \frac{\underline{f}_0}{g_0} < \frac{\lambda_k}{\frac{\lambda_k}{g_0+\underline{f}_0} g_0} - \frac{\underline{f}_0}{g_0} = \frac{\lambda_k(g_0+\underline{f}_0)}{\lambda_k g_0} - \frac{\lambda_k \underline{f}_0}{\lambda_k g_0} = 1$$

and

$$\frac{\lambda_k}{\chi g_\infty} - \frac{\overline{f}_\infty}{g_\infty} > \frac{\lambda_k}{\frac{\lambda_k}{g_\infty+\overline{f}_\infty} g_\infty} - \frac{\overline{f}_\infty}{g_\infty} = \frac{\lambda_k(g_\infty+\overline{f}_\infty)}{\lambda_k g_\infty} - \frac{\lambda_k \overline{f}_\infty}{\lambda_k g_\infty} = 1.$$

The case in which $\frac{\lambda_k}{g_\infty+\underline{f}_\infty} < \chi < \frac{\lambda_k}{g_0+\overline{f}_0}$ can be considered in a similar way. The proof of this theorem is complete. □

Step 4 of the proof of Theorem 3.1 makes it possible to obtain other conditions for the existence of solutions to problem (1.1)–(1.5) contained in the sets $\mathcal{S}_k^+$ and $\mathcal{S}_k^-$ for some $k \in \mathbb{N}$.

Theorem 3.2. *Let $g_0 > -\underline{f}_0$, $-\overline{f}_\infty < g_\infty \le -\underline{f}_\infty$, and for some $k \in \mathbb{N}$ the following condition is satisfied:*

$$\frac{\lambda_k}{g_0+\underline{f}_0} < \chi < \frac{\lambda_k}{g_\infty+\overline{f}_\infty}.$$

Then the statement of Theorem 3.1 holds.

Theorem 3.3. *Let $g_0 > -\underline{f}_0$, $g_\infty \le -\overline{f}_\infty$, and for some $k \in \mathbb{N}$ the following condition is satisfied:*

$$\chi > \frac{\lambda_k}{g_0+\underline{f}_0}.$$

Then the statement of Theorem 3.1 holds.

Theorem 3.4. *Let $-\overline{f}_0 < g_0 \le -\underline{f}_0$, $g_\infty > -\underline{f}_\infty$, and for some $k \in \mathbb{N}$ the following condition is satisfied:*

$$\frac{\lambda_k}{g_\infty+\underline{f}_\infty} < \chi < \frac{\lambda_k}{g_0+\overline{f}_0}.$$

Then the statement of Theorem 3.1 holds.

Theorem 3.5. *Let $g_0 \le -\overline{f}_0$, $g_\infty > -\underline{f}_\infty$, and for some $k \in \mathbb{N}$ the following condition is satisfied:*

$$\chi > \frac{\lambda_k}{g_\infty+\underline{f}_\infty}.$$

Then the statement of Theorem 3.1 holds.

The proofs of these theorems are similar to that of Step 4 of Theorem 3.1.

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