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Convergence of weak solutions of elliptic problems with datum in L^1

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Abstract. Motivated by the Q-condition result proven by Arcoya and Boccardo in [*J. Funct. Anal.* **268**(2015), No. 5, 1153–1166], we analyze the behaviour of the weak solutions $\{u_\varepsilon\}$ of the problems

$$\begin{cases} -\Delta_p u_\varepsilon + \varepsilon |f(x)| u_\varepsilon = f(x) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

when ε tends to 0. Here, Ω denotes a bounded open set of $\mathbb{R}^N$ ($N \geq 2$), $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian operator ($1 < p < \infty$) and $f(x)$ is an $L^1(\Omega)$ function.

We show that this sequence converges in some sense to u, the entropy solution of the problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In the semilinear case, we prove stronger results provided the weak solution of that problem exists.

Keywords: nonlinear elliptic equations, entropy solution, Q-condition.

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1 Introduction

In this paper we develop a new method to approach solutions (in a broad sense that we will discuss later) of a problem with data only in L^1 that does not require the approximation of such data by more regular functions. Specifically, we consider the following boundary value problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + b(x)g(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

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where Ω is a bounded open set of $\mathbb{R}^N$ ($N \geq 2$) and $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear Leray–Lions operator, i.e., it is a Carathéodory function such that for every $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ ($\xi \neq \eta$), and for almost every $x \in \Omega$ satisfies

$$a(x,s,\xi)\xi \geq \alpha|\xi|^p, \tag{1.1}$$

$$|a(x,s,\xi)| \leq h(x) + \beta|\xi|^{p-1}, \tag{1.2}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0, \tag{1.3}$$

where $1 < p < \infty$, $h(x) \in L^{p'}(\Omega)$ and $\alpha, \beta > 0$. With respect to the coefficient $b(x)$ of the lower order term and to the datum $f(x)$, it assumed that

$$0 \leq b(x) \in L^1(\Omega), \qquad f(x) \in L^1(\Omega), \tag{1.4}$$

and $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying that

$$g \text{ is increasing, odd and } \lim_{s \to +\infty} g(s) = +\infty. \tag{1.5}$$

A simple model of function g for the reader may be $g(s) = |s|^{\gamma-1}s$ for $\gamma > 0$.

We remark that the problem (P) under the previous hypotheses (1.1), (1.2), (1.3), (1.4) and (1.5) does not always have a solution in the usual sense when f belongs to $L^1(\Omega)$. Moreover, in the case in which the solution of the problem (P) exists with a right-hand side in $L^1(\Omega)$ it is not necessarily bounded; in fact, it may not even be in the $W^{1,1}_{\mathrm{loc}}(\Omega)$ space when $p \leq 2 - \frac{1}{N}$. Motivated by this, many authors started to study if there was a more general concept of solution in which existence and uniqueness were guaranteed; see, for example, the paper [6], where they use the concept of renormalized solution, or [5], where the concept of entropy solution is introduced.

Nevertheless, under some extra conditions the existence of a weak solution of (P) can be ensured. In [2] (see also [1]), the authors proved that if there exists certain relation between the coefficient $b(x)$ of the lower order term and the datum $f(x)$, then the existence of a bounded weak solution is granted even if $f(x)$ only belongs to $L^1(\Omega)$. Concretely, they showed that if the so-called Q-condition is satisfied, i.e., if there exists some $Q > 0$ such that

$$|f(x)| \leq Qb(x), \tag{1.6}$$

then the problem (P) has a unique weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, they also gave an $L^\infty(\Omega)$-estimate for u, namely

$$\|u\|_\infty \leq g^{-1}(Q).$$

Therefore, they put in evidence that this interplay between the coefficients provides a regularizing effect on the problem (P). After the publication of these works, several number of papers studying this kind of regularizing effects given by the interplay between coefficients in other types of problems were published, such as [3,4], giving rise to a prolific and original line of modern research.

Motivated by this result, in this paper we approach the problem (P) in such a way that the resulting approximated problems satisfy the relation (1.6) and we study the convergence of the sequence of solutions. Concretely, we consider the following approximated elliptic problems

$$\begin{cases} -\operatorname{div}(a(x,u_n,\nabla u_n)) + \left[b(x) + \frac{1}{n}|f(x)|\right]g(u_n) = f(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_n}$$

Observe that the coefficients of these problems satisfy the relation (1.6) since

$$|f(x)| \le n\big[b(x) + \tfrac{1}{n}|f(x)|\big],$$

so, if we assume the hypotheses (1.1), (1.2), (1.3), (1.4) and (1.5), the results of [2] provide for each $n \in \mathbb{N}$ the existence of a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of (P_n) which also satisfies that

$$\|u_n\|_\infty \le g^{-1}(n). \tag{1.7}$$

The purpose of this paper will be to study the behaviour of the sequence $\{u_n\}$ when n goes to ∞. We stress that similar studies can be done on other problems for which existence or regularity results have been proven thanks to some Q-condition type hypothesis. Therefore, this paper can be the beginning of a productive line of research.

The main result, stated below, is related with the entropy solution of (P), whose existence and uniqueness is guaranteed thanks to the results of [5]. We also point out that the proof of our theorem is, in fact, an alternative existence proof to the one given in [5], where the major difference between both are the approximate problems considered.

Theorem 1.1. *Suppose that $a(x,s,\xi)$ satisfies (1.1), (1.2) and (1.3), that $b(x)$ and $f(x)$ verify (1.4) and that g satisfies (1.5). Then the solution of (P) in the sense of Definition 2.5 exists and the sequence $\{u_n\}$ of weak solutions of (P_n) converges in measure to that solution.*

Note that the sequence of weak solutions $\{u_n\}$ of (P_n), in general, cannot converge weakly in $W_0^{1,p}(\Omega)$ because, in this case, that would imply the existence of a weak solution of (P). Recall that this type of solution (see Definition 2.4) does not always exist for problem (P).

In the semilinear case, i.e., when $p = 2$, we study if this stronger convergence can be proved as long as the weak solution of (P) exists. For this purpose, we consider the linear operator $a(x,s,\xi) = M(x)\xi$, where $M(x)$ is a symmetric bounded elliptic matrix, i.e., there exist $\alpha, \beta > 0$ such that

$$\alpha|\xi|^2 \le M(x)\xi\xi, \tag{1.8}$$

$$|M(x)| \le \beta \tag{1.9}$$

for every $\xi \in \mathbb{R}^N$ and for almost every x in Ω.

The mentioned result for the semilinear case is the following one.

Theorem 1.2. *Suppose that $a(x,s,\xi) = M(x)\xi$ with $M(x)$ a symmetric matrix satisfying (1.8) and (1.9). Assume also that $b(x)$ and $f(x)$ verify (1.4) and that g satisfies (1.5). If the weak solution $u \in H_0^1(\Omega)$ of (P) exists and it is in $L^\infty(\Omega)$, then $\{u_n\}$, the sequence of weak solutions of (P_n), verifies that*

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

We stress that, unlike Theorem 1.1, this theorem is not an existence result since we are assuming that the weak solution of (P) exists.

In order to prove these results we will follow the next structure in the work. In Section 2 we state the theorem of [2] in which our study is motivated, we take a brief review of the Marcinkiewicz spaces, we remind the concept of entropy solution of (P) and we give other preliminary results. In Section 3 we prove Theorem 1.1, the main result of this paper. Finally, in Section 4, we deal with the semilinear case and we give the proof of Theorem 1.2.

2 Preliminaries

First of all, we state here the result of [2] that has motivated this research and then we indicate the key of the proof. As we will see, the tools which are used in the proof are not excessively sophisticated, so the approach we adopt in this paper is elemental.

Theorem 2.1 ([2]). *Suppose that $a(x,s,\xi)$ satisfies (1.1), (1.2) and (1.3), that $b(x)$ and $f(x)$ verify (1.4) and that g satisfies (1.5). If the relation (1.6) between $b(x)$ and $f(x)$ is verified, then there exists a unique $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ weak solution of (P) which also satisfies that*

$$\|u\|_\infty \le g^{-1}(Q).$$

Remark 2.2. The key of the proof is to obtain an a priori $L^\infty(\Omega)$-estimate. The idea is to approximate (P) in such a way that its coefficients still satisfy the Q-condition (1.6), and this condition allows us to prove the uniform boundedness in $L^\infty(\Omega)$ of the sequence of approximated solutions.

Formally speaking, this $L^\infty(\Omega)$-estimate is obtained by taking as test function in (P) the mapping $G_k(u) := \max\{\min\{u+k, 0\}, u-k\}$ with $k > 0$. In particular, using (1.1) and (1.6) we get that

$$\alpha \int_\Omega |\nabla G_k(u)|^p + \int_\Omega b(x)g(u)G_k(u) \le \int_\Omega f(x)G_k(u) \le \int_\Omega Qb(x)|G_k(u)|,$$

i.e., that

$$\alpha \int_\Omega |\nabla G_k(u)|^p + \int_\Omega b(x)\big[|g(u)| - Q\big]|G_k(u)| \le 0.$$

Observe that g^{-1} exists thanks to (1.5) and that we can choose $k = g^{-1}(Q)$ in the above inequality to get that the second integral is nonnegative and, as a consequence, it is deduced that $g^{-1}(Q)$ is an a priori bound in $L^\infty(\Omega)$. $\square$

In several parts of this paper we work with the Marcinkiewicz spaces. For the convenience of the reader, we recall here their definition and some of their properties. For $0 < q < \infty$, we denote by $\mathcal{M}^q(\Omega)$ the set of measurable functions $v\colon \Omega \to \mathbb{R}$ such that there exists $C > 0$ satisfying that

$$\operatorname{meas}\{|v| > k\} \le \frac{C}{k^q}, \qquad \forall k > 0. \tag{2.1}$$

This space is a complete quasi-normed space with the quasi-norm

$$\|v\|^q_{\mathcal{M}^q(\Omega)} = \inf\{C > 0 : (2.1) \text{ holds}\}.$$

We also recall that, since Ω is bounded, then

$$\mathcal{M}^{q_2}(\Omega) \hookrightarrow L^{q_1}(\Omega) \hookrightarrow \mathcal{M}^{q_1}(\Omega)$$

for $0 < q_1 < q_2 < \infty$.

Related with these spaces we state the following lemma whose proof can be found in [5, Lemma 4.1]. For any $k > 0$ we set $T_k(s) = \min\{k, \max\{s, -k\}\}$.

Lemma 2.3 ([5]). *Let $u\colon \Omega \to \mathbb{R}$ be a function such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$ and*

$$\frac{1}{k}\int_{\{|u|<k\}} |\nabla u|^p \le M$$

for some constant $M > 0$ and for every $k > 0$. Then $u \in \mathcal{M}^{p_1}(\Omega)$ for $p_1 = \frac{N(p-1)}{N-p}$ if $1 < p < N$ and for every $p_1 > 1$ if $p \geq N$. More precisely, there exists $C = C(M, N, p) > 0$ such that

$$\text{meas}\{|u| > k\} \leq \frac{C}{k^{p_1}}, \qquad \forall k > 0.$$

We also recall here the concepts of weak solution and entropy solution of (P).

Definition 2.4. A function $u\colon \Omega \to \mathbb{R}$ is a *weak solution* of the problem (P) if $u \in W_0^{1,p}(\Omega)$, $b(x)g(u) \in L^1(\Omega)$ and

$$\int_\Omega a(x, u, \nabla u)\nabla\varphi + \int_\Omega b(x)g(u)\varphi = \int_\Omega f(x)\varphi$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Definition 2.5. A function $u\colon \Omega \to \mathbb{R}$ is an *entropy solution* of (P) if $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$, $b(x)g(u) \in L^1(\Omega)$ and

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - \varphi) + \int_\Omega b(x)g(u)T_k(u - \varphi) = \int_\Omega f(x)T_k(u - \varphi)$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and every $k > 0$.

Observe that the concept of entropy solution is more general than the concept of weak solution, i.e., every weak solution is an entropy solution. Although the reciprocal is not true in general, if an entropy solution of (P) is in $W_0^{1,p}(\Omega)$, then is also a weak solution of (P) (see [5, Corollary 4.3]).

Regarding the uniqueness, both types of solutions are unique (see [5, Theorem 5.1]). However, unlike the weak solution, which may not exist when $p \leq 2 - \frac{1}{N}$, it was proved in [5, Theorem 6.1] that the entropy solution of (P) always exists.

Finally, we end this section with a convergence lemma that we will use throughout this paper.

Lemma 2.6. *Suppose that $a(x, s, \xi)$ satisfies (1.1), (1.2) and (1.3), that $b(x)$ and $f(x)$ verify (1.4) and that g satisfies (1.5). If the sequence $\{u_n\}$ of weak solutions of (P_n) is bounded in $\mathcal{M}^q(\Omega)$ for some $q > 0$ and satisfies that $u_n \to u$ a.e. in Ω for some function u, then*

$$b(x)g(u_n) \to b(x)g(u) \quad \text{in } L^1(\Omega).$$

Proof. Let $\psi_{k,\delta}\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$\psi_{k,\delta}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq k, \\ \frac{1}{\delta}(s - k) & \text{if } k < s < k + \delta, \\ 1 & \text{if } s \geq k + \delta, \\ -\psi_{k,\delta}(-s) & \text{if } s < 0. \end{cases}$$

Taking $\psi_{k,\delta}(u_n) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in (P_n) and dropping two nonnegative terms we obtain that

$$\int_\Omega b(x)g(u_n)\psi_{k,\delta}(u_n) \leq \int_\Omega |f(x)|\,|\psi_{k,\delta}(u_n)|,$$

what implies that

$$\int_{\{k+\delta\le|u_n|\}} b(x)|g(u_n)| \le \int_{\{k\le|u_n|\}} |f(x)|.$$

If $\delta \to 0$, Fatou Lemma gives

$$\int_{\{k\le|u_n|\}} b(x)|g(u_n)| \le \int_{\{k\le|u_n|\}} |f(x)|.$$

We claim that $\{b(x)g(u_n)\}$ is uniformly integrable. Fix $\varepsilon > 0$. Since $b(x)$ is nonnegative by (1.4) and g is increasing and odd by (1.5), we deduce from the above inequality that for every measurable set $E \subset \Omega$ we have

$$\begin{aligned}\int_E b(x)|g(u_n)| &= \int_{E\cap\{|u_n|\le k\}} b(x)|g(u_n)| + \int_{E\cap\{k\le|u_n|\}} b(x)|g(u_n)| \\ &\le g(k)\int_E b(x) + \int_{\{k\le|u_n|\}} |f(x)|.\end{aligned}$$

On the one hand, since $f(x) \in L^1(\Omega)$, thanks to the absolute continuity of the integral there exists some $\delta' > 0$ such that if $E \subset \Omega$ is a measurable set with $\mathrm{meas}(E) < \delta'$ then $\int_E |f(x)| < \frac{\varepsilon}{2}$. As $\{u_n\}$ is bounded in $\mathcal{M}^q(\Omega)$, we can fix $k > 0$ large enough such that $\mathrm{meas}\{|u_n| \ge k\} \le \delta'$ for every $n \in \mathbb{N}$. Thus,

$$\int_{\{k\le|u_n|\}} |f(x)| \le \frac{\varepsilon}{2}, \qquad \forall n \in \mathbb{N}.$$

On the other hand, since $b(x) \in L^1(\Omega)$, again by the absolute continuity of the integral there exists some $\delta > 0$ such that $E \subset \Omega$ is a measurable set with $\mathrm{meas}(E) < \delta$ then

$$\int_E b(x) < \frac{\varepsilon}{2g(k)}.$$

In this way, we have that if $E \subset \Omega$ is a measurable set with $\mathrm{meas}(E) < \delta$ then

$$\int_E b(x)|g(u_n)| \le g(k)\int_E b(x) + \int_{\{k\le|u_n|\}} |f(x)| < \varepsilon, \qquad \forall n \in \mathbb{N}.$$

Therefore, the sequence $\{b(x)g(u_n)\}$ is uniformly integrable. As we also have that this sequence $b(x)g(u_n) \to b(x)g(u)$ a.e. in Ω, we can apply Vitali's Theorem (since $\mathrm{meas}(\Omega) < \infty$) to conclude that $b(x)g(u) \in L^1(\Omega)$ and that

$$b(x)g(u_n) \to b(x)g(u) \quad \text{in } L^1(\Omega). \qquad \square$$

3 Convergence to the entropy solution

In this section we give the proof of Theorem 1.1.

Proof of Theorem 1.1. First, let us remember that as u_n are weak solutions of (P_n), then for every $n \in \mathbb{N}$ and for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ we have that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right]g(u_n)\varphi = \int_\Omega f(x)\varphi. \tag{3.1}$$

Now we begin with the proof.

Step 1. $\{u_n\}$ is bounded on some Marcinkiewicz space.

Taking $T_k(u_n) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in (3.1) we obtain for every $n \in \mathbb{N}$ and for every $k > 0$ that

$$\int_{\{|u_n| \le k\}} a(x, u_n, \nabla u_n) \nabla T_k(u_n) + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n) T_k(u_n) = \int_\Omega f(x) T_k(u_n).$$

Observe that the second integral is nonnegative since $g(s)s \ge 0$ for every $s \in \mathbb{R}$ by (1.5) and that we can apply (1.1) on the first integral since $a(x, u_n, \nabla u_n) = a(x, T_k(u_n), \nabla T_k(u_n))$ on the set $\{|u_n| < k\}$. So, from the above equality we deduce that

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p = \alpha \int_{\{|u_n| < k\}} |\nabla u_n|^p \le \int_\Omega f(x) T_k(u_n) \le k \|f\|_1, \qquad \forall n \in \mathbb{N},\ \forall k > 0. \tag{3.2}$$

Thus, we can apply Lemma 2.3 to assure that there exists a constant $C > 0$ depending only of N, p, α and f such that

$$\operatorname{meas}\{|u_n| > k\} \le C k^{-\frac{N(p-1)}{N-p}}, \tag{3.3}$$

for every $n \in \mathbb{N}$ and every $k > 0$. As a consequence, we deduce that $\{u_n\}$ is bounded on the space $\mathcal{M}^{p_1}(\Omega)$ with $p_1 = \frac{N(p-1)}{N-p}$.

Step 2. $\{u_n\}$ converges in measure to some function u.

To show that $\{u_n\}$ converges in measure it suffices to show that it is Cauchy in measure. Let $\varepsilon > 0$ and let $t > 0$. As

$$\{|u_n - u_m| > t\} \subseteq \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\},$$

then

$$\operatorname{meas}\{|u_n - u_m| > t\} \le \operatorname{meas}\{|u_n| > k\} + \operatorname{meas}\{|u_m| > k\} + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > t\}.$$

Thanks to (3.3), we can fix $k_0 > 0$ large enough to obtain that

$$\operatorname{meas}\{|u_n| > k_0\} < \frac{\varepsilon}{3}, \qquad \forall n \in \mathbb{N}.$$

By (3.2), we deduce that $\{T_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$ for every $k > 0$. Thus, for every fixed $k > 0$ there exists a subsequence $\{u_{\sigma_k(n)}\}$ of $\{u_n\}$ such that $\{T_k(u_{\sigma_k(n)})\}$ is Cauchy in $L^p(\Omega)$. Using the Cantor's diagonal argument, we can build a subsequence $\{u_{\sigma(n)}\}$ of $\{u_n\}$ such that $\{T_k(u_{\sigma(n)})\}$ is Cauchy in $L^p(\Omega)$ for every $k > 0$. For the sake of simplicity, we still denote $\{u_{\sigma(n)}\}$ by $\{u_n\}$.

So, since $\{T_{k_0}(u_n)\}$ is a Cauchy sequence in $L^p(\Omega)$, there exists $n_0 \in \mathbb{N}$ such that

$$\operatorname{meas}\{|T_{k_0}(u_n) - T_{k_0}(u_m)| > t\} \le t^{-p} \int_\Omega |T_{k_0}(u_n) - T_{k_0}(u_m)|^p < \frac{\varepsilon}{3}, \qquad \forall m, n \ge n_0.$$

Thus, it is proven that $\{u_n\}$ is Cauchy in measure and hence there exists some measurable function u such that $u_n \to u$ in measure. As a consequence, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$u_n \to u \quad \text{a.e. in } \Omega.$$

Now, since for $k>0$ fixed the sequence $\{T_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$ by (3.2) and $T_k(u)$ is its only possible almost everywhere limit because of the continuity of T_k, we can conclude that

$$
\begin{aligned}
T_k(u_n) &\rightharpoonup T_k(u) && \text{in } W_0^{1,p}(\Omega),\\
T_k(u_n) &\to T_k(u) && \text{in } L^p(\Omega),\\
T_k(u_n) &\to T_k(u) && \text{a.e. in } \Omega.
\end{aligned}
$$

Observe that this implies that $T_k(u)\in W_0^{1,p}(\Omega)$ for every $k>0$.

Step 3. $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$ for every $k>0$.

Following the ideas of [8], in order to obtain the strong convergence of the truncations in the $W_0^{1,p}(\Omega)$ space we choose

$$
w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))
$$

with $h>k>0$ as test function in (3.1). See that if we set $M=4k+h$ then we have that $\nabla w_n = 0$ on the set $\{|u_n|>M\}$. Thus, we can write

$$
\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n + \int_\Omega \left[b(x)+\tfrac{1}{n}|f(x)|\right]g(u_n)w_n = \int_\Omega f(x)w_n. \tag{3.4}
$$

Now, we split the first integral on the sets $\{|u_n|<k\}$ and $\{|u_n|\geq k\}$. On the one hand, observing that $\{|T_k(u_n)-T_k(u)|\leq 2k\}=\Omega$, that $\nabla T_k(u_n)=0$ on the set $\{|u_n|\geq k\}$ and that $a(x,s,0)=0$ by (1.1), we obtain that

$$
\begin{aligned}
&\int_{\{|u_n|<k\}} a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n\\
&\qquad = \int_{\{|u_n|<k\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_{2k}(T_k(u_n)-T_k(u))\\
&\qquad = \int_{\{|u_n|<k\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla (T_k(u_n)-T_k(u))\\
&\qquad = \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla (T_k(u_n)-T_k(u)).
\end{aligned} \tag{3.5}
$$

On the other hand, using (1.1) we deduce that

$$
\begin{aligned}
&a(x,T_M(u_n),\nabla T_M(u_n))\nabla (G_h(u_n)-T_k(u))\\
&\qquad = a(x,T_M(u_n),\nabla T_M(u_n))\nabla G_h(u_n) - a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_k(u)\\
&\qquad \geq -a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_k(u)
\end{aligned}
$$

and, thus, we have

$$
\begin{aligned}
&\int_{\{|u_n|\geq k\}} a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n\\
&\qquad = \int_{\{|u_n|\geq k\}\cap\{|G_h(u_n)+k+T_k(u)|\leq 2k\}} a(x,T_M(u_n),\nabla T_M(u_n))\nabla (G_h(u_n)-T_k(u))\\
&\qquad \geq -\int_{\{|u_n|\geq k\}} |a(x,T_M(u_n),\nabla T_M(u_n))|\,|\nabla T_k(u)|.
\end{aligned} \tag{3.6}
$$

From equations (3.5) and (3.6) we deduce that

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))$$
$$\leq \int_{\{|u_n|\geq k\}} |a(x, T_M(u_n), \nabla T_M(u_n))|\, |\nabla T_k(u)| + \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))\nabla w_n.$$

Adding $-\int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u))$ to both sides of the previous inequality and using (3.4) we obtain that

$$\begin{aligned}\int_\Omega &[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))]\nabla(T_k(u_n) - T_k(u))\\ &\leq \int_{\{|u_n|\geq k\}} |a(x, T_M(u_n), \nabla T_M(u_n))|\, |\nabla T_k(u)|\\ &\quad - \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n) w_n + \int_\Omega f(x) w_n\\ &\quad - \int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u)).\end{aligned} \tag{3.7}$$

Our next step will be taking limits when $n \to \infty$ on the above inequality. First, see that as $|a(x, T_M(u_n), \nabla T_M(u_n))|$ is bounded in $L^{p'}(\Omega)$ by (1.2) and (3.2), and as $\chi_{\{|u_n|\geq k\}}|\nabla T_k(u)|$ converges strongly to zero in $L^p(\Omega)$ by Lebesgue Theorem, then

$$\lim_{n\to\infty} \int_{\{|u_n|\geq k\}} |a(x, T_M(u_n), \nabla T_M(u_n))|\, |\nabla T_k(u)| = 0. \tag{3.8}$$

Secondly, since $\{b(x)g(u_n)\}$ is bounded in $L^1(\Omega)$ by Lemma 2.6 and since $\{\frac{1}{n}|f(x)|g(u_n)\}$ is also bounded in $L^1(\Omega)$ because $\frac{1}{n}|f(x)g(u_n)| \leq |f(x)|$ for every $n \in \mathbb{N}$ by (1.5) and (1.7), Lebesgue Theorem easily implies that

$$\begin{aligned}\lim_{n\to\infty} &\left(- \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n) w_n + \int_\Omega f(x) w_n \right)\\ &= \int_\Omega [-b(x)g(u) + f(x)] T_{2k}(u - T_h(u)).\end{aligned} \tag{3.9}$$

Finally, since $a(x, T_k(u_n), \nabla T_k(u)) \to a(x, T_k(u), \nabla T_k(u))$ strongly in $L^{p'}(\Omega)$ by (1.2) and by Lebesgue Theorem, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $L^p(\Omega)$, we deduce that

$$\lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u)) = 0. \tag{3.10}$$

Observe that the first integral of (3.7) is nonnegative by (1.3). So if we take limits when $n \to \infty$ in (3.7) and we apply (3.8), (3.9) and (3.10) we obtain that

$$\begin{aligned}0 &\leq \lim_{n\to\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))]\nabla(T_k(u_n) - T_k(u))\\ &\leq \int_\Omega [-b(x)g(u) + f(x)] T_{2k}(u - T_h(u)).\end{aligned} \tag{3.11}$$

Now, see that $b(x)g(u) \in L^1(\Omega)$ by Lemma 2.6, so Lebesgue Theorem implies that

$$\lim_{h\to\infty} \int_\Omega [-b(x)g(u) + f(x)] T_{2k}(u - T_h(u)) = 0$$

and thus we can take limits when $h \to \infty$ in (3.11) to assure that

$$\lim_{n\to\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))]\nabla(T_k(u_n) - T_k(u)) = 0.$$

This allows us to apply Lemma 5 of [7] to conclude that

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega) \text{ for every } k > 0.$$

Step 4. u is the entropy solution of (P).

Let us take $T_k(u_n - \varphi)$ with $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$ as test function in (3.1). Observe that if we define $L = k + \|\varphi\|_\infty$, then we have that $\nabla T_k(u_n - \varphi) = 0$ on the set $\{|u_n| > L\}$, so we can write

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi) = \int_\Omega a(x, T_L(u_n), \nabla T_L(u_n))\nabla T_k(u_n - \varphi)$$

and thus (3.1) with this test function can be rewritten as

$$\begin{aligned} &\int_\Omega a(x, T_L(u_n), \nabla T_L(u_n))\nabla T_k(u_n - \varphi) \\ &\qquad + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n) T_k(u_n - \varphi) = \int_\Omega f(x) T_k(u_n - \varphi). \end{aligned} \tag{3.12}$$

Since $T_L(u_n) \to T_L(u)$ strongly in $W_0^{1,p}(\Omega)$, then we have that $\nabla T_L(u_n) \to \nabla T_L(u)$ a.e. in Ω and, as a consequence of (1.2) and Lebesgue Theorem, we have that

$$a(x, T_L(u_n), \nabla T_L(u_n)) \to a(x, T_L(u), \nabla T_L(u)) \quad \text{in } L^{p'}(\Omega).$$

As we also have that $\nabla T_k(u_n - \varphi) \to \nabla T_k(u - \varphi)$ in $L^p(\Omega)$, we can assure that

$$\begin{aligned} \int_\Omega a(x, T_L(u_n), \nabla T_L(u_n))\nabla T_k(u_n - \varphi) &\to \int_\Omega a(x, T_L(u), \nabla T_L(u))\nabla T_k(u - \varphi) \\ &= \int_\Omega a(x, u, \nabla u)\nabla T_k(u - \varphi). \end{aligned}$$

If we use that $b(x)g(u_n) \to b(x)g(u)$ in $L^1(\Omega)$ by Lemma 2.6 and that $\frac{1}{n}|f(x)|g(u_n) \to 0$ in $L^1(\Omega)$ thanks to the (1.7) estimate, we can easily pass to the limit in (3.12) to obtain that

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - \varphi) + \int_\Omega b(x) g(u) T_k(u - \varphi) = \int_\Omega f(x) T_k(u - \varphi),$$

so we can conclude that u is the entropy solution of (P). Finally, observe that due to the uniqueness of the entropy solution we can assert that the whole original sequence $\{u_n\}$ converges in measure to u. □

4 The semilinear case

In this section we prove the Theorem 1.2 and we give some additional remarks.

Proof of Theorem 1.2. First, let us show that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Taking $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (P_n) and in (P), we deduce that

$$\begin{aligned}\int_\Omega M(x)\nabla u_n \nabla u_n + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n)u_n &= \int_\Omega f(x)u_n \\ &= \int_\Omega M(x)\nabla u \nabla u_n + \int_\Omega b(x)g(u)u_n.\end{aligned}$$

Since $g(s)s \geq 0$ for every $s \in \mathbb{R}$ by (1.5), then the term $\int_\Omega \frac{1}{n}|f(x)|g(u_n)u_n$ is nonnegative and we can drop it to obtain that

$$\int_\Omega M(x)\nabla u_n \nabla u_n + \int_\Omega b(x)g(u_n)u_n \leq \int_\Omega M(x)\nabla u \nabla u_n + \int_\Omega b(x)g(u)u_n.$$

We can rewrite this expression as

$$\begin{aligned}&\int_\Omega M(x)\nabla\left(u_n - \frac{u}{2}\right)\nabla\left(u_n - \frac{u}{2}\right) - \frac{1}{4}\int_\Omega M(x)\nabla u \nabla u \\ &\quad + \int_\Omega b(x)[g(u_n) - g(u)](u_n - u) + \int_\Omega b(x)g(u_n)u - \int_\Omega b(x)g(u)u \leq 0.\end{aligned}$$

Observe that we have used here the symmetry of the matrix $M(x)$ to obtain the identity $M(x)\nabla u_n \nabla u = M(x)\nabla u \nabla u_n$.

Now, as $b(x) \geq 0$ and g is increasing by (1.5), then the term $\int_\Omega b(x)[g(u_n) - g(u)](u_n - u)$ is nonnegative and we can drop it. If also we apply the ellipticity condition (1.8) of $M(x)$, we obtain that

$$\alpha \int_\Omega \left|\nabla\left(u_n - \frac{u}{2}\right)\right|^2 \leq \frac{1}{4}\int_\Omega M(x)\nabla u \nabla u - \int_\Omega b(x)g(u_n)u + \int_\Omega b(x)g(u)u. \tag{4.1}$$

Arguing as in the beginning of the proof of the Theorem 1.1, we can deduce that $\{u_n\}$ is bounded in some Marcinkiewicz space and thus we can apply Lemma 2.6 to assert that $\{b(x)g(u_n)\}$ is bounded in $L^1(\Omega)$. Thanks to this and to the fact that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $b(x) \in L^1(\Omega)$, $M(x)$ is bounded by (1.9) and g is continuous, we can assure that the right hand side of (4.1) is bounded.

As a consequence, we obtain that $\{u_n - \frac{u}{2}\}$ is bounded in $H_0^1(\Omega)$ and, since $u \in H_0^1(\Omega)$, we can deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thanks to this bound there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function $v \in H_0^1(\Omega)$ such that $u_n \rightharpoonup v$ in $H_0^1(\Omega)$ and $u_n \to v$ a.e. in Ω.

Now, if we bear in mind that $b(x)g(u_n) \to b(x)g(v)$ in $L^1(\Omega)$ by Lemma 2.6 and that $\frac{1}{n}|f(x)g(u_n)| \leq |f(x)| \in L^1(\Omega)$ by (1.7) estimate, we can easily pass to the limit in

$$\int_\Omega M(x)\nabla u_n \nabla \varphi + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n)\varphi = \int_\Omega f(x)\varphi, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

to obtain that

$$\int_\Omega M(x)\nabla v \nabla \varphi + \int_\Omega b(x)g(v)\varphi = \int_\Omega f(x)\varphi, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

and thus it is proven that $v = u$, i.e., that v is the weak solution of (P). Moreover, due to the uniqueness of the solution u we can affirm that the whole original sequence $\{u_n\}$ converges weakly in $H_0^1(\Omega)$ to u. □

Observe that if we take $b(x) = 0$ in (P), then the assumption $u \in L^\infty(\Omega)$ is not necessary in the proof of this theorem. This allows us to state the following result.

Theorem 4.1. *Suppose that $a(x, s, \xi) = M(x)\xi$ with $M(x)$ a symmetric matrix satisfying* (1.8) *and* (1.9). *Assume also that $b(x) = 0$, that $f(x)$ verifies* (1.4) *and that g satisfies* (1.5). *If the weak solution $u \in H_0^1(\Omega)$ of* (P) *exists, then $\{u_n\}$, the sequence of weak solutions of* (P_n) *given by Theorem* 2.1, *verifies that*

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

To end this paper, we state a remark related with the case in which f is a nonnegative function.

Remark 4.2. If $f \geq 0$ the proofs are easier and stronger results can be proven. This is mainly due to two facts: $\{u_n\}$ is nonnegative and increasing. The monotony of $\{u_n\}$ assures the existence of its a.e. limit and, by Theorem 1.1, this a.e. limit must be u, the entropy solution of (P).

Observe that this implies that $u_n \leq u$ a.e. in Ω for every $n \in \mathbb{N}$ and thus the assumption $u \in L^\infty(\Omega)$ on Theorem 1.2 implies that $\{u_n\}$ is bounded in $L^\infty(\Omega)$. This allows us not only to prove that theorem in simpler way, but also to show that

$$u_n \to u \quad \text{in } H_0^1(\Omega).$$

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References

[1] D. Arcoya, L. Boccardo, Regularizing effect of the interplay between coefficients in some elliptic equations, *J. Funct. Anal.* **268**(2015), No. 5, 1153–1166. https://doi.org/10.1016/j.jfa.2014.11.011; MR3304596; Zbl 1317.35082

[2] D. Arcoya, L. Boccardo, Regularizing effect of L^q interplay between coefficients in some elliptic equations, *J. Math. Pures Appl. (9)* **111**(2018), 106–125. https://doi.org/10.1016/j.matpur.2017.08.001; MR3760750; Zbl 1390.35067

[3] D. Arcoya, L. Boccardo, Regularizing effect of two hypotheses on the interplay between coefficients in some Hamilton–Jacobi equations, *Adv. Nonlinear Stud.* **21**(2021), No. 2, 251–260. https://doi.org/10.1515/ans-2021-2126; MR4250466; Zbl 1472.35123

[4] D. Arcoya, L. Boccardo, L. Orsina, Regularizing effect of the interplay between coefficients in some nonlinear Dirichlet problems with distributional data, *Ann. Mat. Pura Appl. (4)* **199**(2020), No. 5, 1909–1921. https://doi.org/10.1007/s10231-020-00949-8; MR4142856; Zbl 1453.35081

[5] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22**(1995), No. 2, 241–273. MR1354907; Zbl 0866.35037

[6] L. Boccardo, D. Giachetti, J. I. Díaz, F. Murat, Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms, *J. Differential Equations* **106**(1993), No. 2, 215–237. `https://doi.org/10.1006/jdeq.1993.1106`; MR1251852; Zbl 0803.35046

[7] L. Boccardo, F. Murat, J.-P. Puel, Existence of bounded solutions for nonlinear elliptic unilateral problems, *Ann. Mat. Pura Appl. (4)* **152**(1988), 183–196. `https://doi.org/10.1007/BF01766148`; MR980979; Zbl 0687.35042

[8] C. Leone, A. Porretta, Entropy solutions for nonlinear elliptic equations in L^1, *Nonlinear Anal.* **32**(1998), No. 3, 325–334. `https://doi.org/10.1016/S0362-546X(96)00323-9`; MR1610574; Zbl 1155.35352