On some classes of solvable difference equations related to iteration processes

Stevo Stević[✉]

Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III,
11000 Beograd, Serbia

Department of Medical Research, China Medical University Hospital, China Medical University,
Taichung 40402, Taiwan, Republic of China

Abstract. We present several classes of nonlinear difference equations solvable in closed form, which can be obtained from some known iteration processes, and for some of them we give some generalizations by presenting methods for constructing them. We also conduct several analyses and give many comments related to the difference equations and iteration processes.

Keywords: difference equation, iteration process, equations solvable in a closed form.

2020 Mathematics Subject Classification: 39A45.

1   Introduction

The sets of natural numbers, nonnegative integers, integers, real numbers and complex numbers, we denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, whereas the notation $l = \overline{s,t}$, when $s, t \in \mathbb{Z}$ and $s \le t$ is used instead of writing $s \le l \le t$, $l \in \mathbb{Z}$. By C_j^n, $n \in \mathbb{N}$ and $j = \overline{0,n}$, we denote the binomial coefficients. Recall that

$$C_j^n = \frac{n!}{j!(n-j)!},$$

where we regard that $0! = 1$ (some information on the coefficients can be found, e.g., in [4, 22, 32, 34, 43]).

Difference equations and systems naturally appear in many areas of science and mathematics [9, 12–14, 18, 19, 22, 26, 27, 29, 34, 43, 51, 59]. The problem of finding formulas for their solutions in closed form appeared long time ago, and was treated by many known mathematicians such as D. Bernoulli, de Moivre, Euler, Lagrange and Laplace (see, e.g., [9, 13–15, 17, 23–28]). Unfortunately, for a great majority of the equations and systems it is impossible to find such

✉ Email: sstevic@ptt.rs; sscite1@gmail.com

formulas, especially if they are nonlinear. In [1,10,16,17,19,22,30,31,33–35] can be found some classical solvable nonlinear difference equations, as well as systems of difference equations.

Several classes of solvable nonlinear difference equations can be obtained by using some known iteration processes. Some of them can be found, for example, in [11,12,55,56].

Motivated by some recent investigations on solvability of difference equations and systems of difference equations (see, e.g., [3,40,42,53,54,57,58] and the references therein) and some examples in [12], we have studied recently connections between some difference equations obtained from known iteration processes and their solvability. Related equations and topics such as finding invariants and studying equations obtained from solvable ones can be found in [5–8,21,36–39,46,47,49,50,52].

Here we continue the investigation of solvability of difference equations and their relationships with known iteration processes. We deal with some equations of the form

$$x_{n+1} = \widehat{f}(x_n), \quad n \in \mathbb{N}_0,$$

the autonomous difference equation of first order.

First we show that the Newton–Raphson iteration process for finding roots of quadratic equations produces a solvable nonlinear difference equation extending a known example of such a difference equation. Recall that the Newton–Raphson iteration process is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

(see, e.g., [12,18]), where f is a given function.

Based on it and another known difference equation, we present a related class of solvable nonlinear difference equations. Then we present a solvable class of nonlinear difference equations generalizing two known ones which are obtained by the Newton–Raphson iteration process for calculating reciprocals. We also present an interesting method for constructing a class of solvable nonlinear difference equations generalizing a solvable equation obtained from the Halley iteration process for finding square roots. We also conduct several analyses and give many comments related to solvable nonlinear difference equations and iteration processes.

2   Some analyses and main results

In this section we conduct some analyses related to the relationships between solvable difference equations and some known iteration processes, and state and prove our main results.

2.1   Newton–Raphson iteration process for quadratic equations and solvability

Let

$$f(x) = x^2 + px + q, \tag{2.1}$$

be a quadratic function.

By using function (2.1) in (1.1) we get

$$x_{n+1} = x_n - \frac{x_n^2 + px_n + q}{2x_n + p}, \quad n \in \mathbb{N}_0,$$

that is,

$$x_{n+1} = \frac{x_n^2 - q}{2x_n + p}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

If a solution to equation (2.2) converges to a point x^*, then it is clear that x^* must be equal to one of the zeros of function (2.1).

From the numerical point of view the interesting case is when $q \neq 0$ (if $q = 0$, then the roots of (2.1) are obviously 0 and $-p$). Assume additionally that $p^2 \neq 4q$. Then the function has two different zeros, say, a and b, and equation (2.2) can be rewritten in the form

$$x_{n+1} = \frac{x_n^2 - ab}{2x_n - a - b}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

First, assume that $a \neq b$. We consider the cases $a + b = 0$ and $a + b \neq 0$ separately.

Case $a + b = 0$. In this case we have $b = -a$. Hence, equation (2.3) becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a^2}{x_n}\right), \quad n \in \mathbb{N}_0. \tag{2.4}$$

It is well known that the equation is solvable in closed form [12, 22], and that its general solution is given by

$$x_n = a\frac{1 + \left(\frac{x_0 - a}{x_0 + a}\right)^{2^n}}{1 - \left(\frac{x_0 - a}{x_0 + a}\right)^{2^n}}, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Recall that the difference equation in (2.4) serves for calculating a square root of number a^2.

Case $a + b \neq 0$. From (2.3) and by some simple calculations, it follows that

$$x_{n+1} - a = \frac{x_n^2 - 2ax_n + a^2}{2x_n - a - b}, \quad n \in \mathbb{N}_0, \tag{2.6}$$

and

$$x_{n+1} - b = \frac{x_n^2 - 2bx_n + b^2}{2x_n - a - b}, \quad n \in \mathbb{N}_0. \tag{2.7}$$

From (2.6) and (2.7) we have

$$\frac{x_{n+1} - a}{x_{n+1} - b} = \left(\frac{x_n - a}{x_n - b}\right)^2, \quad n \in \mathbb{N}_0,$$

and consequently

$$\frac{x_n - a}{x_n - b} = \left(\frac{x_0 - a}{x_0 - b}\right)^{2^n}, \quad n \in \mathbb{N}_0,$$

from which it easily follows that

$$x_n = \frac{b\left(\frac{x_0 - a}{x_0 - b}\right)^{2^n} - a}{\left(\frac{x_0 - a}{x_0 - b}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0. \tag{2.8}$$

The sequence defined in (2.8) is a solution to equation (2.3). Indeed, let

$$y_n := \left(\frac{x_0 - a}{x_0 - b}\right)^{2^n}, \quad n \in \mathbb{N}_0.$$

Then we have

$$\frac{x_n^2 - ab}{2x_n - a - b} = \frac{(\frac{by_n - a}{y_n - 1})^2 - ab}{2\frac{by_n - a}{y_n - 1} - a - b} = \frac{b^2 y_n^2 - 2aby_n + a^2 - aby_n^2 + 2aby_n - ab}{(y_n - 1)(2by_n - 2a - (a + b)y_n + a + b)}$$

$$= \frac{b(b - a)y_n^2 - a(b - a)}{(y_n - 1)(b - a)(y_n + 1)} = \frac{by_n^2 - a}{y_n^2 - 1} = \frac{b(\frac{x_0 - a}{x_0 - b})^{2^{n+1}} - a}{(\frac{x_0 - a}{x_0 - b})^{2^{n+1}} - 1}$$

$$= x_{n+1}$$

as claimed.

Remark 2.1. Note that from (2.8) with $b = -a$ is obtained formula (2.5).

From (2.8) we easily obtain the following corollary.

Corollary 2.2. *Consider equation (2.3) where $a \neq b$ and $ab \neq 0$. Then the following statements are true.*

(a) If $\left|\frac{x_0 - a}{x_0 - b}\right| < 1$, then $\lim_{n \to +\infty} x_n = a$.

(b) If $\left|\frac{x_0 - a}{x_0 - b}\right| > 1$, then $\lim_{n \to +\infty} x_n = b$.

(c) If $\frac{x_0 - a}{x_0 - b} = -1$, that is, $x_0 = \frac{a+b}{2}$, then x_1 is not defined.

Remark 2.3. Note that the case

$$\frac{x_0 - a}{x_0 - b} = 1$$

is excluded, since we assume $a \neq b$.

Case $a = b$. If $a = b$ and $x_0 = a$, then since in this case equation (2.3) becomes

$$x_{n+1} = \frac{x_n^2 - a^2}{2(x_n - a)}, \quad n \in \mathbb{N}_0, \tag{2.9}$$

we have that x_1 is not defined, so that in this case the solution to the equation is not well-defined.

If $x_{n_0} = a$ for some $n_0 \in \mathbb{N}$, and

$$x_j \neq a, \quad j = \overline{0, n_0 - 1},$$

then from (2.9) we have

$$a = x_{n_0} = \frac{x_{n_0-1}^2 - a^2}{2(x_{n_0-1} - a)} = \frac{x_{n_0-1} + a}{2},$$

and consequently $x_{n_0-1} = a$, which is a contradiction. Therefore, if $x_0 \neq a$ we have that

$$x_n \neq a \quad \text{for} \quad n \in \mathbb{N}_0. \tag{2.10}$$

Hence, if $a = b$ and $x_0 \neq a$, then from (2.9) and (2.10) we have that equation (2.3) becomes

$$x_{n+1} = \frac{x_n}{2} + \frac{a}{2}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$x_n = \frac{x_0}{2^n} + a\left(1 - \frac{1}{2^n}\right), \quad n \in \mathbb{N}_0, \tag{2.11}$$

(for the original source see [25]; see also [10, 19, 22, 34]).

From (2.11) we easily obtain the following corollary.

Corollary 2.4. *Consider equation (2.3) where $a = b$ and $a \neq 0$. Then every solution to equation (2.3) such that $x_0 \neq a$ converges to a.*

Remark 2.5. Equation (2.3) appeared in [55] but we did not consider it, nor did we formulate any of the above results in the case.

Since we assume that $q = ab \neq 0$, the above analysis excluded the case. However, it is of some interest to consider equation (2.3) also in this case.

Case $ab = 0$. First note that, due to the symmetry of equation (2.3) with respect to parameters a and b, we may assume $b = 0$. In this case the difference equation becomes

$$x_{n+1} = \frac{x_n^2}{2x_n - a}, \tag{2.12}$$

for $n \in \mathbb{N}_0$.

If $a = 0$, then we have

$$x_{n+1} = \frac{x_n^2}{2x_n}, \quad n \in \mathbb{N}_0. \tag{2.13}$$

Hence, if $x_0 \neq 0$ we get

$$x_n = \frac{x_0}{2^n}, \quad n \in \mathbb{N}_0,$$

showing the solvability of equation (2.12) in this case. If $x_0 = 0$, then from (2.13) we see that x_1 is not defined. Therefore, the solution to equation (2.13) is also not well-defined.

Now assume that $a \neq 0$. If $x_0 = 0$, then a simple inductive argument shows that

$$x_n = 0, \quad n \in \mathbb{N}_0. \tag{2.14}$$

If $x_{n_1} = 0$ for some $n_1 \in \mathbb{N}$, and

$$x_j \neq 0, \quad j = \overline{0, n_1 - 1}, \tag{2.15}$$

then from (2.12) we have $x_{n_1-1} = 0$, which is a contradiction. From (2.14) and (2.15) we see that when $x_0 \neq 0$ we have that $x_n \neq 0$ for all $n \in \mathbb{N}_0$ for which x_n is defined. Hence, we can use the change of variables

$$x_n = \frac{1}{y_n}, \quad n \in \mathbb{N}_0, \tag{2.16}$$

and obtain the equation

$$y_{n+1} = y_n(2 - ay_n), \quad n \in \mathbb{N}_0.$$

It is well known that general solution to the equation is given by

$$y_n = \frac{1 - (1 - ay_0)^{2^n}}{a}, \quad n \in \mathbb{N}_0,$$

(see, e.g., [11, 12]).

Hence, we have that the general solution to equation (2.12) in this case is given by the formula

$$x_n = \frac{ax_0^{2^n}}{x_0^{2^n} - (x_0 - a)^{2^n}}, \tag{2.17}$$

for $n \in \mathbb{N}_0$.

Remark 2.6. Formula (2.17) can be also obtained from the formula (2.8) with $b = 0$. Indeed, the above consideration in the case $a + b \neq 0$ also holds in the case when $b = 0$. Note also that if $b = 0$, then $a + b = a \neq 0$. Hence, all the conditions there are satisfied if $a \neq 0$ and $b = 0$.

Remark 2.7. The change of variables (2.16) is a basic one and frequently appears in the literature (see, e.g. [4,51]). One of the basic examples of difference equations where it is applied is the following

$$x_{n+1} = \frac{a_n x_n}{b_n + c_n x_n}, \quad n \in \mathbb{N}_0,$$

which, by the change of variables, is transformed to a nonhomogeneous linear difference equation of first order, which is theoretically solvable (this was shown first by Lagrange [26], then by another method by Laplace [27]; see, also [10,16,19,34]). For some related changes of variables see, e.g., [40,53,57] and the related references therein.

2.2   A relative of equation (2.4)

The difference equation

$$x_{n+1} = \frac{2x_n}{x_n^2 + 1}, \quad n \in \mathbb{N}_0, \tag{2.18}$$

is another known difference equation. The long-term behaviour of its solutions can be studied by using standard methods to the governing function

$$f(t) = \frac{2t}{t^2 + 1}, \quad t \in \mathbb{R},$$

(see, e.g., [4, Problems 9.34, 9.35]).

However, the equation is also solvable. Indeed, first note that if x^* is an equilibrium of equation (2.18), then it is easy to see that

$$x^* \in \{-1, 0, 1\}.$$

Since

$$x_{n+1} - 1 = -\frac{(x_n - 1)^2}{x_n^2 + 1}$$

and

$$x_{n+1} + 1 = \frac{(x_n + 1)^2}{x_n^2 + 1}$$

for $n \in \mathbb{N}_0$, we have

$$\frac{x_{n+1} - 1}{x_{n+1} + 1} = -\left(\frac{x_n - 1}{x_n + 1}\right)^2, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$\frac{x_n - 1}{x_n + 1} = -\left(\frac{x_0 - 1}{x_0 + 1}\right)^{2^n}, \quad n \in \mathbb{N},$$

and finally

$$x_n = \frac{1 - \left(\frac{x_0 - 1}{x_0 + 1}\right)^{2^n}}{1 + \left(\frac{x_0 - 1}{x_0 + 1}\right)^{2^n}}, \quad n \in \mathbb{N}. \tag{2.19}$$

Remark 2.8. Solvability of equation (2.18) is not so surprising. Namely, note that by using the change of variables (2.16) from equation (2.18) it is obtained equation (2.4) with $a = 1$.

By using the change of variables in (2.16) in equation (2.2) we obtain the equation

$$y_{n+1} = \frac{p y_n^2 + 2 y_n}{1 - q y_n^2}, \quad n \in \mathbb{N}_0. \tag{2.20}$$

Let $p = -(a + b)$ and $q = ab$, then equation (2.20) becomes

$$y_{n+1} = \frac{-(a + b) y_n^2 + 2 y_n}{1 - ab y_n^2}, \quad n \in \mathbb{N}_0. \tag{2.21}$$

If $a + b \neq 0$, then from (2.8) we obtain

$$y_n = \frac{\left(\frac{1 - a y_0}{1 - b y_0}\right)^{2^n} - 1}{b \left(\frac{1 - a y_0}{1 - b y_0}\right)^{2^n} - a}, \quad n \in \mathbb{N}_0, \tag{2.22}$$

whereas if $a = b$, then from (2.11) we obtain

$$y_n = \frac{y_0 2^n}{1 + a y_0 (2^n - 1)}, \quad n \in \mathbb{N}_0. \tag{2.23}$$

From (2.22) we obtain the following corollary.

Corollary 2.9. *Consider equation (2.21) where $a \neq b$ and $ab \neq 0$. Then for well-defined solutions of the equation the following statements are true.*

(a) If $\left|\frac{1 - a y_0}{1 - b y_0}\right| < 1$, then $\lim_{n \to +\infty} y_n = \frac{1}{a}$.

(b) If $\left|\frac{1 - a y_0}{1 - b y_0}\right| > 1$, then $\lim_{n \to +\infty} y_n = \frac{1}{b}$.

(c) If $\frac{1 - a y_0}{1 - b y_0} = -1$, that is, $y_0 = \frac{2}{a + b}$, then $y_n = 0$, $n \in \mathbb{N}$.

(d) If $y_0 = 0$, then $y_n = 0$, $n \in \mathbb{N}_0$.

Remark 2.10. Note that if $y_0 \neq 0$, the case

$$\frac{1 - a y_0}{1 - b y_0} = 1$$

is excluded, since we assume $a \neq b$.

From (2.21) and (2.23) we obtain the following corollary.

Corollary 2.11. *Consider equation (2.21) where $a = b \neq 0$. Then every solution to equation (2.21) such that $y_0 \neq 0$, $y_0 \neq 1/a$, and*

$$y_0 \neq \frac{1}{a(1 - 2^n)}, \quad n \in \mathbb{N},$$

converges to $\frac{1}{a}$.

Remark 2.12. Note that if $a = b$ and $y_0 = 0$, then $y_n = 0$ for every $n \in \mathbb{N}$.

Remark 2.13. Note that if $a = b$ and $y_0 = 1/a$ or

$$y_0 = \frac{1}{a(1 - 2^n)}$$

for some $n \in \mathbb{N}$, then y_n is not defined, and consequently the corresponding solution to equation (2.21).

2.3   Newton–Raphson iteration process for calculating reciprocals

It is well known that if we apply the Newton-Raphson iteration process to

$$f(x) = 1 - \frac{1}{ax} \tag{2.24}$$

where $a \neq 0$, we obtain the equation

$$x_{n+1} = 2x_n - ax_n^2, \quad n \in \mathbb{N}_0. \tag{2.25}$$

Recall that the equation is solvable in closed form [11, 12, 18, 55].

If we apply the iteration process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)}{2f'(x_n)} \left(\frac{f(x_n)}{f'(x_n)} \right)^2, \quad n \in \mathbb{N}_0$$

to the function in (2.24) we obtain the equation

$$x_{n+1} = 3x_n - 3ax_n^2 + a^2 x_n^3, \quad n \in \mathbb{N}_0, \tag{2.26}$$

see, e.g., [18], where it is suggested to show that the relation holds

$$\frac{1}{a} - x_{n+1} = a^2 \left(\frac{1}{a} - x_n \right)^3, \quad n \in \mathbb{N}_0. \tag{2.27}$$

From (2.27) we see that the relation (2.26) is also solvable in closed form. Indeed, let

$$y_n := \frac{1}{a} - x_n, \tag{2.28}$$

then from (2.26) we have

$$y_n = a^2 y_{n-1}^3, \quad n \in \mathbb{N},$$

which is a simple product-type difference equation (for some examples of such difference equations and systems of equations, see, e.g., [54, 58] and the related references therein).

By iterating the last relation we get

$$y_n = a^2 (a^2 y_{n-2}^3)^3 = a^{2(1+3)} y_{n-2}^{3^2}.$$

By a simple inductive argument we obtain

$$y_n = a^{2\sum_{j=0}^{n-1} 3^j} y_0^{3^n} = a^{3^n - 1} y_0^{3^n}, \quad n \in \mathbb{N}_0,$$

from which along with (2.28) it follows that

$$x_n = \frac{1}{a} - a^{3^n - 1} \left(\frac{1}{a} - x_0 \right)^{3^n}, \quad n \in \mathbb{N}_0. \tag{2.29}$$

Remark 2.14. The matrix counterpart of equation (2.25)

$$X_{n+1} = (2I - AX_n)X_n, \quad n \in \mathbb{N}_0,$$

is the Schultz iteration process [48] which has been studied a lot.

2.4 A generalization of equations (2.25) and (2.26)

Equations (2.25) and (2.26) are, among other things, obtained from two known iteration processes by employing them to the function in (2.24). Here we show that a sequence of iteration processes, which can be used for calculating reciprocals and containing relations (2.25) and (2.26), can be obtained in a simple way. Moreover, we show that they all are solvable in closed form.

If in the difference equation

$$y_{n+1} = y_n^k, \quad n \in \mathbb{N}_0, \tag{2.30}$$

where $k \in \mathbb{N} \setminus \{1\}$ is a fixed number, we use the change of variables

$$y_n = 1 - a x_n, \quad n \in \mathbb{N}_0, \tag{2.31}$$

where $a \neq 0$, we have

$$1 - a x_{n+1} = \sum_{j=0}^{k} C_j^k (-a)^j x_n^j, \quad n \in \mathbb{N}_0,$$

so after some simple calculation we obtain

$$x_{n+1} = \sum_{j=1}^{k} C_j^k (-a)^{j-1} x_n^j, \quad n \in \mathbb{N}_0. \tag{2.32}$$

From (2.32) for each $k \in \mathbb{N} \setminus \{1\}$ we obtain a difference equation which can be used for calculating reciprocals.

Now note that from (2.30) we have

$$y_n = y_0^{k^n}, \quad n \in \mathbb{N}_0. \tag{2.33}$$

By using (2.33) in (2.31) we get

$$x_n = \frac{1 - (1 - a x_0)^{k^n}}{a}, \quad n \in \mathbb{N}_0. \tag{2.34}$$

Formula (2.34) shows that equation (2.32) is also solvable in closed form.

Remark 2.15. Note that if in equation (2.32) we take $k = 2$, then we obtain equation (2.25), whereas if we take $k = 3$, then we obtain equation (2.26). This means that the difference equation is a natural generalization of the equations (2.25) and (2.26).

Remark 2.16. The matrix counterparts of equations (2.32) have been also studied considerably. Our literature review shows that the topic has been quite popular among scientists working on numerical mathematics for a long time, and it seems that such iteration processes are rediscovered from time to time. There are also some operator counterparts of equations (2.25), (2.26) and (2.32) (see, for example, [2, 41] and the references therein). So, the facts mentioned in this subsection should be folklore. Nevertheless, the above explanation suggests a natural way for constructing the matrix and operator iteration processes. From (2.30) we also see how is naturally obtained an iteration process whose rate of the convergence has a given order (for the notion see, e.g., [12, 18]).

2.5   A relative to equation (2.32)

By using change of variables (2.16) in equation (2.32) we obtain the equation

$$y_{n+1} = \frac{y_n^k}{\sum_{j=1}^k C_j^k (-a)^{j-1} y_n^{k-j}}, \quad n \in \mathbb{N}_0,$$

that is,

$$y_{n+1} = \frac{a y_n^k}{y_n^k - (y_n - a)^k}, \quad n \in \mathbb{N}_0. \tag{2.35}$$

Hence, from (2.34) we have that the general solution to equation (2.35) is given by

$$y_n = \frac{a y_0^{k^n}}{y_0^{k^n} - (y_0 - a)^{k^n}}, \quad n \in \mathbb{N}_0.$$

For example, if $k = 3$, then equation (2.35) becomes

$$y_{n+1} = \frac{y_n^3}{3 y_n^2 - 3 a y_n + a^2}, \quad n \in \mathbb{N}_0,$$

and its general solution is

$$y_n = \frac{a y_0^{3^n}}{y_0^{3^n} - (y_0 - a)^{3^n}}, \quad n \in \mathbb{N}_0.$$

2.6   Newton–Raphson method for polynomials of the third degree and solvability

Here we conduct some analyses regarding solvability of difference equations obtained by applying the Newton–Raphson iteration process to polynomials of the third degree, and generalise a class of solvable difference equations by presenting a method for constructing the generalization.

Difference equations can be used for calculating roots of some functions, but it is quite a rare situation that they are solvable in closed form. For example, if we want to calculate a root of the function

$$f(x) = x^3 - x$$

(we can easily find all of them by an elementary method), by using the Newton-Raphson process we get the equation

$$x_{n+1} = x_n - \frac{x_n^3 - x_n}{3 x_n^2 - 1} = \frac{2 x_n^3}{3 x_n^2 - 1}, \quad n \in \mathbb{N}_0. \tag{2.36}$$

The equation frequently appears in the literature (see [20, 44]), and this explains how it can be obtained, which is one of the reasons why we mention the equation. Another reason is connected to the method used in dealing with equation (2.3).

Namely, from (2.36) and some calculations we get

$$x_{n+1} - 1 = \frac{(x_n - 1)^2 (2 x_n + 1)}{3 x_n^2 - 1}$$

and

$$x_{n+1} + 1 = \frac{(x_n + 1)^2 (2 x_n - 1)}{3 x_n^2 - 1}$$

from which it follows that

$$\frac{x_{n+1} - 1}{x_{n+1} + 1} = \left(\frac{x_n - 1}{x_n + 1}\right)^2 \frac{2x_n + 1}{2x_n - 1}.$$

However, the natural change of variables

$$y_n = \frac{x_n - 1}{x_n + 1}$$

cannot show the solvability of relation (2.36).

Let us analyse the general case. If we apply the Newton–Raphson iteration process to an arbitrary polynomial of the third order

$$p_3(t) = t^3 + pt^2 + qt + r \tag{2.37}$$

we get

$$x_{n+1} = x_n - \frac{x_n^3 + px_n^2 + qx_n + r}{3x_n^2 + 2px_n + q} = \frac{2x_n^3 + px_n^2 - r}{3x_n^2 + 2px_n + q}, \tag{2.38}$$

for $n \in \mathbb{N}_0$.

If a, b and c are the roots of (2.37), then (2.38) can be written in the form

$$x_{n+1} = \frac{2x_n^3 - (a + b + c)x_n^2 + abc}{3x_n^2 - 2(a + b + c)x_n + ab + bc + ca}, \quad n \in \mathbb{N}_0, \tag{2.39}$$

and by some calculations we have

$$x_{n+1} - a = \frac{2x_n^3 - (4a + b + c)x_n^2 + 2a(a + b + c)x_n - a^2(b + c)}{3x_n^2 - 2(a + b + c)x_n + ab + bc + ca}, \tag{2.40}$$

for $n \in \mathbb{N}_0$.

Let

$$q_3(t) = 2t^3 - (4a + b + c)t^2 + 2a(a + b + c)t - a^2(b + c).$$

Then, a direct calculation shows that $q_3(a) = 0$, from which it follows that

$$q_3(t) = (t - a)(2t^2 - (2a + b + c)t + a(b + c)) = (t - a)^2(2t - (b + c)).$$

Hence (2.40) can be written as follows

$$x_{n+1} - a = \frac{(x_n - a)^2(2x_n - (b + c))}{3x_n^2 - 2(a + b + c)x_n + ab + bc + ca}, \quad n \in \mathbb{N}_0. \tag{2.41}$$

Since the root of (2.37) we chose was arbitrary, we see that from (2.41) the following relations also hold

$$x_{n+1} - b = \frac{(x_n - b)^2(2x_n - (a + c))}{3x_n^2 - 2(a + b + c)x_n + ab + bc + ca}, \quad n \in \mathbb{N}_0, \tag{2.42}$$

$$x_{n+1} - c = \frac{(x_n - c)^2(2x_n - (a + b))}{3x_n^2 - 2(a + b + c)x_n + ab + bc + ca}, \quad n \in \mathbb{N}_0. \tag{2.43}$$

From (2.41)–(2.43), we have

$$\frac{x_{n+1} - a}{x_{n+1} - b} = \left(\frac{x_n - a}{x_n - b}\right)^2 \frac{2x_n - (b + c)}{2x_n - (a + c)}, \tag{2.44}$$

$$\frac{x_{n+1} - b}{x_{n+1} - c} = \left(\frac{x_n - b}{x_n - c}\right)^2 \frac{2x_n - (a + c)}{2x_n - (a + b)}, \tag{2.45}$$

$$\frac{x_{n+1} - c}{x_{n+1} - a} = \left(\frac{x_n - c}{x_n - a}\right)^2 \frac{2x_n - (a + b)}{2x_n - (b + c)}, \tag{2.46}$$

for $n \in \mathbb{N}_0$.

From (2.44)–(2.46) we see that we can obtain a solvable difference equation if $a + b$, $b + c$ and $c + a$ takes some of the values in the set $\{2a, 2b, 2c\}$. However, it is not difficult to see that in all the cases we get $a = b = c$, so that the equations (2.44)–(2.46) become trivial.

This analysis shows that the method used in solving equation (2.3) cannot be applied to equation (2.39). Nevertheless, there are some equations of the form

$$x_{n+1} = \frac{x_n^3 + px_n^2 + qx_n + r}{sx_n^2 + ux_n + v}, \quad n \in \mathbb{N}_0, \tag{2.47}$$

which are solvable in closed form, but are obtained by using some other iteration processes.

Example 2.17. The difference equation [20, 33, 45]

$$x_{n+1} = \frac{x_n^3 + 3ax_n}{3x_n^2 + a}, \quad n \in \mathbb{N}_0. \tag{2.48}$$

is used for finding a square root of number a. It is interesting that the difference equation is solvable in closed form. See [56] where a class/sequence of solvable difference equations for finding square roots is presented. Beside this, it is also interesting that the equation can be obtained, for example, from the Halley iteration process [18]

$$x_{n+1} = x_n - \frac{2f'(x_n)f(x_n)}{2f'(x_n)^2 - f''(x_n)f(x_n)}, \quad n \in \mathbb{N}_0,$$

applied to the function

$$f(t) = x^2 - a. \tag{2.49}$$

The fact was not mentioned in [56].

A detailed analysis of the method for solving equation (2.48) given in [56], shows that one of the most important facts used in the method is that the following relations hold

$$t^3 + 3at - \sqrt{a}(3t^2 + a) = (t - \sqrt{a})^3$$

and

$$t^3 + 3at + \sqrt{a}(3t^2 + a) = (t + \sqrt{a})^3.$$

Hence it is of interest to see for which values of parameters p, q, r, s, u and v the following identities hold

$$t^3 + pt^2 + qt + r - a(st^2 + ut + v) = (t - a)^3 \tag{2.50}$$

and

$$t^3 + pt^2 + qt + r - d(st^2 + ut + v) = (t - d)^3 \tag{2.51}$$

for some given numbers a and d such that $a \neq d$.

From (2.50) and (2.51) we obtain the following nonlinear algebraic system of equations

$$p - as = -3a, \quad p - ds = -3d, \tag{2.52}$$
$$q - au = 3a^2, \quad q - du = 3d^2, \tag{2.53}$$
$$r - av = -a^3, \quad r - dv = -d^3. \tag{2.54}$$

From (2.52) we have

$$-s(a - d) = -3(a - d)$$

from which along with the assumption $a \neq d$, it follows that $s = 3$. By using it in (2.52) we get $p = 0$.

From (2.53) we have

$$-u(a - d) = 3(a - d)(a + d) \tag{2.55}$$

and

$$2q = 3(a^2 + d^2) + (a + d)u. \tag{2.56}$$

From (2.55) along with the assumption $a \neq d$, it follows that $u = -3(a + d)$. By using it in (2.56) we get $q = -3ad$.

From (2.54) we have

$$v(a - d) = (a - d)(a^2 + ad + d^2) \tag{2.57}$$

and

$$2r = -(a^3 + d^3) + (a + d)v. \tag{2.58}$$

From (2.57) along with the assumption $a \neq d$, it follows that $v = a^2 + ad + d^2$. By using it in (2.58) we get $r = ad(a + d)$.

This analysis suggests that the following special case of equation (2.47)

$$x_{n+1} = \frac{x_n^3 - 3adx_n + ad(a + d)}{3x_n^2 - 3(a + d)x_n + a^2 + ad + d^2}, \quad n \in \mathbb{N}_0, \tag{2.59}$$

is solvable. Indeed, the following theorem holds.

Theorem 2.18. *The equation* (2.59), *where* $a, d \in \mathbb{C}$ *are such that* $a \neq d$ *is solvable in closed form.*

Proof. From (2.59) and some simple calculation we have

$$x_{n+1} - a = \frac{(x_n - a)^3}{3x_n^2 - 3(a + d)x_n + a^2 + ad + d^2}, \quad n \in \mathbb{N}_0, \tag{2.60}$$

and

$$x_{n+1} - d = \frac{(x_n - d)^3}{3x_n^2 - 3(a + d)x_n + a^2 + ad + d^2}, \quad n \in \mathbb{N}_0. \tag{2.61}$$

From (2.60) and (2.61) we have

$$\frac{x_{n+1} - a}{x_{n+1} - d} = \left(\frac{x_n - a}{x_n - d}\right)^3, \quad n \in \mathbb{N}_0,$$

from which it easily follows that

$$\frac{x_n - a}{x_n - d} = \left(\frac{x_0 - a}{x_0 - d}\right)^{3^n}, \quad n \in \mathbb{N}_0,$$

and consequently

$$x_n = \frac{d\left(\frac{x_0 - a}{x_0 - d}\right)^{3^n} - a}{\left(\frac{x_0 - a}{x_0 - d}\right)^{3^n} - 1}, \quad n \in \mathbb{N}_0. \tag{2.62}$$

By some calculations it is checked that (2.62) is a solution to equation (2.59). $\qquad \square$

Remark 2.19. Note that if $d = -a$ equation (2.59) reduces to the equation (2.48) where a is replaced by a^2, from which its solvability again follows.

2.7 Difference equations obtained by polynomials of the fourth degree

In [56] we have also shown that the difference equation

$$x_{n+1} = \frac{x_n^4 + 6ax_n^2 + a^2}{4x_n^3 + 4ax_n}, \quad n \in \mathbb{N}_0, \tag{2.63}$$

is solvable in closed form.

An interesting problem is to try to find a generalization of equation (2.63) by using the method above applied to equation (2.47), which is also solvable in closed form. The following equation

$$x_{n+1} = \frac{x_n^4 + px_n^3 + qx_n^2 + rx_n + s}{\alpha x_n^3 + \beta x_n^2 + \gamma x_n + \delta}, \quad n \in \mathbb{N}_0, \tag{2.64}$$

is a natural generalization of equation (2.63).

Following (2.50) and (2.51), it is of interest to see for which values of parameters $p, q, r, s,$ α, β, γ and δ the following identities hold

$$t^4 + pt^3 + qt^2 + rt + s - a(\alpha t^3 + \beta t^2 + \gamma t + \delta) = (t - a)^4 \tag{2.65}$$

and

$$t^4 + pt^3 + qt^2 + rt + s - d(\alpha t^3 + \beta t^2 + \gamma t + \delta) = (t - d)^4 \tag{2.66}$$

for some given numbers a and d such that $a \neq d$.

From (2.65) and (2.66) we obtain the following nonlinear algebraic system of equations

$$p - \alpha a = -4a, \qquad p - \alpha d = -4d, \tag{2.67}$$
$$q - \beta a = 6a^2, \qquad q - \beta d = 6d^2, \tag{2.68}$$
$$r - \gamma a = -4a^3, \qquad r - \gamma d = -4d^3, \tag{2.69}$$
$$s - \delta a = a^4, \qquad s - \delta d = d^4. \tag{2.70}$$

From (2.67) we have

$$-\alpha(a - d) = -4(a - d)$$

from which along with the assumption $a \neq d$, it follows that $\alpha = 4$. By using it in (2.67) we get $p = 0$.

From (2.68) we have

$$-\beta(a - d) = 6(a - d)(a + d) \tag{2.71}$$

and

$$2q = 6(a^2 + d^2) + (a + d)\beta. \tag{2.72}$$

From (2.71) along with the assumption $a \neq d$, it follows that $\beta = -6(a + d)$. By using it in (2.72) we get $q = -6ad$.

From (2.69) we have

$$-\gamma(a - d) = -4(a - d)(a^2 + ad + d^2) \tag{2.73}$$

and

$$2r = -4(a^3 + d^3) + (a + d)\gamma. \tag{2.74}$$

From (2.73) along with the assumption $a \neq d$, it follows that $\gamma = 4(a^2 + ad + d^2)$. By using it in (2.74) we get $r = 4ad(a + d)$.

From (2.70) we have

$$-\delta(a - d) = (a - d)(a^3 + a^2d + ad^2 + d^3) \tag{2.75}$$

and

$$2s = a^4 + d^4 + (a + d)\delta. \tag{2.76}$$

From (2.75) along with the assumption $a \neq d$, it follows that $\delta = -(a^3 + a^2d + ad^2 + d^3)$. By using it in (2.76) we get $s = -ad(a^2 + ad + d^2)$.

From the analysis we obtain the following result.

Theorem 2.20. *The equation*

$$x_{n+1} = \frac{x_n^4 - 6adx_n^2 + 4ad(a + d)x_n - ad(a^2 + ad + d^2)}{4x_n^3 - 6(a + d)x_n^2 + 4(a^2 + ad + d^2)x_n - (a^3 + a^2d + ad^2 + d^3)}, \tag{2.77}$$

for $n \in \mathbb{N}_0$, where $a, d \in \mathbb{C}$ are such that $a \neq d$ is solvable in closed form.

Proof. From (2.77) and some simple calculation we have

$$x_{n+1} - a = \frac{(x_n - a)^4}{4x_n^3 - 6(a + d)x_n^2 + 4(a^2 + ad + d^2)x_n - (a^3 + a^2d + ad^2 + d^3)}, \tag{2.78}$$

for $n \in \mathbb{N}_0$, and

$$x_{n+1} - d = \frac{(x_n - d)^4}{4x_n^3 - 6(a + d)x_n^2 + 4(a^2 + ad + d^2)x_n - (a^3 + a^2d + ad^2 + d^3)}, \tag{2.79}$$

for $n \in \mathbb{N}_0$.

Using the relations in (2.78) and (2.79) we have

$$\frac{x_{n+1} - a}{x_{n+1} - d} = \left(\frac{x_n - a}{x_n - d}\right)^4, \quad n \in \mathbb{N}_0.$$

Hence

$$\frac{x_n - a}{x_n - d} = \left(\frac{x_0 - a}{x_0 - d}\right)^{4^n}, \quad n \in \mathbb{N}_0,$$

and consequently

$$x_n = \frac{d\left(\frac{x_0 - a}{x_0 - d}\right)^{4^n} - a}{\left(\frac{x_0 - a}{x_0 - d}\right)^{4^n} - 1}, \quad n \in \mathbb{N}_0, \tag{2.80}$$

which is the general solution to equation (2.77). $\qquad\square$

Remark 2.21. Note that if $d = -a$ equation (2.77) reduces to the equation (2.63) where a is replaced by a^2, from which its solvability again follows.

2.8   Difference equations obtained by polynomials of the fifth degree

In [56] we have also shown that the difference equation

$$x_{n+1} = \frac{x_n^5 + 10ax_n^3 + 5a^2x_n}{5x_n^4 + 10ax_n^2 + a^2}, \quad n \in \mathbb{N}_0, \tag{2.81}$$

is solvable in closed form.

Our aim is to find a generalization of equation (2.81) similar to equation (2.47), which is also solvable in closed form. The following equation

$$x_{n+1} = \frac{x_n^5 + px_n^4 + qx_n^3 + rx_n^2 + sx_n + u}{\alpha x_n^4 + \beta x_n^3 + \gamma x_n^2 + \delta x_n + \eta}, \quad n \in \mathbb{N}_0, \tag{2.82}$$

is a natural generalization of (2.81).

We find the values of parameters p, q, r, s, u, α, β, γ, δ and η such that the following identities hold

$$t^5 + pt^4 + qt^3 + rt^2 + st + u - a(\alpha t^4 + \beta t^3 + \gamma t^2 + \delta t + \eta) = (t - a)^5 \tag{2.83}$$

and

$$t^5 + pt^4 + qt^3 + rt^2 + st + u - d(\alpha t^4 + \beta t^3 + \gamma t^2 + \delta t + \eta) = (t - d)^5 \tag{2.84}$$

for some given numbers a and d such that $a \neq d$.

From (2.83) and (2.84) we have

$$
\begin{aligned}
p - \alpha a &= -5a, & p - \alpha d &= -5d, & (2.85) \\
q - \beta a &= 10a^2, & q - \beta d &= 10d^2, & (2.86) \\
r - \gamma a &= -10a^3, & r - \gamma d &= -10d^3, & (2.87) \\
s - \delta a &= 5a^4, & s - \delta d &= 5d^4, & (2.88) \\
u - \eta a &= -a^5, & u - \eta d &= -d^5. & (2.89)
\end{aligned}
$$

From (2.85) it follows that

$$-\alpha(a - d) = -5(a - d)$$

from which along with the assumption $a \neq d$, it follows that $\alpha = 5$. From this and (2.85) we get $p = 0$.

From (2.86) we have

$$-\beta(a - d) = 10(a - d)(a + d) \tag{2.90}$$

and

$$2q = 10(a^2 + d^2) + (a + d)\beta. \tag{2.91}$$

From (2.90) along with the assumption $a \neq d$, it follows that $\beta = -10(a + d)$. By using it in (2.91) we get $q = -10ad$.

From (2.87) we have

$$-\gamma(a - d) = -10(a - d)(a^2 + ad + d^2) \tag{2.92}$$

and

$$2r = -10(a^3 + d^3) + (a + d)\gamma. \tag{2.93}$$

From (2.92) along with the assumption $a \neq d$, it follows that $\gamma = 10(a^2 + ad + d^2)$. By using it in (2.93) we get $r = 10ad(a + d)$.

From (2.88) we have

$$-\delta(a - d) = 5(a - d)(a^3 + a^2d + ad^2 + d^3) \tag{2.94}$$

and

$$2s = 5a^4 + 5d^4 + (a + d)\delta. \tag{2.95}$$

From (2.94) along with the assumption $a \neq d$, it follows that $\delta = -5(a^3 + a^2d + ad^2 + d^3)$. By using it in (2.95) we get $s = -5ad(a^2 + ad + d^2)$.

From (2.89) we have

$$-\eta(a - d) = -(a - d)(a^4 + a^3d + a^2d^2 + ad^3 + d^4) \tag{2.96}$$

and

$$2u = -(a^5 + d^5) + (a + d)\delta. \tag{2.97}$$

From (2.96) along with the assumption $a \neq d$, it follows that $\eta = a^4 + a^3d + a^2d^2 + ad^3 + d^4$. By using it in (2.97) we get $u = ad(a^3 + a^2d + ad^2 + d^3)$.

From the analysis we obtain the following result.

Theorem 2.22. *Let*

$$p_4(t) = 5t^4 - 10(a+d)t^3 + 10(a^2+ad+d^2)t^2 - 5(a^3+a^2d+ad^2+d^3)t + a^4+a^3d+a^2d^2+ad^3+d^4.$$

Then the equation

$$x_{n+1} = \frac{x_n^5 - 10adx_n^3 + 10ad(a+d)x_n^2 - 5ad(a^2+ad+d^2)x_n + ad(a^3+a^2d+ad^2+d^3)}{p_4(x_n)}, \tag{2.98}$$

for $n \in \mathbb{N}_0$, where $a, d \in \mathbb{C}$ are such that $a \neq d$, is solvable in closed form.

Proof. From (2.98) we have

$$x_{n+1} - a = \frac{(x_n - a)^5}{p_4(x_n)}, \tag{2.99}$$

for $n \in \mathbb{N}_0$, and

$$x_{n+1} - d = \frac{(x_n - d)^5}{p_4(x_n)}, \tag{2.100}$$

for $n \in \mathbb{N}_0$.

Employing (2.99) and (2.100) it follows that

$$\frac{x_{n+1} - a}{x_{n+1} - d} = \left(\frac{x_n - a}{x_n - d}\right)^5, \quad n \in \mathbb{N}_0.$$

Hence

$$\frac{x_n - a}{x_n - d} = \left(\frac{x_0 - a}{x_0 - d}\right)^{5^n}, \quad n \in \mathbb{N}_0,$$

and consequently

$$x_n = \frac{d\left(\frac{x_0-a}{x_0-d}\right)^{5^n} - a}{\left(\frac{x_0-a}{x_0-d}\right)^{5^n} - 1}, \quad n \in \mathbb{N}_0, \tag{2.101}$$

finishing the proof. $\square$

Remark 2.23. Note that if $d = -a$ equation (2.98) reduces to the equation (2.81) where a is replaced by a^2, implying its solvability.

2.9   A generalization of equations (2.63) and (2.81)

A natural question is if above theorems can be generalized to a more general difference equation. Although, at the first sight, the problem looks technically quite complex, it is interesting that the method used in the proofs of the above theorems can be also employed for finding the corresponding class of difference equations solvable in closed form, which are of the form

$$x_{n+1} = \frac{x_n^k + a_1 x_n^{k-1} + \cdots + a_{k-1} x_n + a_k}{b_0 x_n^{k-1} + b_1 x_n^{k-2} + \cdots + b_{k-2} x_n + b_{k-1}}, \quad n \in \mathbb{N}_0, \tag{2.102}$$

where $k \in \mathbb{N}$ and the coefficients

$$a_j, \ j = \overline{1,k}, \quad \text{and} \quad b_l, \ l = \overline{0,k-1}, \tag{2.103}$$

are complex numbers.

We want to find the values of the coefficients in (2.103) such that the following identities hold

$$t^k + a_1 t^{k-1} + \cdots + a_j t^{k-j} + \cdots + a_{k-1} t + a_k \\ - a(b_0 t^{k-1} + b_1 t^{k-2} + \cdots + b_{j-1} t^{k-j} + \cdots + b_{k-2} t + b_{k-1}) = (t-a)^k \tag{2.104}$$

and

$$t^k + a_1 t^{k-1} + \cdots + a_j t^{k-j} + \cdots + a_{k-1} t + a_k \\ - d(b_0 t^{k-1} + b_1 t^{k-2} + \cdots + b_{j-1} t^{k-j} + \cdots + b_{k-2} t + b_{k-1}) = (t-d)^k \tag{2.105}$$

for some given numbers a and d such that $a \neq d$.

From (2.104) and (2.105) we obtain the following nonlinear algebraic system of equations

$$\begin{aligned} a_1 - ab_0 &= C_1^k(-a), & a_1 - db_0 &= C_1^k(-d) \\ &\vdots & &\vdots \\ a_j - ab_{j-1} &= C_j^k(-a)^j, & a_j - db_{j-1} &= C_j^k(-d)^j, \\ &\vdots & &\vdots \\ a_k - ab_{k-1} &= C_k^k(-a)^k, & a_k - db_{k-1} &= C_k^k(-d)^k. \end{aligned} \tag{2.106}$$

From (2.106) we have

$$-(a-d)b_{j-1} = C_j^k((-a)^j - (-d)^j) \tag{2.107}$$

and

$$2a_j = C_j^k(-1)^j(a^j + d^j) + (a+d)b_{j-1}, \tag{2.108}$$

$j = \overline{1,k}$.

From (2.107) and since $a \neq d$ we obtain

$$b_{j-1} = C_j^k(-1)^{j+1}\frac{a^j - d^j}{a - d}, \quad j = \overline{1,k}. \tag{2.109}$$

By using (2.109) in (2.108) we have

$$2a_j = C_j^k(-1)^j(a^j + d^j) + (a+d)C_j^k(-1)^{j+1}\frac{a^j - d^j}{a - d},$$

$j = \overline{1,k}$, from which it follows that

$$a_j = adC_j^k(-1)^{j+1}\frac{a^{j-1} - d^{j-1}}{a - d}, \quad j = \overline{1,k}. \tag{2.110}$$

Remark 2.24. Note that from (2.110) with $j = 1$ it follows that $a_1 = 0$, whereas from (2.109) with $j = 1$ it follows that $b_0 = C_1^k = k$. Further, from (2.109) and (2.110) it follows that

$$a_j = adb_{j-1}\frac{a^{j-1} - d^{j-1}}{a^j - d^j}, \quad j = \overline{2,k}.$$

Now we formulate and prove the general result.

Theorem 2.25. *Let equation* (2.102) *be such that the coefficients* a_j, $j = \overline{1,k}$, *and* b_l, $l = \overline{0,k-1}$, *are given by* (2.109) *and* (2.110), *where* $a, d \in \mathbb{C}$ *are such that* $a \neq d$. *Then the equation is solvable in closed form.*

Proof. Let

$$p_{k-1}(t) = b_0 t^{k-1} + b_1 t^{k-2} + \cdots + b_{k-2}t + b_{k-1}.$$

Then from (2.102) and the choice of the coefficients a_j, $j = \overline{1,k}$, and b_l, $l = \overline{0,k-1}$ (see (2.104) and (2.105)), we have

$$x_{n+1} - a = \frac{(x_n - a)^k}{p_{k-1}(x_n)}, \tag{2.111}$$

for $n \in \mathbb{N}_0$, and

$$x_{n+1} - d = \frac{(x_n - d)^k}{p_{k-1}(x_n)}, \tag{2.112}$$

for $n \in \mathbb{N}_0$.

From (2.111) and (2.112) we have

$$\frac{x_{n+1} - a}{x_{n+1} - d} = \left(\frac{x_n - a}{x_n - d}\right)^k, \quad n \in \mathbb{N}_0.$$

Hence

$$\frac{x_n - a}{x_n - d} = \left(\frac{x_0 - a}{x_0 - d}\right)^{k^n}, \quad n \in \mathbb{N}_0,$$

and finally

$$x_n = \frac{d\left(\frac{x_0-a}{x_0-d}\right)^{k^n} - a}{\left(\frac{x_0-a}{x_0-d}\right)^{k^n} - 1}, \quad n \in \mathbb{N}_0,$$

as claimed. $\square$

References

[1] D. Adamović, Solution to problem 194, *Mat. Vesnik* **23**(1971), 236–242.

[2] M. Altman, An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space, *Pacific J. Math.* **10**(1960), 1107–1113. https://doi.org/10.2140/pjm.1960.10.1107

[3] I. Bajo, E. Liz, Global behaviour of a second-order nonlinear difference equation, *J. Difference Equ. Appl.* **17**(2011), No. 10, 1471–1486. https://doi.org/10.1080/10236191003639475

[4] M. I. Bashmakov, B. M. Bekker, V. M. Gol'hovoi, *Zadachi po matematike. Algebra i analiz* (in Russian), Nauka, Moskva, 1982.

[5] K. Berenhaut, J. Foley, S. Stević, Boundedness character of positive solutions of a max difference equation, *J. Difference Equ. Appl.* **12**(2006), No. 12, 1193–1199. https://doi.org/10.1080/10236190600949766

[6] K. Berenhaut, J. Foley, S. Stević, The global attractivity of the rational difference equation $y_n = 1 + (y_{n-k}/y_{n-m})$, *Proc. Amer. Math. Soc.* **135**(2007), No. 4, 1133–1140. https://doi.org/10.1090/S0002-9939-06-08580-7

[7] K. Berenhaut, S. Stević, The behaviour of the positive solutions of the difference equation $x_n = A + (x_{n-2}/x_{n-1})^p$, *J. Difference Equ. Appl.* **12**(2006), No. 9, 909–918. https://doi.org/10.1080/10236190600836377

[8] L. Berg, S. Stević, On the asymptotics of the difference equation $y_n(1 + y_{n-1} \cdots y_{n-k+1}) = y_{n-k}$, *J. Difference Equ. Appl.* **17**(2011), No. 4, 577–586. https://doi.org/10.1080/10236190903203820

[9] D. Bernoulli, Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur (in Latin), *Commentarii Acad. Petropol. III,* 1728 (1732), 85–100.

[10] G. Boole, *A treatise on the calculus of finite differences*, Third Edition, Macmillan and Co., London, 1880.

[11] B. P. Demidovich, *Sbornik zadach i uprazhneniy po matematicheskomu analizu* (in Russian), 8th Edition, Nauka, Moskva, 1972.

[12] B. P. Demidovich, I. A. Maron, *Computational mathematics*, Mir Publishers, Moscow, 1973.

[13] A. de Moivre, *Miscellanea analytica de seriebus et quadraturis* (in Latin), J. Tonson & J.Watts, Londini, 1730.

[14] A. de Moivre, *The doctrine of chances*, 3rd edition, Strand Publishing, London, 1756.

[15] L. Euler, *Introductio in analysin infinitorum, tomus primus* (in Latin), Lausannae, 1748.

[16] T. Fort, Finite differences and difference equations in the real domain, Oxford Univ. Press, London, 1948. Zbl 0030.11902

[17] J. F. W. Herschel, *A collection of examples of the applications of the calculus of finite differences*, Printed J. Smith, Cambridge, 1820.

[18] F. B. Hildebrand, *Introduction to numerical analysis*, 2nd Ed., Dover Publication Inc., Mineola, New York, 1987. Zbl 4044987

[19] C. Jordan, *Calculus of finite differences*, 3rd edition, Chelsea Publishing Company, New York, 1965. Zbl 3248272

[20] W. A. KOSMALA, *A friendly introduction to analysis*, 2nd Ed. Pearson, Upper Saddle River, New Jersey, 2004.

[21] W. A. KOSMALA, C. TEIXEIRA, More on the difference equation $y_{n+1} = (p + y_{n-1})/(qy_n + y_{n-1})$, *Appl. Anal.* **81**(2002), No. 1, 143–151. https://doi.org/10.1080/0003681021000021114

[22] V. A. KRECHMAR, *A problem book in algebra*, Mir Publishers, Moscow, 1974.

[23] S. F. LACROIX, *Traité des differénces et des séries* (in French), J. B. M. Duprat, Paris, 1800.

[24] S. F. LACROIX, *An elementary treatise on the differential and integral calculus, with an appendix and notes by J. Herschel*, J. Smith, Cambridge, 1816.

[25] J.-L. LAGRANGE, Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes (in French), *Miscellanea Taurinensia* **1**(1759), 33–42 (*Lagrange Œuvres* **1**(1867), 23–36).

[26] J.-L. LAGRANGE, *Œuvres*, Vol. 2, Gauthier-Villars, Paris, 1868.

[27] P. S. LAPLACE, Recherches sur l'intégration des équations différentielles aux différences finies et sur leur usage dans la téorie des hasards (in French), *Mémoires de l' Académie Royale des Sciences de Paris* 1773, **7**(1776) (Laplace Œuvres, **8**(1891), 69–197).

[28] P. S. LAPLACE, Sur le calcul des fonctions génératrices (in French), *Journal de l'École Polytechnique* **8**(1810), No. 15, 229–265.

[29] H. LEVY, F. LESSMAN, *Finite difference equations*, Dover Publications, New York, 1992. MR1217083

[30] A. A. MARKOFF, *Differenzenrechnung* (in German), Teubner, Leipzig, 1896.

[31] L. M. MILNE-THOMSON, *The calculus of finite differences*, MacMillan and Co., London, 1933.

[32] D. S. MITRINOVIĆ, *Matematička indukcija binomna formula kombinatorika* (in Serbian) [Mathematical induction binomial formula combinatorics], Gradjevinska Knjiga, Beograd, 1980.

[33] D. S. MITRINOVIĆ, D. D. ADAMOVIĆ, *Nizovi i redovi* (in Serbian) [Sequences and series], Naučna Knjiga, Beograd, Serbia, 1971.

[34] D. S. MITRINOVIĆ, J. D. KEČKIĆ, *Metodi izračunavanja konačnih zbirova* (in Serbian) [Methods for calculating finite sums], Naučna Knjiga, Beograd, 1984.

[35] N. E. NÖRLUND, *Vorlesungen über Differenzenrechnung* (in German), Berlin, Springer, 1924. https://doi.org/10.1007/978-3-642-50824-0

[36] G. PAPASCHINOPOULOS, C. J. SCHINAS, On the behavior of the solutions of a system of two nonlinear difference equations, *Comm. Appl. Nonlinear Anal.* **5**(1998), No. 2, 47–59. Zbl 1725641

[37] G. PAPASCHINOPOULOS, C. J. SCHINAS, Invariants for systems of two nonlinear difference equations, *Differential Equations Dynam. Systems* **7**(1999), 181–196. Zbl 1644224

[38] G. Papaschinopoulos, C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations, *Nonlinear Anal.* **46**(2001), 967–978. https://doi.org/10.1016/S0362-546X(00)00146-2

[39] G. Papaschinopoulos, C. J. Schinas, G. Stefanidou, On a *k*-order system of Lyness-type difference equations, *Adv. Difference Equ.* **2007**, Article ID 31272, 13 pp. https://doi.org/10.1155/2007/31272

[40] G. Papaschinopoulos, G. Stefanidou, Asymptotic behavior of the solutions of a class of rational difference equations, *Inter. J. Difference Equ.* **5**(2010), No. 2, 233–249. MR2771327

[41] W. V. Petryshyn, On the inversion of matrices and linear operators, *Proc. Amer. Math. Soc.* **16**(1965), 893–901. https://doi.org/10.2307/2035578

[42] M. H. Rhouma, The Fibonacci sequence modulo π, chaos and some rational recursive equations, *J. Math. Anal. Appl.* **310**(2005), 506–517. https://doi.org/10.1016/j.jmaa.2005.02.038

[43] J. Riordan, *Combinatorial identities*, John Wiley & Sons Inc., New York–London–Sydney, 1968. Zbl 0194.00502

[44] V. A. Sadovnichy, A. A. Grigorian, S. V. Konyagin, *Zadachi studencheskih matematicheskih olimpiad* (in Russian), Izdatel'stvo Moskovskogo Universiteta, 1987.

[45] V. A. Sadovnichy, A. S. Podkolzin, *Zadachi studencheskih olimpiad po matematike* (in Russian), Moskva, Nauka, 1978.

[46] C. Schinas, Invariants for difference equations and systems of difference equations of rational form, *J. Math. Anal. Appl.* **216**(1997), 164–179. https://doi.org/10.1006/jmaa.1997.5667

[47] C. Schinas, Invariants for some difference equations, *J. Math. Anal. Appl.* **212**(1997), 281–291. https://doi.org/10.1006/jmaa.1997.5499

[48] G. Schulz, Iterative Berechnung der reziproken Matrix, *Z. Angew. Math. Mech.* **13**(1933), 57–59. https://doi.org/10.1002/zamm.19330130111

[49] S. Stević, A global convergence results with applications to periodic solutions, *Indian J. Pure Appl. Math.* **33**(2002), No. 1, 45–53. Zbl 1724040

[50] S. Stević, On the recursive sequence $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$ II, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **10**(2003), No. 6, 911–916. Zbl 1051.39012

[51] S. Stević, A short proof of the Cushing–Henson conjecture, *Discrete Dyn. Nat. Soc.* Vol. **2006**, Article ID 37264, 5 pp. https://doi.org/10.1155/DDNS/2006/37264

[52] S. Stević, Boundedness character of a class of difference equations, *Nonlinear Anal.* **70**(2009), 839–848. https://doi.org/{10.1016/j.na.2008.01.014}

[53] S. Stević, On the system of difference equations $x_n = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$, *Appl. Math. Comput.* **219**(2013), 4755–4764. https://doi.org/10.1016/j.amc.2012.10.092

[54] S. Stević, Solvability of a general class of two-dimensional hyperbolic-cotangent-type systems of difference equations, *Adv. Difference Equ.* **2019**, Article No. 294, 34 pp. https://doi.org/10.1186/s13662-019-2233-y

[55] S. Stević, On a class of solvable difference equations generalizing an iteration process for calculating reciprocals, *Adv. Difference Equ.* **2021**, Article No. 205, 14 pp. https://doi.org/10.1186/s13662-021-03366-0

[56] S. Stević, On a class of recursive relations for calculating square roots of numbers, *J. Appl. Math. Comput.*, in press. https://doi.org/10.1007/s12190-022-01780-5

[57] S. Stević, J. Diblik, B. Iričanin, Z. Šmarda, On a solvable system of rational difference equations, *J. Difference Equ. Appl.* **20**(2014), No. 5–6, 811–825. https://doi.org/10.1080/10236198.2013.817573

[58] S. Stević, B. Iričanin, Z. Šmarda, Solvability of a close to symmetric system of difference equations, *Electron. J. Differential Equations* **2016**, Article No. 159, 1–13. Zbl 1344.39004

[59] N. N. Vorobiev, *Fibonacci numbers*, Birkhäuser, Basel, 2002. https://doi.org/10.1007/978-3-0348-8107-4