Global bifurcation of positive solutions for a class of superlinear elliptic systems

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Abstract. We are concerned with the global bifurcation of positive solutions for semi-linear elliptic systems of the form

$$
\begin{cases}
-\Delta u = \lambda f(u,v) & \text{in } \Omega, \\
-\Delta v = \lambda g(u,v) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter, $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with smooth boundary $\partial\Omega$. We establish the existence of an unbounded branch of positive solutions, emanating from the origin, which is bounded in positive λ-direction. The nonlinearities f, $g \in C^1(\mathbb{R} \times \mathbb{R}, (0, \infty))$ are nondecreasing for each variable and have superlinear growth at infinity. The proof of our main result is based upon bifurcation theory. In addition, as an application for our main result, when f and g subject to the upper growth bound, by a technique of taking superior limit for components, then we may show that the branch must bifurcate from infinity at $\lambda = 0$.

Keywords: elliptic systems, positive solutions, superlinear growth, bifurcation.

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1 Introduction

Let B be the unit ball in $\mathbb{R}^N$. D. D. Joseph and T. S. Lundgren [12] considered

$$
\begin{cases}
-\Delta u = \lambda e^u, & x \in B, \\
u = 0, & x \in \partial B
\end{cases}
\tag{1.1}
$$

and found a very interesting phenomenon that the behaviour of the connected component of positive solutions of (1.1) heavily depends on the dimension N, see Figure 1.1 below.

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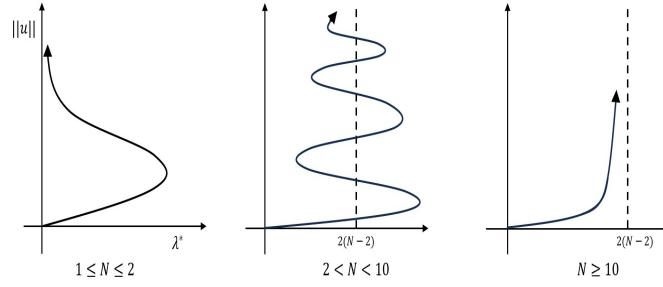
Figure 1.1: Global continua for (1.1) depend on N

Fourth order analogue of (1.1), a biharmonic elliptic problem

$$\begin{cases} \Delta^2 u = \lambda e^u, & x \in B, \\ u = |\nabla u| = 0, & x \in \partial B \end{cases} \tag{1.2}$$

has been extensively studied by several authors, see G. Arioli, F. Gazzola, H.-C. Grunau, E. Mitidieri [2] and A. Ferrero, H.-C. Grunau [8] and the references therein.

For elliptic systems, Ph. Clément, D. G. de Figueiredo and E. Mitidieri [7] investigated the existence of positive solution of a Dirichlet problem for

$$-\Delta u = f(v), \qquad -\Delta v = f(u) \tag{1.3}$$

in a bounded convex domain Ω of $\mathbb{R}^N$ with smooth boundary. Furthermore, the authors considered the existence of nontrivial solutions for the biharmonic equation subject to Navier boundary conditions. Namely

$$\Delta^2 u = g(u) \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial \Omega.$$

This problem is a special case of (1.3) when $f(v) = v$.

Recently, M. Chhetri, P. Girg [6] considered the elliptic system

$$\begin{cases} -\Delta u = \lambda \hat{f}(v) & \text{in } \Omega, \\ -\Delta v = \lambda \hat{g}(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial \Omega, \end{cases} \tag{1.4}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter and $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with $C^{2,\eta}$-boundary $\partial \Omega$ for some $\eta \in (0,1)$. The nonlinearities $\hat{f}$, $\hat{g} : \mathbb{R} \to (0, +\infty)$ are nondecreasing continuous functions and have superlinear growth at infinity, i.e.

$$\lim_{s \to +\infty} \frac{\hat{f}(s)}{s} = +\infty = \lim_{s \to +\infty} \frac{\hat{g}(s)}{s}.$$

Then the authors established the global structure of positive solutions for system (1.4).

Of course the natural question is whether or not we may show the global structure of positive solutions for the more general system

$$\begin{cases} -\Delta u = \lambda f(u,v) & \text{in } \Omega, \\ -\Delta v = \lambda g(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial \Omega, \end{cases} \tag{1.5}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter. We make the following assumptions throughout the paper.

(H1) f, $g \in C^1(\mathbb{R} \times \mathbb{R}, (0, +\infty))$ are nondecreasing for each variable and there exists a $\tau > 0$, satisfy

$$\min\left\{\frac{\partial f}{\partial s}(s,t), \frac{\partial f}{\partial t}(s,t)\right\} > \frac{f(s,t)}{s+t} \quad \text{for } (s,t) \in \mathbb{R}^2 \backslash B_\tau,$$

$$\min\left\{\frac{\partial g}{\partial s}(s,t), \frac{\partial g}{\partial t}(s,t)\right\} > \frac{g(s,t)}{s+t} \quad \text{for } (s,t) \in \mathbb{R}^2 \backslash B_\tau,$$

where $B_\tau := \{(s,t) \in \mathbb{R}^2 : |s|^2 + |t|^2 \leq \tau\}$;

(H2)
$$\lim_{s+t \to +\infty} \frac{f(s,t)}{s+t} = \lim_{s+t \to +\infty} \frac{g(s,t)}{s+t} = +\infty.$$

Notice that the functions satisfying (H1)–(H2) are easy to illustrate, for example $f(s,t) = (s+t)^5 + 1$, $g(s,t) = (s+t)^3 + 2$.

For system of equations with $\lambda = 1$, see [9–11] for $N = 2$ and [1,3,4] for $N \geq 3$, where existence results were discussed, but no any information about the *Connectivity Properties* of positive solution set are provided. In fact, these positive solutions of (1.5) may not lie on one bifurcating set.

It is the purpose of this paper to show the existence of a unbounded component of positive solutions of (1.5) by use bifurcation theory and a technique of taking *superior limit for components*, see [15,16].

In order to better state our result, we will briefly introduce the following notations that were defined in more detail in [6] and extend the use of these definitions to all systems throughout the paper.

Let
$$E := [W_0^{1,2}(\Omega) \cap W^{2,r}(\Omega)]^2 \quad \text{and} \quad X := [L^r(\Omega)]^2$$

be Banach spaces endowed with norms

$$\|(w_1, w_2)\|_E := \|w_1\|_{W^{2,r}(\Omega)} + \|w_2\|_{W^{2,r}(\Omega)} \quad \text{and} \quad \|(w_1, w_2)\|_X := \|w_1\|_{L^r(\Omega)} + \|w_2\|_{L^r(\Omega)},$$

respectively for $r > N$. By assumption $r > N$, $W^{2,r}(\Omega)$ is continuously imbedded into $C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$. Thus there exists $\xi_* > 0$ such that $\|\omega\|_{L^\infty(\Omega)} \leq \xi_* \|\omega\|_{W^{2,r}(\Omega)}$ holds for all $\omega \in W^{2,r}(\Omega)$. By a solution of (1.5) we mean $(\lambda, (u,v)) \in \mathbb{R} \times E$ which solves (1.5) in the strong sense, that is, $(u,v) \in W^{2,r}(\Omega) \times W^{2,r}(\Omega)$ and $(\lambda, (u,v))$ satisfies (1.5) almost everywhere in Ω. Now define $\mathcal{S} := \{(\lambda, (u,v)) \in \mathbb{R} \times E : (\lambda, (u,v)) \text{ solution of (1.5)}\}$.

Definition 1.1 ([6]).

(1) By a *continuum* of solutions of (1.5) we mean a subset of $\mathcal{S}$ which is closed and connected.

(2) By a *component* of solutions set $\mathcal{S}$ we mean a continuum which is maximal with respect to inclusion ordering.

(3) $\lambda_\infty \in \mathbb{R}$ is *a bifurcation point from infinity* if the solution set $\mathcal{S}$ contains a sequence $(\lambda_n, (u_n, v_n))$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n, v_n)\|_E \to +\infty$ as $n \to +\infty$.

The main result of the paper is the following.

Theorem 1.2. *Let (H1)–(H2) hold, then there exists an unbounded component $\mathcal{C} \subset \mathcal{S}$ satisfying the following:*

(a) *For any $\lambda \in (0, \lambda^*)$, $(\lambda, (u, v)) \in \mathcal{C}$ is positive, i.e. $u > 0$ and $v > 0$.*

(b) *If $\lambda = 0$, then $(0, (0, 0))$ is the unique element belonging to $\mathcal{C}$.*

(c) $\mathrm{Proj}_{\lambda \in [0, +\infty)} \mathcal{C} := \{\lambda \in [0, +\infty) : \exists\, (u, v) \in E \text{ with } (\lambda, (u, v)) \in \mathcal{C}\} \subset [0, \lambda^*)$.

(d) *There exists a sequence of positive solutions $\{(\lambda_n, (u_n, v_n))\} \in \mathcal{C}$ satisfying $\lambda_n \in (0, \lambda^*)$ for all $n \in \mathbb{N}$ and $\lim_{n \to +\infty} \|(u_n, v_n)\|_E \to +\infty$.*

Remark 1.3. In the special case Ω is convex, M. Chhetri, P. Girg [6] show that the only bifurcation point of positive solutions of (1.4) from infinity with $\hat{f}(v) = v^p$, $\hat{g}(u) = u^q$ at $\lambda = 0$ under the critical hyperbola condition

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}. \tag{1.6}$$

The proof of this result in [6] is deeply depend on the uniform priori bound. For system (1.4), condition (1.6) is optimal for obtaining the priori estimate when $\hat{f}(v) = v^p$, $\hat{g}(u) = u^q$. But for the more general system (1.5), the conditions for obtaining a priori bound are more complicated, and the proof will be more difficult if we want to obtain the similar results. We give the specific proof in Section 5.

The rest of paper is arranged as follows: In Section 2 we present the nonexistence result of (1.5). Section 3 is devoted to asymptotically positively homogeneous system by using a global continuation principle. In Section 4, we prove Theorem 1.2. In final section, as an application of Theorem 1.2, by applying some priori estimates, see [17], we attempt to understand the structure of the resulting continua of positive solutions.

2   Statement of the nonexistence result

Let $\mu_1 > 0$ be the principal eigenvalue of

$$\begin{cases} -\Delta \varphi = \mu \varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\varphi_1 \in W_0^{1,2}(\Omega)$ be the corresponding eigenfunction, then $\frac{\partial \varphi_1}{\partial \vec{n}} < 0$ on Ω, where $\vec{n}$ is the outward unit normal on $\partial\Omega$. Without loss of generality, we normalize the eigenfunction such that $\varphi_1 > 0$ in Ω.

We shall prove the nonexistence result.

Theorem 2.1. *Suppose there exist a_1, a_2, α_1, $\alpha_2 > 0$ such that*

$$f(s, t) > a_1(s + t) + \alpha_1, \qquad g(s, t) > a_2(s + t) + \alpha_2, \qquad \forall\, (s, t) \in \mathbb{R}^2. \tag{2.1}$$

Then for $\lambda \geq \lambda^* := \frac{\mu_1}{2a^*}$, there are no solutions for (1.5), where $a^* := \min\{a_1, a_2\}$.

Remark 2.2. We notice that in order to obtain the nonexistence result of solutions for (1.5), it suffices to show that f and g satisfy (2.1), which is weaker than (H2).

Proof. By (H1), since f, g are positive functions, all solutions $(\lambda, (u, v))$ of (1.5) with $\lambda > 0$ must satisfy $u, v > 0$ in Ω by the maximum principle. Let $(\lambda, (u, v))$ be a solution of (1.5) with $\lambda > 0$. Then

$$-\Delta(u + v) > \lambda(a_1(u + v) + \alpha_1) + \lambda(a_2(u + v) + \alpha_2)$$

$$> 2\lambda a^*(u + v) + 2\lambda \alpha^* = 2\lambda a^*\left(u + v + \frac{\alpha^*}{a^*}\right) \quad \text{in } \Omega,$$

where $a^* := \min\{a_1, a_2\}$, $\alpha^* := \min\{\alpha_1, \alpha_2\}$. Therefore, we have

$$-\Delta\left(u + v + \frac{\alpha^*}{a^*}\right) > 2\lambda a^*\left(u + v + \frac{\alpha^*}{a^*}\right) \quad \text{in } \Omega.$$

Denoting $w := u + v + \frac{\alpha^*}{a^*}$, we see that $w > 0$ on $\overline{\Omega}$ and

$$-\Delta w > 2\lambda a^* w \quad \text{in } \Omega.$$

Since $-\Delta\varphi_1 = \mu_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$. We have

$$\int_\Omega (\varphi_1 \Delta w - w \Delta\varphi_1)dx < \int_\Omega (-2\lambda a^* w\varphi_1 + w\mu_1\varphi_1)dx = (-2\lambda a^* + \mu_1)\int_\Omega (w\varphi_1)dx. \quad (2.2)$$

On the other hand, since $\varphi_1 = 0$ on $\partial\Omega$, $\inf_{\partial\Omega} w > 0$ and $\frac{\partial\varphi_1}{\partial\vec{n}} < 0$ on $\partial\Omega$, we get

$$\int_\Omega (\varphi_1 \Delta w - w\Delta\varphi_1)dx = \int_{\partial\Omega} (\varphi_1 \nabla w - w\nabla\varphi_1)\cdot \vec{n}\, dS = -\int_{\partial\Omega} w\nabla\varphi_1 \cdot \vec{n}\, dS. \quad (2.3)$$

Then we have

$$-\int_{\partial\Omega} w\nabla\varphi_1 \cdot \vec{n}\, dS \geq -\inf_{\partial\Omega} w \int_{\partial\Omega} \frac{\partial\varphi_1}{\partial\vec{n}}dS > 0. \quad (2.4)$$

It follows from (2.2), (2.3) and (2.4) that for (u, v) to be a solution of (1.5) for $\lambda > 0$, we must have $\lambda < \frac{\mu_1}{2a^*}$. Therefore, (1.5) has no solution for $\lambda \geq \lambda^* := \frac{\mu_1}{2a^*}$. □

3 Asymptotically positively homogeneous system

In order to discuss the auxiliary result, we mention some properties of the following eigenvalue problem

$$\begin{cases} -\Delta w_1 = \lambda[a_{11}(x)w_1 + a_{12}(x)w_2] & \text{in } \Omega, \\ -\Delta w_2 = \lambda[a_{21}(x)w_1 + a_{22}(x)w_2] & \text{in } \Omega, \\ w_1 = w_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where $a_{ij} : \overline{\Omega} \to (0, \infty)$ are continuous function(i, j = 1, 2). It follows from [14, Theorem 4.1] that (3.1) has exactly one positive principal eigenvalue λ_1 and associated eigenfunction (χ_1, ψ_1) is positive in Ω.

Remark 3.1. If a_{ij} = constant (i, j = 1, 2), the principal eigenvalue of (3.1) and the corresponding eigenfunction are related to μ_1 and φ_1 (μ_1 and φ_1 are defined in the Section 2.)

For example, let $a_{11} = 2$, $a_{12} = 9$, $a_{21} = 4$, $a_{22} = 2$, the principal eigenvalue of (3.1) is $\frac{\mu_1}{8}$ and the associated eigenfunction is $(\frac{3}{2}\varphi_1, \varphi_1)$. The detailed calculation method is shown in Appendix 2 of [5].

Now, let us consider an asymptotically positively homogeneous system

$$\begin{cases} -\Delta u = \lambda[a_{11}(x)u^+ + a_{12}(x)v^+] + \lambda\widetilde{f}(u,v) & \text{in } \Omega, \\ -\Delta v = \lambda[a_{21}(x)u^+ + a_{22}(x)v^+] + \lambda\widetilde{g}(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $s^+ := \max\{s, 0\}$, $a_{ij}(i, j = 1, 2)$ are as in (3.1) and $\lambda \in \mathbb{R}$ is the bifurcation parameter. $\widetilde{f}, \widetilde{g} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(F1) $\widetilde{f}, \widetilde{g}$ are continuous and bounded functions;

(F2) $a_{11}s^+ + a_{12}t^+ + \widetilde{f}(s,t) > 0$, $a_{21}s^+ + a_{22}t^+ + \widetilde{g}(s,t) > 0$ for all $(s,t) \in \mathbb{R} \times \mathbb{R}$.

By a solution of (3.2) we mean $(\lambda, (u,v)) \in \mathbb{R} \times E$ which solves (3.2) in the strong sense. Now let $\mathcal{T} := \{(\lambda, (u,v)) : (\lambda, (u,v)) \text{ solution of } (3.2)\}$. We shall prove the following bifurcation result.

Theorem 3.2. *Let (F1)–(F2) hold. Then λ_1 is the only bifurcation point from infinity for (3.2). Moreover, there exists a component $\mathcal{X} \subset \mathcal{T}$ bifurcating from infinity at λ_1 and satisfies:*

(i) for $\lambda > 0$ and $(\lambda, (u,v)) \in \mathcal{X}$, then $u > 0$ and $v > 0$;

(ii) for $\lambda = 0$, $(u,v) = (0,0)$ is the unique solution of (3.2) and $(0, (0,0)) \in \mathcal{X}$;

(iii) $\text{Proj}_\lambda \mathcal{X} := \{\lambda \in \mathbb{R} : \exists (u,v) \in E \text{ with } (\lambda, (u,v)) \in \mathcal{X}\}$ is bounded from above and unbounded from below.

To prove this theorem, we use a variant of Krasnoselskii's necessary condition for bifurcation from infinity (Lemma 3.4), Theorem 2.1 and the global continuation principle of Leray and Schauder (Lemma 3.3) below.

Lemma 3.3 ([19]). *Let Y be a Banach space with $Y \neq \{0\}$ and let $F : Y \to Y$ be compact. Then the solution component $\hat{\mathcal{C}} \subset \mathbb{R} \times Y$ of the equation*

$$x = \lambda F(x)$$

which contains $(0,0) \in \mathbb{R} \times Y$ is unbounded as are both subsets

$$\hat{\mathcal{C}}_\pm := \hat{\mathcal{C}} \cap (\mathbb{R}_\pm \times Y),$$

where $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0]$.

System (3.2) is equivalent to

$$(u,v) = \lambda L^+(u,v) + \lambda H(u,v), \tag{3.3}$$

where $L^+ : E \to E$ is defined by

$$(u,v) \mapsto (-\Delta)^{-1}(a_{11}(x)u^+ + a_{12}(x)v^+, a_{21}(x)u^+ + a_{22}(x)v^+)$$

and $H : E \to E$ is defined by

$$(u,v) \mapsto (-\Delta)^{-1}(\widetilde{f}(u,v), \widetilde{g}(u,v)).$$

Notice that L^+ is not linear but both L^+ and H are continuous and compact. Moreover, since $\widetilde{f}$ and $\widetilde{g}$ are bounded, H satisfies

$$\lim_{\|(u,v)\|_E \to +\infty} \frac{\|H(u,v)\|_E}{\|(u,v)\|_E} = 0. \tag{3.4}$$

For asymptotically linear problem, a necessary condition for bifurcation from infinity was established in [13]. Inspired by this work, we prove the following lemma to show that the unique possible bifurcation point from infinity for (3.3) is λ_1.

Lemma 3.4. *If λ_∞ is a bifurcation point from infinity for (3.3), then $\lambda_\infty = \lambda_1$. Moreover, for any sequence $(\lambda_j, (u_j, v_j)) \in \mathbb{R} \times E$ with $\lambda_j \to \lambda_1$ and $\|(u_j, v_j)\|_E \to +\infty$ as $j \to +\infty$. There exists a subsequence $(\lambda_{j_k}, (u_{j_k}, v_{j_k}))$ of $(\lambda_j, (u_j, v_j))$ such that*

$$\lim_{j_k \to +\infty} \frac{(u_{j_k}, v_{j_k})}{\|(u_{j_k}, v_{j_k})\|_E} = \frac{(\chi_1, \psi_1)}{\|(\chi_1, \psi_1)\|_E}, \tag{3.5}$$

where the convergence is in $C^{1,\eta}(\overline{\Omega}) \times C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$.

Proof. Now by the same argument in the proof of [6, Proposition 3.1], with obvious changes, we may deduce the desired results. Let $(\lambda_j, (u_j, v_j)) \in \mathbb{R} \times E$ be solutions of (3.2) such that $\|(u_j, v_j)\|_E \to +\infty$ and $\lambda_j \to \lambda_\infty$. Then $(\widehat{u}_j, \widehat{v}_j) = \frac{(u_j, v_j)}{\|(u_j, v_j)\|_E}$ satisfies

$$\widehat{u}_j = \lambda_j(-\Delta)^{-1}\left(a_{11}(x)\widehat{u}_j^+ + a_{12}(x)\widehat{v}_j^+ + \frac{\widetilde{f}(u_j, v_j)}{\|(u_j, v_j)\|_E}\right),$$

$$\widehat{v}_j = \lambda_j(-\Delta)^{-1}\left(a_{21}(x)\widehat{u}_j^+ + a_{22}(x)\widehat{v}_j^+ + \frac{\widetilde{g}(u_j, v_j)}{\|(u_j, v_j)\|_E}\right),$$

or equivalently satisfied

$$(\widehat{u}_j, \widehat{v}_j) = \lambda_j L^+(\widehat{u}_j, \widehat{v}_j) + \lambda_j \frac{H(u_j, v_j)}{\|(u_j, v_j)\|_E}.$$

It then follows from (3.4) that the right hand side is bounded in X (independent of j). Hence $\|\widehat{u}_j\|_{W^{2,r}(\Omega)}$ and $\|\widehat{v}_j\|_{W^{2,r}(\Omega)}$ are bounded (independent of j) and so are $\|\widehat{u}_j\|_{C^{1,\eta}(\overline{\Omega})}$ and $\|\widehat{v}_j\|_{C^{1,\eta}(\overline{\Omega})}$ for some $\eta \in (0,1)$. Since $C^{1,\eta}(\overline{\Omega}) \hookrightarrow C^{1,\eta'}(\overline{\Omega})$ compactly for $\eta' \in (0,\eta)$, passing to subsequences, $\widehat{u}_j \to \widehat{u}$, $\widehat{v}_j \to \widehat{v}$ in $C^{1,\eta'}(\overline{\Omega})$. Therefore, $(\lambda_\infty, (\widehat{u}, \widehat{v}))$ satisfies

$$-\Delta\widehat{u} = \lambda_\infty[a_{11}(x)\widehat{u}^+ + a_{12}(x)\widehat{v}^+] \quad \text{in } \Omega, \tag{3.6}$$

$$-\Delta\widehat{v} = \lambda_\infty[a_{21}(x)\widehat{u}^+ + a_{22}(x)\widehat{v}^+] \quad \text{in } \Omega, \tag{3.7}$$

$$\widehat{u} = \widehat{v} = 0 \quad \text{on } \partial\Omega.$$

Suppose $\lambda_\infty \leq 0$. Since $\widehat{u}^+ \geq 0$ and $\widehat{v}^+ \geq 0$, it follows by applying the maximum principle to (3.6) that $\widehat{u} \equiv 0$ and hence repeating the same argument using (3.7) we get $\widehat{v} \equiv 0$ as well. This leads to a contradiction since $\|(\widehat{u}, \widehat{v})\|_E = 1$.

For $\lambda_\infty > 0$, we distinguish two cases: $\widehat{v}^+ \equiv 0$ and $\widehat{v}^+ \not\equiv 0$. In the first case, if $\widehat{u}^+ \equiv 0$, from (3.6), using the maximum principle, a contradiction as before. If $\widehat{u}^+ \not\equiv 0$, then it follows from (3.7) and $a_{21} > 0$ that $\widehat{v} > 0$ in Ω. However, this contradicts $\widehat{v}^+ \equiv 0$.

In the case $\widehat{v}^+ \not\equiv 0$, we may get $\widehat{u}^+ \not\equiv 0$ in Ω from (3.6) by the maximum principle, which in turn implies $\widehat{u} > 0$ and $\widehat{v} > 0$ in Ω from (3.6) and (3.7) by the maximum principle again. Thus $\lambda_\infty > 0$ and $\widehat{u}$, $\widehat{v} > 0$ in Ω satisfy the linear eigenvalue problem (3.1). However, we already discussed that (3.1) has precisely one eigenvalue λ_1 with componentwise positive eigenfunction (χ_1, ψ_1). Therefore, it must be that $\lambda_\infty = \lambda_1$ and

$$(\widehat{u}, \widehat{v}) = \frac{(\chi_1, \psi_1)}{\|(\chi_1, \psi_1)\|_E}. \qquad \square$$

Now we will complete the proof of Theorem 3.2.

Proof. (3.3) satisfies the hypotheses of Lemma 3.3 with $F := L^+ + H$. Then there exist unbounded continua

$$\mathcal{X}_\pm \subset \widehat{\mathcal{T}} := \{(\lambda, (u,v)) \in \mathbb{R} \times E : (\lambda, (u,v)) \text{ is a solution of (3.2)}\}$$

containing $(0, (0,0))$. By Theorem 2.1,

$$\mathcal{X}_+ \subset ([0, \lambda^*) \times E)$$

and thus $\mathcal{X}_+$ must be unbounded in the Banach space E-direction. Then $\mathcal{X} := \mathcal{X}_+ \cup \mathcal{X}_-$ is a component containing $(0, (0,0))$. By Lemma 3.4, λ_1 is the only bifurcation point from infinity from (3.3) and $\mathcal{X}_+$ is unbounded in the E-direction, hence $\mathcal{X}_+$ must bifurcate from infinity at λ_1. By similar argument in [6], we will verify that $\mathcal{X}$ satisfies the properties (i)-(iii).

It follows from assumption (F2) that u, $v > 0$ in Ω whenever $(\lambda, (u,v)) \in \mathcal{X}$ and $\lambda > 0$. This implies part (i). For $\lambda = 0$, $(u,v) = (0,0)$ is the only solution of (3.2) and $(0, (0,0)) \in \mathcal{X}$. Hence part (ii) holds. Applying Lemma 3.3, we see that $\mathcal{X}_-$ must be unbounded in $\mathbb{R} \times E$. However, by part (ii) and the fact that λ_1 is the unique bifurcation point from infinity for (3.3), we see that $\mathcal{X}_-$ must be unbounded in the negative λ-direction. Hence $(-\infty, \lambda_1) \subset \text{Proj}_\lambda \mathcal{X}$. $\qquad \square$

4 Proof of main result

For $n \in \mathbb{N}$, let

$$\mathcal{A}_n^f := \{(s,t) \in \mathbb{R}_+^2 : f(s,t) = n(s+t)\},$$

where $\mathbb{R}_+ := [0, +\infty)$. We shall show that $\mathcal{A}_n^f$ contains a curve Γ_n^f which can be (globally) parametrized as the graph of a decreasing function.

First, we state and prove several preliminary results.

Lemma 4.1. *There exists $n_0 > 0$ such that for any $n > n_0$, $\mathcal{A}_n^f \neq \emptyset$.*

Proof. It follows from (H1) and (H2) that there exists $n_0 > 0$ such that for any $n > n_0$, there exists $(s_n^*, s_n^*) \in \mathbb{R}_+^2$ with $f(s_n^*, s_n^*) = 2ns_n^*$. Consequently, $\mathcal{A}_n^f \neq \emptyset$. $\qquad \square$

For given $\theta \in [0, +\infty)$, let

$$t = \theta s, \quad s \in [0, +\infty).$$

Obviously, (H1) and (H2) imply that $\lim\limits_{s \to \infty} \frac{f(s, \theta s)}{s + \theta s} = \infty$. For any $n > n_0$, denote

$$\mathcal{A}_{n,\theta}^f := \left\{(s, \theta s) : (s, \theta s) \in \mathcal{A}_n^f\right\}.$$

Fix $n > n_0$, analogous to proof of Lemma 4.1, it is easy to see $\mathcal{A}_{n,\theta}^f \neq \emptyset$.

Lemma 4.2. *Fix $n > n_0$, there exists $M_n > 0$ independent of $\theta \in [0, +\infty)$ such that*

$$\sup \left\{ s + \theta s : (s, \theta s) \in \mathcal{A}_n^f \right\} \leq M_n.$$

Proof. Suppose on the contrary that there exists a sequence $\{(s_k, \theta_k s_k)\} \in \mathcal{A}_n^f$ such that $\lim_{k \to \infty}(s_k + \theta_k s_k) = \infty$. Then it follows from (H2) that

$$\lim_{(s_k + \theta_k s_k) \to +\infty} \frac{f(s_k, \theta_k s_k)}{s_k + \theta_k s_k} = +\infty.$$

This contradicts the fact that $f(s_k, \theta_k s_k) = n(s_k + \theta_k s_k)$. $\qquad\qquad\square$

Fix $\theta \in [0, +\infty)$, define

$$\gamma_n(\theta) := \max \left\{ s \in \mathbb{R}_+ : (s, \theta s) \in \mathcal{A}_{n,\theta}^f \right\}.$$

Lemma 4.3. *For any $M > 0$ and $\theta \in [0, +\infty)$, there exists $n_1 > n_0 > 0$ such that $(\gamma_n(\theta))^2 + (\theta \gamma_n(\theta))^2 > M$ for any $n > n_1$.*

Proof. Suppose on the contrary that there exists a sequence $\{(\theta_n, \gamma_n(\theta_n))\}$ such that $(\gamma_n(\theta_n))^2 + (\theta_n \gamma_n(\theta_n))^2$ is bounded for any $n > n_1$. After taking a subsequence if necessary, we have

$$(\theta_n, \gamma_n(\theta_n)) \to (\theta_*, \gamma_*) \quad \text{as } n \to \infty$$

in $\mathbb{R}_+^2$. Since $(\gamma_n(\theta_n), \theta_n \gamma_n(\theta_n)) \in \mathcal{A}_n^f$, then

$$f(\gamma_n(\theta_n), \theta_n \gamma_n(\theta_n)) = n(\gamma_n(\theta_n) + \theta_n \gamma_n(\theta_n)).$$

It is easy to verify that

$$f(\gamma_*, \theta_* \gamma_*) = n(\gamma_* + \theta_* \gamma_*) \to \infty \quad \text{as } n \to \infty,$$

this contradicts the fact that $f(\gamma_*, \theta_* \gamma_*)$ is bounded since $\theta_* \in [0, +\infty)$. $\qquad\square$

For $(s, t) \in \mathbb{R}_+^2$, denote
$$F(s, t) := f(s, t) - n(s + t).$$

Let
$$\widetilde{s} := \sup\{s \in \mathbb{R}_+ : (s, 0) \in \mathcal{A}_{n,0}^f\}.$$

Then Lemma 4.2 implies $\widetilde{s} < \infty$.

Lemma 4.4. *There exists $n_2 > n_1 > 0$ such that for $n > n_2$, there exists a decreasing function $t = \Gamma_n^f(s)$ for $s \in (0, \widetilde{s})$, which joins the point $(\widetilde{s}, 0)$ to a point $(0, \widehat{t})$ for some $\widehat{t} < \infty$.*

Proof. For given $\theta \in (0, +\infty)$, we know that $(\gamma_n(\theta), \theta \gamma_n(\theta)) \in \mathcal{A}_n^f$. By (H1), we have

$$\min \left\{ f_t(\gamma_n(\theta), \theta \gamma_n(\theta)), f_s(\gamma_n(\theta), \theta \gamma_n(\theta)) \right\} > \frac{f(\gamma_n(\theta), \theta \gamma_n(\theta))}{\gamma_n(\theta) + \theta \gamma_n(\theta)}$$

if n sufficiently large. Therefore,

$$
\begin{aligned}
F_t(\gamma_n(\theta), \theta\gamma_n(\theta)) &= f_t(\gamma_n(\theta), \theta\gamma_n(\theta)) - n \\
&> \frac{f(\gamma_n(\theta), \theta\gamma_n(\theta))}{\gamma_n(\theta) + \theta\gamma_n(\theta)} - n \\
&= \frac{n(\gamma_n(\theta) + \theta\gamma_n(\theta))}{\gamma_n(\theta) + \theta\gamma_n(\theta)} - n \\
&= 0
\end{aligned}
$$

for sufficiently large n. By similar argument, we can obtain $F_s(\gamma_n(\theta), \theta\gamma_n(\theta)) > 0$ for sufficiently large n. Consequently, applying the implicit function existence theorem, there exists a unique curve $t = \Gamma_n^f(s)$ in $(\gamma_n(\theta) - \delta, \gamma_n(\theta) + \delta)$ for sufficiently small $\delta > 0$, and

$$
(\Gamma_n^f(s))' = -\frac{F_s(s,t)}{F_t(s,t)} = -\frac{f_s(s,t) - n}{f_t(s,t) - n} < 0
$$

for sufficiently large n. Thus there exists $n_2 > n_1 > 0$ such that for $n > n_2$, $\Gamma_n^f(s)$ is a decreasing function for $s \in (\gamma_n(\theta) - \delta, \gamma_n(\theta) + \delta)$. By the standard extension method, we may get a decreasing function $\Gamma_n^f(\cdot)$ defined on $(0, \widetilde{s})$.

By Lemma 4.2, we have $\{(\gamma_n(\theta), \theta\gamma_n(\theta)) : \theta \in (0, +\infty)\} \subset \mathcal{A}_n^f$ is bounded. This together with the fact that $\Gamma_n^f(s)$ is decreasing, we can deduce that $\lim_{\theta \to 0+}(\gamma_n(\theta), \theta\gamma_n(\theta)) = (\widehat{s}, 0)$ for $\widehat{s} \in (0, +\infty)$ and $\lim_{\theta \to +\infty}(\gamma_n(\theta), \theta\gamma_n(\theta)) = (0, \widehat{t})$ for some $\widehat{t} < \infty$. Obviously, we have $\widehat{s} = \widetilde{s}$ by the definition of $\widetilde{s}$. □

It is easy to see that there exists n_*, n^* such that for all $n \geq n_*$, Γ_n^f divide $\mathbb{R}_+^2$ into two parts

$$
\mathbb{R}_+^2 = \Omega_n^f \cup \Gamma_n^f \cup U_n^f, \quad \Omega_n^f \cap U_n^f = \varnothing,
$$

and for all $n \geq n^*$, Γ_n^g divide $\mathbb{R}_+^2$ into two parts

$$
\mathbb{R}_+^2 = \Omega_n^g \cup \Gamma_n^g \cup U_n^g, \quad \Omega_n^g \cap U_n^g = \varnothing,
$$

where Ω_n^f, Ω_n^g are bounded, and U_n^f, U_n^g are unbounded.

4.1 Approximation problems

Fix $n \in \mathbb{N}$ and define $f_n(t,s)$, $g_n(t,s) : \mathbb{R}^2 \to (0, \infty)$ by

$$
f_n(s,t) = \begin{cases} f(s,t), & (s,t) \in \Omega_n^f, \\ n(s+t), & (s,t) \in U_n^f \cup \Gamma_n^f, \end{cases}
$$

$$
g_n(s,t) = \begin{cases} g(s,t), & (s,t) \in \Omega_n^g, \\ n(s+t), & (s,t) \in U_n^g \cup \Gamma_n^g. \end{cases}
$$

Then f_n and g_n are continuous functions on $\mathbb{R}^2$.

For each $n \in \mathbb{N}$, we consider the following problem

$$
\begin{cases} -\Delta u = \lambda f_n(u,v) & \text{in } \Omega, \\ -\Delta v = \lambda g_n(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}
$$

which approaches (1.5) as $n \to \infty$. We will use Theorem 3.2 to treat (4.1) and thus we rewrite (4.1) in the form of (3.2) as

$$\begin{cases} -\Delta u = \lambda[nu^+ + nv^+] + \lambda \widetilde{f}_n(u,v) & \text{in } \Omega, \\ -\Delta v = \lambda[nu^+ + nv^+] + \lambda \widetilde{g}_n(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (4.2)$$

where

$$\widetilde{f}_n(s,t) := f_n(s,t) - ns^+ - nt^+, \qquad \widetilde{g}_n(s,t) := g_n(s,t) - ns^+ - nt^+.$$

We note that $\widetilde{f}_n$ and $\widetilde{g}_n$ are bounded in $\mathbb{R}^2$. Indeed, since f_n is nondecreasing for each variable and $f_n(s,t) = f(s,t) > 0$ for $(s,t) \in \overline{\Omega}_n^f$, we get

$$|\widetilde{f}_n(s,t)| \leq \sup_{(s,t)\in\mathbb{R}^2} |f_n(s,t) - n(s^+ + t^+)| \leq \sup_{(s,t)\in\Omega_n^f} |f_n(s,t) - n(s^+ + t^+)| + f(0,0) = \text{constant},$$

where the constant is independent of s, t and depends on n. We can repeat the same argument for $\widetilde{g}_n$. Since $f_n(s,t)$, $g_n(s,t) > 0$, it is easy to see that (4.2) satisfies the hypotheses of Theorem 3.2 with $a_{11} = n$, $a_{12} = n$, $a_{21} = n$, $a_{22} = n$, $\widetilde{f} = \widetilde{f}_n$, $\widetilde{g} = \widetilde{g}_n$, and $\lambda_1 = \lambda_{1,A}$, where

$$A = \begin{pmatrix} n & n \\ n & n \end{pmatrix}.$$

Then by Theorem 3.2, $\lambda_{1,A}$ is the unique bifurcation point from infinity for (4.2) and there exists a component $\mathcal{C}_n$ of positive solutions of (4.2) bifurcating from infinity at $\lambda_{1,A}$ satisfying the properties (i)–(iii) of Theorem 3.2. In particular, $(0,(0,0)) \in \mathcal{C}_n$, $\mathcal{C}_n$ is bounded above by $\lambda^* \times E$ (λ^* is as in Theorem 2.1) and $\mathcal{C}_n$ does not cross $\{0\} \times \mathbb{E}$ except through the point $(0,(0,0))$.

4.2 Passing to the limit

We first state some properties of the superior limit of a certain infinity collection of connected sets.

Definition 4.5 ([18]). Let X be a Banach space and $\{C_n : n = 1,2,\dots\}$ be a certain infinite collection of subsets of X. Then the superior limit $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in X : \exists\{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\}.$$

Lemma 4.6 ([15]). *Let X be a Banach space and let $\{C_n\}$ be a family of closed connected subsets of X. Assume that:*

(i) *there exist $z_n \in C_n$, $n = 1,2,\dots$, and $z^* \in X$, such that $z_n \to z^*$;*

(ii) *$r_n = \sup\{\|x\| : x \in C_n\} = \infty$;*

(iii) *for every $R > 0$, $(\bigcup_{n=1}^{\infty}) \cap B_R$ is a relatively compact set of X, where $B_R = \{x \in X : \|x\| \leq R\}$.*

Then there exists an unbounded component $\mathcal{C}$ in $\mathcal{D}$ and $z^* \in \mathcal{C}$.

By means of the corresponding auxiliary equations (4.2), we obtained a sequence of unbounded components $\mathcal{C}_n$, and this enables us to find an unbounded component $\mathcal{C}$ satisfying

$$\mathcal{C} \subset \limsup_{n \to \infty} \mathcal{C}_n.$$

It following from the existence of Γ_n^f and Γ_n^g that $f_n(s,t) = f(t,s)$, $g_n(s,t) = g(t,s)$ for $n \to \infty$. Thus $(\lambda, (u,v)) \in \mathcal{C}$ solves the original problem (1.5) when $n \to \infty$. Now we verify $\{\mathcal{C}_n\}$ satisfying the assumptions of Lemma 4.6. By the definition of continuum and component, $\mathcal{C}_n$ is closed.

Since all of $\mathcal{C}_n$ contain $(0, (0,0))$, we can choose $z_n \in \mathcal{C}_n$ such that $z_n = (0, (0,0))$ for $n = 1, 2, \dots$ Clearly, $z_n \to z^* = (0, (0,0))$, the assumption (i) of Lemma 4.6 is satisfied.

By unboundedness of $\mathcal{C}_n$, obviously, we have

$$r_n = \sup\{|\lambda| + \|(u,v)\|_E : (\lambda, (u,v)) \in \mathcal{C}_n\} = +\infty.$$

(iii) in Lemma 4.6 can be deduced directly from the Arzelà–Ascoli theorem and the definition of f_n, g_n. Therefore, the superior limit of $\mathcal{C}_n$ contains a component $\mathcal{C}$.

It follows from (H1) that u, $v > 0$ in Ω for $\lambda > 0$ whenever $(\lambda, (u,v)) \in \mathcal{C}$, which establishes part (a), Clearly, $(0, (0,0)) \in \mathcal{C}$, which together with the maximum principle establish part (b). Part (c) follows from Theorem 2.1. By construction of $\mathcal{C}$, there exists a sequence $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$ such that $0 < \lambda_n < \lambda^*$ (λ^* is as in Theorem 2.1) and $u_n > 0$, $v_n > 0$ for all $n \in \mathbb{N}$, and $\|(u_n, v_n)\|_E \to +\infty$ as $n \to +\infty$. Thus $\mathcal{C}$ is unbounded in the Banach space E. This establishes part (d) and completes the proof of Theorem 1.2.

5 Application of Theorem 1.2

The unbounded component $\mathcal{C}$ from Theorem 1.2 may bifurcate from infinity at any or all $\lambda \in [0, \lambda^*]$. Next, As an application of Theorem 1.2, we will show that, under additional assumptions on f and g, the component $\mathcal{C}$ must approach towards the hyperplane $\lambda = 0$ as the norm $\|(u,v)\|_E$ grows large.

5.1 Main result

Theorem 5.1. *Let (H1)–(H2) hold. Assume*

$$f(u,v) \leq C_1(1 + v^{p_1} + u^{p_2}), \qquad u, \; v \geq 0, \; x \in \Omega, \tag{5.1}$$

$$g(u,v) \leq C_1(1 + u^{q_1} + v^{q_2}), \qquad u, \; v \geq 0, \; x \in \Omega, \tag{5.2}$$

$$f(u,v) + g(u,v) \geq \kappa(u+v) - C_1, \qquad u, \; v \geq 0, \; x \in \Omega, \tag{5.3}$$

here p_1, $q_1 > 0$, $p_1 q_1 > 1$, p_2, $q_2 \geq 1$, $C_1 > 0$ and $\kappa > \mu_1$ (μ_1 defined in Section 2). Define

$$\alpha = \frac{2(p_1 + 1)}{p_1 q_1 - 1}, \qquad \beta = \frac{2(q_1 + 1)}{p_1 q_1 - 1}.$$

If

$$\max\{\alpha, \beta\} > N - 1 \tag{5.4}$$

and

$$p_2, \; q_2 < \frac{N+1}{N-1}. \tag{5.5}$$

Then $\lambda_\infty = 0$ is the unique bifurcation point from infinity in $[0, \lambda^*]$, for the component $\mathcal{C} \subseteq \mathcal{S}$ from Theorem 1.2. More specifically,

(i) *There exists a sequence of positive solutions $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$ with $\lambda_n \in [0, \lambda^*)$ for all $n \in \mathbb{N}$ such that $\|(u_n, v_n)\|_E \to +\infty$ and $\lambda_n \to 0^+$ as $n \to +\infty$.*

(ii) *Any sequence $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$ such that $\|(u_n, v_n)\|_E \to +\infty$ as $n \to \infty$ and $\lambda_n > 0$ must satisfy $\lambda_n \to 0^+$ as $n \to +\infty$.*

It is worth noting that for problem (1.5) satisfying (H1), (H2) and (5.1)–(5.5), the solutions $(\lambda, (u, v)) \in \mathbb{R} \times E$.

Lemma 5.2. *Assume (5.1)–(5.5) hold. Let $\lambda_n \in \mathbb{R}$ be a sequence with $\lambda_1 < \lambda^*$ such that $\lambda_n \searrow 0^+$ as $n \to +\infty$. Then for each $n \in \mathbb{N}$, there exists $C_n := C(\lambda_n)$ such that any solution $(\lambda, (u, v))$ of (1.5) satisfies*

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \le C_n, \quad \text{for all } \lambda \in [\lambda_n, \lambda^*)$$

and $C_n \to +\infty$ as $n \to +\infty$.

Proof. We begin by observing that under above hypotheses uniform a priori bound result holds [17, Theorem 1.1] for positive solutions of (1.5). By retracing the proof of [17, Theorem 2.1, Proposition 3.1] with λf and λg (in place of f and g) we will establish the dependence of the uniform bounds on λ.

First, let $b^* > \lambda^*$, we consider the system (1.5) with $\lambda \in [\lambda_n, b^*]$ under the assumptions

$$|\lambda f(u, v)| \le b^* C_1(|v|^{p_1} + |u|^{p_2}) + b^* h_2(x), \qquad u,\ v \in \mathbb{R},\ x \in \Omega, \qquad (5.6)$$

$$|\lambda g(u, v)| \le b^* C_1(|u|^{q_1} + |v|^{q_2}) + b^* h_2(x), \qquad u,\ v \in \mathbb{R},\ x \in \Omega, \qquad (5.7)$$

$$a\lambda f(u, v) + b\lambda g(u, v) \ge \kappa \lambda_n(au + bv) - b^* C_1, \qquad u,\ v \ge 0,\ x \in \Omega, \qquad (5.8)$$

with $p_1,\ q_1 > 0$, $p_1 q_1 > 1$, $p_2,\ q_2 \ge 1$, $h_2 \in L^\gamma(\Omega)$, $\gamma > \frac{N}{2}$, $a,\ b > 0$, $\kappa \lambda_n > \mu_1$ and $C_1 \ge 0$. By [17, Theorem 2.1, Proposition 3.1], we know any nonnegative solution of (1.5) satisfies

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \le C_n, \quad \text{for all } \lambda \in [\lambda_n, b^*].$$

The constant C_n depends only on $p_1,\ q_1,\ p_2,\ q_2,\ \gamma,\ C_1$ and the norms of h_2 in $h_2 \in L^\gamma(\Omega)$.

Next, we show $C_n \to +\infty$ as $n \to +\infty$. In fact, by Theorem 1.2, we know that there exists a sequence of positive solutions $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$ such that $\lambda_n \in (0, \lambda^*)$ for all $n \in \mathbb{N}$ and $\|(u_n, v_n)\|_E \to +\infty$ as $n \to +\infty$.

For $n \in \mathbb{N}$, let

$$\sup\{\|(u, v)\|_E : (\lambda, (u, v)) \in \mathcal{C},\ \lambda_n < \lambda < \lambda_*\} =: B_n,$$

then clearly

$$B_n \le C_n.$$

For $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$, we have

$$\lim_{n \to +\infty} \|(u_n, v_n)\|_E = \infty,$$

therefore,

$$\lim_{n \to +\infty} B_n = \infty,$$

thus

$$\lim_{n \to +\infty} C_n = \infty. \qquad \square$$

Now we will complete the proof of Theorem 5.1.

Proof of Theorem 5.1. By Lemma 5.2, the component $\mathcal{C}$ must bifurcate from infinity at $\lambda = 0$ and by construction $(0, (0,0)) \in \mathcal{C}$. Part (i) follows from the construction of $\mathcal{C}$ and the fact that $\mathcal{C}$ cannot cross the hyperplane $\{0\} \times E$.

Let $\{\lambda_n, (u_n, v_n)\} \in \mathcal{C}$ with $\|(u_n, v_n)\|_E \to +\infty$ as $n \to +\infty$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$. Suppose to the contrary that $\lambda_n \to \tilde{\lambda} > 0$ as $n \to +\infty$. By Lemma 5.2,

$$\|u_n\|_{L^\infty(\Omega)}, \ \|v_n\|_{L^\infty(\Omega)} \leq C_{\tilde{\lambda}} < +\infty$$

for all $\lambda \in [\frac{\tilde{\lambda}}{2}, \lambda^*)$, a contradiction to $\|(u_n, v_n)\|_E \to +\infty$ as $n \to +\infty$. Hence part (ii) follows. This completes the proof of Theorem 5.1. $\square$

5.2 Examples

Let $f(u,v) = (u+v)^\tau + 1$, $g(u,v) = (u+2v)^\tau + 1$, where $\tau \in (1, \frac{N+1}{N-1})$.

It is easy to see f, g satisfy (H1) and (H2). When C_1 is large enough, there exist $\tau < p_2$, $q_2 < \frac{N+1}{N-1}$, p_1, $q_1 > \tau$ and $p_1 q_1 > 1$, such that (5.1)–(5.5) hold. Then there exists an unbounded continuum $\mathcal{C}$ and $\lambda_\infty = 0$ is the unique bifurcation point from infinity in $[0, \lambda^*]$.

Such as, when $N = 3$. Let $f(u,v) = (u+v)^{\frac{3}{2}} + 1$, $g(u,v) = (u+2v)^{\frac{3}{2}} + 1$, then (H1) and (H2) hold. We set p_2, $q_2 = \frac{7}{4}$, $p_1 = \frac{7}{5}$, $q_1 = \frac{16}{7}$, then (5.1)–(5.5) hold.

Acknowledgements

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