



Long time behavior of the solution to a chemotaxis system with nonlinear indirect signal production and logistic source

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Abstract. This paper is devoted to studying the following quasilinear parabolic-elliptic-elliptic chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + au - bu^\gamma, & x \in \Omega, t > 0, \\ 0 = \Delta v - v + w^{\gamma_1}, & x \in \Omega, t > 0, \\ 0 = \Delta w - w + u^{\gamma_2}, & x \in \Omega, t > 0, \end{cases}$$

with homogeneous Neumann boundary conditions in a bounded and smooth domain $\Omega \subset \mathbb{R}^n (n \geq 1)$, where $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and the functions $\varphi, \psi \in C^2([0, \infty))$ satisfy $\varphi(s) \geq a_0(s+1)^\alpha$ and $|\psi(s)| \leq b_0s(1+s)^{\beta-1}$ for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. It is proved that if $\gamma - \beta \geq \gamma_1\gamma_2$, the classical solution of system would be globally bounded. Furthermore, a specific model for $\gamma_1 = 1, \gamma_2 = \kappa$ and $\gamma = \kappa + 1$ with $\kappa > 0$ is considered. If $\beta \leq 1$ and $b > 0$ is large enough, there exist $C_\kappa, \mu_1, \mu_2 > 0$ such that the solution (u, v, w) satisfies

$$\begin{aligned} & \left\| u(\cdot, t) - \left(\frac{b}{a}\right)^{\frac{1}{\kappa}} \right\|_{L^\infty(\Omega)} + \left\| v(\cdot, t) - \frac{b}{a} \right\|_{L^\infty(\Omega)} + \left\| w(\cdot, t) - \frac{b}{a} \right\|_{L^\infty(\Omega)} \\ & \leq \begin{cases} C_\kappa e^{-\mu_1 t}, & \text{if } \kappa \in (0, 1], \\ C_\kappa e^{-\mu_2 t}, & \text{if } \kappa \in (1, \infty), \end{cases} \end{aligned}$$

for all $t \geq 0$. The above results generalize some existing results.

Keywords: chemotaxis system, nonlinear indirect secretion, global boundedness, long time behavior.

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1 Introduction

Chemotaxis is one of the basic physiological reactions of cells or organisms, which refers to the directional movement of biological cells or organisms along the concentration gradient of stimulants under the stimulation of some chemicals in the environment. The establishment of chemotactic mathematical model can be traced back to the pioneering work proposed by Keller and Segel [16] to describe the aggregation of cellular slime molds, which is given by

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + f(u), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x) & x \in \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$, $\tau \in \{0, 1\}$, ν denotes the outward unit normal vector on $\partial\Omega$, $u(x, t)$ denotes the cell density and $v(x, t)$ represents the concentration of the chemical signal. Here, $f(u)$ describes cell proliferation and death, $\nabla \cdot (\varphi(u)\nabla u)$ and $-\nabla \cdot (\psi(u)\nabla v)$ represent self-diffusion and cross-diffusion, respectively. It is well known that chemotaxis research has many important applications in both biology and medicine so that it has been one of the hottest research focuses in applied mathematics nowadays. In the past few decades, a large number of valuable theoretical results have been established. Among them, one of the main issues related to (1.1) is to study whether there is a global in-time bounded solution or when blow-up occurs. For $\tau = 1$, $\varphi(u) = 1$, $\psi(u) = \chi u$ and $f(u) = 0$ with $\chi > 0$, it has been shown that the system (1.1) has globally bounded classical solution when $n = 1$ [24] or $n = 2$ and $\int_{\Omega} u_0 dx < \frac{4\pi}{\chi}$ [5, 23], whereas the system (1.1) has finite time blow-up solution in the case of $n = 2$ and $\int_{\Omega} u_0 dx > \frac{4\pi}{\chi}$ [9, 26] or in the case of $n \geq 3$ [36, 39]. Inter alia, when $f(u) = u - \mu u^2$ with $\mu > 0$, under the restrictions that $\tau = 1$ and Ω is convex, Winkler [40] proved that if the ratio $\frac{\mu}{\chi}$ is sufficiently large, then the unique nontrivial spatially homogeneous equilibrium given by $u = v \equiv \frac{1}{\mu}$ is globally asymptotically stable. Later on, Cao [2] used an approach based on maximal Sobolev regularity and improved Winkler's results without the restrictions $\tau = 1$ and the convexity of Ω . When the chemical substance diffuses much faster than the diffusion of cells, the system (1.1) can be reduced to the simplified parabolic-elliptic model, i.e. $\tau = 0$. Such model was first studied for $\varphi(u) = 1$, $\psi(u) = \chi u$ and $f(u) = 0$ in [14]. Recently, when $f(u) = Au - bu^\alpha$ with $\alpha > 1$, $A \geq 0$ and $b > 0$, in [35], a concept of very weak solutions was introduced, and global existence of such solutions for any nonnegative initial data $u_0 \in L^1(\Omega)$ was proved under the assumption that $\alpha > 2 - \frac{1}{n}$, moreover, boundedness properties of the constructed solutions were studied by Winkler. Thereafter various variants of (1.1) have been considered by many other scholars [6, 11, 31, 34]. In general, diffusion functions $\varphi(u)$ and $\psi(u)$ may not be linear forms, such as diffusion in porous media and volume filling effect. When $\varphi(u), \psi(u)$ are nonlinear and $f(u) = 0$ or $f(u) \neq 0$, a lot of scholars have studied the finite time blow-up of solution and the existence of globally bounded classical solution to system (1.1). We refer the readers to [8, 12, 13, 37, 38] for more details.

With regard to the system (1.1), the term of chemotaxis signal production v is produced directly by the cell density u . However, the mechanism of signal production might be very complex in realistic biological processes. On the one hand, the signal generation usually undergoes intermediate stages, i.e. signal v is not produced directly by cells u , but is governed

by some other signal substances w . The related models can be described as

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + f(u), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + w, & x \in \Omega, t > 0, \\ \tau w_t = \Delta w - w + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (1.2)$$

where u, v, w represent the density of cells, the density of chemical substances and the concentration of indirect signal, respectively. Such problem has been widely studied in recent years. For $\tau = 1, \varphi(u) = 1, \psi(u) = u$ and $f(u) = \mu(u - u^\gamma)$, the authors in [46] proved that if $\gamma > \frac{n}{4} + \frac{1}{2}$, then the system possesses globally bounded classical solution. Moreover, if μ is large enough, the solution (u, v, w) converges to $(1, 1, 1)$ in L^∞ -norm as $t \rightarrow \infty$. When φ and ψ satisfy some nonlinear conditions and smoothness conditions, it also has been showed that the solution to system (1.2) is globally bounded in [30]. Recently, the authors in [18] have studied the system (1.2) for $\tau = 0$, where $\varphi(s) \geq a_0(s+1)^\alpha$ and $|\psi(s)| \leq b_0s(1+s)^{\beta-1}$ for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. They have proved that the nonnegative classical solution to (1.2) is global in time and bounded. In addition, if μ satisfy some suitable conditions, the solution (u, v, w) converges to $(1, 1, 1)$ in L^∞ -norm as $t \rightarrow \infty$. More relevant results on the system with indirect signal production can refer to [10, 19].

One the other hand, the signal generation may be in a nonlinear form, which is given by

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + f(u), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + g(u), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (1.3)$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded, smooth domain. When $\tau = 0, \varphi(u) = 1, \psi(u) = \chi u, f(u) = au - bu^\theta$ and $g(u) = u^\kappa$ with $\chi, b, \kappa > 0, a \in \mathbb{R}$ and $\theta > 1$, Xiang [44] obtained the global existence and boundedness of solution for (1.3) under either $\kappa + 1 < \max\{\theta, 1 + \frac{2}{n}\}$ or $\theta = \kappa + 1, b \geq \frac{(\kappa n - 2)}{\kappa n} \chi$. Besides, they studied the dynamical behavior of the solution on the interactions among nonlinear cross-diffusion, generalized logistic source and signal production. In addition, When $\tau = 1, \varphi(u) = 1, \psi(u) = \chi u, f(u) = 0$ and $g(u) \in C^1([0, \infty))$ satisfying $0 \leq g(u) \leq Ku^\alpha$ with some constants $K, \alpha > 0$, Liu and Tao [21] proved that the classical solution of the system (1.3) is globally bounded if $0 < \alpha < \frac{2}{n}$. When the second equation degenerates into an elliptic equation (i.e. $\tau = 0$), $\varphi(u) = 1, \psi(u) = \chi u, f(u) = 0$ and v is replaced by $\mu(t) = \frac{1}{|\Omega|} \int_\Omega g(u), g(u) \geq ku^k$ for all $u \geq 0$ with some $k > 0$, Winkler [43] derived a blow-up critical exponent $k = \frac{2}{n}$, which asserted that the radially symmetric solution blows up in finite time if the parameter k satisfies $k > \frac{2}{n}$. Conversely, when $k < \frac{2}{n}$, they proved that there exists suitable initial value such that the system has globally bounded classical solution. Later on, the authors in [45] considered the case $f(u) = \lambda u - \mu u^\alpha$ with $\lambda, \mu > 0$ and $\alpha > 1$, and they generalized the blow-up results developed in [43] with $k + 1 > \alpha (\frac{2}{n} + 1)$. Intuitively, the existing literatures show that the logistics source (i.e. $f(u) = \lambda u - \mu u^\alpha$ with $\lambda, \mu > 0$ and $\alpha > 1$) and its possibly damping behavior have important influences on the behavior of the solution. For instance, the strong logistic damping (i.e. μ is suitably large) may ensure the system has globally bounded classical solution, especially in higher-dimensional case. More precisely, when $\alpha = 2$, Tello and Winkler [29] proved that for all suitably regular

initial data, the system had a unique globally bounded classical solution if $\mu > \max\{0, \frac{n-2}{n}\chi\}$. Afterwards, Cao and Zheng [3] proved that such global solution to a quasilinear system (1.3) is also known to exist for all nonnegative and smooth initial data if μ is suitably large. However, ‘‘logistic source’’ does not always prevent chemotactic collapse. When $\alpha = 2$, such assertion was verified in [41] for one-dimensional case by Winkler, and also could be found in [15] for higher-dimensional setting. Recently, Winkler [42] obtained a condition on initial data to ensure the occurrence of finite-time blow-up to system (1.3) for

$$\alpha < \begin{cases} \frac{7}{6}, & \text{if } n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)}, & \text{if } n \geq 5. \end{cases}$$

Some boundedness or blow-up results to variants of system (1.3) can also be found in [20, 22, 25, 32, 33, 47].

Among the existing literatures, it is not difficult to find that there are very few papers to study the chemotaxis system, where chemical signal production is not only indirect but also nonlinear. Based on the complexity of biological process, such signal production mechanism could be more in line with the actual situation. Inspired by the above works, in this paper, we are concerned with the following system

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + au - bu^\gamma, & x \in \Omega, t > 0, \\ 0 = \Delta v - v + w^{\gamma_1}, & x \in \Omega, t > 0, \\ 0 = \Delta w - w + u^{\gamma_2}, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, ν denotes the outward unit normal vector on $\partial\Omega$, the parameters satisfy $a, b, \gamma_2 > 0, \gamma_1 \geq 1$ and $\gamma > 1$, and $\varphi(u), \psi(u)$ are self-diffusion and cross-diffusion functions, respectively. Since from a physical point of view, the equation modeling the migration of cells should rather be regarded as nonlinear diffusion [27]. Thus, here we assume that the diffusion functions $\varphi, \psi \in C^2[0, \infty)$ fulfill

$$\varphi(s) \geq a_0(s+1)^\alpha \quad (1.5)$$

and

$$|\psi(s)| \leq b_0s(s+1)^{\beta-1}, \quad (1.6)$$

for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$.

The main purpose of the present paper is to explore the interplay of nonlinear diffusion functions φ, ψ and logistic source term $au - bu^\gamma$ as well as nonlinear indirect signal production mechanism for system (1.4). To the best of our knowledge, studying the fully parabolic chemotaxis system need to use the method of variation-of-constants formula and heat semigroup, which can not be applied to the system (1.4). In this paper, we shall use a different method to reveal the influence of nonlinear diffusion functions φ, ψ and logistic source term $au - bu^\gamma$ as well as nonlinear indirect signal production mechanism on the dynamical behavior of the solution to system (1.4).

Firstly, we state our boundedness result to system (1.4) as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded and smooth domain. Assume that $a, b, \gamma_2 > 0, \gamma > 1, \gamma_1 \geq 1$ and functions $\varphi, \psi \in C^2[0, \infty)$ with $\varphi(s) \geq a_0(s+1)^\alpha$ and $|\psi(s)| \leq b_0s(s+1)^{\beta-1}$ for*

all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. If $\gamma - \beta \geq \gamma_1 \gamma_2$, then for any nonnegative initial data $0 \neq u_0 \in C(\bar{\Omega})$, the system (1.4) admits a unique nonnegative classical solution (u, v, w) belonging to $C[(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))]$. Moreover, the solution of system (1.4) is bounded in $\Omega \times (0, \infty)$, namely, there exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad (1.7)$$

for all $t > 0$.

In contrast to the boundedness criterion obtained in [18], the boundedness condition in Theorem 1.1 is more generalized involving nonlinear diffusion and logistic source term as well as nonlinear indirect signal production mechanism.

From the viewpoint of biological evolution, it has profound theoretical and practical significance to study the long time behavior of chemotaxis system. Based on [7, 18, 44], we have also studied the long time behavior of solution to a special case (see system (3.1) in Section 3) of system 1.1 (i.e. $\gamma_1 = 1, \gamma_2 = \kappa$ and $\gamma = \kappa + 1$ with $\kappa > 0$). Here, it should be pointed out that from the above Theorem 1.1 if $\beta \leq 1$, the corresponding system has globally bounded classical solution for this case. Thus, from Theorem 1.1, there exists $R > 0$ independent of $a, b, \alpha, \beta, a_0, b_0$ and κ such that

$$u(x, t) \leq R \quad (1.8)$$

holds on $\bar{\Omega} \times [0, \infty)$. Moreover, we can also find $\lambda > 0$ independent of a, b, a_0, b_0 and κ such that

$$(u + 1)^{2\beta - \alpha - 2} \leq \lambda \quad (1.9)$$

holds on $\bar{\Omega} \times [0, \infty)$.

Therefore, the second conclusion of this paper can be stated as

Theorem 1.2. Let $0 \leq u_0 \in C(\bar{\Omega})$ and $a, b, \kappa > 0$. Assume that functions $\varphi, \psi \in C^2[0, \infty)$ with $\varphi(s) \geq a_0(s + 1)^\alpha$ and $|\psi(s)| \leq b_0 s(s + 1)^{\beta-1}$ for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. If $\beta \leq 1$ and

$$\begin{cases} b > \frac{b_0}{4} \sqrt{\frac{\lambda a}{a_0}}, & \kappa \in (0, 1], \\ b > \frac{\lambda b_0^2 \left[(\kappa - 1)R^\kappa + \sqrt{(\kappa - 1)^2 R^{2\kappa} + \frac{64a_0 a \Omega}{\lambda b_0^2}} \right]}{32a_0}, & \kappa \in (1, \infty), \end{cases} \quad (1.10)$$

then there exists $C_\kappa > 0$ large enough such that the classical solution (u, v, w) to system (3.1) satisfies

$$\left\| u(\cdot, t) - \left(\frac{b}{a} \right)^{\frac{1}{\kappa}} \right\|_{L^\infty(\Omega)} + \left\| v(\cdot, t) - \frac{b}{a} \right\|_{L^\infty(\Omega)} + \left\| w(\cdot, t) - \frac{b}{a} \right\|_{L^\infty(\Omega)} \leq \begin{cases} C_\kappa e^{-\mu_1 t}, & \kappa \in (0, 1], \\ C_\kappa e^{-\mu_2 t}, & \kappa \in (1, \infty), \end{cases}$$

for all $t \geq 0$, where

$$\mu_1 = \frac{\kappa a}{(n + 2)b^2} \left(b^2 - \frac{\lambda a b_0^2}{16a_0} \right) \quad (1.11)$$

and

$$\mu_2 = \frac{\kappa \left(\frac{a}{b} \right)^{\frac{2-\kappa}{\kappa}}}{(n + 2)} \left\{ b - \frac{\lambda b_0^2}{16a_0} \left[\frac{a}{b} + (\kappa - 1)R^\kappa \right] \right\}, \quad (1.12)$$

with $R > 0$ and $\lambda > 0$ defined in (1.8) and (1.9), respectively.

The results in Theorem 1.2 are similar to those in [44, Theorem 5.1(i)], but more general, since self-diffusion, cross-diffusion and indirect secretion mechanism are involved. We need to modify the method in [44] to overcome the difficulties from these terms (see (3.10) and (3.25) in the proof of Lemma 3.2). In addition, our conclusion in Theorem 1.2 can also be seen as an extension of [7] or [18]. Comparing with [7], in Theorem 1.2, we calculate the exponential convergence rate explicitly in terms of the model parameters with diffusion functions, generalized logistic source and nonlinear indirect secretion. But in [7], the convergence rate estimates were derived but not stated explicitly (see [7, Theorem 1]) for special logistic source and linear secretion. Comparing with [18], since our model is nonlinear indirect production, we have to divide the range of κ into $(0, 1]$ and $(1, +\infty)$ to construct different functionals $A(t)$ and $H(t)$ (see Lemma 3.2) to prove Theorem 1.2.

Remark 1.3. It is relevant to point out that by the limitation of the method, we also have no idea the long time behavior of solution to system (1.4) for generalized parameters γ_1, γ_2 and γ satisfying the condition in Theorem 1.1.

The outline of this paper is as follows. In Section 2, the global existence and boundedness of classical solution to (1.4) is proved. In Section 3, by applying the method of energy functional, we obtain that the solution to system (3.1) exponentially converges to the point $((\frac{a}{b})^{\frac{1}{\kappa}}, \frac{a}{b}, \frac{a}{b})$ as $t \rightarrow \infty$.

2 Global existence and boundedness

In this section, we will obtain the existence and boundedness of globally classical solution to system (1.4). At the beginning, we give a statement on the local existence of classical solutions. The proof depends on the Schauder fixed theorem. We omit it for brevity and refer the readers to [30] for more details.

Lemma 2.1. *Let $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded and smooth domain. Assume that $\varphi, \psi \in C^2[0, \infty)$ satisfy (1.5) and (1.6), respectively. For any nonnegative initial data $0 \neq u_0 \in C(\bar{\Omega})$, there exists $T_{\max} \in (0, \infty]$ such that the system (1.4) admits a unique nonnegative classical solution (u, v, w) belonging to $C[(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))]$ in $\Omega \times (0, T_{\max})$ with*

$$u, v, w \geq 0 \quad \text{in } \bar{\Omega} \times (0, T_{\max}). \quad (2.1)$$

Furthermore,

$$\text{if } T_{\max} < \infty, \quad \text{then } \limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (2.2)$$

Lemma 2.2. *Let $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and (u, v, w) be a solution of system (1.4). Assume that $\varphi, \psi \in C^2[0, \infty)$ satisfy (1.5) and (1.6). Then for any $\eta_1, \eta_2 > 0$ and $\theta > 1$, there exist $c_0, c_1 > 0$ depending only on $\gamma_1, \gamma_2, \eta_1, \eta_2, \theta$ such that*

$$\int_{\Omega} w^\theta \leq \eta_2 \int_{\Omega} (u+1)^{\gamma_2 \theta} + c_0 \quad (2.3)$$

and

$$\int_{\Omega} v^\theta \leq \eta_1 \eta_2 \int_{\Omega} (u+1)^{\gamma_1 \gamma_2 \theta} + c_1, \quad (2.4)$$

for all $t \in (0, T_{\max})$.

Proof. Integrating the first equation of system (1.4) over Ω , we find

$$\frac{d}{dt} \int_{\Omega} u dx = \int_{\Omega} au - bu^{\gamma} \leq a \int_{\Omega} u - \frac{b}{|\Omega|^{\gamma-1}} \left(\int_{\Omega} u \right)^{\gamma} \quad \text{for all } t \in (0, T_{\max}), \quad (2.5)$$

where we have used Hölder's inequality. Thus, using a standard ODE comparison theory, it shows that

$$\int_{\Omega} u \leq \max \left\{ \int_{\Omega} u_0, \left(\frac{a}{b} \right)^{\frac{1}{\gamma-1}} |\Omega| \right\} \quad \text{for all } t \in (0, T_{\max}). \quad (2.6)$$

Moreover, we can derive directly by integrating the third equation over Ω ,

$$\|w\|_{L^1(\Omega)} = \|u^{\gamma_2}\|_{L^1(\Omega)} \leq \|(u+1)^{\gamma_2}\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \quad (2.7)$$

Multiplying the third equation of system(1.4) with $w^{\theta-1}$ and integrating by parts over Ω , we can get

$$\frac{4(\theta-1)}{\theta^2} \int_{\Omega} |\nabla w^{\frac{\theta}{2}}|^2 + \int_{\Omega} w^{\theta} = \int_{\Omega} u^{\gamma_2} w^{\theta-1} \leq \frac{\theta-1}{\theta} \int_{\Omega} w^{\theta} + \frac{1}{\theta} \int_{\Omega} u^{\gamma_2 \theta} \quad (2.8)$$

by Young's inequality. Hence

$$\|w\|_{L^{\theta}(\Omega)} \leq \|u^{\gamma_2}\|_{L^{\theta}(\Omega)} \leq \|(u+1)^{\gamma_2}\|_{L^{\theta}(\Omega)} \quad \text{for all } t \in (0, T_{\max}) \quad (2.9)$$

and

$$\frac{4(\theta-1)}{\theta} \int_{\Omega} |\nabla w^{\frac{\theta}{2}}|^2 \leq \int_{\Omega} u^{\gamma_2 \theta} \leq \int_{\Omega} (u+1)^{\gamma_2 \theta} \quad \text{for all } t \in (0, T_{\max}). \quad (2.10)$$

By Ehrling's lemma, for any $\eta_2 > 0, \theta > 1$ and function $\phi \in W^{1,2}(\Omega)$, there exists $C_0 = C_0(\eta_2, \theta) > 0$ such that

$$\|\phi\|_{L^2(\Omega)}^2 \leq \eta_2 \|\phi\|_{W^{1,2}(\Omega)}^2 + C_0 \|\phi\|_{L^{\frac{\theta}{\theta-1}}(\Omega)}^{\theta}. \quad (2.11)$$

Let $\phi = w^{\frac{\theta}{2}}$, from (2.7),(2.9) and (2.10), there exists $C_1 = C_1(\eta_2, \theta) > 0$ such that

$$\int_{\Omega} w^{\theta} \leq \eta_2 \int_{\Omega} (u+1)^{\gamma_2 \theta} + C_1 \|(u+1)^{\gamma_2}\|_{L^1(\Omega)}^{\theta}. \quad (2.12)$$

For $\gamma_2 \in (0, 1]$, using Hölder's inequality, one may obtain from (2.6)

$$\|(u+1)^{\gamma_2}\|_{L^1(\Omega)}^{\theta} \leq C_2 \quad (2.13)$$

with $C_2 = C_2(\eta_2, \theta, \gamma_2) > 0$. For $\gamma_2 \in (1, \infty)$, using interpolation inequality and Young's inequality, from (2.6) we deduce

$$\|(u+1)^{\gamma_2}\|_{L^1(\Omega)}^{\theta} \leq \|(u+1)^{\gamma_2}\|_{L^{\theta}(\Omega)}^{\theta \tau} \|(u+1)^{\gamma_2}\|_{L^{\frac{1}{\gamma_2}}(\Omega)}^{\theta(1-\tau)} \leq \eta_2 \int_{\Omega} (u+1)^{\gamma_2 \theta} + C_3 \quad (2.14)$$

where $\tau = \frac{\gamma_2-1}{\gamma_2-\frac{1}{\theta}} \in (0, 1)$ and $C_3 = C_3(\eta_2, \theta, \gamma_2) > 0$. Thus (2.3) is the direct result of combining (2.12)–(2.14). Similarly, multiplying the second equation of system(1.4) with $v^{\theta-1}$, by the same procedure as above, we can obtain for any $\eta_1 > 0$ and $\theta > 1$

$$\int_{\Omega} v^{\theta} \leq \eta_1 \int_{\Omega} w^{\gamma_1 \theta} + C_4 \quad \text{for all } t \in (0, T_{\max}) \quad (2.15)$$

with $C_4 = C_4(\eta_1, \theta, \gamma_1) > 0$. Since $\gamma_1 \geq 1$, we can obtain from (2.3)

$$\int_{\Omega} w^{\gamma_1 \theta} \leq \eta_2 \int_{\Omega} (u+1)^{\gamma_1 \gamma_2 \theta} + C_5 \quad \text{for all } t \in (0, T_{\max}) \quad (2.16)$$

with $C_5 > 0$. Combining (2.15)–(2.16) yields (2.4). This completes the proof of Lemma 2.2. \square

Lemma 2.3. Let $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and (u, v, w) be a solution of system (1.4). Assume that functions $\varphi, \psi \in C^2[0, \infty)$ satisfying (1.5) and (1.6) for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. If $\gamma - \beta \geq \gamma_1 \gamma_2$, then for any $p > \max\{1, 1 - \beta\}$, there exists a constant $C > 0$ such that

$$\int_{\Omega} (u+1)^p \leq C \quad (2.17)$$

for all $t \in (0, T_{\max})$.

Proof. Multiplying the first equation of system (1.4) by $(u+1)^{p-1}$ and integrating by parts over Ω , we derive

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} (u+1)^p &= -(p-1) \int_{\Omega} (u+1)^{p-2} \varphi(u) |\nabla u|^2 + (p-1) \int_{\Omega} (u+1)^{p-2} \psi(u) \nabla u \cdot \nabla v \\ &\quad + a \int_{\Omega} u(u+1)^{p-1} - b \int_{\Omega} u^\gamma (u+1)^{p-1} \end{aligned} \quad (2.18)$$

for all $t \in (0, T_{\max})$. Since φ satisfies (1.5), we can estimate the first term on the right-hand side of (2.18) as

$$\begin{aligned} -(p-1) \int_{\Omega} (u+1)^{p-2} \varphi(u) |\nabla u|^2 &\leq -(p-1) \int_{\Omega} a_0 (1+u)^\alpha (1+u)^{p-2} |\nabla u|^2 \\ &\leq -\frac{4a_0(p-1)}{(p+\alpha)^2} \int_{\Omega} |\nabla (u+1)^{\frac{p+\alpha}{2}}|^2 \end{aligned} \quad (2.19)$$

for all $t \in (0, T_{\max})$. Let $\Psi(u) = \int_0^u (\xi+1)^{p-2} \psi(\xi) d\xi$, thus

$$\nabla \Psi(u) = (u+1)^{p-2} \psi(u) \nabla u \quad (2.20)$$

and

$$|\Psi(u)| \leq \frac{b_0}{\beta+p-1} (u+1)^{\beta+p-1} \quad (2.21)$$

for all $t \in (0, T_{\max})$. From (2.20) and (2.21), we can get

$$\begin{aligned} (p-1) \int_{\Omega} (u+1)^{p-2} \psi(u) \nabla u \cdot \nabla v &= (p-1) \int_{\Omega} \nabla \Psi(u) \cdot \nabla v = -(p-1) \int_{\Omega} \Psi(u) \Delta v \\ &\leq (p-1) \int_{\Omega} \Psi(u) |\Delta v| \leq \frac{b_0(p-1)}{\beta+p-1} \int_{\Omega} (u+1)^{\beta+p-1} |\Delta v| \end{aligned} \quad (2.22)$$

for all $t \in (0, T_{\max})$. By the basic inequality $(u+1)^\gamma < 2^\gamma(u^\gamma+1)$ with $\gamma > 1$, we have

$$-b \int_{\Omega} u^\gamma (u+1)^{p-1} \leq -\frac{b}{2^\gamma} \int_{\Omega} (u+1)^{p+\gamma-1} + b \int_{\Omega} (u+1)^{p-1} \quad (2.23)$$

for all $t \in (0, T_{\max})$. Denoting $m_0 = \max\{a, b\}$, from (2.18)–(2.19) and (2.22)–(2.23), we can get

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} (u+1)^p &\leq -\frac{4a_0(p-1)}{(p+\alpha)^2} \int_{\Omega} |\nabla (u+1)^{\frac{p+\alpha}{2}}|^2 + \frac{b_0(p-1)}{\beta+p-1} \int_{\Omega} (u+1)^{\beta+p-1} |\Delta v| \\ &\quad + m_0 \int_{\Omega} u(u+1)^{p-1} - \frac{b}{2^\gamma} \int_{\Omega} (u+1)^{p+\gamma-1} + m_0 \int_{\Omega} (u+1)^{p-1} \\ &\leq \frac{b_0(p-1)}{\beta+p-1} \int_{\Omega} (u+1)^{\beta+p-1} |v-w^{\gamma_1}| + m_0 \int_{\Omega} (u+1)^p - \frac{b}{2^\gamma} \int_{\Omega} (u+1)^{p+\gamma-1} \\ &\leq \frac{b_0(p-1)}{\beta+p-1} \int_{\Omega} (u+1)^{\beta+p-1} v + \frac{b_0(p-1)}{\beta+p-1} \int_{\Omega} (u+1)^{\beta+p-1} w^{\gamma_1} \\ &\quad + m_0 \int_{\Omega} (u+1)^p - \frac{b}{2^\gamma} \int_{\Omega} (u+1)^{p+\gamma-1} \end{aligned} \quad (2.24)$$

for all $t \in (0, T_{\max})$, where we have made use of the second identity $0 = \Delta v - v + w^{\gamma_1}$ in system (1.4). In the sequel, we estimate (2.24) in two different cases.

Case 1 ($\gamma - \beta > \gamma_1 \gamma_2$). In this case, using Young's inequality, we can derive

$$\int_{\Omega} (u+1)^{\beta+p-1} w^{\gamma_1} \leq \frac{b(\beta+p-1)}{2\gamma+3b_0(p-1)} \int_{\Omega} (u+1)^{p+\gamma-1} + C_6 \int_{\Omega} w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} \quad (2.25)$$

with $C_6 = \left(\frac{2\gamma+3b_0(p-1)}{b(\gamma+p-1)}\right)^{\frac{\gamma+p-1}{\gamma-\beta}}$. Since $\gamma - \beta > \gamma_1 \gamma_2$, with applications of Young's inequality, we get from Lemma 2.2 with $\theta = \frac{p+\gamma-1}{\gamma_2} > 1$

$$\begin{aligned} \int_{\Omega} w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} &\leq \frac{b(\beta+p-1)}{2\gamma+3C_6b_0\eta_2(p-1)} \int_{\Omega} w^{\frac{p+\gamma-1}{\gamma_2}} + C_7 \\ &\leq \frac{b(\beta+p-1)}{2\gamma+3C_6b_0(p-1)} \int_{\Omega} (u+1)^{p+\gamma-1} + C_8, \end{aligned} \quad (2.26)$$

where $C_8 = C_7 + c_0$ with $C_7 = \left(\frac{2\gamma+3C_6b_0\eta_2(p-1)}{b(\beta+p-1)}\right)^{\frac{\gamma_1\gamma_2}{\gamma-\beta-\gamma_1\gamma_2}} |\Omega|$. Similarly, we have

$$\int_{\Omega} (u+1)^{\beta+p-1} v \leq \frac{b(\beta+p-1)}{2\gamma+3b_0(p-1)} \int_{\Omega} (u+1)^{p+\gamma-1} + C_9 \int_{\Omega} v^{\frac{p+\gamma-1}{\gamma-\beta}}. \quad (2.27)$$

Since $\gamma_1 \geq 1$, in view of Young's inequality, we can obtain from Lemma 2.2 with $\theta = \frac{p+\gamma-1}{\gamma-\beta} > 1$

$$\begin{aligned} \int_{\Omega} v^{\frac{p+\gamma-1}{\gamma-\beta}} &\leq \eta_1 \eta_2 \int_{\Omega} (u+1)^{\frac{(p+\gamma-1)\gamma_2}{\gamma-\beta}} + c_1 \\ &\leq \eta_1 \eta_2 \left(\frac{1}{\eta_1 \eta_2} \int_{\Omega} (u+1)^{\frac{p+\gamma-1}{\gamma_1}} + (\eta_1 \eta_2)^{\frac{\gamma-\beta-\gamma_1\gamma_2}{\gamma_1\gamma_2}} |\Omega| \right) + c_1 \\ &\leq \int_{\Omega} (u+1)^{\frac{p+\gamma-1}{\gamma_1}} + (\eta_1 \eta_2)^{1+\frac{\gamma-\beta-\gamma_1\gamma_2}{\gamma_1\gamma_2}} |\Omega| + c_1 \\ &\leq \frac{b(\beta+p-1)}{2\gamma+3b_0(p-1)C_9} \int_{\Omega} (u+1)^{p+\gamma-1} + C_{10} \end{aligned} \quad (2.28)$$

with $C_{10} = \left(\frac{2\gamma+3b_0(p-1)C_9}{b(\gamma+p-1)}\right)^{\frac{1}{\gamma_1-1}} + (\eta_1 \eta_2)^{1+\frac{\gamma-\beta-\gamma_1\gamma_2}{\gamma_1\gamma_2}} |\Omega| + c_1$. Since $\gamma > 1$, using Young's inequality, there exists $C_{11} = \frac{b}{2\gamma+2(m_0+1)}$ such that

$$\int_{\Omega} (u+1)^p \leq C_{11} \int_{\Omega} (u+1)^{p+\gamma-1} + C_{12} \quad (2.29)$$

with $C_{12} = \left(\frac{2\gamma+2(m_0+1)}{b}\right)^{\frac{p}{\gamma-1}} |\Omega|$. Using (2.24)–(2.29), we can obtain

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \int_{\Omega} (u+1)^p + \int_{\Omega} (u+1)^p \\ &\leq \frac{b_0(p-1)}{\beta+p-1} \int_{\Omega} (u+1)^{\beta+p-1} v + \frac{b_0(p-1)}{\beta+p-1} \int_{\Omega} (u+1)^{\beta+p-1} w^{\gamma_1} + (m_0+1) \int_{\Omega} (u+1)^p \\ &\quad - \frac{b}{2\gamma} \int_{\Omega} (u+1)^{p+\gamma-1} \\ &\leq \frac{b_0(p-1)}{\beta+p-1} \left[\frac{b(\beta+p-1)}{2\gamma+2b_0(p-1)} \int_{\Omega} (u+1)^{p+\gamma-1} + C_9 \int_{\Omega} v^{\frac{p+\gamma-1}{\gamma-\beta}} + C_6 \int_{\Omega} w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} \right] \\ &\quad + (m_0+1) \left(C_{11} \int_{\Omega} (u+1)^{p+\gamma-1} + C_{12} \right) - \frac{b}{2\gamma} \int_{\Omega} (u+1)^{p+\gamma-1} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{b_0(p-1)}{\beta+p-1} \left[\frac{b(\beta+p-1)}{2^{\gamma+1}b_0(p-1)} \int_{\Omega} (u+1)^{p+\gamma-1} + C_8 + C_{10} \right] \\
&\quad - \frac{3b}{2^{\gamma+2}} \int_{\Omega} (u+1)^{p+\gamma-1} + C_{12}(m_0+1) \\
&\leq -\frac{b}{2^{\gamma+2}} \int_{\Omega} (u+1)^{p+\gamma-1} + (C_8 + C_{10}) \frac{b_0(p-1)}{\beta+p-1} + C_{12}(m_0+1)
\end{aligned} \tag{2.30}$$

for all $t \in (0, T_{\max})$, which means that

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega} (u+1)^p + p \int_{\Omega} (u+1)^p \\
&\leq -\frac{bp}{2^{\gamma+2}} \int_{\Omega} (u+1)^{p+\gamma-1} + (C_8 + C_{10}) \frac{b_0p(p-1)}{\beta+p-1} + C_{12}p(m_0+1).
\end{aligned} \tag{2.31}$$

Thus we can get the conclusion immediately by the ODE comparison principle.

Case 2 ($\gamma - \beta = \gamma_1\gamma_2$). Recalling (2.25) and (2.27), we know

$$\int_{\Omega} (u+1)^{\beta+p-1} w^{\gamma_1} \leq \frac{b(\beta+p-1)}{2^{\gamma+3}b_0(p-1)} \int_{\Omega} (u+1)^{p+\gamma-1} + C_6 \int_{\Omega} w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} \tag{2.32}$$

and

$$\int_{\Omega} (u+1)^{\beta+p-1} v \leq \frac{b(\beta+p-1)}{2^{\gamma+3}b_0(p-1)} \int_{\Omega} (u+1)^{p+\gamma-1} + C_9 \int_{\Omega} v^{\frac{p+\gamma-1}{\gamma-\beta}}. \tag{2.33}$$

Since $\gamma - \beta = \gamma_1\gamma_2$, for any $\eta_1, \eta_2 > 0$, we can obtain from Lemma 2.2

$$\int_{\Omega} w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} = \int_{\Omega} w^{\frac{(p+\gamma-1)}{\gamma_2}} \leq \eta_2 \int_{\Omega} (u+1)^{p+\gamma-1} + c_0 \tag{2.34}$$

and

$$\int_{\Omega} v^{\frac{p+\gamma-1}{\gamma-\beta}} = \int_{\Omega} v^{\frac{p+\gamma-1}{\gamma_1\gamma_2}} \leq \eta_1\eta_2 \int_{\Omega} (u+1)^{p+\gamma-1} + c_1 \tag{2.35}$$

for all $t \in (0, T_{\max})$. Because of the arbitrariness of η_1 and η_2 , we choose $\eta_2 = \frac{b(\beta+p-1)}{2^{\gamma+3}C_6b_0(p-1)}$ and $\eta_1\eta_2 = \frac{b(\beta+p-1)}{2^{\gamma+3}C_9b_0(p-1)}$ in (2.34) and (2.35), respectively. From (2.24), (2.29) and (2.32)–(2.35), we can obtain

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega} (u+1)^p + p \int_{\Omega} (u+1)^p \\
&\leq -\frac{bp}{2^{\gamma+2}} \int_{\Omega} (u+1)^{p+\gamma-1} + (c_0C_6 + c_1C_9) \frac{b_0p(p-1)}{\beta+p-1} + C_{12}p(m_0+1),
\end{aligned} \tag{2.36}$$

for all $t \in (0, T_{\max})$. Using the ODE comparison principle, we can prove the conclusion. The proof of Lemma 2.3 is completed. \square

Proof of Theorem 1.1. Let $a, b, \gamma_2 > 0$, $\gamma_1 \geq 1$, $\gamma > 1$ and (u, v, w) be a solution of system (1.4). From Lemma 2.3, for any $p > \max\{1, 1 - \beta\}$, there exists $C_{13} > 0$ such that $\|u\|_{L^p(\Omega)} \leq C_{13}$ for all $t \in (0, T_{\max})$. By the elliptic L^p -estimate applied to the second and third equations in system (1.4), we have

$$\|w(\cdot, t)\|_{W^{2,p/\gamma_2}(\Omega)} + \|v(\cdot, t)\|_{W^{2,p/\gamma_1\gamma_2}(\Omega)} \leq C_{14} \tag{2.37}$$

for all $t \in (0, T_{\max})$, with some $C_{14} > 0$. Using the Sobolev imbedding theorem, we can get

$$\|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_{15} \quad (2.38)$$

for all $t \in (0, T_{\max})$, with some $C_{15} > 0$. Thus by standard Alikakos–Moser iteration ([28, Lemma A.1]), we can find a constant $C_{16} > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{16}$$

for all $t \in (0, T_{\max})$, which together with Lemma 2.1 implies that $T_{\max} = \infty$. Hence, by standard elliptic regularity theory, we know that (u, v, w) is a globally bounded classical solution of system (1.4). The proof of Theorem 1.1 is completed. \square

3 Long time behavior of the solution for a specific model

In this section, we shall study the long time behavior of the solution for a specific model (i.e. $\gamma_1 = 1, \gamma_2 = \kappa$ and $\gamma = \kappa + 1$ with $\kappa > 0$) with nonlinear indirect signal production and logistic source as follows

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + u(a - bu^\kappa), & x \in \Omega, t > 0, \\ 0 = \Delta v - v + w, & x \in \Omega, t > 0, \\ 0 = \Delta w - w + u^\kappa, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (3.1)$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded and smooth domain, the parameters $a, b, \kappa > 0$ and functions $\varphi, \psi \in C^2[0, \infty)$ satisfy conditions (1.5) and (1.6), respectively.

Based on Theorem 1.1, it is easy to check that if $\beta \leq 1$, then the system (3.1) admits a unique globally bounded classical solution (u, v, w) . Furthermore, such classical solution (u, v, w) may be strictly positive which can be ensured by choosing some suitable $0 \leq u_0 \in C(\bar{\Omega})$ from Theorem 1.1. Thus let us assume that the classical solution (u, v, w) to system (3.1) is strictly positive throughout the proof of Theorem 1.2. For the convenience, we repeat the description stated in (1.8) and (1.9), i.e. there exists $R > 0$ which does not depend on $a, b, \alpha, \beta, a_0, b_0$ and κ such that

$$0 < u(x, t) \leq R \quad (3.2)$$

holds on $\bar{\Omega} \times [0, \infty)$. Moreover, we can also find $\lambda > 0$ independent of a, b, a_0, b_0 and κ such that

$$(u + 1)^{2\beta - \alpha - 2} \leq \lambda \quad (3.3)$$

holds on $\bar{\Omega} \times [0, \infty)$.

In order to prove Theorem 1.2, we introduce a useful lemma.

Lemma 3.1 (cf. [1, Lemma 3.1.]). *Let $g : (t_0, \infty) \rightarrow [0, \infty)$ be uniformly continuous such that $\int_{t_0}^{\infty} g(t)dt < \infty$ with $t_0 > 0$. Then*

$$g(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.4)$$

The key to prove Theorem 1.2 relies on seeking so-called Lyapunov functional inspired from [1, 7]. In the following, we need to construct appropriate energy functionals to system (3.1), which is prepared for the proof of Theorem 1.2.

Lemma 3.2. Let $0 \leq u_0 \in C(\bar{\Omega})$ and $a, b, \kappa > 0$. Assume that $\varphi, \psi \in C^2[0, \infty)$ satisfy (1.5) and (1.6) with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. If $\beta \leq 1$ and the condition (1.10) in Theorem 1.2 holds, then the solution (u, v, w) has the following L^2 -convergence

$$\int_{\Omega} \left(u - \left(\frac{b}{a} \right)^{\frac{1}{\kappa}} \right)^2 + \int_{\Omega} \left(v - \frac{b}{a} \right)^2 + \int_{\Omega} \left(w - \frac{b}{a} \right)^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.5)$$

Proof. For $\kappa \in (0, 1]$, we define the functional

$$A(t) = \int_{\Omega} u - c - c \ln \left(\frac{u}{c} \right), \quad t > 0, \quad (3.6)$$

for $u > 0$, with $c = \left(\frac{a}{b} \right)^{\frac{1}{\kappa}}$. By taking derivative, we can easily obtain that $a(s) = s - c - c \ln \left(\frac{s}{c} \right)$ with $s > 0$ has global minimum zero at $s = c$. Hence, $A(t) \geq 0$ for all $t \geq 0$.

Using Young's inequality and the fact (3.3), we deduce from the first equation of system (3.1)

$$\begin{aligned} \frac{d}{dt} A(t) &= \int_{\Omega} \frac{u-c}{u} u_t \\ &= \int_{\Omega} \frac{u-c}{u} [\nabla \cdot (\varphi(u) \nabla u - \psi(u) \nabla v) + u(a - bu^{\kappa})] \\ &= -c \int_{\Omega} \varphi(u) \frac{|\nabla u|^2}{u^2} + c \int_{\Omega} \psi(u) \frac{\nabla u \cdot \nabla v}{u^2} - b \int_{\Omega} (u-c)(u^{\kappa} - c^{\kappa}) \\ &\leq -a_0 c \int_{\Omega} (u+1)^{\alpha} \frac{|\nabla u|^2}{u^2} + \frac{a_0 c}{\lambda} \int_{\Omega} (u+1)^{2\beta} \frac{|\nabla u|^2}{u^2} + \frac{\lambda b_0^2 c}{4a_0} \int_{\Omega} |\nabla v|^2 \\ &\quad - b \int_{\Omega} (u-c)(u^{\kappa} - c^{\kappa}) \\ &\leq \frac{\lambda b_0^2 c}{4a_0} \int_{\Omega} |\nabla v|^2 - b \int_{\Omega} (u-c)(u^{\kappa} - c^{\kappa}). \end{aligned} \quad (3.7)$$

Multiplying the third equation in system (3.1) by $w - c^{\kappa}$, we get

$$\int_{\Omega} |\nabla w|^2 = - \int_{\Omega} (w - c^{\kappa})^2 + \int_{\Omega} (w - c^{\kappa})(u^{\kappa} - c^{\kappa}). \quad (3.8)$$

Similarly, multiplying the second equation in system (3.1) by $v - c^{\kappa}$, we derive

$$\int_{\Omega} |\nabla v|^2 = - \int_{\Omega} (v - c^{\kappa})^2 + \int_{\Omega} (v - c^{\kappa})(w - c^{\kappa}). \quad (3.9)$$

Substituting (3.8) and (3.9) into (3.7), by Young's inequality we see

$$\begin{aligned} \frac{d}{dt} A(t) &\leq -b \int_{\Omega} (u-c)(u^{\kappa} - c^{\kappa}) + \frac{\lambda b_0^2 c}{4a_0} \int_{\Omega} |\nabla v|^2 - \frac{\lambda b_0^2 c}{4a_0} \int_{\Omega} |\nabla v|^2 \\ &\quad - \frac{\lambda b_0^2 c}{4a_0} \int_{\Omega} (v - c^{\kappa})^2 + \frac{\lambda b_0^2 c}{4a_0} \int_{\Omega} (v - c^{\kappa})(w - c^{\kappa}) - \frac{\lambda b_0^2 c}{8a_0} \int_{\Omega} |\nabla w|^2 \\ &\quad - \frac{\lambda b_0^2 c}{8a_0} \int_{\Omega} (w - c^{\kappa})^2 + \frac{\lambda b_0^2 c}{8a_0} \int_{\Omega} (w - c^{\kappa})(u^{\kappa} - c^{\kappa}) \\ &\leq -b \int_{\Omega} (u-c)(u^{\kappa} - c^{\kappa}) + \frac{\lambda b_0^2 c}{16a_0} \int_{\Omega} (w - c^{\kappa})^2 - \frac{\lambda b_0^2 c}{8a_0} \int_{\Omega} (w - c^{\kappa})^2 \\ &\quad + \frac{\lambda b_0^2 c}{8a_0} \int_{\Omega} (w - c^{\kappa})(u^{\kappa} - c^{\kappa}) \\ &\leq -b \int_{\Omega} (u-c)(u^{\kappa} - c^{\kappa}) + \frac{\lambda b_0^2 c}{16a_0} \int_{\Omega} (u^{\kappa} - c^{\kappa})^2. \end{aligned} \quad (3.10)$$

For $\kappa \in (0, 1]$, we have the following basic inequality

$$(u^\kappa - c^\kappa)^2 \leq c^{\kappa-1}(u - c)(u^\kappa - c^\kappa). \quad (3.11)$$

Thus, from (3.10) and (3.11), we derive

$$\begin{aligned} \frac{d}{dt} A(t) &\leq -\left(b - \frac{\lambda b_0^2 c^\kappa}{16a_0}\right) \int_{\Omega} (u - c)(u^\kappa - c^\kappa) \\ &= -\delta \int_{\Omega} (u - c)(u^\kappa - c^\kappa), \end{aligned} \quad (3.12)$$

where $\delta = b - \frac{\lambda b_0^2 c^\kappa}{16a_0}$. For any $t_0 \geq 0$, integrating both sides of (3.12) on $[t_0, t]$, one can obtain

$$A(t) - A(t_0) \leq -\delta \int_{t_0}^t \int_{\Omega} (u - c)(u^\kappa - c^\kappa). \quad (3.13)$$

Since $A(t) \geq 0$ and δ is nonnegative ensured by $b > \frac{b_0}{4} \sqrt{\frac{\lambda a}{a_0}}$. Thus

$$\int_{t_0}^t \int_{\Omega} (u - c)(u^\kappa - c^\kappa) \leq \frac{A(t_0)}{\delta} < \infty. \quad (3.14)$$

From Theorem 1.1, we know that (u, v, w) is a globally bounded classical solution. Hence, by standard parabolic regularity for parabolic equations [17], we can find $\sigma \in (0, 1)$ and $C > 0$ such that

$$\|u\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} + \|v\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} + \|w\|_{C^{2+\sigma, 1+\frac{\sigma}{2}}(\bar{\Omega} \times [t, t+1])} \leq C, \quad \forall t \geq 1. \quad (3.15)$$

This clearly implies that $\int_{\Omega} (u - c)(u^\kappa - c^\kappa)$ is globally bounded and uniformly continuous with respect to t . Using (3.11) once again, we can obtain from Lemma 3.1

$$\frac{1}{c^{\kappa-1}} \int_{\Omega} (u^\kappa - c^\kappa)^2 \leq \int_{\Omega} (u - c)(u^\kappa - c^\kappa) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.16)$$

On the other hand, using Young's inequality to (3.8), we get

$$\int_{\Omega} |\nabla w|^2 = -\frac{1}{2} \int_{\Omega} (w - c^\kappa)^2 + \frac{1}{2} \int_{\Omega} (u^\kappa - c^\kappa)^2 \quad (3.17)$$

and so

$$\int_{\Omega} (w - c^\kappa)^2 \leq \int_{\Omega} (u^\kappa - c^\kappa)^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.18)$$

Similarly,

$$\int_{\Omega} |\nabla v|^2 = -\frac{1}{2} \int_{\Omega} (v - c^\kappa)^2 + \frac{1}{2} \int_{\Omega} (w - c^\kappa)^2 \quad (3.19)$$

and so

$$\int_{\Omega} (v - c^\kappa)^2 \leq \int_{\Omega} (w - c^\kappa)^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.20)$$

Define $z(s) = s^{\frac{1}{\kappa}}$. By mean value theorem and (3.2), one may obtain

$$u - c = z(u^\kappa) - z(c^\kappa) = \frac{1}{\kappa} \xi^{\frac{1-\kappa}{\kappa}} (u^\kappa - c^\kappa) \quad (3.21)$$

for some ζ between R^κ and c^κ . Thus

$$\int_{\Omega} (u - c)^2 \leq \frac{1}{\kappa^2} R^{\frac{2(1-\kappa)}{\kappa}} \int_{\Omega} (u^\kappa - c^\kappa)^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.22)$$

Therefore, from (3.18), (3.20) and (3.22), we can get (3.5) for $\kappa \in (0, 1]$.

For $\kappa \in (1, +\infty)$, we define the following functional

$$H(t) = \frac{1}{\kappa} \int_{\Omega} \left(u^\kappa - \frac{a}{b} - \frac{a}{b} \ln \left(\frac{bu^\kappa}{a} \right) \right), \quad t > 0, \quad (3.23)$$

for $u > 0$. We can easily obtain the function $h(s) = s - \frac{a}{b} - \frac{a}{b} \ln \left(\frac{bs}{a} \right)$ has global minimum zero over $(0, \infty)$ at $s = \frac{a}{b}$. Thus

$$H(t) = \frac{1}{\kappa} \int_{\Omega} h(u^\kappa) \geq 0 \quad \text{for all } t \geq 0. \quad (3.24)$$

By Young's inequality, we can obtain from (1.5)–(1.6) and (3.2)–(3.3) that

$$\begin{aligned} \frac{d}{dt} H(t) &= \int_{\Omega} \frac{u^\kappa - \frac{a}{b}}{u} u_t \\ &= \int_{\Omega} \frac{u^\kappa - \frac{a}{b}}{u} [\nabla \cdot (\varphi(u) \nabla u - \psi(u) \nabla v) + u(a - bu^\kappa)] \\ &= -\frac{a}{b} \int_{\Omega} \varphi(u) \frac{|\nabla u|^2}{u^2} + \frac{a}{b} \int_{\Omega} \psi(u) \frac{\nabla u \cdot \nabla v}{u^2} - (\kappa - 1) \int_{\Omega} u^{\kappa-2} \varphi(u) |\nabla u|^2 \\ &\quad + (\kappa - 1) \int_{\Omega} u^{\kappa-2} \psi(u) \nabla u \cdot \nabla v - b \int_{\Omega} \left(u^\kappa - \frac{a}{b} \right)^2 \\ &\leq -\frac{aa_0}{b} \int_{\Omega} (u+1)^\alpha \frac{|\nabla u|^2}{u^2} + \frac{ab_0}{b} \int_{\Omega} (u+1)^{\beta-1} \frac{\nabla u \cdot \nabla v}{u} - (\kappa - 1) \int_{\Omega} u^{\kappa-2} \varphi(u) |\nabla u|^2 \\ &\quad + (\kappa - 1) \int_{\Omega} u^{\kappa-2} \psi(u) \nabla u \cdot \nabla v - b \int_{\Omega} \left(u^\kappa - \frac{a}{b} \right)^2 \\ &\leq \frac{\lambda ab_0^2}{4ba_0} \int_{\Omega} |\nabla v|^2 - (\kappa - 1) \int_{\Omega} \left(\sqrt{\varphi(u)} u^{\frac{\kappa}{2}-1} \nabla u - \frac{\psi(u)}{2\sqrt{\varphi(u)}} u^{\frac{\kappa}{2}-1} \nabla v \right)^2 \\ &\quad + \frac{\kappa - 1}{4} \int_{\Omega} \frac{\psi^2(u)}{\varphi(u)} u^{\kappa-2} |\nabla v|^2 - b \int_{\Omega} \left(u^\kappa - \frac{a}{b} \right)^2 \\ &\leq \frac{\lambda b_0^2}{4a_0} \left[\frac{a}{b} + (\kappa - 1) R^\kappa \right] \int_{\Omega} |\nabla v|^2 - b \int_{\Omega} \left(u^\kappa - \frac{a}{b} \right)^2 \\ &= \vartheta \int_{\Omega} |\nabla v|^2 - b \int_{\Omega} \left(u^\kappa - \frac{a}{b} \right)^2 \end{aligned} \quad (3.25)$$

where $\vartheta = \frac{\lambda b_0^2}{4a_0} \left[\frac{a}{b} + (\kappa - 1) R^\kappa \right]$. Multiplying the second equation in system (3.1) by $(v - \frac{a}{b})$, we have

$$\int_{\Omega} |\nabla v|^2 = - \int_{\Omega} \left(v - \frac{a}{b} \right)^2 + \int_{\Omega} \left(v - \frac{a}{b} \right) \left(w - \frac{a}{b} \right). \quad (3.26)$$

Similarly, for the third equation, we get

$$\int_{\Omega} |\nabla w|^2 = - \int_{\Omega} \left(w - \frac{a}{b} \right)^2 + \int_{\Omega} \left(w - \frac{a}{b} \right) \left(u^\kappa - \frac{a}{b} \right). \quad (3.27)$$

Combining (3.26), (3.27) with (3.25) and using Young's inequality, we obtain

$$\begin{aligned}
 \frac{d}{dt}H(t) &\leq -\vartheta \int_{\Omega} \left(v - \frac{a}{b}\right)^2 + \vartheta \int_{\Omega} \left(v - \frac{a}{b}\right) \left(w - \frac{a}{b}\right) - b \int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2 \\
 &\quad - \frac{\vartheta}{2} \int_{\Omega} |\nabla w|^2 - \frac{\vartheta}{2} \int_{\Omega} \left(w - \frac{a}{b}\right)^2 + \frac{\vartheta}{2} \int_{\Omega} \left(w - \frac{a}{b}\right) \left(u^{\kappa} - \frac{a}{b}\right) \\
 &\leq \frac{\vartheta}{4} \int_{\Omega} \left(w - \frac{a}{b}\right)^2 - b \int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2 - \frac{\vartheta}{2} \int_{\Omega} \left(w - \frac{a}{b}\right)^2 \\
 &\quad + \frac{\vartheta}{4} \int_{\Omega} \left(w - \frac{a}{b}\right)^2 + \frac{\vartheta}{4} \int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2 \\
 &= -\epsilon \int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2
 \end{aligned} \tag{3.28}$$

where $\epsilon = b - \frac{\vartheta}{4}$. By the assumption (1.10) in Theorem 1.2, we know that $\epsilon > 0$. Then for any $t_0 \geq 0$, an integration of the inequality (3.28) from t_0 to t entails

$$H(t) - H(t_0) \leq -\epsilon \int_{t_0}^t \int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2. \tag{3.29}$$

Thus the nonnegativity of H yields

$$\int_{t_0}^{\infty} \int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2 \leq \frac{H(t_0)}{\epsilon} < \infty. \tag{3.30}$$

From Lemma 3.1, the global boundedness and uniform continuity of $\int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2$ in t entails

$$\int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{3.31}$$

A simple use of Young's inequality to (3.27) immediately shows

$$\int_{\Omega} |\nabla w|^2 \leq -\frac{1}{2} \int_{\Omega} \left(w - \frac{a}{b}\right)^2 + \frac{1}{2} \int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2 \tag{3.32}$$

and so

$$\int_{\Omega} \left(w - \frac{a}{b}\right)^2 \leq \int_{\Omega} \left(u^{\kappa} - \frac{a}{b}\right)^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{3.33}$$

Similarly, we have

$$\int_{\Omega} |\nabla v|^2 \leq -\frac{1}{2} \int_{\Omega} \left(v - \frac{a}{b}\right)^2 + \frac{1}{2} \int_{\Omega} \left(w - \frac{a}{b}\right)^2. \tag{3.34}$$

Thus

$$\int_{\Omega} \left(v - \frac{a}{b}\right)^2 \leq \int_{\Omega} \left(w - \frac{a}{b}\right)^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{3.35}$$

Since $\kappa \in (1, \infty)$, then there exists a constant $M > 0$ such that

$$M = \sup_{z \in (0, \infty)} \frac{\left(z - \left(\frac{a}{b}\right)^{\frac{1}{\kappa}}\right)^2}{\left(z^{\kappa} - \frac{a}{b}\right)^2} < \infty. \tag{3.36}$$

Therefore

$$\int_{\Omega} \left(u - \left(\frac{a}{b} \right)^{\frac{1}{\kappa}} \right)^2 \leq M \int_{\Omega} \left(u^{\kappa} - \frac{a}{b} \right)^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.37)$$

This completes the proof of L^2 -convergence of the solution to system (3.1). \square

Proof of Theorem 1.2. In view of the Gagliardo–Nirenberg inequality [4], we conclude from (3.5), (3.15), (3.22) and (3.37) that

$$\begin{aligned} \left\| u(\cdot, t) - \left(\frac{a}{b} \right)^{\frac{1}{\kappa}} \right\|_{L^{\infty}(\Omega)} &\leq C_{GN} \left\| u(\cdot, t) - \left(\frac{a}{b} \right)^{\frac{1}{\kappa}} \right\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \left\| u(\cdot, t) - \left(\frac{a}{b} \right)^{\frac{1}{\kappa}} \right\|_{L^2}^{\frac{2}{n+2}} \\ &\leq C \left\| u(\cdot, t) - \left(\frac{a}{b} \right)^{\frac{1}{\kappa}} \right\|_{L^2}^{\frac{2}{n+2}} \\ &\leq C_{\kappa} \left\| u^{\kappa}(\cdot, t) - \frac{a}{b} \right\|_{L^2}^{\frac{2}{n+2}} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (3.38)$$

For $\kappa \in (0, 1]$, by the L'Hospital rule, we get

$$\lim_{u \rightarrow c} \frac{a(u)}{(u-c)(u^{\kappa} - c^{\kappa})} = \lim_{u \rightarrow c} \frac{u - c - c \ln(\frac{u}{c})}{(u-c)(u^{\kappa} - c^{\kappa})} = \frac{1}{2\kappa c^{\kappa}}, \quad c = \left(\frac{a}{b} \right)^{\kappa}. \quad (3.39)$$

Based on (3.38) and (3.39), we choose $t_1 > 0$ such that

$$\frac{1}{4\kappa c^{\kappa}} (u-c)(u^{\kappa} - c^{\kappa}) \leq a(u) \leq \frac{1}{\kappa c^{\kappa}} (u-c)(u^{\kappa} - c^{\kappa}), \quad t \geq t_1, \quad (3.40)$$

and so

$$\frac{1}{4\kappa c^{\kappa}} \int_{\Omega} (u-c)(u^{\kappa} - c^{\kappa}) \leq A(t) \leq \frac{1}{\kappa c^{\kappa}} \int_{\Omega} (u-c)(u^{\kappa} - c^{\kappa}), \quad t \geq t_1. \quad (3.41)$$

Using (3.12) and (3.41), we get

$$\frac{d}{dt} A(t) \leq -\delta \kappa c^{\kappa} A(t), \quad t \geq t_1, \quad (3.42)$$

thus

$$A(t) \leq A(t_1) e^{-\delta \kappa c^{\kappa} (t-t_1)}, \quad t \geq t_1. \quad (3.43)$$

From (3.11), (3.38), (3.41) and (3.42), we can deduce

$$\begin{aligned} \|u(\cdot, t) - c\|_{L^{\infty}(\Omega)} &\leq C_{\kappa} \|u^{\kappa}(\cdot, t) - c^{\kappa}\|_{L^2}^{\frac{2}{n+2}} \\ &\leq C_{\kappa} \left[\int_{\Omega} (u^{\kappa} - c^{\kappa})^2 \right]^{\frac{1}{n+2}} \\ &\leq C_{\kappa} \left[\int_{\Omega} c^{\kappa-1} (u-c)(u^{\kappa} - c^{\kappa}) \right]^{\frac{1}{n+2}} \\ &\leq C_{\kappa} \left[4\kappa c^{2\kappa-1} A(u) \right]^{\frac{1}{n+2}} \\ &\leq C_{\kappa} (4\kappa c^{2\kappa-1} A(t_1))^{\frac{1}{n+2}} e^{-\frac{\kappa \delta c^{\kappa} (t-t_1)}{n+2}}, \quad t \geq t_1. \end{aligned} \quad (3.44)$$

Repeating the similar steps for w and v , we can obtain from (3.18), (3.19) and (3.44)

$$\|w(\cdot, t) - c^\kappa\|_{L^\infty(\Omega)} \leq C_\kappa \left(4\kappa c^{2\kappa-1} A(t_1)\right)^{\frac{1}{n+2}} e^{-\frac{\kappa\delta c^\kappa(t-t_1)}{n+2}}, \quad t \geq t_1 \quad (3.45)$$

and

$$\|v(\cdot, t) - c^\kappa\|_{L^\infty(\Omega)} \leq C_\kappa \left(4\kappa c^{2\kappa-1} A(t_1)\right)^{\frac{1}{n+2}} e^{-\frac{\kappa\delta c^\kappa(t-t_1)}{n+2}}, \quad t \geq t_1. \quad (3.46)$$

For $\kappa \in (1, \infty)$, using the the L'Hospital rule, we deduce

$$\lim_{u \rightarrow c} \frac{h(u^\kappa)}{(u^\kappa - c^\kappa)^{2\kappa}} = \lim_{z \rightarrow c^\kappa} \frac{z - c^\kappa - c^\kappa \ln\left(\frac{z}{c^\kappa}\right)}{(z - c^\kappa)^{2\kappa}} = \frac{c^{\kappa-2}}{2\kappa}. \quad (3.47)$$

From (3.38) and (3.47), we pick $t_2 \geq 0$ such that

$$\frac{c^{\kappa-2}}{4\kappa} \int_\Omega (u^\kappa - c^\kappa)^2 \leq H(t) \leq \frac{c^{\kappa-2}}{\kappa} \int_\Omega (u^\kappa - c^\kappa)^2, \quad t \geq t_2. \quad (3.48)$$

Using (3.28) and (3.48), we get

$$\frac{d}{dt} H(t) \leq -\epsilon\kappa c^{2-\kappa} H(t), \quad t \geq t_2, \quad (3.49)$$

which implies

$$H(t) \leq H(t_2) e^{-\epsilon\kappa c^{2-\kappa}(t-t_2)}, \quad t \geq t_2. \quad (3.50)$$

From (3.38), (3.48) and (3.50), we infer that

$$\begin{aligned} \|u(\cdot, t) - c\|_{L^\infty(\Omega)} &\leq C_\kappa \|u^\kappa(\cdot, t) - c^\kappa\|_{L^2}^{\frac{2}{n+2}} \\ &\leq C_\kappa (4\kappa c^{2\kappa-1} H(t_2))^{\frac{1}{n+2}} e^{-\frac{\epsilon\kappa c^{2-\kappa}(t-t_2)}{n+2}}, \quad t \geq t_2. \end{aligned} \quad (3.51)$$

Analogously, taking (3.33), (3.35) and (3.51) into account, we can obtain

$$\|w(\cdot, t) - c^\kappa\|_{L^\infty(\Omega)} \leq C_\kappa (4\kappa c^{2\kappa-1} H(t_2))^{\frac{1}{n+2}} e^{-\frac{\epsilon\kappa c^{2-\kappa}(t-t_2)}{n+2}}, \quad t \geq t_2 \quad (3.52)$$

and

$$\|v(\cdot, t) - c^\kappa\|_{L^\infty(\Omega)} \leq C_\kappa (4\kappa c^{2\kappa-1} H(t_2))^{\frac{1}{n+2}} e^{-\frac{\epsilon\kappa c^{2-\kappa}(t-t_2)}{n+2}}, \quad t \geq t_2. \quad (3.53)$$

Finally, plugging δ and ϵ into (3.44)–(3.46) and (3.51)–(3.53), we take C_κ large enough and then complete the proof of Theorem 1.2. \square

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