# A Dynamic Survey of Graph Labeling

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**Abstract**

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the late 1960s. In the intervening years dozens of graph labelings techniques have been studied in over 1000 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

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# 1 Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [973] in 1967, or one given by Graham and Sloane [484] in 1980. Rosa [973] called a function $f$ a *$\beta$-valuation* of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [475] subsequently called such labelings *graceful* and this is now the popular term. Rosa introduced $\beta$-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, $\beta$-valuations originated as a means of attacking the conjecture of Ringel [962] that $K_{2n+1}$ can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with $n$ edges. Although an unpublished result of Erdős says that most graphs are not graceful (cf. [484]), most graphs that have some sort of regularity of structure are graceful. Sheppard [1059] has shown that there are exactly $q!$ gracefully labeled graphs with $q$ edges. Rosa [973] has identified essentially three reasons why a graph fails to be graceful: (1) $G$ has "too many vertices" and "not enough edges," (2) $G$ "has too many edges," and (3) $G$ "has the wrong parity." An infinite class of graphs that are not graceful for the second reason is given in [229]. As an example of the third condition Rosa [973] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles $C_{4n+1}$ and $C_{4n+2}$ are not graceful.

Acharya [12] proved that every graph can be embedded as an induced subgraph of a graceful graph and a connected graph can be embedded as an induced subgraph of a graceful connected graph. Acharya, Rao, and Arumugam [30] proved: every triangle-free graph can be embedded as an induced subgraph of a triangle-free graceful graph; every planar graph can be embedded as an induced subgraph of a planar graceful graph; and every tree can be embedded as an induced subgraph of a graceful tree. These results demonstrate that there is no forbidden subgraph characterization of these particular kinds of graceful graphs.

Harmonious graphs naturally arose in the study by Graham and Sloane [484] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph $G$ with $q$ edges to be *harmonious* if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When $G$ is a tree, exactly one label may be used on two vertices. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number of edges $q$ and the degree of every vertex is divisible by $2^k$ then $q$ is divisible by $2^{k+1}$. Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [812] have generalized this condition as follows: if a harmonious graph with $q$ edges has degree sequence $d_1, d_2, \ldots, d_p$ then $\gcd(d_1, d_2, \ldots d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph. More generally, Sethuraman and Elumalai [1029] have shown that any given set of graphs $G_1, G_2, \ldots, G_t$ can be embedded in a graceful or harmonious graph. Determining whether a graph has a harmonious labeling was shown to be NP-complete by Auparajita, Dulawat, and Rathore in 2001 (see [683]).

In the early 1980s Bloom and Hsu [238], [239],[223] extended graceful labelings to directed graphs by defining a graceful labeling on a directed graph $D(V, E)$ as a one-to-one map $\theta$ from $V$ to $\{0, 1, 2, \ldots, |E|\}$ such that $\theta(y) - \theta(x) \pmod{(|E| + 1)}$ is distinct for every edge $xy$ in $E$. Graceful labelings of directed graphs also arose in the characterization of finite neofields by Hsu and Keedwell [560], [561]. Graceful labelings of directed graphs was the subject of Marr's 2007

Ph.D. dissertation [843]. In [843] and [844] Marr presents results of graceful labelings of directed paths, stars, wheels, and umbrellas.

Over the past four decades in excess of 1200 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes of graphs have been done by several authors and in some cases the same terminology is used for different concepts. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two labeling methods, include [217], [234], [655], [442], and [444]. The book edited by Acharya, Arumugam, and Rosa [17] includes a variety of labeling methods that we do not discuss in this survey. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings, and neofields is discussed in [238] and [239]. The connection between graceful labelings and perfect systems of difference sets is given in [220]. Bloom and Hsu [240] extended the notion of graceful labeling to directed graphs. (See also [283]). Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, secret sharing schemes,and models for constraint programming over finite domains–see [235], [236], [1159], [942], [1113], [1112], [1102] and [858] for details. Terms and notation not defined below follow that used in [304] and [442].

## 2 Graceful and Harmonious Labelings

### 2.1 Trees

The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers. Kotzig [564] has called the effort to prove it a "disease." Among the trees known to be graceful are: caterpillars [973] (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [564], [1323] and [611]; trees with diameter at most 5 [1323] and [557]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [221], [939]; rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity [282]; rooted trees with diameter $D$ where every vertex has even degree except for one root and the leaves in level $\lfloor D/2 \rfloor$ [176]; rooted trees with diameter $D$ where every vertex has even degree except for one root and the leaves, which are in level $\lfloor D/2 \rfloor$ [176]; rooted trees with diameter $D$ where every vertex has even degree except for one root, the vertices in level $\lfloor D/2 \rfloor - 1$, and the leaves which are in level $\lfloor D/2 \rfloor$ [176]; the graph obtained by identifying the endpoints any number of paths of a fixed length except for the case that the length has the form $4r+1$, $r > 1$ and the number of paths is of the form $4m$ with $m > r$ [1000]; regular bamboo trees [1000] (a rooted tree consisting of branches of equal length the endpoints of which are identified with end points of stars of equal size); and olive trees [924], [3] (a rooted tree consisting of $k$ branches, where the $i$th branch is a path of length $i$); Bahls, Lake, and Wertheim [160] proved that spiders for which the lengths of every path from the center to a leaf differ by at most one are graceful. (A *spider* is a tree that has at most one vertex (called the *center*) of degree greater than 2.) Motivated by Horton's work [555], in 2010 Fang [405] used a deterministic back-tracking algorithm to prove that all trees with at most 35 vertices are graceful. In 2011 Fang [406] used a hybrid algorithm that involved probabilistic backtracking, tabu searching, and constraint programming satisfaction to verify that every tree with at most 31 vertices is harmonious.

Aldred, Širáň and Širáň [52] have proved that the number of graceful labelings of $P_n$ grows at least as fast as $(5/3)^n$. They mention that this fact has an application to topological graph theory. One such application was provided by Goddyn, Richter, and and Širáň [473] who used graceful labelings of paths on $2s+1$ vertices ($s \geq 2$) to obtain $2^{2s}$ cyclic oriented triangular embeddings of the complete graph on $12s+7$ vertices. The Aldred, Širáň and Širáň bound was improved by Adamaszek [36] to $(2.37)^n$ with the aid of a computer. Cattell [291] has shown that when finding a graceful labeling of a path one has almost complete freedom to choose a particular label $i$ for any given vertex $v$. In particular, he shows that the only cases of $P_n$ when this cannot be done are when $n \equiv 3 \pmod 4$ or $n \equiv 1 \pmod{12}$, $v$ is in the smaller of the two partite sets of vertices, and $i = (n-1)/2$.

In [397] and [398] Eshghi and Azimi [397] discuss a programming model for finding graceful labelings of large graphs. The computational results show that the models can easily solve the graceful labeling problems for large graphs. They used this method to verify that all trees with 30, 35, or 40 vertices are graceful. Stanton and Zarnke [1133] and Koh, Rogers, and Tan [656], [657], [660] gave methods for combining graceful trees to yield larger graceful trees. Rogers in [971] and Koh, Tan, and Rogers in [659] provide recursive constructions to create graceful trees. Burzio and Ferrarese [269] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. In 1979 Bermond [217] conjectured that lobsters are graceful (a *lobster* is a tree with the property that the removal of the endpoints leaves a

caterpillar). Morgan [880] has shown that all lobsters with perfect matchings are graceful.

A *Skolem sequence* of order $n$ is a sequence $s_1, s_2, \ldots, s_{2n}$ of $2n$ terms such that, for each $k \in \{1, 2, \ldots, n\}$, there exist exactly two subscripts $i(k)$ and $j(k)$ with $s_{i(k)} = s_{j(k)} = k$ and $|i(k) - j(k)| = k$. A Skolem sequence of order $n$ exists if and only if $n \equiv 0$ or $1 \pmod 4$. Morgan [881] has used Skolem sequences to construct classes of graceful trees. Morgan and Rees [882] used Skolem and Hooked-Skolem sequences to generate classes of graceful lobsters.

Mishra and Panigrahi [874] and [918] found classes of graceful lobsters of diameter at least five. They show other classes of lobsters are graceful in [875] and [876]. In [1033] Sethuraman and Jesintha [1033] explores how one can generate graceful lobsters from a graceful caterpillar while in [1037] and [1038] (see also [592]) they show how to generate graceful trees from a graceful star. More special cases of Bermond's conjecture have been done by Ng [896], by Wang, Jin, Lu, and Zhang [1233], Abhyanker [2], and by Mishra and Panigrahi [875]. Whether or not lobsters are harmonious seems to have attracted no attention thus far.

Barrientos [195] defines a *y-tree* as a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point. He proves that graphs obtained from a $y$-tree $T$ by replacing every edge $e_i$ of $T$ by a copy of $K_{2,n_i}$ in such a way that the ends of $e_i$ are merged with the two independent vertices of $K_{2,n_i}$ after removing the edge $e_i$ from $T$ are graceful.

Sethuraman and Jesintha [1034], [1035] and [1036] (see also [592]) proved that rooted trees obtained by identifying one of the end vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. They also proved that rooted trees obtained by identifying either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. In [1034], [1035], and [1036] (see also [592] and [593]) Sethuraman and Jesintha prove that all rooted trees in which every level contains pendant vertices and the degrees of the internal vertices in the same level are equal are graceful. Kanetkar and Sane [623] show that trees formed by identifying one end vertex of each of six or fewer paths whose lengths determine an arithmetic progression are graceful.

Chen, Lü, and Yeh [310] define a *firecracker* as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a *banana tree* as a graph obtained by connecting a vertex $v$ to one leaf of each of any number of stars ($v$ is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Before Sethuraman and Jesintha [1040] and [1039] (see also [592]) proved that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful, various kinds of bananas trees had been shown to be graceful by Bhat-Nayak and Deshmukh [224], by Murugan and Arumugam [890], [888] and by Vilfred [1211].

Consider a set of caterpillars, having equal diameter, in which one of the penultimate vertices has arbitrary degree and all the other internal vertices including the other penultimate vertex is of fixed even degree. Jesintha and Sethuraman [595] call the rooted tree obtained by merging an end-vertex adjacent to the penultimate vertex of fixed even degree of each caterpillar a *arbitrarily fixed generalized banana tree*. They prove that such trees are graceful. From this it follows that all banana trees are graceful and all generalized banana trees are graceful.

Zhenbin [1325] has shown that graphs obtained by starting with any number of identical stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful. He also shows that graphs

obtained by starting with any two stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful. In [594] Jesintha and Sethuraman use a method of Hrnciar and Havier [557] to generate graceful trees from a graceful star with $n$ edges.

Aldred and McKay [51] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious has been shown by Graham and Sloane [484]. In a paper published in 2004 Krishnaa [681] claims to proved that all trees have both graceful and harmonious labelings. However, her proofs were flawed.

Using a variant of the Matrix Tree Theorem, Whitty [1256] specifies an $n \times n$ matrix of indeterminates whose determinant is a multivariate polynomial that enumerates the gracefully labelled $(n+1)$-vertex trees. Whitty also gives a bijection between gracefully labelled graphs and rook placements on a chessboard on the Möbius strip.

Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. More specialized results about trees are contained in [217], [234], [655], [827], [276], [610], and [974]. In [377] Edwards and Howard provide a lengthy survey paper on graceful trees.

## 2.2 Cycle-Related Graphs

Cycle-related graphs have been a major focus of attention. Rosa [973] showed that the $n$-cycle $C_n$ is graceful if and only if $n \equiv 0$ or 3 (mod 4) and Graham and Sloane [484] proved that $C_n$ is harmonious if and only if $n \equiv 1$ or 3 (mod 4). Wheels $W_n = C_n + K_1$ are both graceful and harmonious – [428], [554] and [484]. As a consequence we have that a subgraph of a graceful (harmonious) graph need not be graceful (harmonious). The *$n$-cone* (also called the *$n$-point suspension* of $C_m$) $C_m + \overline{K_n}$ has been shown to be graceful when $m \equiv 0$ or 3 (mod 12) by Bhat-Nayak and Selvam [230]. When $n$ is even and $m$ is 2, 6 or 10 (mod 12) $C_m + \overline{K_n}$ violates the parity condition for a graceful graph. Bhat-Nayak and Selvam [230] also prove that the following cones are graceful: $C_4 + \overline{K_n}, C_5 + \overline{K_2}, C_7 + \overline{K_n}, C_9 + \overline{K_2}, C_{11} + \overline{K_n}$ and $C_{19} + \overline{K_n}$. The *helm* $H_n$ is the graph obtained from a wheel by attaching a pendant edge at each vertex of the $n$-cycle. Helms have been shown to be graceful [79] and harmonious [471], [808], [809] (see also [812], [1022], [799], [349] and [952]). Koh, Rogers, Teo, and Yap, [658] define a *web* graph as one obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. They asked whether such graphs are graceful. This was proved by Kang, Liang, Gao, and Yang [626]. Yang has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendant points to the new cycle. In his notation, $W(2,n)$ is the web graph whereas $W(t,n)$ is the *generalized web* with $t$ $n$-cycles. Yang has shown that $W(3,n)$ and $W(4,n)$ are graceful (see [626]), Abhyanker and Bhat-Nayak [4] have done $W(5,n)$ and Abhyanker [2] has done $W(t,5)$ for $5 \leq t \leq 13$. Gnanajothi [471] has shown that webs with odd cycles are harmonious. Seoud and Youssef [1022] define a *closed helm* as the graph obtained from a helm by joining each pendant vertex to form a cycle and a *flower* as the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm. They prove that closed helms and flowers are harmonious when the cycles are odd. A *gear graph* is obtained from the wheel $W_n$ by adding a vertex between every pair of adjacent vertices of the $n$-cycle. In 1984 Ma and Feng [832] proved all gears are graceful while in a Master's thesis in 2006 Chen [311] proved all gears are harmonious. Liu [808] has shown that if two or more vertices are inserted between

every pair of vertices of the $n$-cycle of the wheel $W_n$, the resulting graph is graceful. Liu [806] has also proved that the graph obtain from a gear graph by attaching one or more pendant edges to each vertex between the vertices of the $n$-cycle is graceful.

Abhyanker [2] has investigated various unicyclic (that is, graphs with exactly one cycle) graphs. He proved that the unicyclic graphs obtained by identifying one vertex of $C_4$ with the root of the olive tree with $2n$ branches and identifying an adjacent vertex on $C_4$ with the end point of the path $P_{2n-2}$ are graceful. He showed that if one attaches any number of pendent edges to these unicyclic graphs at the vertex of $C_4$ that is adjacent to the root of the olive tree but not adjacent to the end vertex of the attached path, the resulting graphs are graceful. Likewise, Abhyanker proved that the graph obtained by deleting the branch of length 1 from an olive tree with $2n$ branches and identifying the root of the edge deleted tree with a vertex of a cycle of the form $C_{2n+3}$ is graceful. He also has a number of results similar to these.

Delorme, Maheo, Thuillier, Koh, and Teo [352] and Ma and Feng [831] showed that any cycle with a chord is graceful. This was first conjectured by Bodendiek, Schumacher, and Wegner [243], who proved various special cases. In 1985 Koh and Yap [661] generalized this by defining a *cycle with a $P_k$-chord* to be a cycle with the path $P_k$ joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when $k = 3$ and conjectured that all cycles with a $P_k$-chord are graceful. This was proved for $k \geq 4$ by Punnim and Pabhapote in 1987 [943]. Chen [316] obtained the same result except for three cases which were then handled by Gao [499]. In 2005, Sethuraman and Elumalai [1028] defined a *cycle with parallel $P_k$-chords* as a graph obtained from a cycle $C_n$ $(n \geq 6)$ with consecutive vertices $v_0, v_1, \ldots, v_{n-1}$ by adding disjoint paths $P_k$, $(k \geq 3)$, between each pair of nonadjacent vertices $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_iv_{n-i}, \ldots, v_\alpha v_\beta$ where $\alpha = \lfloor n/2 \rfloor - 1$ and $\beta = \lfloor n/2 \rfloor + 2$ if $n$ is odd or $\beta = \lfloor n/2 \rfloor + 1$ if $n$ is even. They proved that every cycle $C_n$ $(n \geq 6)$ with parallel $P_k$-chords is graceful for $k = 3, 4, 6, 8$, and 10 and they conjecture that the cycle $C_n$ with parallel $P_k$-chords is graceful for all even $k$. Xu [1274] proved that all cycles with a chord are harmonious except for $C_6$ in the case where the distance in $C_6$ between the endpoints of the chord is 2. The gracefulness of cycles with consecutive chords have also been investigated. For $3 \leq p \leq n - r$, let $C_n(p, r)$ denote the $n$-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the $r$ chords $v_1v_p, v_1v_{p+1}, \ldots, v_1v_{p+r-1}$ have been added. Koh and Punnin [651] and Koh, Rogers, Teo, and Yap [658] have handled the cases $r = 2, 3$ and $n - 3$ where $n$ is the length of the cycle. Goh and Lim [474] then proved that all remaining cases are graceful. Moreover, Ma [830] has shown that $C_n(p, n - p)$ is graceful when $p \equiv 0, 3 \pmod 4$ and Ma, Liu, and Liu [833] have proved other special cases of these graphs are graceful. Ma also proved that if one adds to the graph $C_n(3, n - 3)$ any number $k_i$ of paths of length 2 from the vertex $v_1$ to the vertex $v_i$ for $i = 2, \ldots, n$, the resulting graph is graceful. Chen [316] has shown that apart from four exceptional cases, a graph consisting of three independent paths joining two vertices of a cycle is graceful. This generalizes the result that a cycle plus a chord is graceful. Liu [805] has shown that the $n$-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the chords $v_1v_k$ and $v_1v_{k+2}$ $(2 \leq k \leq n - 3)$ are adjoined is graceful.

In [350] Deb and Limaye use the notation $C(n, k)$ to denote the cycle $C_n$ with $k$ cords sharing a common endpoint called the *apex*. For certain choices of $n$ and $k$ there is a unique $C(n, k)$ graph and for other choices there is more than one graph possible. They call these *shell-type* graphs and they call the unique graph $C(n, n-3)$ a *shell*. Notice that the shell $C(n, n-3)$ is the same as the fan $F_{n-1} = P_{n-1} + K_1$. Deb and Limaye define a *multiple shell* to be a collection of edge disjoint shells that have their apex in common. A multiple shell is said to be *balanced* with width $w$ if every shell has order $w$ or every shell has order $w$ or $w + 1$. Deb and Limaye [350]

have conjectured that all multiple shells are harmonious, and have shown that the conjecture is true for the balanced double shells and balanced triple shells. Yang, Xu, Xi, and Qiao [1295] proved the conjecture is true for balanced quadruple shells.

Sethuraman and Dhavamani [1025] use $H(n,t)$ to denote the graph obtained from the cycle $C_n$ by adding $t$ consecutive chords incident with a common vertex. If the common vertex is $u$ and $v$ is adjacent to $u$, then for $k \geq 1,\ n \geq 4$, and $1 \leq t \leq n-3$, Sethuraman and Dhavamani denote by $G(n,t,k)$ the graph obtained by taking the union of $k$ copies of $H(n,k)$ with the edge $uv$ identified. They conjecture that every graph $G(n,t,k)$ is graceful. They prove the conjecture for the case that $t = n-3$.

For $i = 1, 2, \ldots, n$ let $v_{i,1}, v_{i,2}, \ldots, v_{i,2m}$ be the successive vertices of $n$ copies of $C_{2m}$. Sekar [1000] defines a *chain of cycles* $C_{2m,n}$ as the graph obtained by identifying $v_{i,m}$ and $v_{i+1,m}$ for $i = 1, 2, \ldots, n-1$. He proves that $C_{6,2k}$ and $C_{8,n}$ are graceful for all $k$ and all $n$. Barrientos [198] proved that all $C_{8,n}$, $C_{12,n}$, and $C_{6,2k}$ are graceful.

Truszczyński [1185] studied unicyclic graphs and proved several classes of such graphs are graceful. Among these are what he calls dragons. A *dragon* is formed by joining the end point of a path to a cycle (Koh, et al. [658] call these *tadpoles*; Kim and Park [647] call them *kites*). This work led Truszczyński to conjecture that all unicyclic graphs except $C_n$, where $n \equiv 1$ or 2 (mod 4), are graceful. Guo [498] has shown that dragons are graceful when the length of the cycle is congruent to 1 or 2 (mod 4). Lu [826] uses $C_n^{+(m,t)}$ to denote the graph obtained by identifying one vertex of $C_n$ with one endpoint of $m$ paths each of length $t$. He proves that $C_n^{+(1,t)}$ (a tadpole) is not harmonious when $a+t$ is odd and $C_n^{+(2m,t)}$ is harmonious when $n = 3$ and when $n = 2k+1$ and $t = k-1, k+1$ or $2k-1$. In his Master's thesis, Doma [364] investigates the gracefulness of various unicyclic graphs where the cycle has up to 9 vertices. Because of the immense diversity of unicyclic graphs, a proof of Truszczyński's conjecture seems out of reach in the near future.

Cycles that share a common edge or a vertex have received some attention. Murugan and Arumugan [889] have shown that books with $n$ pentagonal pages (i.e., $n$ copies of $C_5$ with an edge in common) are graceful when $n$ is even and not graceful when $n$ is odd. Lu [826] uses $\Theta(C_m)^n$ to denote the graph that make from $n$ copies of $C_m$ that share an edge (an $n$ page book with $m$-polygonal pages). He proves $\Theta(C_{2m+1})^{2n+1}$ is harmonious for all $m$ and $n$; $\Theta(C_{4m+2})^{4n+1}$ and $\Theta(C_{4m})^{4n+3}$ are not harmonious for all $m$ and $n$. Xu [1274] proved that $\Theta(C_m)^2$ is harmonious except when $m = 3$. ($\Theta(C_m)^2$ is isomorphic to $C_{2(m-1)}$ with a chord "in the middle.")

Let $C_n^{(t)}$ denote the one-point union of $t$ cycles of length $n$. Bermond, Brouwer, and Germa [218] and Bermond, Kotzig, and Turgeon [220]) proved that $C_3^{(t)}$ (that is, the *friendship graph* or *Dutch $t$-windmill*) is graceful if and only if $t \equiv 0$ or 1 (mod 4) while Graham and Sloane [484] proved $C_3^{(t)}$ is harmonious if and only if $t \not\equiv 2$ (mod 4). Koh, Rogers, Lee, and Toh [652] conjecture that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0$ or 3 (mod 4). Yang and Lin [1287] have proved the conjecture for the case $n = 5$ and Yang, Xu, Xi, Li, and Haque [1293] did the case $n = 7$. Xu, Yang, Li and Xi [1278] did the case $n = 11$. Xu, Yang, Han and Li [1279] did the case $n = 13$. Qian [947] verifies this conjecture for the case that $t = 2$ and $n$ is even and Yang, Xu, Xi, and Li [1294] did the case $n = 9$. Figueroa-Centeno, Ichishima, and Muntaner-Batle [415] have shown that if $m \equiv 0$ (mod 4) then the one-point union of 2, 3, or 4 copies of $C_m$ admits a special kind of graceful labeling called an $\alpha$-labeling (see Section 3.1) and if $m \equiv 2$ (mod 4), then the one-point union of 2 or 4 copies of $C_m$ admits an $\alpha$-labeling. Bodendiek, Schumacher, and Wegner [249] proved that the one-point union of any two cycles is graceful when the number of

edges is congruent to 0 or 3 modulo 4. (The other cases violate the necessary parity condition.) Shee [1055] has proved that $C_4^{(t)}$ is graceful for all $t$. Seoud and Youssef [1020] have shown that the one-point union of a triangle and $C_n$ is harmonious if and only if $n \equiv 1 \pmod 4$ and that if the one-point union of two cycles is harmonious then the number of edges is divisible by 4. The question of whether this latter condition is sufficient is open. Figueroa-Centeno, Ichishima, and Muntaner-Batle [415] have shown that if $G$ is harmonious then the one-point union of an odd number of copies of $G$ using the vertex labeled 0 as the shared point is harmonious. Sethuraman and Selvaraju [1047] have shown that for a variety of choices of points, the one-point union of any number of non-isomorphic complete bipartite graphs is graceful. They raise the question of whether this is true for all choices of the common point.

Another class of cycle-related graphs is that of triangular cacti. The *block-cutpoint graph* of a graph $G$ is a bipartite graph in which one partite set consists of the cut vertices of $G$, and the other has a vertex $b_i$ for each block $B_i$ of $G$. A *block* of a graph is a maximal connected subgraph that has no cut-vertex. A *triangular cactus* is a connected graph all of whose blocks are triangles. A *triangular snake* is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, \ldots, n-1$). Rosa [975] conjectured that all triangular cacti with $t \equiv 0$ or $1 \pmod 4$ blocks are graceful. (The cases where $t \equiv 2$ or $3 \pmod 4$ fail to be graceful because of the parity condition.) Moulton [883] proved the conjecture for all triangular snakes. A proof of the general case (i.e., all triangular cacti) seems hopelessly difficult. Liu and Zhang [812] gave an incorrect proof that triangular snakes with an odd number of triangles are harmonious whereas triangular snakes with $n \equiv 2 \pmod 4$ triangles are not harmonious. Xu [1275] subsequently proved that triangular snakes are harmonious if and only if the number of triangles is not congruent to 2 (mod 4).

A *double triangular snake* consists of two triangular snakes that have a common path. That is, a double triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, \ldots, n-1$ and to a new vertex $u_i$ for $i = 1, 2, \ldots, n-1$. Xi, Yang, and Wang [1271] proved that all double triangular snakes are harmonious.

For any graph $G$ defining $G$-snake analogous to triangular snakes, Sekar [1000] has shown that $C_n$-snakes are graceful when $n \equiv 0 \pmod 4$ ($n \geq 8$) and when $n \equiv 2 \pmod 4$ and the number of $C_n$ is even. Gnanajothi [471, pp. 31-34] had earlier shown that quadrilateral snakes are graceful. Grace [482] has proved that $K_4$-snakes are harmonious. Rosa [975] has also considered analogously defined quadrilateral and pentagonal cacti and examined small cases. Yu, Lee, and Chin [1314] showed that $Q_2$-and $Q_3$-snakes are graceful and, when the number of blocks is greater than 1, $Q_2-$, $Q_3$- and $Q_4$-snakes are harmonious.

Barrientos [189] calls a graph a *$kC_n$-snake* if it is a connected graph with $k$ blocks whose block-cutpoint graph is a path and each of the $k$ blocks is isomorphic to $C_n$. (When $n > 3$ and $k > 3$ there is more than one $kC_n$-snake.) If a $kC_n$-snake where the path of minimum length that contains all the cut-vertices of the graph has the property that the distance between any two consecutive cut-vertices is $\lfloor n/2 \rfloor$ it is called *linear*. Barrientos proves that $kC_4$-snakes are graceful and that the linear $kC_6$-snakes are graceful when $k$ is even. He further proves that $kC_8$-snakes and $kC_{12}$-snakes are graceful in the cases where the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph are all even and that certain cases of $kC_{4n}$-snakes and $kC_{5n}$-snakes are graceful (depending on the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph).

Several people have studied cycles with pendant edges attached. Frucht [428] proved that any cycle with a pendant edge attached at each vertex (i.e., a *crown*) is graceful (see also [562]). If $G$ has order $n$, the *corona of $G$ with $H$*, $G \odot H$ is the graph obtained by taking one copy of $G$ and $n$ copies of $H$ and joining the $i$th vertex of $G$ with an edge to every vertex in the $i$th copy of $H$. Barrientos [194] also proves: if $G$ is a graceful graph of order $m$ and size $m-1$, then $G \odot nK_1$ and $G + nK_1$ are graceful; if $G$ is a graceful graph of order $p$ and size $q$ with $q > p$, then $(G \cup (q+1-p)K_1) \odot nK_1$ is graceful; and all unicyclic graphs other than a cycle for which the deletion of any edge from the cycle results in a caterpillar are graceful. In [420] Figueroa-Centeno, R. Ichishima, Muntaner-Batle, Oshima introduce two methods for constructing graceful unicyclic graphs that have cycle length congruent to 0 or 3 (mod 4). Their results subsume all existing ones on graceful unicyclic graphs with cycle length congruent to 0 or 3 (mod 4).

In [191] Barrientos proved that helms (graphs obtained from a wheel by attaching one pendant edge to each vertex) are graceful. Grace [481] showed that an odd cycle with one or more pendant edges at each vertex is harmonious and conjectured that $C_{2n} \odot K_1$, an even cycle with one pendant edge attached at each vertex, is harmonious. This conjecture has been proved by Liu and Zhang [811], Liu [808] and [809], Hegde [524], Huang [563], and Bu [258]. Sekar [1000] has shown that the graph $C_m \odot P_n$ obtained by attaching the path $P_n$ to each vertex of $C_m$ is graceful. For any $n \geq 3$ and any $t$ with $1 \leq t \leq n$, let $C_n^{+t}$ denote the class of graphs formed by adding a single pendant edge to $t$ vertices of a cycle of length $n$. Ropp [972] proved that for every $n$ and $t$ the class $C_n^{+t}$ contains a graceful graph. Gallian and Ropp [442] conjectured that for all $n$ and $t$, all members of $C_n^{+t}$ are graceful. This was proved by Qian [947] and by Kang, Liang, Gao, and Yang [626]. Of course, such graphs are just a special case of the aforementioned conjecture of Truszczyński that all unicyclic graphs except $C_n$ for $n \equiv 1$ or 2 (mod 4) are graceful. Sekar [1000] proved that the graph obtained by identifying an endpoint of a star with a vertex of a cycle is graceful. Lu [826] shows that the graph obtained by identifying each vertex of an odd cycle with a vertex disjoint copy of $C_{2m+1}$ is harmonious if and only if $m$ is odd.

For given cycle $C_n$ with $n \equiv 0$ or 3 (mod 4) and a family of trees $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, let $u_i$ and $v_i, 1 \leq i \leq n$, be fixed vertices of $C_n$ and $T_i$, respectively. Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [420] provide two construction methods that generate a graceful labeling of the the unicyclic graphs obtained from $C_n$ and $\mathcal{T}$ by amalgamating them at each $u_i$ and $v_i$. Their results encompass all previously known results for unicyclic graphs whose cycle length is 0 or 3 (mod 4) and considerably extend the known classes of graceful unicyclic graphs.

Solairaju and Chithra [1118] defined three classes of graphs obtained by connecting copies of $C_4$ in various ways. Denote the four consecutive vertices of $i$th copy of $C_4$ by $v_{i,1}, v_{i,2}, v_{i,3}, v_{i_4}$. They show that the graphs obtained by identifying $v_{i,4}$ with $v_{i+1,2}$ for $i = 1, 2, \ldots, n-1$ is graceful; the graphs obtained by joining $v_{i,4}$ with $v_{i+1,2}$ for $i = 1, 2, \ldots, n-1$ by an edge is graceful; and the graphs obtained by joining $v_{i,4}$ with $v_{i+1,2}$ for $i = 1, 2, \ldots, n-1$ with a path of length 2 is graceful.

In a paper published in 1985 Bloom and Hsu [240] say a directed graph $D$ with $e$ edges has a graceful labeling $\theta$ if for each vertex $v$ there is a vertex labeling $\theta$ that assigns each vertex a distinct integer from 0 to $e$ such that for each directed edge $(u, v)$ the integers $\theta(v) - \theta(u)$ mod $(e+1)$ are distinct and nonzero . They conjectured that digraphs whose underlying graphs are wheels and that have all directed edges joining the hub and the rim in the same direction and all directed edges in the same direction are graceful. This conjecture was proved in 2009 by Hegde and Shivarajkumarn [544].

## 2.3 Product Related Graphs

Graphs that are cartesian products and related graphs have been the subject of many papers. That planar grids, $P_m \times P_n$, are graceful was proved by Acharya and Gill [24] in 1978 although the much simpler labeling scheme given by Maheo [839] in 1980 for $P_m \times P_2$ readily extends to all grids. In 1980 Graham and Sloane [484] proved ladders, $P_m \times P_2$, are harmonious when $m > 2$ and in 1992 Jungreis and Reid [621] showed that the grids $P_m \times P_n$ are harmonious when $(m, n) \neq (2, 2)$. A few people have looked at graphs obtained from planar grids in various ways. Kathiresan [630] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful and that graphs obtained by appending an edge to each vertex of a ladder are graceful [632]. Acharya [15] has shown that certain subgraphs of grid graphs are graceful. Lee [704] defines a *Mongolian tent* as a graph obtained from $P_m \times P_n$, $n$ odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex. A *Mongolian village* is a graph formed by successively amalgamating copies of Mongolian tents with the same number of rows so that adjacent tents share a column. Lee proves that Mongolian tents and villages are graceful. A *Young tableau* is a subgraph of $P_m \times P_n$ obtained by retaining the first two rows of $P_m \times P_n$ and deleting vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a nonincreasing sequence. Lee and Ng [724] have proved that all Young tableaus are graceful. Lee [704] has also defined a variation of Mongolian tents by adding an extra vertex above the top row of a Young tableau and joining every other vertex of that row to the extra vertex. He proves these graphs are graceful.

*Prisms* are graphs of the form $C_m \times P_n$. These can be viewed as grids on cylinders. In 1977 Bodendiek, Schumacher, and Wegner [243] proved that $C_m \times P_2$ is graceful when $m \equiv 0 \pmod 4$. According to the survey by Bermond [217], Gangopadhyay and Rao Hebbare did the case that $m$ is even about the same time. In a 1979 paper, Frucht [428] stated without proof that he had done all $C_m \times P_2$. A complete proof of all cases and some related results were given by Frucht and Gallian [431] in 1988.

In 1992 Jungreis and Reid [621] proved that all $C_m \times P_n$ are graceful when $m$ and $n$ are even or when $m \equiv 0 \pmod 4$. They also investigated the existence of a stronger form of graceful labeling called an $\alpha$-labeling (see Section 3.1) for graphs of the form $P_m \times P_n, C_m \times P_n$, and $C_m \times C_n$ (see also [444]).

Yang and Wang have shown that the prisms $C_{4n+2} \times P_{4m+3}$ [1292], $C_n \times P_2$ [1290], and $C_6 \times P_m (m \geq 2)$ (see [1292]) are graceful. Singh [1089] proved that $C_3 \times P_n$ is graceful for all $n$. In their 1980 paper Graham and Sloane [484] proved that $C_m \times P_n$ is harmonious when $n$ is odd and they used a computer to show $C_4 \times P_2$, the cube, is not harmonious. In 1992 Gallian, Prout, and Winters [446] proved that $C_m \times P_2$ is harmonious when $m \neq 4$. In 1992, Jungreis and Reid [621] showed that $C_4 \times P_n$ is harmonious when $n \geq 3$. Huang and Skiena [565] have shown that $C_m \times P_n$ is graceful for all $n$ when $m$ is even and for all $n$ with $3 \leq n \leq 12$ when $m$ is odd. Abhyanker [2] proved that the graphs obtained from $C_{2m+1} \times P_5$ by adding a pendent edge to each vertex of an outer cycle is graceful.

*Torus grids* are graphs of the form $C_m \times C_n$ $(m > 2, n > 2)$. Very little success has been achieved with these graphs. The graceful parity condition is violated for $C_m \times C_n$ when $m$ and $n$ are odd and the harmonious parity condition [484, Theorem 11] is violated for $C_m \times C_n$ when $m \equiv 1, 2, 3 \pmod 4$ and $n$ is odd. In 1992 Jungreis and Reid [621] showed that $C_m \times C_n$ is graceful when $m \equiv 0 \pmod 4$ and $n$ is even. A complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

There has been some work done on prism-related graphs. Gallian, Prout, and Winters [446] proved that all prisms $C_m \times P_2$ with a single vertex deleted or single edge deleted are graceful and harmonious. The *Möbius ladder* $M_n$ is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of $P_n$. In 1989 Gallian [441] showed that all Möbius ladders are graceful and all but $M_3$ are harmonious. Ropp [972] has examined two classes of prisms with pendant edges attached. He proved that all $C_m \times P_2$ with a single pendant edge at each vertex are graceful and all $C_m \times P_2$ with a single pendant edge at each vertex of one of the cycles are graceful.

Another class of cartesian products that has been studied is that of books and "stacked" books. The *book* $B_m$ is the graph $S_m \times P_2$ where $S_m$ is the star with $m+1$ vertices. In 1980 Maheo [839] proved that the books of the form $B_{2m}$ are graceful and conjectured that the books $B_{4m+1}$ were also graceful. (The books $B_{4m+3}$ do not satisfy the graceful parity condition.) This conjecture was verified by Delorme [351] in 1980. Maheo [839] also proved that $L_n \times P_2$ and $B_{2m} \times P_2$ are graceful. Both Grace [480] and Reid (see [445]) have given harmonious labelings for $B_{2m}$. The books $B_{4m+3}$ do not satisfy the harmonious parity condition [484, Theorem 11]. Gallian and Jungreis [445] conjectured that the books $B_{4m+1}$ are harmonious. Gnanajothi [471] has verified this conjecture by showing $B_{4m+1}$ has an even stronger form of labeling – see Section 4.1. Liang [784] also proved the conjecture. In 1988 Gallian and Jungreis [445] defined a *stacked book* as a graph of the form $S_m \times P_n$. They proved that the stacked books of the form $S_{2m} \times P_n$ are graceful and posed the case $S_{2m+1} \times P_n$ as an open question. The *n-cube* $K_2 \times K_2 \times \cdots \times K_2$ ($n$ copies) was shown to be graceful by Kotzig [671]—see also [839]. Although Graham and Sloane [484] used a computer in 1980 to show that the 3-cube is not harmonious (see also [919]), Ichishima and Oshima [571] proved that the $n$-cube $Q_n$ has a stronger form of harmonious labeling (see Section 4.1) for $n \geq 4$.

In 1986 Reid [961] found a harmonious labeling for $K_4 \times P_n$. Petrie and Smith [927] have investigated graceful labelings of graphs as an exercise in constraint programming satisfaction. They have shown that $K_m \times P_n$ is graceful for $(m,n) = (4,2), (4,3), (4,4), (4,5),$ (see also [956]) and (5,2) but is not graceful for $(3,3)$ and $(6,2)$. Redl [956] also proved that $K_4 \times P_n$ is graceful for $n = 1, 2, 3, 4$ and 5 using a constraint programming approach. Their labeling for $K_5 \times P_2$ is the unique graceful labeling. They also considered the graph obtained by identifying the hubs of two copies of $W_n$. The resulting graph is not graceful when $n = 3$ but is graceful when $n$ is 4 and 5. Smith and Puget [1113] has used a computer search to prove that $K_m \times P_2$ is not graceful for $m = 7, 8, 9,$ and 10. She conjectures that $K_m \times P_2$ is not graceful for $m > 5$. Redl [956] asks if all graphs of the form $K_4 \times P_n$ are graceful.

For a bipartite graph $G$ with partite sets $X$ and $Y$ let $G'$ be a copy of $G$ and $X'$ and $Y'$ be copies of $X$ and $Y$. Lee and Liu [718] define the *mirror graph*, $M(G)$, of $G$ as the disjoint union of $G$ and $G'$ with additional edges joining each vertex of $Y$ to its corresponding vertex in $Y'$. The case that $G = K_{m,n}$ is more simply denoted by $M(m,n)$. They proved that for many cases $M(m,n)$ has a stronger form of graceful labeling (see §3.1 for details).

The *composition* $G_1[G_2]$ is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1,y_1),(x_2,y_2)|\ x_1x_2 \in E(G_1)$ or $x_1 = x_2$ and $y_1y_2 \in E(G_2)\}$. The *symmetric product* $G_1 \oplus G_2$ of graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1,y_1),(x_2,y_2)|\ x_1x_2 \in E(G_1)$ or $y_1y_2 \in E(G_2)$ but not both$\}$. Seoud and Youssef [1021] have proved that $P_n \oplus \overline{K_2}$ is graceful when $n > 1$ and $P_n[P_2]$ is harmonious for all $n$. They also observe that the graphs $C_m \oplus C_n$ and $C_m[C_n]$ violate the parity conditions for graceful and harmonious graphs when $m$ and $n$ are odd.

## 2.4 Complete Graphs

The questions of the gracefulness and harmoniousness of the complete graphs $K_n$ have been answered. In each case the answer is positive if and only if $n \leq 4$ ([475], [1088], [484], [223]). Both Rosa [973] and Golomb [475] proved that the complete bipartite graphs $K_{m,n}$ are graceful while Graham and Sloane [484] showed they are harmonious if and only if $m$ or $n = 1$. Aravamudhan and Murugan [73] have shown that the complete tripartite graph $K_{1,m,n}$ is both graceful and harmonious while Gnanajothi [471, pp. 25–31] has shown that $K_{1,1,m,n}$ is both graceful and harmonious and $K_{2,m,n}$ is graceful. Some of the same results have been obtained by Seoud and Youssef [1016] who also observed that when $m, n$, and $p$ are congruent to 2 (mod 4), $K_{m,n,p}$ violates the parity conditions for harmonious graphs. Beutner and Harborth [223] give graceful labelings for $K_{1,m,n}, K_{2,m,n}, K_{1,1,m,n}$ and conjecture that these and $K_{m,n}$ are the only complete multipartite graphs that are graceful. They have verified this conjecture for graphs with up to 23 vertices via computer.

Beutner and Harborth [223] also show that $K_n - e$ ($K_n$ with an edge deleted) is graceful only if $n \leq 5$; any $K_n - 2e$ ($K_n$ with two edges deleted) is graceful only if $n \leq 6$; and any $K_n - 3e$ is graceful only if $n \leq 6$. They also determine all graceful graphs of the form $K_n - G$ where $G$ is $K_{1,a}$ with $a \leq n-2$ and where $G$ is a matching $M_a$ with $2a \leq n$.

The *windmill* graph $K_n^{(m)}$ ($n > 3$) consists of $m$ copies of $K_n$ with a vertex in common. A necessary condition for $K_n^{(m)}$ to be graceful is that $n \leq 5$ – see [658]. Bermond [217] has conjectured that $K_4^{(m)}$ is graceful for all $m \geq 4$. The gracefulness of $K_4^{(m)}$ is equivalent to the existence of a $(12m+1, 4, 1)$-perfect difference family, which are known to exit for $m \leq 1000$ (see [565], [1], [1239], and [461]). Bermond, Kotzig, and Turgeon [220] proved that $K_n^{(m)}$ is not graceful when $n = 4$ and $m = 2$ or 3, and when $m = 2$ and $n = 5$. In 1982 Hsu [559] proved that $K_4^{(m)}$ is harmonious for all $m$. Graham and Sloane [484] conjectured that $K_n^{(2)}$ is harmonious if and only if $n = 4$. They verified this conjecture for the cases that $n$ is odd or $n = 6$. Liu [799] has shown that $K_n^{(2)}$ is not harmonious if $n = 2^a p_1^{a_1} \cdots p_s^{a_s}$ where $a, a_1, \ldots, a_s$ are positive integers and $p_1, \ldots, p_s$ are distinct odd primes and there is a $j$ for which $p_j \equiv 3 \pmod 4$ and $a_j$ is odd. He also shows that $K_n^{(3)}$ is not harmonious when $n \equiv 0 \pmod 4$ and $3n = 4^e(8k+7)$ or $n \equiv 5 \pmod 8$. Koh, Rogers, Lee, and Toh [652] and Rajasingh and Pushpam [953] have shown that $K_{m,n}^{(t)}$, the one-point union of $t$ copies of $K_{m,n}$, is graceful. Sethuraman and Selvaraju [1043] have proved that the one-point union of graphs of the form $K_{2,m_i}$ for $i = 1, 2, \ldots, n$, where the union is taken at a vertex from the partite set with exactly 2 vertices is graceful if at most two of the $m_i$ are equal. They conjecture that the restriction that at most two of the $m_i$ are equal is not necessary. Koh, Rogers, Lee, and Toh [658] introduced the notation $B(n, r, m)$ for the graph consisting of $m$ copies of $K_n$ with a $K_r$ in common ($n \geq r$). (We note that Guo [499] has used the notation $B(n, r, m)$ to denote the graph obtained by joining opposite endpoints of three disjoint paths of lengths $n, r$ and $m$.) Bermond [217] raised the question: "For which $m, n$, and $r$ is $B(n, r, m)$ graceful?" Of course, the case $r = 1$ is the same as $K_n^{(m)}$. For $r > 1$, $B(n, r, m)$ is graceful in the following cases: $n = 3$, $r = 2$, $m \geq 1$ [653]; $n = 4$, $r = 2$, $m \geq 1$ [351]; $n = 4$, $r = 3$, $m \geq 1$ (see [217]), [653]. Seoud and Youssef [1016] have proved $B(3, 2, m)$ and $B(4, 3, m)$ are harmonious. Liu [798] has shown that if there is a prime $p$ such that $p \equiv 3 \pmod 4$ and $p$ divides both $n$ and $n-2$ and the highest power of $p$ that divides $n$ and $n-2$ is odd, then $B(n, 2, 2)$ is not graceful. Smith and Puget [1113] has shown that up to symmetry, $B(5, 2, 2)$ has a unique graceful labeling; $B(n, 3, 2)$ is not graceful for $n = 6, 7, 8, 9$,

and 10; $B(6,3,3)$ and $B(7,3,3)$ are not graceful; and $B(5,3,3)$ is graceful. Combining results of Bermond and Farhi [219] and Smith and Puget [1113] show that $B(n,2,2)$ is not graceful for $n > 5$. Lu [826] obtained the following results: $B(m,2,3)$ and $B(m,3,3)$ are not harmonious when $m \equiv 1 \pmod 8$; $B(m,4,2)$ and $B(m,5,2)$ are not harmonious when $m$ satisfies certain special conditions; $B(m,1,n)$ is not harmonious when $m \equiv 5 \pmod 8$ and $n \equiv 1,2,3 \pmod 4$; $B(2m+1,2m,2n+1) \cong K_{2m} + \overline{K_{2n+1}}$ is not harmonious when $m \equiv 2 \pmod 4$.

More generally, Bermond and Farhi [219] have investigated the class of graphs consisting of $m$ copies of $K_n$ having exactly $k$ copies of $K_r$ in common. They proved such graphs are not graceful for $n$ sufficiently large compared to $r$. Barrientos [195] proved that the graph obtained by performing the one-point union of any collection of the complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_t,n_t}$, where each $K_{m_i,n_i}$ appears at most twice and $\gcd(n_1, n_2, \ldots, n_t) = 1$, is graceful.

Sethuraman and Elumalai [1027] have shown that $K_{1,m,n}$ with a pendent edge attached to each vertex is graceful and Jirimutu [613] has shown that the graph obtained by attaching a pendant edge to every vertex of $K_{m,n}$ is graceful (see also [62]). In [1041] Sethuraman and Kishore determine the graceful graphs that are the union of $n$ copies of $K_4$ with $i$ edges deleted for $1 \le i \le 5$ and with one edge in common. The only cases that are not graceful are those graphs where the members of the union are $C_4$ for $n \equiv 3 \pmod 4$ and where the members of the union are $P_2$. They conjecture that these two cases are the only instances of edge induced subgraphs of the union of $n$ copies of $K_4$ with one edge in common that are not graceful. Sethuraman and Selvaraju [1049] have shown that union of any number of copies of $K_4$ with an edge deleted and one edge in common is harmonious.

Clemens, Coulibaly, Garvens, Gonnering, Lucas, and Winters [339] investigated the gracefulness of the one-point and two-point unions of graphs. They show the following graphs are graceful: the one-point union of an end vertex of $P_n$ and $K_4$; the graph obtained by taking the one-point union of $K_4$ with one end vertex of $P_n$ and the one-point union of the other end vertex of $P_n$ with the central vertex of $K_{1,r}$; the graph obtained by taking the one-point union of $K_4$ with one end vertex of $P_n$ and the one-point union of the other end of $P_n$ with a vertex from the partite set of order 2 of $K_{2,r}$; the graph obtained from the graph just described by appending any number of edges to the other vertex of the partite set of order 2; the two-point union of the two vertices of the partite set of order 2 in $K_{2,r}$ and two vertices from $K_4$; and the graph obtained from the graph just described by appending any number of edges to one of the vertices from the partite set of order 2.

## 2.5 Disconnected Graphs

There have been many papers dealing with graphs that are not connected. For any graph $G$ the graph $mG$ denotes the disjoint union of $m$ copies of $G$. In 1975 Kotzig [670] investigated the gracefulness of the graphs $rC_s$. When $rs \equiv 1$ or $2 \pmod 4$, these graphs violate the gracefulness parity condition. Kotzig proved that when $r = 3$ and $4k > 4$, then $rC_{4k}$ has a stronger form of graceful labeling called $\alpha$-labeling (see §3.1) whereas when $r \ge 2$ and $s = 3$ or 5, $rC_s$ is not graceful. In 1984 Kotzig [672] once again investigated the gracefulness of $rC_s$ as well as graphs that are the disjoint union of odd cycles. For graphs of the latter kind he gives several necessary conditions. His paper concludes with an elaborate table that summarizes what was then known about the gracefulness of $rC_s$. M. He [513] has shown that graphs of the form $2C_{2m}$ and graphs obtained by connecting two copies of $C_{2m}$ with an edge are graceful. Cahit [279] has

shown that $rC_s$ is harmonious when $r$ and $s$ are odd and Seoud, Abdel Maqsoud, and Sheehan [1007] noted that when $r$ or $s$ is even, $rC_s$ is not harmonious. Seoud, Abdel Maqsoud, and Sheehan [1007] proved that $C_n \cup C_{n+1}$ is harmonious if and only if $n \geq 4$. They conjecture that $C_3 \cup C_{2n}$ is harmonious when $n \geq 3$. This conjecture was proved when Yang, Lu, and Zeng [1288] showed that all graphs of the form $C_{2j+1} \cup C_{2n}$ are harmonious except for $(n, j) = (2, 1)$. As a consequence of their results about super edge-magic labelings (see §5.2) Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [419] have that $C_3 \cup C_n$ is harmonious if and only if $n \geq 6$ and $n$ is even.

In 1978 Kotzig and Turgeon [675] proved that $mK_n$ is graceful if and only if $m = 1$ and $n \leq 4$. Liu and Zhang [812] have shown that $mK_n$ is not harmonious for $n$ odd and $m \equiv 2 \pmod 4$ and is harmonious for $n = 3$ and $m$ odd. They conjecture that $mK_3$ is not harmonious when $m \equiv 0 \pmod 4$. Bu and Cao [259] give some sufficient conditions for the gracefulness of graphs of the form $K_{m,n} \cup G$ and they prove that $K_{m,n} \cup P_t$ and the disjoint union of complete bipartite graphs are graceful under some conditions.

Recall a *Skolem sequence* of order $n$ is a sequence $s_1, s_2, \ldots, s_{2n}$ of $2n$ terms such that, for each $k \in \{1, 2, \ldots, n\}$, there exist exactly two subscripts $i(k)$ and $j(k)$ with $s_{i(k)} = s_{j(k)} = k$ and $|i(k) - j(k)| = k$. (A Skolem sequence of order $n$ exists if and only if $n \equiv 0$ or $1 \pmod 4$). Abrham [6] has proved that any graceful 2-regular graph of order $n \equiv 0 \pmod 4$ in which all the component cycles are even or of order $n \equiv 3 \pmod 4$, with exactly one component an odd cycle, can be used to construct a Skolem sequence of order $n + 1$. Also, he showed that certain special Skolem sequences of order $n$ can be used to generate graceful labelings on certain 2-regular graphs.

The graph $H_n$ obtained from the cycle with consecutive vertices $u_1, u_2, \ldots, u_n$ $(n \geq 6)$ by adding the chords $u_2u_n, u_3u_{n-1}, \ldots, u_\alpha u_\beta$, where $\alpha = (n-1)/2$ for all $n$ and $\beta = (n-1)/2 + 3$ if $n$ is odd or $\beta = n/2 + 2$ if $n$ is even is called the *cycle with parallel chords*. In A. Elumalai and G. Sethuraman [382] prove the following: for odd $n \geq 5$, $H_n \cup K_{p,q}$ is graceful; for even $n \geq 6$ and $m = (n-2)/2$ or $m = n/2$ $H_n \cup K_{1,m}$ is graceful; for $n \geq 6$, $H_n \cup P_m$ is graceful, where $m = n$ or $n - 2$ depending on $n \equiv 1$ or $3 \pmod 4$ or $m \equiv n - 1$ or $n - 3$ depending on $n \equiv 0$ or $2 \pmod 4$.

In 1985 Frucht and Salinas [432] conjectured that $C_s \cup P_n$ is graceful if and only if $s + n \geq 7$ and they proved the conjecture for the case that $s = 4$. Frucht [430] did the case the $s = 3$ and the case that $s = 2n + 1$. Bhat-Nayak and Deshmukh [227] also did the case $s = 3$ and they have done the cases of the form $C_{2x+1} \cup P_{x-2\theta}$ where $1 \leq \theta \leq \lfloor (x-2)/2 \rfloor$ [228]. Choudum and Kishore [330] have done the cases where $s \geq 5$ and $n \geq (s+5)/2$ and Kishore [650] did the case $s = 5$. Gao and Liang [449] have done the following cases: $s > 4, n = 2$ (see also [448]); $s = 4k, n = k + 2, n = k + 3, n = 2k + 2; s = 4k + 1, n = 2k, n = 3k - 1, n = 4k - 1; s = 4k + 2, n = 3k, n = 3k + 1, n = 4k + 1; s = 4k + 3, n = 2k + 1, n = 3k, n = 4k$. Seoud, Abdel Maqsoud, and Sheehan [1009] did the case that $s = 2k$ $(k \geq 3)$ and $n \geq k + 1$ as well as the cases where $s = 6, 8, 10, 12$ and $n \geq 2$. Shimazu [1060] has handled the cases that $s \geq 5$ and $n = 2, s \geq 4$ and $n = 3$ and $s = 2n + 2$ and $n \geq 2$. Liang [785] has done the following cases: $s = 4k, n = k+2, k+3, 2k+1, 2k+2, 2k+3, 2k+4, 2k+5; s = 4k-1, n = 2k, 3k-1, 4k-1; s = 4k+2, n = 3k, 3k+1, 4k+1; s = 4k+3, n = 2k+1, 3k, 4k$. Youssef [1303] proved that $C_5 \cup S_n$ is graceful if and only if $n = 1$ or 2 and that $C_6 \cup S_n$ is graceful if and only if $n$ is odd or $n = 2$ or 4.

Seoud and Youssef [1023] have shown that $K_5 \cup K_{m,n}, K_{m,n} \cup K_{p,q}$ $(m, n, p, q \geq 2), K_{m,n} \cup K_{p,q} \cup K_{r,s}$ $(m, n, p, q, r, s \geq 2,$ $(p, q) \neq (2, 2))$, and $pK_{m,n}$ $(m, n \geq 2, (m, n) \neq (2, 2))$ are

graceful. They also prove that $C_4 \cup K_{1,n}$ $(n \neq 2)$ is not graceful whereas Choudum and Kishore [332], [650] have proved that $C_s \cup K_{1,n}$ is graceful for every $s \geq 7$ and $n \geq 1$. Lee, Quach, and Wang [738] established the gracefulness of $P_s \cup K_{1,n}$. Seoud and Wilson [1015] have shown that $C_3 \cup K_4, C_3 \cup C_3 \cup K_4$, and certain graphs of the form $C_3 \cup P_n$ and $C_3 \cup C_3 \cup P_n$ are not graceful. Abrham and Kotzig [11] proved that $C_p \cup C_q$ is graceful if and only if $p+q \equiv 0$ or 3 (mod 4). Zhou [1327] proved that $K_m \cup K_n$ $(n > 1, m > 1)$ is graceful if and only if $\{m, n\} = \{4, 2\}$ or $\{5, 2\}$. (C. Barrientos has called to my attention that $K_1 \cup K_n$ is graceful if and only if $n = 3$ or 4.) Shee [1054] has shown that graphs of the form $P_2 \cup C_{2k+1}$ $(k > 1)$, $P_3 \cup C_{2k+1}$, $P_n \cup C_3$, and $S_n \cup C_{2k+1}$ all satisfy a condition that is a bit weaker than harmonious. Bhat-Nayak and Deshmukh [225] have shown that $C_{4t} \cup K_{1,4t-1}$ and $C_{4t+3} \cup K_{1,4t+2}$ are graceful. Section 3.1 includes numerous families of disconnected graphs that have a stronger form of graceful labelings.

In considering graceful labelings of the disjoint unions of two or three stars with $e$ edges Yang and Wang [1291] permitted the vertex labels to range from 0 to $e+1$ and 0 to $e+2$, respectively. With these definitions of graceful, they proved that $S_m \cup S_n$ is graceful if and only if $m$ or $n$ is even and that $S_m \cup S_n \cup S_k$ is graceful if and only if at least one of $m, n$, or $k$ is even $(m > 1, n > 1, k > 1)$.

Seoud and Youssef [1019] investigated the gracefulness of specific families of the form $G \cup K_{m,n}$. They obtained the following results: $C_3 \cup K_{m,n}$ is graceful if and only if $m \geq 2$ and $n \geq 2$; $C_4 \cup K_{m,n}$ is graceful if and only if $(m, n) \neq (1, 1)$; $C_7 \cup K_{m,n}$ and $C_8 \cup K_{m,n}$ are graceful for all $m$ and $n$; $mK_3 \cup nK_{1,r}$ is not graceful for all $m, n$ and $r$; $K_i \cup K_{m,n}$ is graceful for $i \leq 4$ and $m \geq 2, n \geq 2$ except for $i = 2$ and $(m, n) = (2, 2)$; $K_5 \cup K_{1,n}$ is graceful for all $n$; $K_6 \cup K_{1,n}$ is graceful if and only if $n$ is not 1 or 3. Youssef [1306] completed the characterization of the graceful graphs of the form $C_n \cup K_{p,q}$ where $n \equiv 0$ or 3 (mod 4) by showing that for $n > 8$ and $n \equiv 0$ or 3 (mod 4), $C_n \cup K_{p,q}$ is graceful for all $p$ and $q$ (see also [193]). Note that when $n \equiv 1$ or 2 (mod 4) certain cases of $C_n \cup K_{p,q}$ violate the parity condition for gracefulness.

For $i = 1, 2, \ldots, m$ let $v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ be a 4-cycle. Yang and Pan [1286] define $F_{k,4}$ to be the graph obtained by identifying $v_{i,3}$ and $v_{i+1,1}$ for $i = 1, 2, \ldots, k-1$. They prove that $F_{m_1,4} \cup F_{m_2,4} \cup \cdots \cup F_{m_n,4}$ is graceful for all $n$. Pan and Lu [916] have shown that $(P_2 + \overline{K_n}) \cup K_{1,m}$ and $(P_2 + \overline{K_n}) \cup T_n$ are graceful.

Barrientos [193] has shown the following graphs are graceful: $C_6 \cup K_{1,2n+1}$; $\bigcup_{i=1}^{t} K_{m_i,n_i}$ for $2 \leq m_i < n_i$; and $C_m \cup \bigcup_{i=1}^{t} K_{m_i,n_i}$ for $2 \leq m_i < n_i, m \equiv 0$ or 3 (mod 4), $m \geq 11$.

Youssef [1304] has shown that if $G$ is harmonious then $mG$ and $G^m$ are harmonious for all odd $m$. He asks the question of whether $G$ is harmonious implies $G^m$ is harmonious when $m \equiv 0$ (mod 4).

## 2.6 Joins of Graphs

A number of classes of graphs that are the join of graphs have been shown to be graceful or harmonious. Acharya [12] proved that if $G$ is a connected graceful graph, then $G + \overline{K_n}$ is graceful. Redl [956] showed that the double cone $C_n + \overline{K_2}$ is graceful for $n = 3, 4, 5, 7, 8, 9, 11$. That $C_n + \overline{K_2}$ is not graceful for $n \equiv 2$ (mod 4) follows that Rosa's parity condition. Redl asks what other double cones are graceful. Reid [961] proved that $P_n + \overline{K_t}$ is harmonious. Sethuraman and Selvaraju [1048] have shown that $P_n + K_2$ is harmonious. They ask whether $S_n + P_n$ or $P_m + P_n$ is harmonious. Of course, wheels are of the form $C_n + K_1$ and are graceful and harmonious. In 2006 Chen [311] proved that multiple wheels $nC_m + K_1$ are harmonious for all $n \not\equiv 0$ mod 4. She believes that the $n \not\equiv 0$ (mod 4) case is also harmonious. Chen also proved that if $H$ has at

least one edge, $H + K_1$ is harmonious, and $n$ is odd, then $nH + K$ is harmonious.

Shee [1054] has proved $K_{m,n} + K_1$ is harmonious and observed that various cases of $K_{m,n} + K_t$ violate the harmonious parity condition in [484]. Liu and Zhang [812] have proved that $K_2 + K_2 + \cdots + K_2$ is harmonious. Yuan and Zhu [1316] proved that $K_{m,n} + K_2$ is graceful and harmonious. Gnanajothi [471, pp. 80–127] obtained the following: $C_n + \overline{K_2}$ is harmonious when $n$ is odd and not harmonious when $n \equiv 2, 4, 6 \pmod 8$; $S_n + \overline{K_t}$ is harmonious; and $P_n + \overline{K_t}$ is harmonious. Balakrishnan and Kumar [178] have proved that the join of $\overline{K}_n$ and two disjoint copies of $K_2$ is harmonious if and only if $n$ is even. Ramírez-Alfonsín [955] has proved that if $G$ is graceful and $|V(G)| = |E(G)| = e$ and either 1 or $e$ is not a vertex label then $G + \overline{K_t}$ is graceful for all $t$.

Seoud and Youssef [1021] have proved: the join of any two stars is graceful and harmonious; the join of any path and any star is graceful; and $C_n + \overline{K_t}$ is harmonious for every $t$ when $n$ is odd. They also prove that if any edge is added to $K_{m,n}$ the resulting graph is harmonious if $m$ or $n$ is at least 2. Deng [353] has shown certain cases of $C_n + \overline{K_t}$ are harmonious. Seoud and Youssef [1018] proved: the graph obtained by appending any number of edges from the two vertices of degree $n \geq 2$ in $K_{2,n}$ is not harmonious; dragons $D_{m,n}$ (i.e., $P_m$ is appended to $C_n$) are not harmonious when $m + n$ is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has odd order. Youssef [1303] has shown that if $G$ is a graceful graph with $p$ vertices and $q$ edges with $p = q + 1$, then $G + S_n$ is graceful.

Sethuraman and Elumalai [1031] have proved that for every graph $G$ with $p$ vertices and $q$ edges the graph $G + K_1 + \overline{K_m}$ is graceful when $m \geq 2^p - p - 1 - q$. As a corollary they deduce that every graph is a vertex induced subgraph of a graceful graph. Balakrishnan and Sampathkumar [179] ask for which $m \geq 3$ is the graph $mK_2 + \overline{K_n}$ graceful for all $n$. Bhat-Nayak and Gokhale [229] have proved that $2K_2 + \overline{K_n}$ is not graceful. Youssef [1303] has shown that $mK_2 + \overline{K_n}$ is graceful if $m \equiv 0$ or $1 \pmod 4$ and that $mK_2 + \overline{K_n}$ is not graceful if $n$ is odd and $m \equiv 2$ or $3 \pmod 4$. Ma [829] proved that if $G$ is a graceful tree then, $G + K_{1,n}$ is graceful. Amutha and Kathiresan [62] proved that the graph obtained by attaching a pendent edge to each vertex of $2K_2 + \overline{K_n}$ is graceful.

Wu [1265] proves that if $G$ is a graceful graph with $n$ edges and $n + 1$ vertices then the join of $G$ and $\overline{K_m}$ and the join of $G$ and any star are graceful. Wei and Zhang [1249] proved that for $n \geq 3$ the disjoint union of $P_1 + P_n$ and a star, the disjoint union of $P_1 + P_n$ and $P_1 + P_{2n}$, and the disjoint union of $P_2 + \overline{K_n}$ and a graceful graph with $n$ edges are graceful. More technical results on disjoint unions and joins are given in [1248],[1249], [1250],[1247], and [285].

## 2.7 Miscellaneous Results

It is easy to see that $P_n^2$ is harmonious [481] while a proof that $P_n^2$ is graceful has been given by Kang, Liang, Gao, and Yang [626]. ($P_n^k$, the $k$th *power of* $P_n$, is the graph obtained from $P_n$ by adding edges that join all vertices $u$ and $v$ with $d(u, v) = k$.) This latter result proved a conjecture of Grace [481]. Seoud, Abdel Maqsoud, and Sheehan [1007] proved that $P_n^3$ is harmonious and conjecture that $P_n^k$ is not harmonious when $k > 3$. The same conjecture was made by Fu and Wu [434]. However, Youssef [1310] has proved that $P_8^4$ is harmonious and $P_n^k$ is harmonious when $k$ is odd. Yuan and Zhu [1316] proved that $P_n^{2k}$ is harmonious when $1 \leq k \leq (n-1)/2$. Selvaraju [1001] has shown that $P_n^3$ and the graphs obtained by joining the centers of any two stars with the end vertices of the path of length $n$ in $P_n^3$ are harmonious.

Cahit [279] proves that the graphs obtained by joining $p$ disjoint paths of a fixed length

$k$ to single vertex are harmonious when $p$ is odd and when $k = 2$ and $p$ is even. Gnanajothi [471, p. 50] has shown that the graph that consists of $n$ copies of $C_6$ that have exactly $P_4$ in common is graceful if and only if $n$ is even. For a fixed $n$, let $v_{i1}, v_{i2}, v_{i3}$ and $v_{i4}$ $(1 \leq i \leq n)$ be consecutive vertices of $n$ 4-cycles. Gnanajothi [471, p. 35] also proves that the graph obtained by joining each $v_{i1}$ to $v_{i+1,3}$ is graceful for all $n$ and the generalized Petersen graph $P(n,k)$ is harmonious in all cases (see also [743]). Recall $P(n,k)$, where $n \geq 5$ and $1 \leq k \leq n$, has vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_i a_{i+1} \mid i = 0, 1, \ldots, n-1\} \cup \{a_i b_i \mid i = 0, 1, \ldots, n-1\} \cup \{b_i b_{i+k} \mid i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo $n$ [1246]. The standard Petersen graph is $P(5,2)$.) Redl [956] has used a constraint programming approach to show that $P(n,k)$ is graceful for $n = 5, 6, 7, 8, 9$, and 10. In [1202] and [1204] (see [1205] for a correction) Vietri proved that $P(8t,3)$ and $P(8t+4,3)$ are graceful for all $t$. She conjectures that the graphs $P(8t,3)$ have a stronger form a graceful labeling called an $\alpha$-labeling (see §3.1). The gracefulness of the generalized Petersen graphs is an open problem. A conjecture in the graph theory book by Chartrand and Lesniak [304, p. 266] that graceful graphs with arbitrarily large chromatic numbers do not exist was shown to be false by Acharya, Rao, and Arumugam [30] (see also Mahmoody [841]).

Sethuraman and Selvaraju [1042] define a graph $H$ to be a *supersubdivision* of a graph $G$, if every edge $uv$ of $G$ is replaced by $K_{2,m}$ ($m$ may vary for each edge) by identifying $u$ and $v$ with the two vertices in $K_{2,m}$ that form the partite set with exactly two members. Sethuraman and Selvaraju prove that every supersubdivision of a path is graceful and every cycle has some supersubdivision that is graceful. They conjecture that every supersubdivision of a star is graceful and that paths and stars are the only graphs for which every supersubdivision is graceful. Barrientos [195] disproved this latter conjecture by proving that every supersubdivision of a $y$-trees is graceful (recall a $y$-tree is obtained from a path by appending an edge to a vertex of a path adjacent to an end point). Barrientos asks if paths and $y$-trees are the only graphs for which every supersubdivision is graceful. This seems unlikely to be the case. The conjecture that every supersubdivision of a star is graceful was proved by Kathiresan and Amutha [634]. In [1046] Sethuraman and Selvaraju prove that every connected graph has some supersubdivision that is graceful. They pose the question as to whether this result is valid for disconnected graphs. They also ask if there is any graph other than $K_{2,m}$ that can be used to replace an edge of a connected graph to obtain a supersubdivision that is graceful. In [1045] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a strong form of graceful labeling called an $\alpha$-labeling (see §3.1 for the definition).

Kathiresan [631] uses the notation $P_{a,b}$ to denote the graph obtained by identifying the end points of $b$ internally disjoint paths each of length $a$. He conjectures that $P_{a,b}$ is graceful except when $a$ is odd and $b \equiv 2 \pmod 4$ and proves the conjecture for the case that $a$ is even and $b$ is odd. Liang and Zuo [789] proved that the graph $P_{a,b}$ is graceful when both $a$ and $b$ are even. Sekar [1000] has shown that $P_{a,b}$ is graceful when $a \neq 4r+1$, $r > 1, b = 4m$, and $m > r$. Yang (see [1289]) proved that $P_{a,b}$ is graceful when $a = 3, 5, 7$, and 9 and $b$ is odd and when $a = 2, 4, 6$, and 8 and $b$ is even (see [1289]). Yang, Rong, and Xu [1289] proved that $P_{a,b}$ is graceful when $a = 10, 12$, and 14 and $b$ is even. Kathiresan also shows that the graph obtained by identifying a vertex of $K_n$ with any noncenter vertex of the star with $2^{n-1} - n(n-1)/2$ edges is graceful.

For a family of graphs $G_1(u_1, u_2), G_2(u_2, u_3), \ldots, G_m(u_m, u_{m+1})$ where $u_i$ and $u_{i+1}$ are vertices in $G_i$ Cheng, Yao, Chen, and Zhang [320] define a *graph-block chain* $H_m$ as the graph obtained by identifying $u_{i+1}$ of $G_i$ with $u_{i+1}$ of $G_{i+1}$ for $i = 1, 2, \ldots, m$. They denote this graph

by $H_m = G_1(u_1, u_2) \oplus G_2(u_2, u_3) \oplus \cdots \oplus G_m(u_m, u_{m+1})$. The case where each $G_i$ has the form $P_{a_i,b_i}$ they call a *path-block chain*. The vertex $u_1$ is called the *initial vertex* of $H_m$. They define a *generalized spider* $S_m^*$ as a graph obtained by starting with an initial vertex $u_0$ and $m$ path-block graphs and join $u_0$ with each initial vertex of each of the path-block graphs. Similarly, they define a *generalized caterpillar* $T_m^*$ as a graph obtained by starting with $m$ path-block chains $H_1, H_2, \ldots, H_m$ and a caterpillar $T$ with $m$ isolated vertices $v_1, v_2, \ldots, v_m$ and join each $v_i$ with the initial vertex of each $H_i$. They prove several classes of path-block chains, generalized spiders, and generalized caterpillars are graceful.

The graph $T_n$ with $3n$ vertices and $6n-3$ edges is defined as follows. Start with a triangle $T_1$ with vertices $v_{1,1}, v_{1,2}$ and $v_{1,3}$. Then $T_{i+1}$ consists of $T_i$ together with three new vertices $v_{i+1,1}, v_{i+1,2}, v_{i+1,3}$ and edges $v_{i+1,1}v_{i,2}$, $v_{i+1,1}v_{i,3}$, $v_{i+1,2}v_{i,1}$, $v_{i+1,2}v_{i,3}$, $v_{i+1,3}v_{i,1}$, $v_{i+1,3}v_{i,2}$. Gnanajothi [471] proved that $T_n$ is graceful if and only if $n$ is odd. Sekar [1000] proved $T_n$ is graceful when $n$ is odd and $T_n$ with a pendant edge attached to the starting triangle is graceful when $n$ is even.

For a graph $G$, the *splitting graph of* $G$, $S^1(G)$, is obtained from $G$ by adding for each vertex $v$ of $G$ a new vertex $v^1$ so that $v^1$ is adjacent to every vertex that is adjacent to $v$. Sekar [1000] has shown that $S^1(P_n)$ is graceful for all $n$ and $S^1(C_n)$ is graceful for $n \equiv 0, 1 \pmod 4$.

The total graph $T(P_n)$ has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in $P_n$. Balakrishnan, Selvam, and Yegnanarayanan [180] have proved that $T(P_n)$ is harmonious.

For any graph $G$ with vertices $v_1, \ldots, v_n$ and a vector $\mathbf{m} = (m_1, \ldots, m_n)$ of positive integers the corresponding *replicated graph*, $R_{\mathbf{m}}(G)$, of $G$ is defined as follows. For each $v_i$ form a stable set $S_i$ consisting of $m_i$ new vertices $i = 1, 2, \ldots, n$ (a *stable* set $S$ consists of a set of vertices such that there is not an edge $v_iv_j$ for all pairs $v_i, v_j$ in $S$); two stable sets $S_i, S_j, i \neq j$, form a complete bipartite graph if each $v_iv_j$ is an edge in $G$ and otherwise there are no edges between $S_i$ and $S_j$. Ramírez-Alfonsín [955] has proved that $R_{\mathbf{m}}(P_n)$ is graceful for all $\mathbf{m}$ and all $n > 1$ (see §3.2 for a stronger result) and that $R_{(m,1,\ldots,1)}(C_{4n}), R_{(2,1,\ldots,1)}(C_n)$ $(n \geq 8)$ and, $R_{(2,2,1,\ldots,1)}(C_{4n})$ $(n \geq 12)$ are graceful.

For any permutation $f$ on $1, \ldots, n$, the *$f$-permutation* graph on a graph $G$, $P(G, f)$, consists of two disjoint copies of $G$, $G_1$ and $G_2$, each of which has vertices labeled $v_1, v_2, \ldots, v_n$ with $n$ edges obtained by joining each $v_i$ in $G_1$ to $v_{f(i)}$ in $G_2$. In 1983 Lee (see [779]) conjectured that for all $n > 1$ and all permutations on $1, 2, \ldots, n$, the permutation graph $P(P_n, f)$ is graceful. Lee, Wang, and Kiang [779] proved that $P(P_{2k}, f)$ is graceful when $f = (12)(34)\cdots(k, k+1)\cdots(2k-1, 2k)$. They conjectured that if $G$ is a graceful nonbipartite graph with $n$ vertices, then for any permutation $f$ on $1, 2, \ldots, n$, the permutation graph $P(G, f)$ is graceful. Fan and Liang [404] have shown that if $f$ is a permutation in $S_n$ where $n \geq 2(m-1) + 2l$ then the permutation graph $P(P_n, f)$ is graceful if the disjoint cycle form of $f$ is $\prod_{k=0}^{l-1}(m+2k, m+2k+1)$, and if $n \geq 2(m-1) + 4l$ the permutation graph $P(P_n, f)$ is graceful the disjoint cycle form of $f$ is $\prod_{k=0}^{l-1}(m+4k, m+4k+2)(m+4k+1, m+4k+3)$. Some families of graceful permutation graphs are given in [711], [786], and [503].

Gnanajothi [471, p. 51] calls a graph $G$ *bigraceful* if both $G$ and its line graph are graceful. She shows the following are bigraceful: $P_m$; $P_m \times P_n$; $C_n$ if and only if $n \equiv 0, 3 \pmod 4$; $S_n$; $K_n$ if and only if $n \leq 3$; and $B_n$ if and only if $n \equiv 3 \pmod 4$. She also shows that $K_{m,n}$ is not bigraceful when $n \equiv 3 \pmod 4$. (Gangopadhyay and Hebbare [447] used the term "bigraceful" to mean a bipartite graceful graph.) Murugan and Arumugan [887] have shown that graphs obtained from $C_4$ by attaching two disjoint paths of equal length to two adjacent vertices are

bigraceful.

Several well-known isolated graphs have been examined. Graceful labelings have been found for the Petersen graph [428], the cube [453], the icosahedron and the dodecahedron. Graham and Sloane [484] showed that all of these except the cube are harmonious. Winters [1260] verified that the Grőtzsch graph (see [251, p. 118]), the Heawood graph (see [251, p. 236]), and the Herschel graph (see [251, p. 53]) are graceful. Graham and Sloane [484] determined all harmonious graphs with at most five vertices. Seoud and Youssef [1020] did the same for graphs with six vertices.

A number of authors have investigated the gracefulness of the directed graphs obtained from copies of directed cycles $\vec{C}_m$ that have a vertex in common or have an edge in common. A digraph $D(V, E)$ is said to be *graceful* if there exists an injection $f: V(G) \to \{0, 1, \ldots, |E|\}$ such that the induced function $f': E(G) \to \{1, 2, \ldots, |E|\}$ that is defined by $f'(u, v) = (f(v) - f(u)) \pmod{|E| + 1}$ for every directed edge $uv$ is a bijection. The notations $n \cdot \vec{C}_m$ and $n - \vec{C}_m$ are used to denote the digraphs obtained from $n$ copies of $\vec{C}_m$ with exactly one point in common and the digraphs obtained from $n$ copies of $\vec{C}_m$ with exactly one edge in common. Du and Sun [374] proved that a necessary condition for $n - \vec{C}_m$ to be graceful is that $mn$ is even and that $n \cdot \vec{C}_m$ is graceful when $m$ is even. They conjectured that $n \cdot \vec{C}_m$ is graceful for any odd $m$ and even $n$. In a series of papers it was proved that $n - \vec{C}_m$ is graceful for $m = 5, 7, 9, \ldots, 23$ and $n$ even. Finally, Jirimutu, Xu, Feng, and Bao [618] proved the conjecture of Du and Sun [374] that $n \cdot \vec{C}_m$ is graceful for any odd $m$ and even $n$. Xu, Jirimutu, Wang, and Min [1276] prove that $n - \vec{C}_m$ is graceful for $m = 4, 6, 8, 10$ and even $n$.

Marr [844] and [843] summarizes previously known results on graceful directed graphs and presents some new results on directed paths, stars, wheels, and umbrellas.

In 2009 Zak [1319] defined the following generalization of harmonious labelings. For a graph $G(V, E)$ and a positive integer $t \geq |E|$ a function $h$ from $V(G)$ to $Z_t$ (the additive group of integers modulo $t$) is called a *$t$-harmonious labeling* of $G$ if $h$ is injective for $t \geq |V|$ or surjective for $t < |V|$, and $h(u) + h(v) \neq h(x) + h(y)$ for all distinct edges $uv$ and $xy$. The smallest such $t$ for which $G$ has a $t$-harmonious labeling is called the *harmonious order of $G$* . Obviously, a graph $G(V, E)$with $|E| \geq |V|$ is harmonious if and only if the harmonious order of $G$ is $|E|$. Zak determines the harmonious order of complete graphs, complete bipartite graphs, even cycles, some cases of $P_n^k$, and $2nK_3$. He presents some results about the harmonious order of the Cartesian products of graphs, the disjoint union of copies of a given graph, and gives an upper bound for the harmonious order of trees. He conjectures that the harmonious order of a tree of order $n$ is $n + o(n)$.

For a graph with $e$ edges Vietri [1205] generalizes the notion of a graceful labeling by allowing the vertex labels to be real numbers in the interval $[0, e]$. For a simple graph $G(V, E)$ she defines an injective map $\gamma$ from $V$ to $[0, e]$ to be a *real-graceful* labeling of $G$ provided that

$$\sum 2^{\gamma(u)-\gamma(v)} + 2^{\gamma(v)-\gamma(u)} = 2^{e+1} - 2^{-e} - 1,$$

where the sum is taken over all edges $uv$. In the case that the labels are integers, she shows that a real-graceful labeling is equivalent to a graceful labeling. In contrast to the case for graceful labelings, she shows that the cycles $C_{4t+1}$ and $C_{4t+2}$ have real-graceful labelings. She also shows that the non-graceful graphs $K_5, K_6$ and $K_7$ have real-graceful labelings. With one exception, her real-graceful labels are integers.

## 2.8 Summary

The results and conjectures discussed above are summarized in the tables following. The letter G after a class of graphs indicates that the graphs in that class are known to be graceful; a question mark indicates that the gracefulness of the graphs in the class is an open problem; we put a question mark after a "G" if the graphs have been conjectured to be graceful. The analogous notation with the letter H is used to indicate the status of the graphs with regard to being harmonious. The tables impart at a glimpse what has been done and what needs to be done to close out a particular class of graphs. Of course, there is an unlimited number of graphs one could consider. One wishes for some general results that would handle several broad classes at once but the experience of many people suggests that this is unlikely to occur soon. The Graceful Tree Conjecture alone has withstood the efforts of scores of people over the past four decades. Analogous sweeping conjectures are probably true but appear hopelessly difficult to prove.

Table 1: **Summary of Graceful Results**

| *Graph* | *Graceful* |
|---|---|
| trees | G if $\leq 35$ vertices [405]<br>G if symmetrical [221]<br>G if at most 4 end-vertices [564]<br>G? Ringel-Kotzig<br>G caterpillars [973]<br>G? lobsters [217] |
| cycles $C_n$ | G iff $n \equiv 0, 3 \pmod 4$ [973] |
| wheels $W_n$ | G [428], [554] |
| helms (see §2.2) | G [79] |
| webs (see §2.2) | G [626] |
| gears (see §2.2) | G [832] |
| cycles with $P_k$-chord (see §2.2) | G [352], [831], [661], [943] |
| $C_n$ with $k$ consecutive chords (see §2.2) | G if $k = 2, 3, n-3$ [651], [658] |
| unicyclic graphs | G? iff G $\neq C_n, n \equiv 1, 2 \pmod 4$ [1185] |
| $P_n^k$ | G if $k = 2$ [626] |
| $C_n^{(t)}$ (see §2.2) | $n = 3$ G iff $t \equiv 0, 1 \pmod 4$<br>[218], [220]<br>G? if $nt \equiv 0, 3 \pmod 4$ [652]<br>G if $n = 6, t$ even [652]<br>G if $n = 4, t > 1$ [1055]<br>G if $n = 5, t > 1$ [1287]<br>G if $n = 7$ and $t \equiv 0, 3 \pmod 4$ [1293]<br>G if $n = 9$ and $t \equiv 0, 3 \pmod 4$ [1294]<br>G if $t = 2$ $n \not\equiv 1 \pmod 4$ [947], [249]<br>G if $n = 11$ [1278] |

Table 1: **Summary of Graceful Results continued**

| *Graph* | *Graceful* |
|---|---|
| triangular snakes (see §2.2) | G iff number of blocks $\equiv 0, 1 \pmod 4$ [883] |
| $K_4$-snakes (see §2.2) | ? |
| quadrilateral snakes (see §2.2) | G [471], [947] |
| crowns $C_n \odot K_1$ | G [428] |
| $C_n \odot P_k$ | G [1000] |
| grids $P_m \times P_n$ | G [24] |
| prisms $C_m \times P_n$ | G if $n = 2$ [431], [1290]<br>G if $m$ even [565]<br>G if $m$ odd and $3 \leq n \leq 12$ [565]<br>G if $m = 3$ [1089]<br>G if $m = 6$ see [1292]<br>G if $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$ [1292] |
| $K_m \times P_n$ | G if $(m, n) = (4, 2), (4, 3), (4, 4), (4, 5), (5, 2)$[1113]<br>not G if $(m, n) = (3, 3), (6, 2), (7, 2), (8, 2), (9, 2)(10, 2)$ [1113]<br>not G? for $(m, 2)$ with $m > 5$ [1113] |
| $K_{m,n} \odot K_1$ | G [613] |
| torus grids $C_m \times C_n$ | G if $m \equiv 0 \pmod 4$, $n$ even [621]<br>not G if $m, n$ odd (parity condition) |
| vertex-deleted $C_m \times P_n$ | G if $n = 2$ [446] |
| edge-deleted $C_m \times P_n$ | G if $n = 2$ [446] |
| Möbius ladders $M_n$ (see §2.3) | G [441] |
| stacked books $S_m \times P_n$ (see §2.3) | $n = 2$, G iff $m \not\equiv 3 \pmod 4$ [839], [351], [445]<br>G if $m$ even [445] |
| $n$-cube $K_2 \times K_2 \times \cdots \times K_2$ | G [671] |

Table 1: **Summary of Graceful Results continued**

| *Graph* | *Graceful* |
|---|---|
| $K_4 \times P_n$ | G if $n = 2, 3, 4, 5$ [927] |
| $K_n$ | G iff $n \leq 4$ [475], [1088] |
| $K_{m,n}$ | G [973], [475] |
| $K_{1,m,n}$ | G [73] |
| $K_{1,1,m,n}$ | G [471] |
| windmills $K_n^{(m)} (n > 3)$ (see §2.4) | G if $n = 4, m \leq 1000$ [565],[1],[1239],[461]<br>G? if $n = 4, m \geq 4$ [217]<br>not G if $n = 4, m = 2, 3$ [217]<br>not G if $(m, n) = (2, 5)$ [220]<br>not G if $n > 5$ [658] |
| $B(n, r, m)$ $r > 1$ (see §2.4) | G if $(n, r) = (3, 2), (4, 3)$ [653], (4,2) [351]<br>G $(n, r, m) = (5, 2, 2)$ [1113]<br>not G for $(n, 2, 2)$ for $n > 5$ [219], [1113] |
| $mK_n$ (see §2.5) | G iff $m = 1, n \leq 4$ [675] |
| $C_s \cup P_n$ | ? G iff $s + n \geq 7$ [432]<br>G if $s = 3$ [430], $s = 4$ [432], $s = 5$ [650]<br>G if $s > 4, n = 2$ [449]<br>G if $s = 2n + 1$ [430]<br>G if $s = 2k, n \geq k + 1$ [1009] |
| $C_p \cup C_q$ | G iff $p + q \equiv 0, 3 \pmod 4$ [11] |

Table 1: **Summary of Graceful Results continued**

| *Graph* | *Graceful* |
|---|---|
| $C_n \cup K_{p,q}$ | for $n > 8$ G iff $n \equiv 0,3 \pmod 4$ [1306]<br>G $C_6 \times K_{1,2n+1}$ [193]<br>G $C_3 \times K_{m,n}$ iff $m, n \geq 2$ [1019]<br>G $C_4 \times K_{m,n}$ iff $(m,n) \neq (1,1)$[1019]<br>G $C_7 \times K_{m,n}$ [1019]<br>G $C_8 \times K_{m,n}$ [1019] |
| $K_i \cup K_{m,n}$ | G [193] |
| $\bigcup_{i=1}^{t} K_{m_i,n_i}$ | G $2 \leq m_i < n_i$ [193] |
| $C_m \cup \bigcup_{i=1}^{t} K_{m_i,n_i}$ | G $2 \leq m_i < n_i$,<br>$m \equiv 0$ or $3 \pmod 4$, $m \geq 11$ [193] |
| $G + \overline{K_t}$ | G for connected $G$ |
| double cones $C_n + \overline{K_2}$ | G for $n = 3,4,5,7,8,9,11,12$<br>not G for $n \equiv 2 \pmod 4$ |
| $t$-point suspension $C_n + \overline{K_t}$ | G if $n \equiv 0$ or $3 \pmod{12}$ [230]<br>not G if $t$ is even and $n \equiv 2,6,10 \pmod{12}$<br>G if $n = 4,7,11$ or $19$ [230]<br>G if $n = 5$ or $9$ and $t = 2$ [230] |
| $P_n^2$ (see §2.7) | G [710] |
| Petersen $P(n,k)$ (see §2.7) | G for $n = 5,6,7,8,9,10$ [956], $(n,k) = (8t,3)$[1202] |

Table 2: **Summary of Harmonious Results**

| *Graph* | *Harmonious* |
|---|---|
| trees | H if $\leq 26$ vertices [51]<br>H? [484]<br>H caterpillars [484]<br>? lobsters |
| cycles $C_n$ | H iff $n \equiv 1, 3 \pmod 4$ [484] |
| wheels $W_n$ | H [484] |
| helms (see §2.2) | H [471], [809] |
| webs (see §2.2) | H if cycle is odd |
| gears (see §2.2) | H [311] |
| cycles with $P_k$-chord (see §2.2) | ? |
| $C_n$ with $k$ consecutive chords (see §2.2) | ? |
| unicyclic graphs<br>$P_n^k$ | ?<br>H if $k = 2$ [481], $k = 3$ [1007]<br>H if $k$ is even and $k/2 \leq (n-1)/2$ [1316] |
| $C_n^{(t)}$ (see §2.2) | $n = 3$ H iff $t \not\equiv 2 \pmod 4$ [484]<br>H if $n = 4,\ t > 1$ [1055] |
| triangular snakes (see §2.2) | H if number of blocks is odd [1275]<br>not H if number of blocks $\equiv 2$<br>(mod 4) [1275] |
| $K_4$-snakes (see §2.2) | H [482] |
| quadrilateral snakes (see §2.2) | ? |
| crowns $C_n \odot K_1$ | H [481], [811] |
| grids $P_m \times P_n$ | H iff $(m, n) \neq (2, 2)$ [621] |

Table 2: **Summary of Harmonious Results continued**

| *Graph* | *Harmonious* |
|---|---|
| prisms $C_m \times P_n$ | H if $n = 2, m \neq 4$ [446]<br>H if $n$ odd [484]<br>H if $m = 4$ and $n \geq 3$ [621] |
| torus grids $C_m \times C_n$, | H if $m = 4$, $n > 1$ [621]<br>not H if $m \not\equiv 0 \pmod 4$ and $n$ odd [621] |
| vertex-deleted $C_m \times P_n$ | H if $n = 2$ [446] |
| edge-deleted $C_m \times P_n$ | H if $n = 2$ [446] |
| Möbius ladders $M_n$ (see §2.3) | H iff $n \neq 3$ [441] |
| stacked books $S_m \times P_n$ (see §2.3) | $n = 2$, H if $m$ even [480], [961]<br>not H $m \equiv 3 \pmod 4$, $n = 2$,<br>(parity condition)<br>H if $m \equiv 1 \pmod 4$, $n = 2$ [471] |
| $n$-cube $K_2 \times K_2 \times \cdots \times K_2$ | H if and only if $n \geq 4$ [571] |
| $K_4 \times P_n$ | H [961] |
| $K_n$ | H iff $n \leq 4$ [484] |
| $K_{m,n}$ | H iff $m$ or $n = 1$ [484] |
| $K_{1,m,n}$ | H [73] |
| $K_{1,1,m,n}$ | H [471] |
| windmills $K_n^{(m)}$ $(n > 3)$ (see §2.4) | H if $n = 4$ [559]<br>$m = 2$, H? iff $n = 4$ [484]<br>not H if $m = 2, n$ odd or 6 [484]<br>not H for some cases $m = 3$ [799] |

Table 2: **Summary of Harmonious Results continued**

| *Graph* | *Harmonious* |
|---|---|
| $B(n,r,m)$ $r>1$ (see §2.4) | $(n,r)=(3,2),(4,3)$ [1016] |
| $mK_n$ (see §2.5) | H $n=3,\ m$ odd [812]<br>not H for $n$ odd, $m\equiv 2 \pmod 4$ [812] |
| $nG$ | H when $G$ is harmonious and $n$ odd [1304] |
| $G^n$ | H when $G$ is harmonious and $n$ odd [1304] |
| $C_s \cup P_n$ | ? |
| fans $F_n = P_n + K_1$ | H [484] |
| $nC_m + K_1$ $n \not\equiv 0 \bmod 4$ | H [311] |
| double fans $P_n + \overline{K_2}$ | H [484] |
| $t$-point suspension $P_n + \overline{K_t}$ of $P_n$ | H [961] |
| $S_m + K_1$ | H [471], [295] |
| $t$-point suspension $C_n + \overline{K_t}$ of $C_n$ | H if $n$ odd and $t=2$ [961], [471]<br>not H if $n \equiv 2,4,6 \pmod 8$ and $t=2$ [471] |
| $P_n^2$ (see §2.7) | H [481], [811] |
| Petersen $P(n,k)$ (see §2.7) | H [471], [743] |

# 3 Variations of Graceful Labelings

## 3.1 $\alpha$-labelings

In 1966 Rosa [973] defined an *$\alpha$-labeling* (or *$\alpha$-valuation*) as a graceful labeling with the additional property that there exists an integer $k$ so that for each edge $xy$ either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. (Other names for such labelings are *balanced*, *interlaced*, and *strongly graceful*.) It follows that such a $k$ must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an $\alpha$-labeling is necessarily bipartite and therefore can not contain a cycle of odd length. Wu [1268] has shown that a necessary condition for a bipartite graph with $n$ edges and degree sequence $d_1, d_2, \ldots, d_p$ to have an $\alpha$-labeling is that the $\gcd(d_1, d_2, \ldots, d_p, n)$ divides $n(n-1)/2$.

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an $\alpha$-labeling is a typical approach. (See [295], [481], [310], and [621].) Moreover, Jungreis and Reid [621] showed how sequential labelings of graphs (see Section 4.1) can often be obtained by modifying $\alpha$-labelings of the graphs.

Graphs with $\alpha$-labelings have proved to be useful in the development of the theory of graph decompositions. Rosa [973], for instance, has shown that if $G$ is a graph with $q$ edges and has an $\alpha$-labeling, then for every natural number $p$, the complete graph $K_{2qp+1}$ can be decomposed into copies of $G$ in such a way that the automorphism group of the decomposition itself contains the cyclic group of order $p$. In the same vein El-Zanati and Vanden Eynden [385] proved that if $G$ has $q$ edges and admits an $\alpha$-labeling then $K_{qm,qn}$ can be partitioned into subgraphs isomorphic to $G$ for all positive integers $m$ and $n$. Although a proof of Ringel's conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have $\alpha$-labelings are easy to construct (see [973]). Kotzig [669] has shown however that almost all trees have $\alpha$-labelings.

As to which graphs have $\alpha$-labelings, Rosa [973] observed that the $n$-cycle has an $\alpha$-labeling if and only if $n \equiv 0 \pmod 4$ whereas $P_n$ always has an $\alpha$-labeling. Other familiar graphs that have $\alpha$-labelings include caterpillars [973], the $n$-cube [668], Möbius ladders $M_n$ when $n$ is odd (see §2.3) for the definition) [923], $B_{4n+1}$ (i.e., books with $4n+1$ pages) [445], $C_{2m} \cup C_{2m}$ and $C_{4m} \cup C_{4m} \cup C_{4m}$ for all $m > 1$ [670], $C_{4m} \cup C_{4m} \cup C_{4n}$ for all $(m,n) \neq 1,1)$ [399], $P_n \times Q_n$ [839], $K_{1,2k} \times Q_n$ [839], $C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$ [695], $C_{4m} \cup C_{4n+2} \cup C_{4r+2}, C_{4m} \cup C_{4n} \cup C_{4r}$ when $m+n \leq r$ [11], $C_{4m} \cup C_{4n} \cup C_{4r} \cup C_{4s}$ when $m \geq n+r+s$ [7], $C_{4m} \cup C_{4n} \cup C_{4r+2} \cup C_{4s+2}$ when $m \geq n+r+s+1$ [7], $((m+1)^2+1)C_4$ for all $m$ [1326], $k^2C_4$ for all $k$ [1326], and $(k^2+k)C_4$ for all $k$ [1326]. Abrham and Kotzig [9] have shown that $kC_4$ has an $\alpha$-labeling for $1 \leq k \leq 10$ and that if $kC_4$ has an $\alpha$-labeling then so does $(4k+1)C_4, (5k+1)C_4$ and $(9k+1)C_4$. Eshghi [394] proved that $5C_{4k}$ has an $\alpha$-labeling for all $k$. In [399] Eshghi and Carter show several families of graphs of the form $C_{4n_1} \cup C_{4n_2} \cup \cdots \cup C_{4n_k}$ have $\alpha$-labelings. Pei-Shan Lee [703] proved that $C_6 \times P_{2t+1}$ and gear graphs have $\alpha$-labelings. He raises the question of whether $C_{4m+2} \times P_{2t+1}$ has an $\alpha$-labeling for all $m$.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [415] have shown that if $m \equiv 0 \pmod 4$ then the one-point union of 2, 3, or 4 copies of $C_m$ admits an $\alpha$-labeling, and if $m \equiv 2 \pmod 4$ then the one-point union of 2 or 4 copies of $C_m$ admits an $\alpha$-labeling. They conjecture that the one-point union of $n$ copies of $C_m$ admits an $\alpha$-labeling if and only if $mn \equiv 0 \pmod 4$.

In his 2001 Ph. D. thesis Selvaraju [1001] investigated the one-point union of complete bipartite graphs. He proves that the one-point unions of the following forms have an $\alpha$-labeling:

$K_{m,n_1}$ and $K_{m,n_2}$; $K_{m_1,n_1}$, $K_{m_2,n_2}$, and $K_{m_3,n_3}$ where $m_1 \leq m_2 \leq m_3$ and $n_1 < n_2 < n_3$; $K_{m_1,n}$, $K_{m_2,n}$, and $K_{m_3,n}$ where $m_1 < m_2 < m_3 \leq 2n$.

Zhile [1326] uses $C_m(n)$ to denote the connected graph all of whose blocks are $C_m$ and whose block-cutpoint-graph is a path. He proves that for all positive integers $m$ and $n$, $C_{4m}(n)$ has an $\alpha$-labeling but $C_m(n)$ does not have an $\alpha$-labeling when $m$ is odd.

Abrham and Kotzig [11] have proved that $C_m \cup C_n$ has an $\alpha$-labeling if and only if both $m$ and $n$ are even and $m + n \equiv 0 \pmod 4$. Kotzig [670] has also shown that $C_4 \cup C_4 \cup C_4$ does not have an $\alpha$-labeling. He asked if $n = 3$ is the only integer such that the disjoint union of $n$ copies of $C_4$ does not have an $\alpha$-labeling. This was confirmed by Abrham and Kotzig in [10]. Eshghi [393] proved that every 2-regular bipartite graph with 3 components has an $\alpha$-labeling if and only if the number of edges is a multiple of four except for $C_4 \cup C_4 \cup C_4$. In [396] Eshghi gives more results on the existence of $\alpha$-labelings for various families of disjoint union of cycles.

Jungreis and Reid [621] investigated the existence of $\alpha$-labelings for graphs of the form $P_m \times P_n, C_m \times P_n$, and $C_m \times C_n$ (see also [444]). Of course, the cases involving $C_m$ with $m$ odd are not bipartite, so there is no $\alpha$-labeling. The only unresolved cases among these three families are $C_{4m+2} \times P_{2n+1}$ and $C_{4m+2} \times C_{4n+2}$. All other cases result in $\alpha$-labelings. Balakrishman [174] uses the notation $Q_n(G)$ to denote the graph $P_2 \times P_2 \times \cdots \times P_2 \times G$ where $P_2$ occurs $n-1$ times. Snevily [1116] has shown that the graphs $Q_n(C_{4m})$ and the cycles $C_{4m}$ with the path $P_n$ adjoined at each vertex have $\alpha$-labelings. He [1117] also has shown that compositions of the form $G[\overline{K_n}]$ (see §2.3 for the definition) have an $\alpha$-labeling whenever $G$ does (see §2.3 for the definition of composition). Balakrishman and Kumar [177] have shown that all graphs of the form $Q_n(G)$ where $G$ is $K_{3,3}, K_{4,4}$, or $P_m$ have an $\alpha$-labeling. Balakrishman [174] poses the following two problems. For which graphs $G$ does $Q_n(G)$ have an $\alpha$-labeling? For which graphs $G$ does $Q_n(G)$ have a graceful labeling?

Rosa [973] has shown that $K_{m,n}$ has an $\alpha$-labeling (see also [190]). Barrientos [190] has shown that for $n$ even the graph obtained from the wheel $W_n$ by attaching a pendant edge at each vertex has an $\alpha$-labeling. In [197] Barrientos shows how to construct graceful graphs that are formed from the one-point union of a tree that has an $\alpha$-labeling, $P_2$, and the cycle $C_n$. In some cases, $P_2$ is not needed. Qian [947] has proved that quadrilateral snakes have $\alpha$-labelings. Yu, Lee, and Chin [1314] showed that $Q_3$-and $Q_3$-snakes have $\alpha$-labelings. Fu and Wu [434] showed that if $T$ is a tree that has an $\alpha$-labeling with partite sets $V_1$ and $V_2$ then the graph obtained from $T$ by joining new vertices $w_1, w_2, \ldots, w_k$ to every vertex of $V_1$ has an $\alpha$-labeling. Similarly, they prove that the graph obtained from $T$ by joining new vertices $w_1, w_2, \ldots, w_k$ to the vertices of $V_1$ and new vertices $u_1, u_2, \ldots, u_t$ to every vertex of $V_2$ has an $\alpha$-labeling. They also prove that if one of the new vertices of either of these two graphs is replaced by a star and every vertex of the star is joined to the vertices of $V_1$ or the vertices of both $V_1$ and $V_2$, the resulting graphs have $\alpha$-labelings. Fu and Wu [434] further show that if $T$ is a tree with an $\alpha$-labeling and the sizes of the two partite sets of $T$ differ at by at most 1, then $T \times P_m$ has an $\alpha$-labeling.

Lee and Liu [718] investigated the mirror graph $M(m,n)$ of $K_{m,n}$ (see §2.3 for the definition) for $\alpha$-labelings. They proved: $M(m,n)$ has an $\alpha$-labeling when $n$ is odd or $m$ is even; $M(1,n)$ has an $\alpha$-labeling when $n \equiv 0 \pmod 4$; $M(m,n)$ does not have an $\alpha$-labeling when $m$ is odd and $n \equiv 2 \pmod 4$, or when $m \equiv 3 \pmod 4$ and $n \equiv 4 \pmod 8$.

Barrientos [191] defines a *chain graph* as one with blocks $B_1, B_2, \ldots, B_m$ such that for every $i$, $B_i$ and $B_{i+1}$ have a common vertex in such a way that the block-cutpoint graph is a path. He shows that if $B_1, B_2, \ldots, B_m$ are blocks that have $\alpha$-labelings then there exists a chain graph

$G$ with blocks $B_1, B_2, \ldots, B_m$ that has an $\alpha$-labeling. He also shows that if $B_1, B_2, \ldots, B_m$ are complete bipartite graphs, then any chain graph $G$ obtained by concatenation of these blocks has an $\alpha$-labeling.

Wu ([1267] and [1269]) has given a number of methods for constructing larger graceful graphs from graceful graphs. Let $G_1, G_2, \ldots, G_p$ be disjoint connected graphs. Let $w_i$ be in $G_i$ for $1 \leq i \leq p$. Let $w$ be a new vertex not in any $G_i$. Form a new graph $\oplus_w(G_1, G_2, \ldots, G_p)$ by adjoining to the graph $G_1 \cup G_2 \cup \cdots \cup G_p$ the edges $ww_1, ww_2, \ldots, ww_p$. In the case where each of $G_1, G_2, \ldots, G_p$ is isomorphic to a graph $G$ that has an $\alpha$-labeling and each $w_i$ is the isomorphic image of the same vertex in $G_i$, Wu shows that the resulting graph is graceful. If $f$ is an $\alpha$-labeling of a graph, the integer $k$ with the property that for any edge $uv$ either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$ is called the *boundary value* or *critical number* of $f$. Wu [1267] has also shown that if $G_1, G_2, \ldots, G_p$ are graphs of the same order and have $\alpha$-labelings where the labelings for each pair of graphs $G_i$ and $G_{p-i+1}$ have the same boundary value for $1 \leq i \leq n/2$, then $\oplus_w(G_1, G_2, \ldots, G_p)$ is graceful. In [1265] Wu proves that if $G$ has $n$ edges and $n+1$ vertices and $G$ has an $\alpha$-labeling with boundary value $\lambda$, where $|n - 2\lambda - 1| \leq 1$, then $G \times P_m$ is graceful for all $m$.

Ajitha, Arumugan, and Germina [56] use a construction of Koh, Tan, and Rogers [660] to create trees with $\alpha$-labelings from smaller trees with graceful labelings. These in turn allows them to generate large classes of trees that have a type of called edge-antimagic labelings (see §6.1).

Mavonicolas and Michael [849] say that trees $\langle T_1, \theta_1, w_1\rangle$ and $\langle T_2, \theta_2, w_2\rangle$ with roots $w_1$ and $w_2$ and $|V(T_1)| = |V(T_2)|$ are *gracefully consistent* if either they are identical or they have $\alpha$-labelings with the same boundary value and $\theta_1(w_1) = \theta_2(w_2)$. They use this concept to show that a number of known constructions of new graceful trees using several identical copies of a given graceful rooted tree can be extended to the case where the copies are replaced by a set of pairwise gracefully consistent trees. In particular, let $\langle T, \theta, w\rangle$ and $\langle T_0, \theta_0, w_0\rangle$ be gracefully labeled trees rooted at $w$ and $w_0$ respectively. They show that the following four constructions are adaptable to the case when a set of copies of $\langle T, \theta, w\rangle$ is replaced by a set of pairwise gracefully consistent trees. When $\theta(w) = |E(T)|$ the garland construction due to Koh, Rogers, and Tan [654] gracefully labels the tree consisting of $h$ copies of $\langle T, w\rangle$ with their roots connected to a new vertex $r$. In the case when $\theta(w) = |E(T)|$ and whenever $uw \in E(T)$ and $\theta(u) \neq 0$, then $vw \in E(T)$ where $\theta(u) + \theta(v) = |E(T)|$, the attachment construction of Koh, Tan and Rogers [660] gracefully labels the tree formed by identifying the roots of $h$ copies of $\langle T, w\rangle$. The $\Delta$-construction given by Koh, Tan and Rogers [660] gracefully labels the tree formed by merging each vertex of $\langle T_0, w_0\rangle$ with the root of a distinct copy of $\langle T, w\rangle$. When $\theta_0(w_0) = |E(T_0)|$, let $N$ be the set of neighbors of $w_0$ and let $x$ be the vertex of $T$ at even distance from $w$ with $\theta(x) = 0$ or $\theta(x) = |E(T)|$. Then the $\Delta_{+1}$-construction Burzio and Ferrarese [269] gracefully labels the tree formed by merging each non-root vertex of $T_0$ with the root of a distinct copy of $\langle T, w\rangle$ so that for each $v \in N$ the edge $vw_0$ is replaced with a new edge $xw_0$ (where $x$ is in the corresponding copy of $T$).

Snevily [1117] says that a graph $G$ *eventually has an $\alpha$-labeling* provided that there is a graph $H$, called a *host* of $G$, which has an $\alpha$-labeling and that the edge set of $H$ can be partitioned into subgraphs isomorphic to $G$. He defines the *$\alpha$-labeling number of $G$* to be $G_\alpha = \min\{t :$ there is a host $H$ of $G$ with $|E(H)| = t|G|\}$. Snevily proved that even cycles have $\alpha$-labeling number at most 2 and he conjectured that every bipartite graph has an $\alpha$-labeling number. This conjecture was proved by El-Zanati, Fu, and Shiue [383]. There are no known examples

of a graph $G$ with $G_\alpha > 2$. In [1117] Snevily conjectured that the $\alpha$-labeling number for a tree with $n$ edges is at most $n$. Shiue and Fu [1082] proved that $\alpha$-labeling number for a tree with $n$ edges and radius $r$ is at most $\lceil r/2 \rceil n$. They also prove that a tree with $n$ edges and radius $r$ decomposes $K_t$ for some $t \leq (r+1)n^2 + 1$.

Given two bipartite graphs $G_1$ and $G_2$ with partite sets $H_1$ and $L_1$ and $H_2$ and $L_2$, respectively, Snevily [1116] defines their *weak tensor product* $G_1 \overline{\bigotimes} G_2$ as the bipartite graph with vertex set $(H_1 \times H_2, L_1 \times L_2)$ and with edge $(h_1, h_2)(l_1, l_2)$ if $h_1 l_1 \in E(G_1)$ and $h_2 l_2 \in E(G_2)$. He proves that if $G_1$ and $G_2$ have $\alpha$-labelings then so does $G_1 \overline{\bigotimes} G_2$. This result considerably enlarges the class of graphs known to have $\alpha$-labelings.

The sequential join of graphs $G_1, G_2, \ldots, G_n$ is formed from $G_1 \cup G_2 \cup \cdots \cup G_n$ by adding edges joining each vertex of $G_i$ with each vertex of $G_{i+1}$ for $1 \leq i \leq n-1$. Lee and Wang [767] have shown that for all $n \geq 2$ and any positive integers $a_1, a_2, \ldots, a_n$ the sequential join of the graphs $\overline{K}_{a_1}, \overline{K}_{a_2}, \ldots, \overline{K}_{a_n}$ has an $\alpha$-labeling.

In [442] Gallian and Ropp conjectured that every graph obtained by adding a single pendant edge to one or more vertices of a cycle is graceful. Qian [947] proved this conjecture and in the case that the cycle is even he shows the graphs have an $\alpha$-labeling. He further proves that for $n$ even any graph obtained from an $n$-cycle by adding one or more pendant edges at some vertices has an $\alpha$-labeling as long as at least one vertex has degree 3 and one vertex has degree 2.

For any tree $T(V, E)$ whose vertices are properly 2-colored Rosa and Širáň [976] define a *bipartite labeling* of $T$ as a bijection $f : V \to \{0, 1, 2, \ldots, |E|\}$ for which there is a $k$ such that whenever $f(u) \leq k \leq f(v)$, then $u$ and $v$ have different colors. They define the *$\alpha$-size* of a tree $T$ as the maximum number of distinct values of the induced edge labels $|f(u) - f(v)|, uv \in E$, taken over all bipartite labelings $f$ of $T$. They prove that the $\alpha$-size of any tree with $n$ edges is at least $5(n+1)/7$ and that there exist trees whose $\alpha$-size is at most $(5n+9)/6$. They conjectured that minimum of the $\alpha$-sizes over all trees with $n$ edges is asymptotically $5n/6$. This conjecture has been proved for trees of maximum degree 3 by Bonnington and Širáň [270]. Heinrich and Hell [547] defined the *gracesize* of a graph $G$ with $n$ vertices as the maximum, over all bijections $f: V(G) \to \{1, 2, \ldots, n\}$, of the number of distinct values $|f(u) - f(v)|$ over all edges $uv$ of $G$. So, from Rosa and Širáň's result, the gracesize of any tree with $n$ edges is at least $5(n+1)/7$.

In [446] Gallian weakened the condition for an $\alpha$-labeling somewhat by defining a *weakly $\alpha$-labeling* as a graceful labeling for which there is an integer $k$ so that for each edge $xy$ either $f(x) \leq k \leq f(y)$ or $f(y) \leq k \leq f(x)$. Unlike $\alpha$-labelings, this condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph; namely, that the vertex with the label $k$ must be on every odd cycle. Gallian, Prout, and Winters [446] showed that the prisms $C_n \times P_2$ with a vertex deleted have $\alpha$-labelings. The same paper reveals that $C_n \times P_2$ with an edge deleted from a cycle has an $\alpha$-labeling when $n$ is even and a weakly $\alpha$-labeling when $n > 3$.

A special case of $\alpha$-labeling called strongly graceful was introduced by Maheo [839] in 1980. A graceful labeling $f$ of a graph $G$ is called *strongly graceful* if $G$ is bipartite with two partite sets $A$ and $B$ of the same order $s$, the number of edges is $2t + s$, there is an integer $k$ with $t - s \leq k \leq t + s - 1$ such that if $a \in A, f(a) \leq k$, and if $b \in B, f(b) > k$, and there is an involution $\pi$ that is an automorphism of $G$ such that: $\pi$ exchanges $A$ and $B$ and the $s$ edges $a\pi(a)$ where $a \in A$ have as labels the integers between $t+1$ and $t+s$. Maheo's main result is that if $G$ is strongly graceful then so is $G \times Q_n$. In particular, she proved that $(P_n \times Q_n) \times K_2$, $B_{2n}$, and $B_{2n} \times Q_n$ have strongly graceful labelings.

in 1999 Broersma and Hoede [255] conjectures that every tree containing a perfect matching

is strongly graceful. Yao, Cheng, Yao, and Zhao [1296] proved that this conjecture is true for every tree with diameter at most 5 and provided a method for constructing strongly graceful trees.

El-Zanati and Vanden Eynden [386] call a strongly graceful labeling a *strong $\alpha$-labeling*. They show that if $G$ has a strong $\alpha$-labeling, then $G \times P_n$ has an $\alpha$-labeling. They show that $K_{m,2} \times K_2$ has a strong $\alpha$-labeling and that $K_{m,2} \times P_n$ has an $\alpha$-labeling. They also show that if $G$ is a bipartite graph with one more vertex than the number of edges, and if $G$ has an $\alpha$-labeling such that the cardinalities of the sets of the corresponding bipartition of the vertices differ by at most 1, then $G \times K_2$ has a strong $\alpha$-labeling and $G \times P_n$ has an $\alpha$-labeling. El-Zanati and Vanden Eynden [386] also note that $K_{3,3} \times K_2$, $K_{3,4} \times K_2$, $K_{4,4} \times K_2$, and $C_{4k} \times K_2$ all have strong $\alpha$-labelings. El-Zanati and Vanden Eynden proved that $K_{m,2} \times Q_n$ has a strong $\alpha$-labeling and that $K_{m,2} \times P_n$ has an $\alpha$-labeling for all $n$. They also prove that if $G$ is a connected bipartite graph with partite sets of odd order such that in each partite set each vertex has the same degree, then $G \times K_2$ does not have a strong $\alpha$-labeling. As a corollary they have that $K_{m,n} \times K_2$ does not have a strong $\alpha$-labeling when $m$ and $n$ are odd.

An $\alpha$-labeling $f$ of a graph $G$ is called *free* by El-Zanati and Vanden Eynden in [387] if the critical number $k$ (in the definition of $\alpha$-labeling) is greater than 2 and if neither 1 nor $k-1$ is used in the labeling. Their main result is that the union of graphs with free $\alpha$-labelings has an $\alpha$-labeling. In particular, they show that $K_{m,n}$, $m > 1$, $n > 2$, has a free $\alpha$-labeling. They also show that $Q_n$, $n \geq 3$, and $K_{m,2} \times Q_n$, $m > 1$, $n \geq 1$, have free $\alpha$-labelings. El-Zanati [personal communication] has shown that the Heawood graph has a free $\alpha$-labeling.

For connected bipartite graphs Grannell, Griggs, and Holroyd [485] introduced a labeling that lies between $\alpha$-labelings and graceful labelings. They call a vertex labeling $f$ of a bipartite graph $G$ with $q$ edges and partite sets $D$ and $U$ *gracious* if $f$ is a bijection from the vertex set of $G$ to $\{0, 1, \ldots, q\}$ such that the set of edge labels induced by $f(u) - f(v)$ for every edge $uv$ with $u \in U$ and $v \in D$ is $\{1, 2, \ldots, q\}$. Thus a gracious labeling of $G$ with partite sets $D$ and $U$ is a graceful labeling in which every vertex in $D$ has a label lower than every adjacent vertex. They verified by computer that every tree of size up to 20 has a gracious labeling. This led them to conjecture that every tree has a gracious labeling. For any $k > 1$ and any tree $T$ Grannell et al. say that $T$ has a *gracious $k$-labeling* if the vertices of $T$ can be partitioned into sets $D$ and $U$ in such a way that there is a function $f$ from the verticies of $G$ to the integers modulo $k$ such that the edge labels induced by $f(u) - f(v)$ where $u \in U$ and $v \in D$ have the following properties: the number of edges labeled with 0 is one less than the number of verticies labeled with 0 and for each nonzero integer $t$ the number of edges labeled with $t$ is the same as the number of verticies labeled with $t$. They prove that every nontrivial tree has a $k$-gracious labeling for $k = 2, 3, 4$, and 5 and that caterpillars are $k$-gracious for all $k \geq 2$.

The same labeling that is called gracious by Grannell, Griggs, and Holroyd is called a *near $\alpha$-labeling* by El-Zanati, Kenig, and Vanden Eynden [384]. The latter prove that if $G$ is a graph with $n$ edges that has a near $\alpha$-labeling then there exists a cyclic $G$-decomposition of $K_{2nx+1}$ for all positive integers $x$ and a cyclic $G$-decomposition of $K_{n,n}$. They further prove that if $G$ and $H$ have near $\alpha$-labelings, then so does their weak tensor product (see earlier part of this section) with respect to the corresponding vertex partitions. They conjecture that every tree has a near $\alpha$-labeling.

Another kind of labelings for trees was introduced by Ringel, Llado, and Serra [964] in an approach to proving their conjecture $K_{n,n}$ is edge-decomposable into $n$ copies of any given tree with $n$ edges. If $T$ is a tree with $n$ edges and partite sets $A$ and $B$, they define a labeling $f$ from

the set of vertices to $\{1, 2, \ldots, n\}$ to be a *bigraceful* labeling of $T$ if $f$ restricted to $A$ is injective, $f$ restricted to $B$ is injective, and the edge labels given by $f(y) - f(x)$ where $yx$ is an edge with $y$ in $B$ and $x$ in $A$ is the set $\{0, 1, 2, \ldots, n-1\}$. (Notice that this terminology conflicts with that given in Section 2.7 In particular, the Ringel, Llado, and Serra bigraceful does not imply the usual graceful.) Among the graphs that they show are bigraceful are: lobsters, trees of diameter at most 5, stars $S_{k,m}$ with $k$ spokes of paths of length $m$, and complete $d$-ary trees for $d$ odd. They also prove that if $T$ is a tree then there is a vertex $v$ and a nonnegative integer $m$ such that the addition of $m$ leaves to $v$ results in a bigraceful tree. They conjecture that all trees are bigraceful.

Table 3 summarizes some of the main results about $\alpha$-labelings. $\alpha$ indicates that the graphs have an $\alpha$-labeling.

Table 3: **Summary of Results on $\alpha$-labelings**

| *Graph* | *$\alpha$-labeling* |
|---|---|
| cycles $C_n$ | $\alpha$ iff $n \equiv 0 \pmod 4$ [973] |
| caterpillars | $\alpha$ [973] |
| $n$-cube | $\alpha$ [668] |
| books $B_{2n},\ B_{4n+1}$ | $\alpha$ [839],[445] |
| Möbius ladders $M_{2k+1}$ | $\alpha$ [923] |
| $C_m \cup C_n$ | $\alpha$ iff $m, n$ are even and $m+n \equiv 0 \pmod 4$[11] |
| $C_{4m} \cup C_{4m} \cup C_{4m}$ $(m > 1)$ | $\alpha$ [670] |
| $C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$ | $\alpha$ [670] |
| $P_n \times Q_n$ | $\alpha$ [839] |
| $B_{2n} \times Q_n$ | $\alpha$ [839] |
| $K_{1,n} \times Q_n$ | $\alpha$ [839] |
| $K_{m,2} \times Q_n$ | $\alpha$ [386] |
| $K_{m,2} \times P_n$ | $\alpha$ [386] |
| $P_2 \times P_2 \times \cdots \times P_2 \times G$ | $\alpha$ when $G = C_{4m},\ P_m,\ K_{3,3},\ K_{4,4}$ [1116] |
| $P_2 \times P_2 \times \cdots \times P_2 \times P_m$ | $\alpha$ [1116] |
| $P_2 \times P_2 \times \cdots \times P_2 \times K_{m,m}$ | $\alpha$ [1116] when $m = 3$ or 4 |
| $G[\overline{K_n}]$ | $\alpha$ when $G$ is $\alpha$ [1117] |

### 3.2 $k$-graceful Labelings

A natural generalization of graceful graphs is the notion of $k$-graceful graphs introduced independently by Slater [1107] in 1982 and by Maheo and Thuillier [840] in 1982. A graph $G$ with $q$ edges is *$k$-graceful* if there is labeling $f$ from the vertices of $G$ to $\{0, 1, 2, \ldots, q+k-1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k+1, \ldots, q+k-1\}$. Obviously, 1-graceful is graceful and it is readily shown that any graph that has an $\alpha$-labeling is $k$-graceful for all $k$. Graphs that are $k$-graceful for all $k$ are sometimes called *arbitrarily graceful*. Ng [895] has shown that there are graphs that are $k$-graceful for all $k$ but do not have an $\alpha$-labeling.

Results of Maheo and Thuillier [840] together with those of Slater [1107] show that: $C_n$ is $k$-graceful if and only if either $n \equiv 0$ or 1 (mod 4) with $k$ even and $k \leq (n-1)/2$, or $n \equiv 3$ (mod 4) with $k$ odd and $k \leq (n^2-1)/2$. Maheo and Thuillier [840] also proved that the wheel $W_{2k+1}$ is $k$-graceful and conjectured that $W_{2k}$ is $k$-graceful when $k \neq 3$ or $k \neq 4$. This conjecture was proved by Liang, Sun, and Xu [788]. Kang [624] proved that $P_m \times C_{4n}$ is $k$-graceful for all $k$. Lee and Wang [765] showed that the graphs obtained from a nontrivial path of even length by joining every other vertex to one isolated vertex (a *lotus*), the graphs obtained from a nontrivial path of even length by joining every other vertex to two isolated vertices (a *diamond*), and the graphs obtained by arranging vertices into a finite number of rows with $i$ vertices in the $i$th row and in every row the $j$th vertex in that row is joined to the $j$th vertex and $j+1$st vertex of the next row (a *pyramid*) are $k$-graceful. Liang and Liu [783] have shown that $K_{m,n}$ is $k$-graceful. Bu, Gao, and Zhang [262] have proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are $k$-graceful for all $k$. Acharya (see [15]) has shown that a $k$-graceful Eulerian graph with $q$ edges must satisfy one of the following conditions: $q \equiv 0$ (mod 4), $q \equiv 1$ (mod 4) if $k$ is even, or $q \equiv 3$ (mod 4) if $k$ is odd. Bu, Zhang, and He [267] have shown that an even cycle with a fixed number of pendant edges adjoined to each vertex is $k$-graceful. Lu, Pan, and Li [828] have proved that $K_{1,m} \cup K_{p,q}$ is $k$-graceful when $k > 1$, and $p$ and $q$ are at least 2. Jirimutu, Bao, and Kong [614] have shown that the graphs obtained from $K_{2,n}$ $(n \geq 2)$ and $K_{3,n}$ $(n \geq 3)$ by attaching $r \geq 2$ edges at each vertex is $k$-graceful for all $k \geq 2$. Seoud and Elsakhawi [1012] proved: paths and ladders are arbitrarily graceful; and for $n \geq 3$, $K_n$ is $k$-graceful if and only if $k = 1$ and $n = 3$ or 4.

Yao, Cheng, Zhongfu, and Yao [1297] have shown: a tree of order $p$ with maximum degree at least $p/2$ is $k$-graceful for some $k$; if a tree $T$ has an edge $u_1u_2$ such that the two components $T_1$ and $T_2$ of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and $d_{T_2}(u_2) \geq |T_2|/2$, then $T$ is $k$-graceful for some positive $k$; if a tree $T$ has two edges $u_1u_2$ and $u_2u_3$ such that the three components $T_1$, $T_2$, and $T_3$ of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$, $d_{T_2}(u_2) \geq |T_2|/2$, and $d_{T_3}(u_3) \geq |T_3|/2$, then $T$ is $k$-graceful for some $k > 1$; and every Skolem-graceful (see §3.4 for the definition) tree is $k$-graceful for all $k \geq 1$. They conjecture that every tree is $k$-graceful for some $k > 1$.

Several authors have investigated the $k$-gracefulness of various classes of subgraphs of grid graphs. Acharya [13] proved that all 2-dimensional polyminoes that are convex and Eulerian are $k$-graceful for all $k$; Lee [704] showed that Mongolian tents and Mongolian villages are $k$-graceful for all $k$ (see §2.3 for the definitions); Lee and K. C. Ng [724] proved that all Young tableaus (see §2.3 for the definitions) are $k$-graceful for all $k$. (A special case of this is $P_n \times P_2$.) Lee and H. K. Ng [724] subsequently generalized these results on Young tableaus to a wider class of planar graphs.

Duan and Qi [375] use $G_t(m_1, n_1; m_2, n_2; \ldots; m_s, n_s)$ to denote the graph composed of the $s$

complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_s,n_s}$ that have only $t$ $(1 \le t \le \min\{m_1, m_2, \ldots, m_s\})$ common vertices but no common edge and $G(m_1, n_1; m_2, n_2)$ to denote the graph composed of the complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}$ with exactly one common edge. They prove that these graphs are $k$-graceful graphs for all $k$.

Let $c, m, p_1, p_2, \ldots, p_m$ be positive integers. For $i = 1, 2, \ldots, m$, let $S_i$ be a set of $p_i + 1$ integers and let $D_i$ be the set of positive differences of the pairs of elements of $S_i$. If all these differences are distinct then the system $D_1, D_2, \ldots, D_m$ is called a *perfect system of difference sets starting at* $c$ if the union of all the sets $D_i$ is $c, c+1, \ldots, c-1+\sum_{i=1}^{m} \binom{p_i+1}{2}$. There is a relationship between $k$-graceful graphs and perfect systems of difference sets. A perfect system of difference sets starting with $c$ describes a $c$-graceful labeling of a graph that is decomposable into complete subgraphs. A survey of perfect systems of difference sets is given in [5].

Acharya and Hegde [27] generalized $k$-graceful labelings to $(k,d)$-*graceful labelings* by permitting the vertex labels to belong to $\{0, 1, 2, \ldots, k + (q-1)d\}$ and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be $\{k, k+d, k+2d, \ldots, k+(q-1)d\}$. They also introduce an analog of $\alpha$-labelings in the obvious way. Notice that a (1,1)-graceful labeling is a graceful labeling and a $(k,1)$-graceful labeling is a $k$-graceful labeling. Bu and Zhang [266] have shown: $K_{m,n}$ is $(k,d)$-graceful for all $k$ and $d$; for $n > 2, K_n$ is $(k,d)$-graceful if and only if $k = d$ and $n \le 4$; if $m_i, n_i \ge 2$ and $\max\{m_i, n_i\} \ge 3$, then $K_{m_1,n_1} \cup K_{m_2,n_2} \cup \cdots \cup K_{m_r,n_r}$ is $(k,d)$-graceful for all $k, d$, and $r$; if $G$ has an $\alpha$-labeling, then $G$ is $(k,d)$-graceful for all $k$ and $d$; a $k$-graceful graph is a $(kd,d)$-graceful graph; a $(kd,d)$-graceful connected graph is $k$-graceful; and a $(k,d)$-graceful graph with $q$ edges that is not bipartite must have $k \le (q-2)d$.

Let $T$ be a tree with adjacent vertices $u_0$ and $v_0$ and pendant vertices $u$ and $v$ such that the length of the path $u_0 - u$ is the same as the length of the path $v_0 - v$. Hegde and Shetty [539] call the graph obtained from $T$ by deleting $u_0v_0$ and joining $u$ and $v$ an *elementary parallel transformation* of $T$. They say that a tree $T$ is a $T_p$*-tree* if it can be transformed into a path by a sequence of elementary parallel transformations. They prove that every $T_p$-tree is $(k,d)$-graceful for all $k$ and $d$ and every graph obtained from a $T_p$-tree by subdividing each edge of the tree is $(k,d)$-graceful for all $k$ and $d$.

Yao, Cheng, Zhongfu, and Yao [1297] have shown: a tree of order $p$ with maximum degree at least $p/2$ is $(k,d)$-graceful for some $k$ and $d$; if a tree $T$ has an edge $u_1u_2$ such that the two components $T_1$ and $T_2$ of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \ge |T_1|/2$ and $T_2$ is a caterpillar, then $T$ is Skolem-graceful (see §3.4 for the definition); if a tree $T$ has an edge $u_1u_2$ such that the two components $T_1$ and $T_2$ of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \ge |T_1|/2$ and $d_{T_2}(u_2) \ge |T_2|/2$, then $T$ is $(k,d)$-graceful for some $k > 1$ and $d > 1$; if a tree $T$ has two edges $u_1u_2$ and $u_2u_3$ such that the three components $T_1$, $T_2$, and $T_3$ of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \ge |T_1|/2$, $d_{T_2}(u_2) \ge |T_2|/2$, and $d_{T_3}(u_3) \ge |T_3|/2$, then $T$ is $(k,d)$-graceful for some $k > 1$ and $d > 1$; and every Skolem-graceful tree is $(k,d)$-graceful for $k \ge 1$ and $d > 0$. They conjecture that every tree is $(k,d)$-graceful for some $k > 1$ and $d > 1$.

Hegde [527] has proved the following: if a graph is $(k,d)$-graceful for odd $k$ and even $d$, then the graph is bipartite; if a graph is $(k,d)$-graceful and contains $C_{2j+1}$ as a subgraph, then $k \le jd(q-j-1)$; $K_n$ is $(k,d)$-graceful if and only if $n \le 4$; $C_{4t}$ is $(k,d)$-graceful for all $k$ and $d$; $C_{4t+1}$ is $(2t,1)$-graceful; $C_{4t+2}$ is $(2t-1,2)$-graceful; and $C_{4t+3}$ is $(2t+1,1)$-graceful.

Hegde [525] calls a $(k,d)$-graceful graph $(k,d)$-*balanced* if it has a $(k,d)$-graceful labeling $f$ with the property that there is some integer $m$ such that for every edge $uv$ either $f(u) \le m$

and $f(v) > m$, or $f(u) > m$ and $f(v) \leq m$. He proves that if a graph is $(1,1)$-balanced then it is $(k,d)$-graceful for all $k$ and $d$ and that a graph is $(1,1)$-balanced graph if and only if it is $(k,k)$-balanced for all $k$. He conjectures that all trees are $(k,d)$-balanced for some values of $k$ and $d$.

Slater [1110] has extended the definition of $k$-graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices, and the countably infinite Dutch windmill is $k$-graceful for all $k$.

More specialized results on $k$-graceful labelings can be found in [704], [724], [727], [1107], [261], [263], [262], and [308].

### 3.3 $\gamma$-Labelings

In 2004 Chartrand, Erwin, VanderJagt, and Zhang [296] define a $\gamma$-labeling of a graph $G$ of size $m$ as a one-to-one function $f$ from the vertices of $G$ to $\{0,1,2,\ldots,m\}$ that induces an edge labeling $f'$ defined by $f'(uv) = |f(u) - f(v)|$ for each edge $uv$. They define the following parameters of a $\gamma$-labeling: $\mathrm{val}(f) = \Sigma f'(e)$ over all edges $e$ of $G$; $\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}$, $\mathrm{val}_{\min}(G) =$
$\min\{\mathrm{val}(f) : f \text{ is a } \gamma - \text{labeling of } G\}$. Among their results are the following:
$\mathrm{val}_{\min}(P_n) = \mathrm{val}_{\max}(P_n) = \lfloor (n^2-2)/2 \rfloor$; $\mathrm{val}_{\min}(C_n) = 2(n-1)$; for $n \geq 4$, $n$ even, $\mathrm{val}_{\max}(C_n) = n(n+2)/2$; for $n \geq 3$, $n$ odd, $\mathrm{val}_{\max}(C_n) = (n-1)(n+3)/2$; $\mathrm{val}_{\min}(K_n) = \binom{n+1}{3}$; for odd $n$, $\mathrm{val}_{\max}(K_n) = (n^2-1)(3n^2-5n+6)/24$; for even $n$, $\mathrm{val}_{\max}(K_n) = n(3n^3-5n^2+6n-4)/24$; for every $n \geq 3$, $\mathrm{val}_{\min}(K_{1,n-1}) = \binom{\lfloor \frac{n+1}{2} \rfloor}{2} + \binom{\lceil \frac{n+1}{2} \rceil}{2}$;
$\mathrm{val}_{\max}(K_{1,n-1}) = \binom{n}{2}$; for a connected graph of order $n$ and size $m$, $\mathrm{val}_{\min}(G) = m$ if and only if $G$ is isomorphic to $P_n$; if $G$ is maximal outerplanar of order $n \geq 2$, $\mathrm{val}_{\min}(G) \geq 3n-5$ and equality occurs if and only if $G = P_n^2$; if $G$ is a connected $r$-regular bipartite graph of order $n$ and size $m$ where $r \geq 2$, then $\mathrm{val}_{\max}(G) = rn(2m-n+2)/4$.

In another paper on $\gamma$-labelings of trees Chartrand, Erwin, VanderJagt, and Zhang [297] prove for $p, q \geq 2$, $\mathrm{val}_{\min}(S_{p,q})$ (that is, the graph obtained by joining the centers of $K_{1,p}$ and $K_{1,q}$ by an edge)$= (\lfloor p/2 \rfloor + 1)^2 + (\lfloor q/2 \rfloor + 1)^2 - (n_p \lfloor p/2 \rfloor + 1)^2 + (n_q \lfloor (q+2)/2 \rfloor + 1)^2)$, where $n_i$ is 1 if $i$ is even and $n_i$ is 0 if $n_i$ is odd; $\mathrm{val}_{\min}(S_{p,q}) = (p^2 + q^2 + 4pq - 3p - 3q + 2)/2$; for a connected graph $G$ of order $n$ at least 4, $\mathrm{val}_{\min}(G) = n$ if and only if $G$ is a caterpillar with maximum degree 3 and has a unique vertex of degree 3; for a tree $T$ of order $n$ at least 4, maximum degree $\Delta$, and diameter $d$, $\mathrm{val}_{\min}(T) \geq (8n + \Delta^2 - 6\Delta - 4d + \delta_\Delta)/4$ where $\delta_\Delta$ is 0 if $\Delta$ is even and $\delta_\Delta$ is 0 if $\Delta$ is odd. They also give a characterization of all trees of order $n$ at least 5 whose minimum value is $n+1$.

### 3.4 Skolem-Graceful Labelings

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see [752]) calls a graph $G$ with $p$ vertices and $q$ edges *Skolem-graceful* if there is an injection from the set of vertices of $G$ to $\{1,2,\ldots,p\}$ such that the edge labels induced by $|f(x)-f(y)|$ for each edge $xy$ are $1,2,\ldots,q$. A necessary condition for a graph to be Skolem-graceful is that $p \geq q+1$. Lee and Wui [780] have shown that a connected graph

is Skolem-graceful if and only if it is a graceful tree. Yao, Cheng, Zhongfu, and Yao [1297] have shown that a tree of order $p$ with maximum degree at least $p/2$ is Skolem-graceful. Although the disjoint union of trees cannot be graceful, they can be Skolem-graceful. Lee and Wui [780] prove that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has even size. In [331] Choudum and Kishore show that the disjoint union of $k$ copies of the star $K_{1,2p}$ is Skolem graceful if $k \leq 4p+1$ and the disjoint union of any number of copies of $K_{1,2}$ is Skolem graceful. For $k \geq 2$, let $St(n_1, n_2, \ldots, n_k)$ denote the disjoint union of $k$ stars with $n_1, n_2, \ldots, n_k$ edges. Lee, Wang, and Wui [773] showed that the 4-star $St(n_1, n_2, n_3, n_4)$ is Skolem-graceful for some special cases and conjectured that all 4-stars are Skolem-graceful. Denham, Leu, and Liu [354] proved this conjecture. Kishore [650] has shown that a necessary condition for $St(n_1, n_2, \ldots, n_k)$ to be Skolem graceful is that some $n_i$ is even or $k \equiv 0$ or 1 (mod 4) (see also [1315] . He conjectures that each one of these conditions is sufficient. Yue, Yuan-sheng, and Xin-hong [1315] show that for $k$ at most 5, a $k$-star is Skolem-graceful if at one star has even size or $k \equiv 0$ or 1 (mod 4). Choudum and Kishore [329] proved that all 5-stars are Skolem graceful.

Lee, Quach, and Wang [738] showed that the disjoint union of the path $P_n$ and the star of size $m$ is Skolem-graceful if and only if $n = 2$ and $m$ is even or $n \geq 3$ and $m \geq 1$. It follows from the work of Skolem [1101] that $nP_2$, the disjoint union of $n$ copies of $P_2$, is Skolem-graceful if and only if $n \equiv 0$ or 1 (mod 4). Harary and Hsu [508] studied Skolem-graceful graphs under the name *node-graceful*. Frucht [430] has shown that $P_m \cup P_n$ is Skolem-graceful when $m + n \geq 5$. Bhat-Nayak and Deshmukh [226] have shown that $P_{n_1} \cup P_{n_2} \cup P_{n_3}$ is Skolem-graceful when $n_1 < n_2 \leq n_3$, $n_2 = t(n_1 + 2) + 1$ and $n_1$ is even and when $n_1 < n_2 \leq n_3$, $n_2 = t(n_1 + 3) + 1$ and $n_1$ is odd. They also prove that the graphs of the form $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_i}$ where $i \geq 4$ are Skolem-graceful under certain conditions. In [358] Deshmukh states the following results: the sum of all the edges on any cycle in a Skolem graceful graph is even; $C_5 \cup K_{1,n}$ if and only if $n = 1$ or 2; $C_6 \cup K_{1,n}$ if and only if $n = 2$ or 4.

Youssef [1303] proved that if $G$ is Skolem-graceful, then $G + \overline{K_n}$ is graceful. In [1307] Youssef shows that that for all $n \geq 2$, $P_n \cup S_m$ is Skolem-graceful if and only if $n \geq 3$ or $n = 2$ and $m$ is even. Yao, Cheng, Zhongfu, and Yao [1297] have shown that if a tree $T$ has an edge $u_1u_2$ such that the two components $T_1$ and $T_2$ of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and $T_2$ is a caterpillar or have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and $d_{T_2}(u_2) \geq |T_2|/2$, then $T$ is Skolem-graceful.

Mendelsohn and Shalaby [861] defined a *Skolem labeled* graph $G(V, E)$ as one for which there is a positive integer $d$ and a function $L: V \to \{d, d+1, \ldots, d+m\}$, satisfying (a) there are exactly two vertices in $V$ such that $L(v) = d + i$, $0 \leq i \leq m$; (b) the distance in $G$ between any two vertices with the same label is the value of the label; and (c) if $G'$ is a proper spanning subgraph of $G$, then $L$ restricted to $G'$ is not a Skolem labeled graph. Note that this definition is different from the Skolem-graceful labeling of Lee, Quach, and Wang. A *hooked Skolem sequence* of order $n$ is a sequence $s_1, s_2, \ldots, s_{2n+1}$ such that $s_{2n} = 0$ and for each $j \in \{1, 2, \ldots, n\}$, there exists a unique $i \in \{1, 2, \ldots, 2n-1, 2n+1\}$ such that $s_i = s_{i+j} = j$. Mendelsohn [860] established the following: any tree can be embedded in a Skolem labeled tree with $O(v)$ vertices; any graph can be embedded as an induced subgraph in a Skolem labeled graph on $O(v^3)$ vertices; for $d = 1$, there is a Skolem labeling or the minimum hooked Skolem (with as few unlabeled vertices as possible) labeling for paths and cycles; for $d = 1$, there is a minimum Skolem labeled graph containing a path or a cycle of length $n$ as induced subgraph. In [860] Mendelsohn and Shalaby prove that the necessary conditions in [861] are sufficient for a Skolem or minimum

hooked Skolem labeling of all trees consisting of edge-disjoint paths of the same length from some fixed vertex. Graham, Pike, and Shalaby [483] obtained various Skolem labeling results for grid graphs. Among them are $P_1 \times P_n$ and $P_2 \times P_n$ have Skolem labelings if and only if $n \equiv 0$ or 1 mod 4; and $P_m \times P_n$ has a Skolem labeling for all $m$ and $n$ at least 3.

### 3.5 Odd-Graceful Labelings

Gnanajothi [471, p. 182] defined a graph $G$ with $q$ edges to be *odd-graceful* if there is an injection $f$ from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge $xy$ is assigned the label $|f(x)-f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. She proved that the class of odd-graceful graphs lies between the class of graphs with $\alpha$-labelings and the class of bipartite graphs by showing that every graph with an $\alpha$-labeling has an odd-graceful labeling and every graph with an odd cycle is not odd-graceful. She also proved the following graphs are odd-graceful: $P_n$; $C_n$ if and only if $n$ is even; $K_{m,n}$; combs $P_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of $P_n$); books; crowns $C_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of $C_n$) if and only if $n$ is even; the disjoint union of copies of $C_4$; the one-point union of copies of $C_4$; $C_n \times K_2$ if and only if $n$ is even; caterpillars; rooted trees of height 2; the graphs obtained from $P_n$ ($n \geq 3$) by adding exactly two leaves at each vertex of degree 2 of $P_n$; the graphs obtained from $P_n \times P_2$ by deleting an edge that joins to end points of the $P_n$ paths; the graphs obtained from a star by adjoining to each end vertex the path $P_3$ or by adjoining to each end vertex the path $P_4$. She conjectures that all trees are odd-graceful and proves the conjecture for all trees with order up to 10. Barrientos [196] has extended this to trees of order up to 12. Eldergill [378] generalized Gnanajothi's result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd-graceful. He also proved that the one-point union of any number of copies of $C_6$ is odd-graceful. Kathiresan [633] has shown that ladders and graphs obtained from them by subdividing each step exactly once are odd-graceful. Barrientos [199] and [196] has proved the following graphs are odd-graceful: every forest whose components are caterpillars; every tree with diameter at most five is odd-graceful; and all disjoint unions of caterpillars. He conjectures that every bipartite graph is odd-graceful. Seoud, Diab, and Elsakhawi [1010] have shown that a connected complete $r$-partite graph is odd-graceful if and only if $r = 2$ and that the join of any two connected graphs is not odd-graceful.

Sekar [1000] has shown the following graphs are odd-graceful: $C_m \odot P_n$ (the graph obtained by identifying an end point of $P_n$ with every vertex of $C_m$) where $n \geq 3$ and $m$ is even; $P_{a,b}$ when $a \geq 2$ and $b$ is odd (see §2.7); $P_{2,b}$ and $b \geq 2$; $P_{4,b}$ and $b \geq 2$; $P_{a,b}$ when $a$ and $b$ are even and $a \geq 4$ and $b \geq 4$; $P_{4r+1,4r+2}$; $P_{4r-1,4r}$; all $n$-polygonal snakes with $n$ even; $C_n^{(t)}$ (see §2.2 for the definition); graphs obtained by beginning with $C_6$ and repeatedly forming the one-point union with additional copies of $C_6$ in succession; graphs obtained by beginning with $C_8$ and repeatedly forming the one-point union with additional copies of $C_8$ in succession; graphs obtained from even cycles by identifying a vertex of the cycle with the endpoint of a star; $C_{6,n}$ and $C_{8,n}$ (see §2.7); the splitting graph of $P_n$ (see §2.7) the splitting graph of $C_n, n$ even; lobsters, banana trees, and regular bamboo trees (see §2.1).

Yao, Cheng, Zhongfu, and Yao [1297] have shown the following: if a tree $T$ has an edge $u_1u_2$ such that the two components $T_1$ and $T_2$ of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and $T_2$ is a caterpillar, then $T$ is odd-graceful; and if a tree $T$ has a vertex of degree at least $|T|/2$, then $T$ is odd-graceful. They conjecture that for trees the properties of being Skolem-graceful

and odd-graceful are equivalent. Recall a banana tree is a graph obtained by starting with any number os stars and connecting one end-vertex from each to a new vertex. Zhenbin [1325] has shown that graphs obtained by starting with any number of stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are odd-graceful.

Gao [450] has proved the following graphs are odd-graceful: the union of any number of paths; the union of any number of stars; the union of any number of stars and paths; $C_m \cup P_n$; $C_m \cup C_n$; and the union of any number of cycles each of which has order divisible by 4.

Acharya, Germina, Princy, and Rao [23] prove that every bipartite graph $G$ can be embedded in an odd-graceful graph $H$. The construction is done in such a way that if $G$ is planar and odd-graceful, then so is $H$.

In [306] Chawathe and Krishna extend the definition of odd-gracefulness to countably infinite graphs and show that all countably infinite bipartite graphs that are connected and locally finite have odd-graceful labelings.

Solairaju and Chithra [1119] defined a graph $G$ with $q$ edges to be *edge-odd graceful* if there is an bijection $f$ from the edges of the graph to $\{1, 3, 5, \ldots, 2q-1\}$ such that, when each vertex is assigned the sum of all the edges incident to it mod $2q$, the resulting vertex labels are distinct. They prove they following graphs are odd-graceful: paths with at least 3 vertices; odd cycles; ladders $P_n \times P_2$ $(n \geq 3)$; stars with an even number of edges; and crowns $C_n \odot K_1$. In [1120] they prove the following graphs have edge-odd graceful labelings: $P_n$ $(n > 1)$ with a pendant edge attached to each vertex (combs); the graph obtained by appending $2n+1$ pendant edges to each endpoint of $P_2$ or $P_3$; and the graph obtained by subdividing each edge of the star $K_{1,2n}$.

### 3.6 Graceful-like Labelings

As a means of attacking graph decomposition problems, Rosa [973] invented another analogue of graceful labelings by permitting the vertices of a graph with $q$ edges to assume labels from the set $\{0, 1, \ldots, q+1\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, \ldots, q-1, q\}$ or $\{1, 2, \ldots, q-1, q+1\}$. He calls these *$\hat{\rho}$-labelings*. Frucht [430] used the term *nearly graceful labeling* instead of *$\hat{\rho}$-labelings*. Frucht [430] has shown that the following graphs have nearly graceful labelings with edge labels from $\{1, 2, \ldots, q-1, q+1\}$: $P_m \cup P_n$; $S_m \cup S_n$; $S_m \cup P_n$; $G \cup K_2$ where $G$ is graceful; and $C_3 \cup K_2 \cup S_m$ where $m$ is even or $m \equiv 3 \pmod{14}$. Seoud and Elsakhawi [1011] have shown that all cycles are nearly graceful. Barrientos [189] proved that $C_n$ is nearly graceful with edge labels $1, 2, \ldots, n-1, n+1$ if and only if $n \equiv 1$ or $2 \pmod 4$. Gao [451] shows that a variation of banana trees is odd graceful and in some cases has a nearly gracefullabeling. Rosa [975] conjectured that triangular snakes with $t \equiv 0$ or $1 \pmod 4$ blocks are graceful and those with $t \equiv 2$ or $3 \pmod 4$ blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [883] proved Rosa's conjecture while introducing the slightly stronger concept of *almost graceful* by permitting the vertex labels to come from $\{0, 1, 2, \ldots, q-1, q+1\}$ while the edge labels are $1, 2, \ldots, q-1, q$, or $1, 2, \ldots, q-1, q+1$. Seoud and Elsakhawi [1011] and [1012] have shown that the following graphs are almost graceful: $C_n$; $P_n + \overline{K_m}$; $P_n + K_{1,m}$; $K_{m,n}$; $K_{1,m,n}$; $K_{2,2,m}$; $K_{1,1,m,n}$; $P_n \times P_3$ $(n \geq 3)$; $K_5 \cup K_{1,n}$; $K_6 \cup K_{1,n}$, and ladders.

The *symmetric product* $G_1 \oplus G_2$ of $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2)\}$ where $u_1u_2$ is an edge in $G_1$ or $v_1v_2$ is an edge in $G_2$ but not both $u_1u_2$ is an edge in $G_1$ and $v_1v_2$ is an edge in $G_2$. In [1012] Seoud and Elsakhawi show that

$P_2 \oplus \overline{K_2}$ $(n \geq 2)$ is arbitrarily graceful.

For a graph $G$ with $p$ vertices, $q$ edges, and $1 \leq k \leq q$, Eshghi [395] defines a *holey $\alpha$-labeling with respect to $k$* as an injective vertex labeling $f$ for which $f(v) \in \{1, 2, \ldots, q+1\}$ for all $v$, $\{|f(u) - f(v)| \mid \text{for all edges } uv\} = \{1, 2, \ldots, k-1, k+1, \ldots, q+1\}$, and there exist an integer $\gamma$ with $0 \leq \gamma \leq q$ such that $\min\{f(u), f(v)\} \leq \gamma \leq \max\{f(u), f(v)\}$. He proves the following: $P_n$ has a holey $\alpha$-labeling with respect to all $k$; $C_n$ has a holey $\alpha$-labeling with respect to $k$ if and only if either $n \equiv 2 \pmod 4$, $k$ is even, and $(n, k) \neq (10, 6)$, or $n \equiv 0 \pmod 4$ and $k$ is odd.

Recall from Section 2.2 that a *$kC_n$-snake* is a connected graph with $k$ blocks whose block-cutpoint graph is a path and each of the $k$ blocks is isomorphic to $C_n$. In addition to his results on the graceful $kC_n$-snakes given in Section 2.2, Barrientos [193] proved that when $k$ is odd the linear $kC_6$-snake is nearly graceful and that $C_m \cup K_{1,n}$ is nearly graceful when $m = 3, 4, 5$, and 6.

Yet another kind of labeling introduced by Rosa in his 1967 paper [973] is a $\rho$-labeling. A *$\rho$-labeling* (or $\rho$-valuation) of a graph is an injection from the vertices of the graph with $q$ edges to the set $\{0, 1, \ldots, 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, \ldots, a_q$, then $a_i = i$ or $a_i = 2q + 1 - i$. Rosa [973] proved that a cyclic decomposition of the edge set of the complete graph $K_{2q+1}$ into subgraphs isomorphic to a given graph $G$ with $q$ edges exists if and only if $G$ has a $\rho$-labeling. (A decomposition of $K_n$ into copies of $G$ is called *cyclic* if the automorphism group of the decomposition itself contains the cyclic group of order $n$.) It is known that every graph with at most 11 edges has a $\rho$-labeling and that all lobsters have a $\rho$-labeling (see [289]). Donovan, El-Zanati, Vanden Eyden, and Sutinuntopas [365] prove that $rC_m$ has a $\rho$-labeling (or a more restrictive labeling) when $r \leq 4$. They conjecture that every 2-regular graph has a $\rho$-labeling. Aguado, El-Zanati, Hake, Stob, and Yayla [40] give a $\rho$-labeling of $C_r \cup C_s \cup C_t$ for each of the cases where $r \equiv 0$, $s \equiv 1$, $t \equiv 1 \pmod 4$; $r \equiv 0$, $s \equiv 3$, $t \equiv 3 \pmod 4$; and $r \equiv 1$, $s \equiv 1$, $t \equiv 3 \pmod 4$; (iv) $r \equiv 1$, $s \equiv 2$, $t \equiv 3 \pmod 4$; (v) $r \equiv 3$, $s \equiv 3$, $t \equiv 3 \pmod 4$. Caro, Roditty, and Schőnheim [289] provide a construction for the adjacency matrix for every graph that has a $\rho$-labeling. They ask the following question: If $H$ is a connected graph having a $\rho$-labeling and $q$ edges and $G$ is a new graph with $q$ edges constructed by breaking $H$ up into disconnected parts, does $G$ also have a $\rho$-labeling? Kézdy [643] defines a *stunted tree* as one whose edges can be labeled with $e_1, e_2, \ldots, e_n$ so that $e_1$ and $e_2$ are incident and, for all $j = 3, 4, \ldots, n$, edge $e_j$ is incident to at least one edge $e_k$ satisfying $2k \leq j - 1$. He uses Alon's "Combinatorial Nullstellensatz" to prove that if $2n + 1$ is prime, then every stunted tree with $n$ edges has a $\rho$-labeling.

In [424] Fronček generalizes the notion of an $\alpha$-labeling by showing that if a graph $G$ on $n$ edges allows a certain type of $\rho$-labeling), called $\alpha_2$-labeling, then for any positive integer $k$ the complete graph $K_{2nk+1}$ can be decomposed into copies of $G$.

In their investigation of cyclic decompositions of complete graphs El-Zanati, Vanden Eynden, and Punnim [389] introduced two kinds of labelings. They say a bipartite graph $G$ with $n$ edges and partite sets $A$ and $B$ has a *$\theta$-labeling* $h$ if $h$ is a one-to-one function from $V(G)$ to $\{0, 1, \ldots, 2n\}$ such that $\{|h(b) - h(a)| \; ab \in E(G), a \in A, b \in B\} = \{1, 2, \ldots, n\}$. They call $h$ a *$\rho^+$-labeling* of $G$ if $h$ is a one-to-one function from $V(G)$ to $\{0, 1, \ldots, 2n\}$ and the integers $h(x) - h(y)$ are distinct modulo $2n + 1$ taken over all ordered pairs $(x, y)$ where $xy$ is an edge in $G$, and $h(b) > h(a)$ whenever $a \in A, b \in B$ and $ab$ is an edge in $G$. Note that $\theta$-labelings are $\rho^+$-labelings and $\rho^+$-labelings are $\rho$-labelings. They prove that if $G$ is a bipartite graph with $n$ edges and a $\rho^+$-labeling, then for every positive integer $x$ there is a cyclic $G$-decomposition of $K_{2nx+1}$. They prove the following graphs have $\rho^+$-labelings: trees of diameter at most 5, $C_{2n}$,

lobsters, and comets (that is, graphs obtained from stars by replacing each edge by a path of some fixed length). They also prove that the disjoint union of graphs with $\alpha$-labelings have a $\theta$-labeling and conjecture that all forests have $\rho$-labelings.

A *$\sigma$-labeling* of $G(V,E)$ is a one-to-one function $f$ from $V$ to $\{0,1,\ldots,2|E|\}$ such that $\{|f(u)-f(v)| \mid uv \in E(G)\} = \{1,2,\ldots,|E|\}$. Such a labeling of $G$ yields cyclic $G$-decompositions of $K_{2n+1}$ and of $K_{2n+2}-F$, where $F$ is a 1-factor of $K_{2n+2}$. El-Zanati and Vanden Eynden (see [39]) have conjectured that that every 2-regular graph with $n$ edges has a $\rho$-labeling and, if $n \equiv 0$ or 3 (mod 4), then every 2-regular graph has a $\sigma$-labeling. Aguado and El-Zanati [39] have proved that the latter conjecture holds when the graph has at most three components.

Given a bipartite graph $G$ with partite sets $X$ and $Y$ and graphs $H_1$ with $p$ vertices and $H_2$ with $q$ vertices, Fronček and Winters [427] define the *bicomposition* of $G$ and $H_1$ and $H_2$, $G[H_1,H_2]$, as the graph obtained from $G$ by replacing each vertex of $X$ by a copy of $H_1$, each vertex of $Y$ by a copy of $H_2$, and every edge $xy$ by a graph isomorphic to $K_{p,q}$ with the partite sets corresponding to the vertices $x$ and $y$. They prove that if $G$ is a bipartite graph with $n$ edges and $G$ has a $\theta$-labeling that maps the vertex set $V = X \cup Y$ into a subset of $\{0,1,2,\ldots,2n\}$, then the bicomposition $G[\overline{K_p},\overline{K_q}]$ has a $\theta$-labeling for every $p,q \geq 1$. As corollaries they have: if a bipartite graph $G$ with $n$ edges and at most $n+1$ vertices has a gracious labeling (see §3.1), then the bicomposition graph $G[\overline{K_p},\overline{K_q}]$ has a gracious labeling for every $p,q \geq 1$, and if a bipartite graph $G$ with $n$ edges has a $\theta$-labeling, then for every $p,q \geq 1$, the bicomposition $G[\overline{K_p},\overline{K_q}]$ decomposes the complete graph $K_{2npq+1}$.

In a paper published in 2009 [388] El-Zannati and Vanden Eynden survey "Rosa-type" labelings. That is, labelings of a graph $G$ that yield cyclic $G$-decompositions of $K_{2n+1}$ or $K_{2nx+1}$ for all natural numbers $x$. The 2009 survey by Fronček [423] includes generalizations of $\rho$- and $\alpha$-labelings that have been used for finding decompositions of complete graphs that are not covered in [388].

Blinco, El-Zanati, and Vanden Eynden [233] call a non-bipartite graph *almost-bipartite* if the removal of some edge results in a bipartite graph. For these kinds of graphs $G$ they call a labeling $f$ a *$\gamma$-labeling* of $G$ if the following conditions are met: $f$ is a $\rho$-labeling; $G$ is tripartite with vertex tripartition $A,B,C$ with $C=\{c\}$ and $\overline{b} \in B$ such that $\{\overline{b},c\}$ is the unique edge joining an element of $B$ to $c$; if $av$ is an edge of $G$ with $a \in A$, then $f(a) < f(v)$; and $f(c) - f(\overline{b}) = n$. (In § 3.3 the term $\gamma$-labeling is used for a different kind of labeling.) They prove that if an almost-bipartite graph $G$ with $n$ edges has a $\gamma$-labeling then there is a cyclic $G$-decomposition of $K_{2nx+1}$ for all $x$. They prove that all odd cycles with more than 3 vertices have a $\gamma$-labeling and that $C_3 \cup C_{4m}$ has a $\gamma$-labeling if and only if $m > 1$. In [268] Bunge, El-Zanati, and Vanden Eynden prove that every 2-regular almost bipartite graph other than $C_3$ and $C_3 \cup C_4$ have a $\gamma$-labeling.

In [233] Blinco, El-Zanati, and Vanden Eynden consider a slightly restricted $\rho^+$-labeling for a bipartite graph with partite sets $A$ and $B$ by requiring that there exists a number $\lambda$ with the property that $\rho^+(a) \leq \lambda$ for all $a \in A$ and $\rho^+(b) > \lambda$ for all $b \in B$. They denote such a labeling by $\rho^{++}$. They use this kind of labeling to show that if $G$ is a 2-regular graph of order $n$ in which each component has even order then there is a cyclic $G$-decomposition of $K_{2nx+1}$ for all $x$. They also conjecture that every bipartite graph has a $\rho$-labeling and every 2-regular graph has a $\rho$-labeling.

Dufour [376] and Eldergill [378] have some results on the decomposition of complete graphs using labeling methods. Balakrishnan and Sampathkumar [179] showed that for each positive integer $n$ the graph $\overline{K_n} + 2K_2$ admits a $\rho$-labeling. Balakrishnan [174] asks if it is true that

$\overline{K_n} + mK_2$ admits a $\rho$-labeling for all $n$ and $m$. Fronček [422] and Fronček and Kubesa [426] have introduced several kinds of labelings for the purpose of proving the existence of special kinds of decompositions of complete graphs into spanning trees.

For $(p,q)$-graphs with $p = q+1$, Frucht [430] has introduced a stronger version of almost graceful graphs by permitting as vertex labels $\{0, 1, \ldots, q-1, q+1\}$ and as edge labels $\{1, 2, \ldots, q\}$. He calls such a labeling *pseudograceful*. Frucht proved that $P_n$ $(n \geq 3)$, combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star), $C_3 \cup P_n$ $(n \neq 3)$, and $C_4 \cup P_n$ $(n \neq 1)$ are pseudograceful whereas $K_{1,n}$ $(n \geq 3)$ is not. Kishore [650] proved that $C_s \cup P_n$ is pseudograceful when $s \geq 5$ and $n \geq (s+7)/2$ and that $C_s \cup S_n$ is pseudograceful when $s = 3, s = 4$, and $s \geq 7$. Seoud and Youssef [1023] and [1019] extended the definition of pseudograceful to all graphs with $p \leq q+1$. They proved that $K_m$ is pseudograceful if and only if $m = 1, 3$, or 4 [1019]; $K_{m,n}$ is pseudograceful when $n \geq 2$, and $P_m + \overline{K_n}$ $(m \geq 2)$ [1023] is pseudograceful. They also proved that if $G$ is pseudograceful, then $G \cup K_{m,n}$ is graceful for $m \geq 2$ and $n \geq 2$ and $G \cup K_{m,n}$ is pseudograceful for $m \geq 2, n \geq 2$ and $(m,n) \neq (2,2)$ [1019]. They ask if $G \cup K_{2,2}$ is pseudograceful whenever $G$ is. Seoud and Youssef [1019] observed that if $G$ is a pseudograceful Eulerian graph with $q$ edges, then $q \equiv 0$ or 3 (mod 4). Youssef [1306] has shown that $C_n$ is pseudograceful if and only if $n \equiv 0$ or 3 (mod 4), and for $n > 8$ and $n \equiv 0$ or 3 (mod 4), $C_n \cup K_{p,q}$ is pseudograceful for all $p, q \geq 2$ except $(p,q) = (2,2)$. Youssef [1303] has shown that if $H$ is pseudograceful and $G$ has an $\alpha$-labeling with $k$ being the smaller vertex label of the edge labeled with 1 and if either $k+2$ or $k-1$ is not a vertex label of $G$, then $G \cup H$ is graceful. In [1307] Youssef shows that if $G$ is $(p,q)$ pseudograceful graph with $p = q+1$, then $G \cup S_m$ is Skolem-graceful. As a corollary he obtains that for all $n \geq 2$, $P_n \cup S_m$ is Skolem-graceful if and only if $n \geq 3$ or $n = 2$ and $m$ is even.

McTavish [857] has investigated labelings of graphs with $q$ edges where the vertex and edge labels are from $\{0, \ldots, q, q+1\}$. She calls these $\tilde{\rho}$-*labelings*. Graphs that have $\tilde{\rho}$-labelings include cycles and the disjoint union of $P_n$ or $S_n$ with any graceful graph.

Frucht [430] has made an observation about graceful labelings that yields nearly graceful analogs of $\alpha$-labelings and weakly $\alpha$-labelings in a natural way. Suppose $G(V,E)$ is a graceful graph with the vertex labeling $f$. For each edge $xy$ in $E$, let $[f(x), f(y)]$ (where $f(x) \leq f(y)$) denote the interval of real numbers $r$ with $f(x) \leq r \leq f(y)$. Then the intersection $\cap[f(x), f(y)]$ over all edges $xy \in E$ is a unit interval, a single point, or empty. Indeed, if $f$ is an $\alpha$-labeling of $G$ then the intersection is a unit interval; if $f$ is a weakly $\alpha$-labeling, but not an $\alpha$-labeling, then the intersection is a point; and, if $f$ is a graceful but not a weakly $\alpha$-labeling, then the intersection is empty. For nearly graceful labelings, the intersection also gives three distinct classes.

Singh and Devaraj [1095] call a graph $G$ with $p$ vertices and $q$ edges *triangular graceful* if there is an injection $f$ from $V(G)$ to $\{0, 1, 2, \ldots, T_q\}$ where $T_q$ is the $q$th triangular number and the labels induced on each edge $uv$ by $|f(u) - f(v)|$ are the first $q$ triangular numbers. They prove the following graphs are triangular graceful: paths, level 2 rooted trees, olive trees (see § 2.1 for the definition), complete $n$-ary trees, double stars, caterpillars, $C_{4n}, C_{4n}$ with pendent edges, the one-point union of $C_3$ and $P_n$, and unicyclic graphs that have $C_3$ as the unique cycle. They prove that wheels, helms, flowers (see §2.2 for the definition) and $K_n$ with $n \geq 3$ are not triangular graceful. They conjecture that all trees are triangular graceful. In [1051] Sethuraman and Venkatesh introduced a new method for combining graceful trees to obtain trees that have $\alpha$-labelings.

Van Bussel [1199] considered two kinds of relaxations of graceful labelings as applied to trees.

He called a labeling *range-relaxed graceful* it is meets the same conditions as a graceful labeling except the range of possible vertex labels and edge labels are not restricted to the number of edges of the graph (the edges are distinctly labeled but not necessarily labeled 1 to $q$ where $q$ is the number of edges). Similarly, he calls a labeling *vertex-relaxed graceful* if it satisfies the conditions of a graceful labeling while permitting repeated vertex labels. He proves that every tree $T$ with $q$ edges has a range-relaxed graceful labeling with the vertex labels in the range $0, 1, \ldots, 2q-d$ where $d$ is the diameter of $T$ and that every tree on $n$ vertices has a vertex-relaxed graceful labeling such that the number of distinct vertex labels is strictly greater than $n/2$.

Sekar [1000] calls an injective function $\phi$ from the vertices of a graph with $q$ edges to $\{0, 1, 3, 4, 6, 7, \ldots, 3(q-1), 3q-2\}$ *one modulo three graceful* if the edge labels induced by labeling each edge $uv$ with $|\phi(u) - \phi(v)|$ is $\{1, 4, 7, \ldots, 3q-2\}$. He proves that the following graphs are one modulo three graceful: $P_m$; $C_n$ if and only if $n \equiv 0 \bmod 4$; $K_{m,n}$; $C_{2n}^{(2)}$ (the one-point union of two copies of $C_{2n}$); $C_n^{(t)}$ for $n = 4$ or 8 and $t > 2$; $C_6^{(t)}$ and $t \geq 4$; caterpillars; stars; lobsters; banana trees; rooted trees of height 2; ladders; the graphs obtained by identifying the endpoints of any number of copies of $P_n$; the graph obtained by attaching pendent edges to each endpoint of two identical stars and then identifying one endpoint from each of these graphs; the graph obtained by identifying a vertex of $C_{4k+2}$ with an endpoint of a star; $n$-polygonal snakes (see §2.2) for $n \equiv 0 \pmod 4$; $n$-polygonal snakes for $n \equiv 2 \pmod 4$ where the number of polygons is even; crowns $C_n \odot K_1$ for $n$ even; $C_{2n} \odot P_m$ ($C_{2n}$ with $P_m$ attached at each vertex of the cycle) for $m \geq 3$; chains of cycles (see §2.2) of the form $C_{4,m}, C_{6,2m}$, and $C_{8,m}$. He conjectures that every one modulo three graceful graph is graceful.

Kathiresan and Amutha [635] define a function $f : V(G) \to \{0, 1, 2, \ldots, F_q\}$ where $F_q$ is the $q$th Fibonacci number, to be *Fibonacci graceful labeling* if the induced edge labeling $\overline{f}(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. If a graph admits a Fibonacci graceful labeling, it is is called a *Fibonacci graceful* graph. They prove the following: $K_n$ is Fibonacci graceful if and only if $n \leq 3$; if an Eulerian graph with $q$ edges is Fibonacci graceful then $q \equiv 0 \pmod 3$; paths are Fibonacci graceful; fans $P_n \odot K_1$ are Fibonacci graceful; squares of paths $P_n^2$ are Fibonacci graceful; and caterpillars are Fibonacci graceful. They define a function $f : V(G) \to \{0, F_1, F_2, \ldots, F_q\}$ where $F_i$ is the $i$th Fibonacci number, to be *super Fibonacci graceful*labeling if the induced labeling $\overline{f}(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. They show that bistars $B_{n,n}$ are Fibonacci graceful but not super Fibonacci graceful for $n \geq 5$; cycles $C_n$ are super Fibonacci graceful if and only if $n \equiv 0 \pmod 3$; and if $G$ is Fibonacci or super Fibonacci graceful then $G \odot K_1$ is Fibonacci graceful.

In [254] Brešar and Klavžar define a natural extension of graceful labelings of certain tree subgraphs of hypercubes. A subgraph $H$ of a graph $G$ is called *isometric* if for every two vertices $u$, $v$ of $H$, there exists a shortest $u$-$v$ path that lies in $H$. The isometric subgraphs of hypercubes are called *partial cubes*. Two edges $xy$, $uv$ of $G$ are in $\Theta$-relation if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. A $\Theta$-relation is an equivalence relation that partitions $E(G)$ into $\Theta$-classes. A $\Theta$-*graceful labeling* of a partial cube $G$ on $n$ vertices is a bijection $f: V(G) \to \{0, 1, \ldots, n-1\}$ such that, under the induced edge labeling, all edges in each $\Theta$-class of $G$ have the same label and distinct $\Theta$-classes get distinct labels. They prove that several classes of partial cubes are $\Theta$-graceful and the Cartesian product of $\Theta$-graceful partial cubes is $\Theta$-graceful. They also show that if there exists a class of partial cubes that contains all trees and every member of the class admits a $\Theta$-graceful labeling then all trees are graceful.

### 3.7 Cordial Labelings

Cahit [274] has introduced a variation of both graceful and harmonious labelings. Let $f$ be a function from the vertices of $G$ to $\{0,1\}$ and for each edge $xy$ assign the label $|f(x)-f(y)|$. Call $f$ a *cordial labeling* of $G$ if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Cahit [275] proved the following: every tree is cordial; $K_n$ is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all $m$ and $n$; the friendship graph $C_3^{(t)}$ (i.e., the one-point union of $t$ 3-cycles) is cordial if and only if $t \not\equiv 2 \pmod 4$; all fans are cordial; the wheel $W_n$ is cordial if and only if $n \not\equiv 3 \pmod 4$ (see also [372]); maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang, and Kwong [686] determine all $m$ and $n$ for which $mK_n$ is cordial. Youssef [1307] proved that every Skolem-graceful graph (see §3.4 for the definition) is cordial. Liu and Zhu [813] proved that a 3-regular graph of order $n$ is cordial if and only if $n \not\equiv 4 \pmod 8$.

A *$k$-angular cactus* is a connected graph all of whose blocks are cycles with $k$ vertices. In [275] Cahit proved that a $k$-angular cactus with $t$ cycles is cordial if and only if $kt \not\equiv 2 \pmod 4$. This was improved by Kirchherr [648] who showed any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to 2 (mod 4). Kirchherr [649] also gave a characterization of cordial graphs in terms of their adjacency matrices. Ho, Lee, and Shee [553] proved: $P_n \times C_{4m}$ is cordial for all $m$ and all odd $n$; the composition $G$ and $H$ is cordial if $G$ is cordial and $H$ is cordial and has odd order and even size (see §2.3 for definition of composition); for $n \geq 4$ the composition $C_n[K_2]$ is cordial if and only if $n \not\equiv 2 \pmod 4$; the Cartesian product of two cordial graphs of even size is cordial. He, Lee, and Shee [552] showed that a unicyclic graph is cordial unless it is $C_{4k+2}$ and that the generalized Petersen graph (see §2.7 for the definition) $P(n,k)$ is cordial if and only if $n \not\equiv 2 \pmod 4$. Du [372] determines the maximal number of edges in a cordial graph of order $n$ and gives a necessary condition for a $k$-regular graph to be cordial. Riskin [965] proved that Möbius ladders $M_n$ (see §2.3 for the definition) are cordial if and only if $n \geq 3$ and $n \not\equiv 2 \pmod 4$. (See also [1012].)

Seoud and Abdel Maqusoud [1006] proved that if $G$ is a graph with $n$ vertices and $m$ edges and every vertex has odd degree, then $G$ is not cordial when $m+n \equiv 2 \pmod 4$. They also prove the following: for $m \geq 2$, $C_n \times P_m$ is cordial except for the case $C_{4k+2} \times P_2$; $P_n^2$ is cordial for all $n$; $P_n^3$ is cordial if and only if $n \neq 4$; and $P_n^4$ is cordial if and only if $n \neq 4, 5$, or 6. Seoud, Diab, and Elsakhawi [1010] have proved the following graphs are cordial: $P_n + P_m$ for all $m$ and $n$ except $(m,n) = (2,2)$; $C_m + C_n$ if $m \not\equiv 0 \pmod 4$ and $n \neq 2 \pmod 4$; $C_n + K_{1,m}$ for $n \not\equiv 3 \pmod 4$ and odd $m$ except $(n,m) = (3,1)$; $C_n + \overline{K_m}$ when $n$ is odd, and when $n$ is even and $m$ is odd; $K_{1,m,n}$; $K_{2,2,m}$; the $n$-cube; books $B_n$ if and only if $n \not\equiv 3 \pmod 4$; $B(3,2,m)$ for all $m$; $B(4,3,m)$ if and only if $m$ is even; and $B(5,3,m)$ if and only if $m \not\equiv 1 \pmod 4$ (see §2.4 for the notation $B(n,r,m)$).

Diab [361] and [362] proved the following graphs are cordial: $C_m + P_n$ if and only if $(m,n) \neq (3,3), (3,2)$, or (3,1); $P_m + K_{1,n}$ if and only if $(m,n) \neq (1,2)$; $P_m \cup K_{1,n}$ if and only if $(m,n) \neq (1,2)$; $C_m \cup K_{1,n}$; $C_m + \overline{K_n}$ for all $m$ and $n$ except $m \equiv 3 \pmod 4$ and $n$ odd, and $m \equiv 2 \pmod 4$ and $n$ even; $C_m \cup \overline{K_n}$ for all $m$ and $n$ except $m \equiv 2 \pmod 4$; $P_m + \overline{K_n}$; $P_m \cup \overline{K_n}$; $P_m^2 \cup P_n^2$ except for $(m,n) = (2,2)$ or (3,3); $P_n^2 + P_m$ except for $(m,n) = (3,1), (3,2), (2,2), (3,3)$ and (4,2); $P_n^2 \cup P_m$ except for $(n,m) = (2,2), (3,3)$ and (4,2); $P_n^2 + C_m$ if and only if $(n,m) \neq (1,3), (2,3)$ and $(3,3)$.

Youssef [1309] has proved the following: If $G$ and $H$ are cordial and one has even size, then

$G \cup H$ is cordial; if $G$ and $H$ are cordial and both have even size, then $G + H$ is cordial; if $G$ and $H$ are cordial and one has even size and either one has even order, then $G + H$ is cordial; $C_m \cup C_n$ is cordial if and only if $m + n \not\equiv 2 \pmod 4$; $mC_n$ is cordial if and only if $mn \not\equiv 2 \pmod 4$; $C_m + C_n$ is cordial if and only if $(m, n) \neq (3, 3)$ and $\{m \pmod 4, n \pmod 4\} \neq \{0, 2\}$; and if $P_n^k$ is cordial, then $n \geq k+1+\sqrt{k-2}$. He conjectures that this latter condition is also sufficient. He confirms the conjecture for $k = 5, 6, 7, 8$, and 9.

In [1308] Youssef obtained the following results: $C_{2k}$ with one pendant edge is not $(2k+1)$-cordial for $k > 1$; $K_n$ is 4-cordial if and only if $n \leq 6$; $C_n^2$ is 4-cordial if and only if $n \not\equiv 2 \pmod 4$; and $K_{m,n}$ is 4-cordial if and only if $n \not\equiv 2 \pmod 4$; He also provides some necessary conditions for a graph to be $k$-cordial.

Lee and Liu [719] have shown that the complete $n$-partite graph is cordial if and only if at most three of its partite sets have odd cardinality (see also [372]). Lee, Lee, and Chang [700] prove the following graphs are cordial: the Cartesian product of an arbitrary number of paths; the Cartesian product of two cycles if and only if at least one of them is even; and the Cartesian product of an arbitrary number of cycles if at least one of them has length a multiple of 4 or at least two of them are even.

Shee and Ho [1056] have investigated the cordiality of the one-point union of $n$ copies of various graphs. For $C_m^{(n)}$, the one-point union of $n$ copies of $C_m$, they prove:

(i) If $m \equiv 0 \pmod 4$, then $C_m^{(n)}$ is cordial for all $n$;

(ii) If $m \equiv 1$ or $3 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(iii) If $m \equiv 2 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if $n$ is even.

For $K_m^{(n)}$, the one-point union of $n$ copies of $K_m$, Shee and Ho [1056] prove:

(i) If $m \equiv 0 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 3 \pmod 4$;

(ii) If $m \equiv 4 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 1 \pmod 4$;

(iii) If $m \equiv 5 \pmod 8$, then $K_m^{(n)}$ is not cordial for all odd $n$;

(iv) $K_4^{(n)}$ is cordial if and only if $n \not\equiv 1 \pmod 4$;

(v) $K_5^{(n)}$ is cordial if and only if $n$ is even;

(vi) $K_6^{(n)}$ is cordial if and only if $n > 2$;

(vii) $K_7^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(viii) $K_n^{(2)}$ is cordial if and only if $n$ has the form $p^2$ or $p^2 + 1$.

For $W_m^{(n)}$, the one-point union of $n$ copies of the wheel $W_m$ with the common vertex being the center, Shee and Ho [1056] show:

(i) If $m \equiv 0$ or $2 \pmod 4$, then $W_m^{(n)}$ is cordial for all $n$;

(ii) If $m \equiv 3 \pmod 4$, then $W_m^{(n)}$ is cordial if $n \not\equiv 1 \pmod 4$;

(iii) If $m \equiv 1 \pmod 4$, then $W_m^{(n)}$ is cordial if $n \not\equiv 3 \pmod 4$.

For all $n$ and all $m > 1$ Shee and Ho [1056] prove $F_m^{(n)}$, the one-point union of $n$ copies of the fan $F_m = P_m + K_1$ with the common point of the fans being the center, is cordial (see also [792]). The *flag* $Fl_m$ is obtained by joining one vertex of $C_m$ to an extra vertex called the *root*. Shee and Ho [1056] show all $Fl_m^{(n)}$, the one-point union of $n$ copies of $Fl_m$ with the common point being the root, are cordial. In his 2001 Ph. D. thesis Selvaraju [1001] proves that the one-point union of any number of copies of a complete bipartite graph is cordial. Benson and Lee [212] have investigated the regular windmill graphs $K_m^{(n)}$ and determined precisely which

ones are cordial for $m < 14$.

Andar, Boxwala, and Limaye [63], [64], and [67] have proved the following graphs are cordial: helms; closed helms; generalized helms obtained by taking a web (see 2.2 for the definitions) and attaching pendent vertices to all the vertices of the outermost cycle in the case that the number cycles is even; flowers (graphs obtained by joining the vertices of degree one of a helm to the central vertex); sunflower graphs (that is, graphs obtained by taking a wheel with the central vertex $v_0$ and the $n$-cycle $v_1, v_2, \ldots, v_n$ and additional vertices $w_1, w_2, \ldots, w_n$ where $w_i$ is joined by edges to $v_i, v_{i+1}$, where $i+1$ is taken modulo $n$); multiple shells (see §2.2); and the one point unions of helms, closed helms, flowers, gears, and sunflower graphs, where in each case the central vertex is the common vertex.

Du [373] proved that the disjoint union of $n \geq 2$ wheels is cordial if and only if $n$ is even or $n$ is odd and the number of vertices of in each cycle is not 0 (mod 4) or $n$ is odd and the number of vertices of in each cycle is not 3 (mod 4).

Elumalai and Sethurman [381] proved: cycles with parallel cords are cordial and $n$-cycles with parallel $P_k$-chords (see §2.2 for the definition) are cordial for any odd positive integer $k$ at least 3 and any $n \not\equiv 2 \pmod 4$ of length at least 4. They call a graph $H$ an *even-multiple subdivision graph* of a graph $G$ if it is obtained from $G$ by replacing every edge $uv$ of $G$ by a pair of paths of even length starting at $u$ and ending at $v$. They prove that every even-multiple subdivision graph is cordial and that every graph is a subgraph of a cordial graph. In [1252] Wen proves that generalized wheels $C_n + mK_1$ are cordial when $m$ is even and $n \not\equiv 2 \pmod 4$ and when $m$ is odd and $n \not\equiv 3 \pmod 4$.

Vaidya, Ghodasara, Srivastav, and Kaneria investigated graphs obtained by joining two identical graphs by a path. They prove: graphs obtained by joining two copies of the same cycle by a path are cordial [1190]; graphs obtained by joining two copies of the same cycle that has two chords with a common vertex with opposite ends of the chords joining two consecutive vertices of the cycle by a path are cordial [1190]; graphs obtained by joining two rim verticies of two copies of the same wheel by a path are cordial [1192]; and graphs obtained by joining two copies of the same Petersen graph by a path are cordial [1192]. They also prove that graphs obtained by replacing one vertex of a star by a fixed wheel or by replacing each vertex of a star by a fixed Petersen graph are cordial [1192]. In [1197] Vaidya, Ghodasara, Srivastav, and Kaneria investigated graphs obtained by joining two identical cycles that have a chord are cordial and the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed cycle with a chord that forms a triangle with two consecutive edges of the cycle and joining each $G_i$ to $G_{i+1}$ $(i = 1, 2, \ldots, n-1)$ by an edge that is incident with the endpoints of the chords in $G_i$ and $G_{i+1}$ are cordial. Vaidya, Dani, Kanani, and Vihol [1187] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of $G_i$ to the center of $G_{i+1}$ $(i = 1, 2, \ldots, n-1)$ by an edge are cordial.

S. Vaidya, K. Kanani, S. Srivastav, and G. Ghodasara [1194] proved: graphs obtained by subdividing every edge of a cycle with exactly two extra edges that are chords with a common endpoint and whose other end points are joined by an edge of the cycle are cordial; graphs obtained by subdividing every edge of the graph obtained by starting with $C_n$ and adding exactly three chords that result in two 3-cycles and a cycle of length $n-3$ are cordial; graphs obtained by subdividing every edge of a Petersen graph are cordial.

Recall the shell $C(n, n-3)$ is the cycle $C_n$ with $n-3$ cords sharing a common endpoint. Vaidya, Dani, Kanani, and Vihol [1188] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed shell and joining common endpoint of the chords of $G_i$ to the common

endpoint of the chords of $G_{i+1}$ $(i = 1, 2, \ldots, n-1)$ by an edge are cordial. Vaidya, Dani, Kanani and Vihol [1195] define $C_n(C_n)$ as the graph obtained by subdividing each edge of $C_n$ and connecting the new $n$ vertices to form a copy of $C_n$ inscribed the original $C_n$. They prove that $C_n(C_n)$ is cordial if $n \neq 2 \pmod 4$; the graphs obtained by starting with copies $G_1, G_2, \ldots, G_k$ of $C_n(C_n)$ the graph obtained by joining a vertex of degree 2 in $G_i$ to a vertex of degree 2 in $G_{i+1}$ $(i = 1, 2, \ldots, n-1)$ by an edge are cordial; and the graphs obtained by joining vertex of degree 2 from one copy of $C_(C_n)$ to a vertex of degree 2 to another copy of $C_(C_n)$ by any finite path are cordial.

In [67] Andar et al. define a *t-ply graph* $P_t(u, v)$ as a graph consisting of $t$ internally disjoint paths joining vertices $u$ and $v$. They prove that $P_t(u, v)$ is cordial except when it is Eulerian and the number of edges is congruent to 2 (mod 4). In [68] Andar, Boxwala, and Limaye prove that the one-point union of any number of plys with an endpoint as the common vertex is cordial if and only if it is not Eulerian and the number of edges is congruent to 2 (mod 4). They further prove that the path union of shells obtained by joining any point of one shell to any point of the next shell is cordial; graphs obtained by attaching a pendant edge to the common vertex of the cords of a shell are cordial; and cycles with one pendant edge are cordial.

For a graph $G$ and a positive integer $t$, Andar, Boxwala, and Limaye [65] define the *t-uniform homeomorph* $P_t(G)$ of $G$ as the graph obtained from $G$ by replacing every edge of $G$ by vertex disjoint paths of length $t$. They prove that if $G$ is cordial and $t$ is odd, then $P_t(G)$ is cordial; if $t \equiv 2 \pmod 4$ a cordial labeling of $G$ can be extended to a cordial labeling of $P_t(G)$ if and only if the number of edges labeled 0 in $G$ is even; and when $t \equiv 0 \pmod 4$ a cordial labeling of $G$ can be extended to a cordial labeling of $P_t(G)$ if and only if the number of edges labeled 1 in $G$ is even. In [66] Ander et al. prove that $P_t(K_{2n})$ is cordial for all $t \geq 2$ and that $P_t(K_{2n+1})$ is cordial if and only if $t \equiv 0 \pmod 4$ or $t$ is odd and $n \not\equiv 2 \pmod 4$, or $t \equiv 2 \pmod 4$ and $n$ is even.

In [68] Andar, Boxwala, and Limaya show that a cordial labeling of $G$ can be extended to a cordial labeling of the graph obtained from $G$ by attaching $2m$ pendant edges at each vertex of $G$. For a binary labeling $g$ of the vertices of a graph $G$ and the induced edge labels given by $g(e) = |g(u) - g(v)|$ let $v_g(j)$ denote the number of vertices labeled with $j$ and $e_g(j)$ denote the number edges labeled with $j$. Let $i(G) = \min\{|e_g(0) - e_g(1)|\}$ taken over all binary labelings $g$ of $G$ with $|v_g(0) - v_g(1)| \leq 1$. Andar et al. also prove that a cordial labeling $g$ of a graph $G$ with $p$ vertices can be extended to a cordial labeling of the graph obtained from $G$ by attaching $2m+1$ pendant edges at each vertex of $G$ if and only if $G$ does not satisfy either of the conditions: (1) $G$ has an even number of edges and $p \equiv 2 \pmod 4$; (2) $G$ has an odd number of edges and either $p \equiv 1 \pmod 4$ with $e_g(1) = e_g(0) + i(G)$ or $n \equiv 3 \pmod 4$ and $e_g(0) = e_g(1) + i(G)$. Andar, Boxwala, and Limaye [69] also prove: if $g$ is a binary labeling of the $n$ vertices of graph $G$ with induced edge labels given by $g(e) = |g(u) - g(v)|$ then $g$ can be extended to a cordial labeling of $G \odot \overline{K_{2m}}$ if and only if $n$ is odd and $i(G) \equiv 2 \pmod 4$; $K_n \odot \overline{K_{2m}}$ is cordial if and only if $n \neq 4 \pmod 8$; $K_n \odot \overline{K_{2m+1}}$ is cordial if and only if $n \neq 7 \pmod 8$; if $g$ is a binary labeling of the $n$ vertices of graph $G$ with induced edge labels given by $g(e) = |g(u) - g(v)|$ then $g$ can be extended to a cordial labeling of $G \odot C_t$ if $t \neq 3 \bmod 4$, $n$ is odd and $e_g(0) = e_g(1)$. For any binary labeling $g$ of a graph $G$ with induced edge labels given by $g(e) = |g(u) - g(v)|$ they also characterize in terms of $i(G)$ when $g$ can be extended to graphs of the form $G \odot \overline{K_{2m+1}}$.

For graphs $G_1, G_2, \ldots, G_n$ $(n \geq 2)$ that are all copies of a fixed graph $G$, Shee and Ho [1057] call a graph obtained by adding an edge from $G_i$ to $G_{i+1}$ for $i = 1, \ldots, n-1$ a *path-union* of $G$ (the resulting graph may depend on how the edges are chosen). Among their results they show

the following graphs are cordial: path-unions of cycles; path-unions of any number of copies of $K_m$ when $m = 4, 6$, or 7; path-unions of three or more copies of $K_5$; and path-unions of two copies of $K_m$ if and only if $m - 2, m$, or $m + 2$ is a perfect square. They also show that there exist cordial path-unions of wheels, fans, unicyclic graphs, Petersen graphs, trees, and various compositions.

Lee and Liu [719] give the following general construction for the forming of cordial graphs from smaller cordial graphs. Let $H$ be a graph with an even number of edges and a cordial labeling such that the vertices of $H$ can be divided into $t$ parts $H_1, H_2, \ldots, H_t$ each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let $G$ be any graph and $G_1, G_2, \ldots, G_t$ be any $t$ subsets of the vertices of $G$. Let $(G, H)$ be the graph that is the disjoint union of $G$ and $H$ augmented by edges joining every vertex in $G_i$ to every vertex in $H_i$ for all $i$. Then $G$ is cordial if and only if $(G, H)$ is. From this it follows that: all generalized fans $F_{m,n} = \overline{K_m} + P_n$ are cordial; the generalized bundle $B_{m,n}$ is cordial if and only if $m$ is even or $n \not\equiv 2 \pmod 4$ ($B_{m,n}$ consists of $2n$ vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ with an edge from $v_i$ to $u_i$ and $2m$ vertices $x_1, x_2, \ldots x_m, y_1, y_2, \ldots, y_m$ with $x_i$ joined to $v_i$ and $y_i$ joined to $u_i$); if $m$ is odd the generalized wheel $W_{m,n} = \overline{K_m} + C_n$ is cordial if and only if $n \not\equiv 3 \pmod 4$. If $m$ is even, $W_{m,n}$ is cordial if and only if $n \not\equiv 2 \pmod 4$; a complete $k$-partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.

Sethuraman and Selvaraju [1049] have shown that certain cases of the union of any number of copies of $K_4$ with one or more edges deleted and one edge in common are cordial. Youssef [1310] has shown that the $k$th power of $C_n$ is cordial for all $n$ when $k \equiv 2 \pmod 4$ and for all even $n$ when $k \equiv 0 \pmod 4$. Ramanjaneyulu, Venkaiah, and Kothapalli [954] give cordial labelings for a family of planar graphs for which each face is a 3-cycle and a family for which each face is a 4-cycle. Acharya, Germina, Princy, and Rao [23] prove that every graph $G$ can be embedded in a cordial graph $H$. The construction is done in such a way that if $G$ is planar or connected, then so is $H$.

Recall from §2.7 that a graph $H$ is a *supersubdivision* of a graph $G$, if every edge $uv$ of $G$ is replaced by $K_{2,m}$ ($m$ may vary for each edge) by identifying $u$ and $v$ with the two vertices in $K_{2,m}$ that form the partite set with exactly two members. Vaidya and Kanani [1193] prove that supersubdivisions of paths and stars are cordial. They also prove that supersubdivisions of $C_n$ are cordial provided that $n$ and the various values for $m$ are odd.

Cahit [280] calls a graph *$H$-cordial* if it is possible to label the edges with the numbers from the set $\{1, -1\}$ in such a way that, for some $k$, at each vertex $v$ the sum of the labels on the edges incident with $v$ is either $k$ or $-k$ and the inequalities $|v(k) - v(-k)| \leq 1$ and $|e(1) - e(-1)| \leq 1$ are also satisfied, where $v(i)$ and $e(j)$ are, respectively, the number of vertices labeled with $i$ and the number of edges labeled with $j$. He calls a graph *$H_n$-cordial* if it is possible to label the edges with the numbers from the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ in such a way that, at each vertex $v$ the sum of the labels on the edges incident with $v$ is in the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ and the inequalities $|v(i) - v(-i)| \leq 1$ and $|e(i) - e(-i)| \leq 1$ are also satisfied for each $i$ with $1 \leq i \leq n$. Among Cahit's results are: $K_{n,n}$ is $H$-cordial if and only if $n > 2$ and $n$ is even; and $K_{m,n}, m \neq n$, is $H$-cordial if and only if $n \equiv 0 \pmod 4$, $m$ is even and $m > 2, n > 2$. Unfortunately, Ghebleh and Khoeilar [468] have shown that other statements in Cahit's paper are incorrect. In particular, Cahit states that $K_n$ is $H$-cordial if and only if $n \equiv 0 \pmod 4$; $W_n$ is $H$-cordial if and only if $n \equiv 1 \pmod 4$; and $K_n$ is $H_2$-cordial if and only if $n \equiv 0 \pmod 4$ whereas Ghebleh and Khoeilar instead prove that $K_n$ is $H$-cordial if and only if $n \equiv 0$ or $3 \pmod 4$ and $n \neq 3$; $W_n$ is $H$-cordial if and only if $n$ is odd; $K_n$ is $H_2$-cordial if $n \equiv 0$ or $3 \pmod 4$; and $K_n$ is not $H_2$-cordial if

$n \equiv 1 \pmod 4$. Ghebleh and Khoeilar also prove every wheel has an $H_2$-cordial labeling. Cahit generalizes the notion of $H$-cordial labelings in [280].

Cahit and Yilmaz [284] call a graph $E_k$-*cordial* if it is possible to label the edges with the numbers from the set $\{0, 1, 2, \ldots, k-1\}$ in such a way that, at each vertex $v$, the sum of the labels on the edges incident with $v$ modulo $k$ satisfies the inequalities $|v(i) - v(j)| \leq 1$ and $|e(i) - e(j)| \leq 1$, where $v(s)$ and $e(t)$ are, respectively, the number of vertices labeled with $s$ and the number of edges labeled with $t$. Cahit and Yilmaz prove the following graphs are $E_3$-cordial: $P_n$ $(n \geq 3)$; stars $S_n$ if and only if $n \not\equiv 1 \pmod 3$; $K_n$ $(n \geq 3)$; $C_n$ $(n \geq 3)$; friendship graphs; and fans $F_n$ $(n \geq 3)$. They also prove that $S_n$ $(n \geq 2)$ is $E_k$-cordial if and only if $n \not\equiv 1 \pmod k$ when $k$ is odd or $n \not\equiv 1 \pmod{2k}$ when $k$ is even and $k \neq 2$.

Bapat and Limaye [187] provide $E_3$-cordial labelings for: $K_n$ $(n \geq 3)$; snakes whose blocks are all isomorphic to $K_n$ where $n \equiv 0$ or $2 \pmod 3$; the one-point union of any number of copies of $K_n$ where $n \equiv 0$ or $2 \pmod 3$; graphs obtained by attaching a copy of $K_n$ where $n \equiv 0$ or $3 \pmod 3$ at each vertex of a path; and $K_m \odot K_n$. Rani and Sridharan [958] proved: for odd $n > 1$ and $k \geq 2$, $P_n \odot K_1$ is $E_k$-cordial; for $n$ even and $n \neq k/2$, $P_n \odot K_1$ is $E_k$-cordial; and certain cases of fans are $E_k$-cordial.

Hovey [556] has introduced a simultaneous generalization of harmonious and cordial labelings. For any Abelian group $A$ (under addition) and graph $G(V, E)$ he defines $G$ to be $A$-*cordial* if there is a labeling of $V$ with elements of $A$ such that for all $a$ and $b$ in $A$ when the edge $ab$ is labeled with $f(a) + f(b)$, the number of vertices labeled with $a$ and the number of vertices labeled $b$ differ by at most one and the number of edges labeled with $a$ and the number labeled with $b$ differ by at most one. In the case where $A$ is the cyclic group of order $k$, the labeling is called $k$-*cordial*. With this definition we have: $G(V, E)$ is harmonious if and only if $G$ is $|E|$-cordial; $G$ is cordial if and only if $G$ is 2-cordial.

Hovey has obtained the following: caterpillars are $k$-cordial for all $k$; all trees are $k$-cordial for $k = 3, 4$, and 5; odd cycles with pendant edges attached are $k$-cordial for all $k$; cycles are $k$-cordial for all odd $k$; for $k$ even, $C_{2mk+j}$ is $k$-cordial when $0 \leq j \leq \frac{k}{2} + 2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not $k$-cordial; $K_m$ is 3-cordial; and, for $k$ even, $K_{mk}$ is $k$-cordial if and only if $m = 1$.

Hovey advances the following conjectures: all trees are $k$-cordial for all $k$; all connected graphs are 3-cordial; and $C_{2mk+j}$ is $k$-cordial if and only if $j \neq k$, where $k$ and $j$ are even and $0 \leq j < 2k$. The last conjecture was verified by Tao [1171]. Tao's result combined with those of Hovey show that for all positive integers $k$ the $n$-cycle is $k$-cordial with the exception that $k$ is even and $n = 2mk + k$. Tao also proved that the crown with $2mk + j$ vertices is $k$-cordial unless $j = k$ is even, and for $4 \leq n \leq k$ the wheel $W_n$ is $k$-cordial unless $k \equiv 5 \pmod 8$ and $n = (k+1)/2$.

In [1045] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph $G$ and successively form supersubdivisions (see §2.7 for the definition) that are cordial in the case that every edge in $G$ is replaced by $K_{2,m}$ where $m$ is even. Sethuraman and Selvaraju [1044] also show that the one-vertex union of any number of copies of $K_{m,n}$ is cordial and that the one-edge union of $k$ copies of shell graphs $C(n, n-3)$ (see §2.2) is cordial for all $n \geq 4$ and all $k$. They conjectured that the one-point union of any number of copies of graphs of the form $C(n_i, n_i - 3)$ for various $n_i \geq 4$ is cordial. This was proved by Yue, Yuansheng, and Liping in [1318]. Riskin [967] claimed that $K_n$ is $Z_2 \times Z_2$-cordial if and only if $n$ is at most 3 and $K_{m,n}$ is $Z_2 \times Z_2$ cordial if and only if $(m, n) \neq (2, 2)$. However, Pechenik and Wise [925] report that the correct statement for $K_{m,n}$ is $K_{m,n}$ is $Z_2 \times Z_2$ cordial if and only if $m$v and $n$ are not both congruent to 2 mod 4.

In [925] Pechenik and Wise investigate $Z_2 \times Z_2$-cordiality of complete bipartite graphs, paths, cycles, ladders, prisms, and hypercubes. They proved that all complete bipartite graphs are $Z_2 \times Z_2$-cordial except $K_{m,n}$ where $m, n \equiv 2 \bmod 4$; all paths are $Z_2 \times Z_2$-cordial except $P_4$ and $P_5$; all cycles are $Z_2 \times Z_2$-cordial except $C_4, C_5, C_k$, where $k \equiv 2 \bmod 4$; and all ladders $P_2 \times P_k$ are $Z_2 \times Z_2$-cordial except $C_4$. They also introduce a generalization of $A$-cordiality involving digraphs and quasigroups, and we show that there are infinitely many $G$-cordial digraphs for every quasigroup $Q$.

Cairnie and Edwards [287] have determined the computational complexity of cordial and $k$-cordial labelings. They prove the conjecture of Kirchherr [649] that deciding whether a graph admits a cordial labeling is NP-complete. As a corollary, this result implies that the same problem for $k$-cordial labelings is NP-complete. They remark that even the restricted problem of deciding whether connected graphs of diameter 2 have a cordial labeling is also NP-complete.

In [303] Chartrand, Lee, and Zhang introduced the notion of uniform cordiality as follows. Let $f$ be a labeling from $V(G)$ to $\{0,1\}$ and for each edge $xy$ define $f^*(xy) = |f(x) - f(y)|$. For $i = 0$ and 1, let $v_i(f)$ denote the number of vertices $v$ with $f(v) = i$ and $e_i(f)$ denote the number of edges $e$ with $f^*(e) = i$. They call a such a labeling $f$ *friendly* if $|v_0(f) - v_1(f)| \leq 1$. A graph $G$ for which every friendly labeling is cordial is called *uniformly cordial*. They prove that a connected graph of order $n \geq 2$ is uniformly cordial if and only if $n = 3$ and $G = K_3$, or $n$ is even and $G = K_{1,n-1}$.

In [965] Riskin introduced two measures of the noncordiality of a graph. He defines the *cordial edge deficiency* of a graph $G$ as the minimum number of edges, taken over all friendly labelings of $G$, needed to be added to $G$ such that the resulting graph is cordial. If a graph $G$ has a vertex labeling $f$ using 0 and 1 such that the edge labeling $f_e$ given by $f_e(xy) = |f(x) - f(y)|$ has the property that the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1, the *cordial vertex deficiency* defined as $\infty$. Riskin proved: the cordial edge deficiency of $K_n$ $(n > 1)$ is $\lfloor \frac{n}{2} \rfloor - 1$; the cordial vertex deficiency of $K_n$ is $j - 1$ if $n = j^2 + \delta$, when $\delta$ is $-2, 0$ or 2, and $\infty$ otherwise. In [965] Riskin determines the cordial edge deficiency and cordial vertex deficiency for the cases when the Möbius ladders and wheels are not cordial. In [966] Riskin determines the cordial edge deficiencies for complete multipartite graphs that are not cordial and obtains a upper bound for their cordial vertex deficiencies.

If $f$ is a binary vertex labeling of a graph $G$ Lee, Liu, and Tan [720] defined a partial edge labeling of the edges of $G$ by $f^*(uv) = 0$ if $f(u) = f(v) = 0$ and $f^*(uv) = 1$ if $f(u) = f(v) = 1$. They let $e_0(G)$ denote the number of edges $uv$ for which $f^*(uv) = 0$ and $e_1(G)$ denote the number of edges $uv$ for which $f^*(uv) = 1$. They say $G$ is *balanced* if it has a friendly labeling $f$ such that if $|e_0(f) - e_1(f)| \leq 1$. In the case that the number of vertices labeled 0 and the number of vertices labeled 1 are equal and the number of edges labeled 0 and the number of edges labeled 1 are equal they say the labeling is *strongly balanced*. They prove: $P_n$ is balanced for all $n$ and is strongly balanced if $n$ is even; $K_{m,n}$ is balanced if and only if $m$ and $n$ are even, $m$ and $n$ are odd and differ by at most 2, or exactly one of $m$ or $n$ is even (say $n = 2t$) and $t \equiv -1, 0, 1 \pmod{|m-n|}$; a $k$-regular graph with $p$ vertices is strongly balanced if and only if $p$ is even and is balanced if and only if $p$ is odd and $k = 2$; and if $G$ is any graph and $H$ is strongly balanced, the composition $G[H]$ (see §2.3 for the definition) is strongly balanced. In [666] Kong, Lee, Seah, and Tang show: $C_m \times P_n$ is balanced if $m$ and $n$ are odd and is strongly balanced if either $m$ or $n$ is even; and $C_m \odot K_1$ is balanced for all $m \geq 3$ and strongly balanced if $m$ is even. They also provide necessary and sufficient conditions for a graph to be balanced or strongly balanced. Lee, Lee, and Ng [698] show that stars are balanced if and only if the number of edges

of the star is at most 4. Kwong, Lee, Lo, and Wang [692] define a graph $G$ to be *uniformly balanced* if $|e_0(f) - e_1(f)| \leq 1$ for every vertex labeling $f$ that satisfies if $|v_0(f) - v_1(f)| \leq 1$. They present several ways to construct families of uniformly balanced graphs. Kim, Lee, and Ng [645] prove the following: for any graph $G$, $mG$ is balanced for all $m$; for any graph $G$, $mG$ is strongly balanced for all even $m$; if $G$ is strongly balanced and $H$ is balanced, then $G \cup H$ is balanced; $mK_n$ is balanced for all $m$ and strongly balanced if and only if $n = 3$ or $mn$ is even; if $H$ is balanced and $G$ is any graph, the $G \times H$ is strongly balanced; if one of $m$ or $n$ is even, then $P_m[P_n]$ is balanced; if both $m$ and $n$ are even, then $P_m[P_n]$ is balanced; and if $G$ is any graph and $H$ is strongly balanced, then the tensor product $G \otimes H$ is strongly balanced.

### 3.8 The Friendly Index–Balance Index

Recall a function $f$ from $V(G)$ to $\{0,1\}$ where for each edge $xy$, $f^*(xy) = |f(x) - f(y)|$, $v_i(f)$ is the number of vertices $v$ with $f(v) = i$, and $e_i(f)$ is the number of edges $e$ with $f^*(e) = i$ is called *friendly* if $|v_0(f) - v_1(f)| \leq 1$. Lee and Ng [726] define the *friendly index set* of a graph $G$ as $\mathrm{FI}(G) = \{|e_0(f) - e_1(f)|$ where $f$ runs over all friendly labelings $f$ of $G\}$. They proved: for any graph $G$ with $q$ edges $\mathrm{FI}(G) \subseteq \{0, 2, 4, \ldots, q\}$ if $q$ is even and $\mathrm{FI}(G) \subseteq \{1, 3, \ldots, q\}$ if $q$ is odd; for $1 \leq m \leq n$, $\mathrm{FI}(K_{m,n}) = \{(m-2i)^2 | \ 0 \leq i \leq \lfloor m/2 \rfloor\}$ if $m+n$ is even; and $\mathrm{FI}(K_{m,n}) = \{i(i+1) | \ 0 \leq i \leq m\}$ if $m + n$ is odd. In [730] Lee and Ng prove the following: $\mathrm{FI}(C_{2n}) = \{0, 4, 8, \ldots, 2n\}$ when $n$ is even; $\mathrm{FI}(C_{2n}) = \{2, 6, 10, \ldots, 2n\}$ when $n$ is odd; and $\mathrm{FI}(C_{2n+1}) = \{1, 3, 5, \ldots, 2n-1\}$. Elumalai [380] defines a *cycle with a full set of chords* as the graph $PC_n$ obtained from $C_n = v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the cords $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}, v_{(n+2)/2}$ when $n$ is even and $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-3)/2}, v_{(n+3)/2}$ when $n$ is odd. Lee and Ng [728] prove: $\mathrm{FI}(PC_{2m+1}) = \{3m-2, 3m-4, 3m-6, \ldots, 0\}$ when $m$ is even and $\mathrm{FI}(PC_{2m+1}) = \{3m-2, 3m-4, 3m-6, \ldots, 1\}$ when $m$ is odd; $\mathrm{FI}(PC_4) = \{1, 3\}$; for $m \geq 3$, $\mathrm{FI}(PC_{2m}) = \{3m-5, 3m-7, 3m-9, \ldots, 1\}$ when $m$ is even; $\mathrm{FI}(PC_{2m}) = \{3m-5, 3m-7, 3m-9, \ldots, 0\}$ when $m$ is odd.

Salehi and Lee [985] determined the friendly index for various classes of trees. Among their results are: for a tree with $q$ edges that has a perfect matching, the friendly index is the odd integers from 1 to $q$ and for $n \geq 2$, $\mathrm{FI}(P_n) = \{n-1-2i | \ 0 \leq i \lfloor (n-1)/2 \rfloor$. Lee and Ng [728] define $PC(n,p)$ as the graph obtained from the cycle $C_n$ with consecutive vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the $p$ cords joining $v_i$ to $v_{n-i}$ for $1 \leq p\lfloor n/2 \rfloor - 1$. They prove $\mathrm{FI}(PC(2m+1, p)) = \{2m+p-1, 2m+p-3, 2m+p-5, \ldots, 1\}$ if $p$ is even and $\mathrm{FI}(PC(2m+1, p)) = \{2m+p-1, 2m+p-3, 2m+p-5, \ldots, 0\}$ if $p$ is odd; $\mathrm{FI}(PC(2m, 1)) = \{2m-1, 2m-3, 2m-5, \ldots, 1\}$; for $m \geq 3$, and $p \geq 2$, $\mathrm{FI}(PC(2m, p)) = \{2m+p-4, 2m+p-6, 2m+p-8, \ldots, 0\}$ when $p$ is even, and $\mathrm{FI}(PC(2m, p)) = \{2m+p-4, 2m+p-6, 2m+p-8, \ldots, 1\}$ when $p$ is odd. More generally, they show that the integers in the friendly index of a cycle with an arbitrary nonempty set of parallel chords form an arithmetic progression with a common difference 2. Shiu and Kwong [1063] determine the friendly index of the grids $P_n \times P_2$. The maximum and minimum friendly indices for $C_m \times P_n$ were given by Shiu and Wong in [1081].

In [729] Lee and Ng prove: for $n \geq 2$, $\mathrm{FI}(C_{2n} \times P_2) = \{0, 4, 8, \ldots, 6n-8, 6n\}$ if $n$ is even and $\mathrm{FI}(C_{2n} \times P_2) = \{2, 6, 10, \ldots, 6n-8, 6n\}$ if $n$ is odd; $\mathrm{FI}(C_3 \times P_2) = \{1, 3, 5\}$; for $n \geq 2$, $\mathrm{FI}(C_{2m+1} \times P_2) = \{6n-1\} \cup \{6n-5-2k|$ where $k \geq 0$ and $6n-5-2k \geq 0\}$; $\mathrm{FI}(M_{4n})$ (here $M_{4n}$ is the Möbius ladder with $4n$ steps) $= \{6n-4-4k|$ where $k \geq 0$ and $6n-4-4k \geq 0\}$; $\mathrm{FI}(M_{4n+2}) = \{6n+3\} \cup \{6n-5-2k|$ where $k \geq 0$ and $6n-5-2k > 0\}$. In [693] Kwong, Lee, and Ng completely determine the friendly index of 2-regular graphs with two components. As a corollary, they show that $C_m \cup C_n$ is cordial if and only if $m + n = 0, 1$ or 3 (mod 4).

Ho, Lee, and Ng [550] determine the friendly index sets of stars and various regular windmills. In [1252] Wen determines the friendly index of generalized wheels $C_n + mK_1$ for all $m > 1$. In [984] Salehi and De determine the friendly index sets of certain caterpillars of diameter 4 and disprove a conjecture of Lee and Ng [730] that the friendly index sets of trees form an arithmetic progression. The maximum and minimum friendly indices for for $C_m \times P_n$ were given by Shiu and Wong in [1081]. Salehi and Bayot [982] have determined the friendly index set of $P_m \times P_n$.

For positive integers $a \leq b \leq c$, Lee, Ng, amd Tong [734] define the *broken wheel* $W(a,b,c)$ with three spokes as the graph obtained from $K_4$ with vertices $u_1, u_2, u_3, c$ by inserting vertices $x_{1,1}, x_{1,2}, \ldots x_{1,a-1}$ along the edge $u_1u_2$, $x_{2,1}, x_{2,2}, \ldots x_{2,b-1}$ along the edge $u_2u_3$, $x_{3,1}, x_{3,2}, \ldots x_{3,c-1}$ along the edge $u_3u_1$. They determine the friendly index set for broken wheels with three spokes.

Lee and Ng [728] define a *parallel chord* of $C_n$ as an edge of the form $v_iv_{n-i}$ $(i < n-1)$ that is not an edge of $C_n$. For $n \geq 6$, they call the cycle $C_n$ with consecutive vertices $v_1, v_2, \ldots, v_n$ and the edges $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}v_{(n+2)/2}$ for $n$ even and $v_2v_{n-1}, v_3v_{n-2}, \ldots, v_{(n-1)/2}v_{(n+3)/2}$ for $n$ odd, $C_n$ with a *full set of parallel chords*. They determine the friendly index of these graphs and show that for any cycle with an arbitrary non-empty set of parallel chords the numbers in its friendly index set form an arithmetic progression with common difference 2.

For a graph $G(V,E)$ and a graph $H$ rooted at one of its vertices $v$, Ho, Lee, and Ng [549] define a *root-union of* $(H,v)$ *by* $G$ as the graph obtained from $G$ by replacing each vertex of $G$ with a copy of the root vertex $v$ of $H$ to which is appended the rest of the structure of $H$. They investigate the friendly index set of the root-union of stars by cycles.

For a graph $G(V,E)$, the *total graph* $T(G)$ of $G$, is the graph with vertex set $V \cup E$ and edge set $E \cup \{(v,uv)|\ v \in V, uv \in E\}$. Note that the total graph of the $n$-star is the friendship graph and the total graph of $P_n$ is a triangular snake. Lee and Ng [725] use $SP(1^n,m)$ to denote the spider with one central vertex joining $n$ isolated vertices and a path of length $m$. They show: $\mathrm{FI}(K_1 + 2nK_2)$ (friendship graph with $2n$ triangles) $= \{2n, 2n-4, 2n-8, \ldots, 0\}$ if $n$ is even; $\{2n, 2n-4, 2n-8, \ldots, 2\}$ if $n$ is odd; $\mathrm{FI}(K_1 + (2n+1)K_2) = \{2n+1, 2n-1, 2n-3, \ldots, 1\}$; for $n$ odd, $\mathrm{FI}(T(P_n)) = \{3n-7, 3n-11, 3n-15, \ldots, z\}$ where $z = 0$ if $n \equiv 1 \pmod 4$ and $z = 2$ if $n \equiv 3 \pmod 4$; for $n$ even, $\mathrm{FI}(T(P_n)) = \{3n-7, 3n-11, 3n-15, \ldots, n+1\} \cup \{n-1, n-3, n-5, \ldots, 1\}$; for $m \leq n-1$ and $m+n$ even, $\mathrm{FI}(T(SP(1^n,m))) = \{3(m+n)-4, 3(m+n)-8, 3(m+n)-12, \ldots, (m+n) \pmod 4\}$; for $m+n$ odd, $\mathrm{FI}(T(SP(1^n,m))) = \{3(m+n)-4, 3(m+n)-8, 3(m+n)-12, \ldots, m+n+2\} \cup \{m+n, m+n-2, m+n-4, \ldots, 1\}$; for $n \geq m$ and $m+n$ even, $\mathrm{FI}(T(SP(1^n,m))) = \{|4k-3(m+n)|\ |(n-m+2)/2 \leq k \leq m+n\}$; for $n \geq m$ and $m+n$ odd, $\mathrm{FI}(T(SP(1^n,m))) = \{|4k-3(m+n)|\ |(n-m+3)/2 \leq k \leq m+n\}$.

Kwong and Lee [689] determine the friendly index any number of copies of $C_3$ that share an edge in common and the friendly index any number of copies of $C_4$ that share an edge in common.

In [646] Kim, Lee, and Ng define the *balance index set* of a graph $G$ as $\{|e_0(f) - e_1(f)|\}$ where $f$ runs over all friendly labelings $f$ of $G$. Zhang, Lee, and Wen [698] investigate the balance index sets for the disjoint union of up to four stars and Zhang, Ho, Lee, and Wen [1320] investigate the balance index sets for trees with diameter at most four. Kwong, Lee, and Sarvate [694] determine the balance index sets for cycles with one pendant edge, flowers, and regular windmills. Lee, Ng, and Tong [733] determine the balance index set of certain graphs obtained by starting with copies of a given cycle and successively identifying one particular vertex of one copy with a particular vertex of the next. For graphs $G$ and $H$ and a bijection $\pi$ from $G$ to $H$, Lee and Su [754] define $\mathrm{Perm}(G,\pi,H)$ as the graph obtaining from the disjoint union of $G$ and $H$ by joining each $v$ in $G$ to $\pi(v)$ with an edge. They determine the balanced index sets of the

disjoint union of cycles and the balanced index sets for graphs of the form $\mathrm{Perm}(G, \pi, H)$ where $G$ and $H$ are regular graphs, stars, paths, and cycles with a chord. They conjecture that the balanced index set for every graph of the form $\mathrm{Perm}(G, \pi, H)$ is an arithmetic progression. Wen [1251] determines the balance index set of the graph that is constructed by identifying the center of a star with one vertex from each of two copies of $C_n$ and provides a necessary and sufficient for such graphs to be balanced. In [756] Lee, Su, and Wang determine the balance index sets of the disjoint union of a variety of regular graphs of the same order. Kwong [687] determines the balanced index sets of rooted trees of height at most 2, thereby settling the problem for trees with diameter at most 4. His method can be used to determine the balance index set of any tree.

In[1064] Shiu and Kwong made a major advance by introducing an easier approach to find the balance index sets of a large number of families of graphs in a unified and uniform manner. They use this method to determine the balance index sets for $r$-regular graphs, amalgamations of $r$-regular graphs, complete bipartite graphs, wheels, one point unions of regular graphs, sun graphs, generalized theta graphs, $m$-ary trees, spiders, grids $P_m \times P_n$, and cylinders $C_m \times P_n$. They provide a formula that enables one to determine the balance index sets of many biregular graphs (that is, graphs with the property that there exist two distinct positive integers $r$ and $s$ such that every vertex has degree $r$ or $s$).

In [1063] Shiu and Kwong define the *full friendly index set* of a graph $G$ as $\{e_0(f) - e_1(f)\}$ where $f$ runs over all friendly labelings of $G$. The full friendly index for $P_2 \times P_n$ is given by Shiu and Kwong in [1063]. The full friendly index of $C_m \times C_n$ is given by Shiu and Ling in [1074].

In [326] and [690] Chopra, Lee and Su and Kwong and Lee introduce a dual of balance index sets as follows. For an edge labeling $f$ using 0 and 1 they define a partial vertex labeling $f^*$ by assigning 0 or 1 to $f^*(v)$ depending on whether there are more 0-edges or 1-edges incident to $v$ and leaving $f^*(v)$ undefined otherwise. For $i = 0$ or 1 and a graph $G(V, E)$, let $e_f(i) = |\{uv \in E : f(uv) = i\}|$ and $v_f(i) = |\{v \in V : f^*(v) = i\}|$. They define the *edge-balance index* of $G$ as $\mathrm{EBI}(G) = \{|v_f(0) - v_f(1)| : \text{the edge labeling } f \text{ satisfies } |e_f(0) - e_f(1)| \leq 1\}$. Among the graphs whose edge-balance index sets have been investigated by Lee and his colleagues are: fans and wheels [326]; generalized theta graphs [690]; flower graphs [691] and [691]; stars, paths, spiders, and double stars [762]; $(p, p+1)$-graphs [759]; prisms and Möbius ladders [1243]; 2-regular graphs, complete graphs [1242]; and the envelope graphs of stars, paths, and cycles [334]. (The *envelope graph* of $G(V, E)$ is the graph with vertex set $V(G) \cup E(G)$ and set $E(G) \cup \{(u, (u, v)) : U \in V, (u, v) \in E)\}$).

Chopra, Lee, and Su [328] prove that the edge-balanced index of the fan $P_3 + K_1$ is $\{0, 1, 2\}$ and edge-balanced index of the fan $P_n + K_1$, $n \geq 4$, is $\{0, 1, 2, \ldots, n-2\}$. They define the broken fan graphs $BF(a, b)$ as the graph with $V(BF(a, b)) = \{c\} \cup \{v_1, \ldots, v_a\} \cup \{u_1, \ldots, u_b\}$ and $E(BF(a, b)) = \{(c, v_i) |\ i = 1, \ldots, a\} \cup \{(c, u_i) |\ 1, \ldots, b\} \cup E(P_a) \cup E(P_b)$ $(a \geq 2$ and $b \geq 2)$. They prove the edge-balance index set of $BF(a, b)$ is $\{0, 1, 2, \ldots, a + b - 4\}$. In [699] Lee, Lee, and Su present a technique that determines the balance index sets of a graph from its degree sequence. In addition, they give an explicit formula giving the exact values of the balance indices of generalized friendship graphs, envelope graphs of cycles, and envelope graphs of cubic trees.

In 1990 Cahit [276] proposed the idea of distributing the vertex and edge labels among $\{0, 1, \ldots, k-1\}$ as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V, E)$ and any positive integer $k$, assign vertex labels from $\{0, 1, \ldots, k-1\}$ so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with $i$ and the number of vertices labeled with $j$ differ by at most

one and the number of edges labeled with $i$ and the number of edges labeled with $j$ differ by at most one. Cahit has called a graph with such an assignment of labels *k-equitable*. Note that $G(V,E)$ is graceful if and only if it is $|E|+1$-equitable and $G(V,E)$ is cordial if and only if it is 2-equitable. Cahit [275] has shown the following: $C_n$ is 3-equitable if and only if $n \not\equiv 3 \pmod 6$; the triangular snake with $n$ blocks is 3-equitable if and only if $n$ is even; the friendship graph $C_3^{(n)}$ is 3-equitable if and only if $n$ is even; an Eulerian graph with $q \equiv 3 \pmod 6$ edges is not 3-equitable; and all caterpillars are 3-equitable [275]. Cahit [275] claimed to prove that $W_n$ is 3-equitable if and only if $n \not\equiv 3 \pmod 6$ but Youssef [1305] proved that $W_n$ is 3-equitable for all $n \geq 4$. Youssef [1303] also proved that if $G$ is a $k$-equitable Eulerian graph with $q$ edges and $k \equiv 2$ or $3 \pmod 4$ then $q \not\equiv k \pmod{2k}$. Cahit conjectures [275] that a triangular cactus with $n$ blocks is 3-equitable if and only if $n$ is even. In [276] Cahit proves that every tree with fewer than five end vertices has a 3-equitable labeling. He conjectures that all trees are $k$-equitable [277]. In 1999 Speyer and Szaniszló [1130] proved Cahit's conjecture for $k = 3$.

Vaidya, Ghodasara, Srivastav, and Kaneria [1191] have shown that the graphs obtained by replacing each vertex of a star by a fixed cycle are 3-equitable. Vaidya, Dani, Kanani and Vihol [1187] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of $G_i$ to the center of $G_{i+1}$ $(i = 1, 2, \ldots, n-1)$ by an edge are 3-equitable. Recall the shell $C(n, n-3)$ is the cycle $C_n$ with $n-3$ cords sharing a common endpoint called the *apex*. Vaidya, Dani, Kanani, and Vihol [1188] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed shell and joining each apex of $G_i$ to the apex of $G_{i+1}$ $(i = 1, 2, \ldots, n-1)$ by an edge are 3-equitable. For a graph $G$ and vertex $v$ of $G$, Vaidya, Dani, Kanani, and Vihol [1189] define the duplication of $v$ as the graph obtained from $G$ by adding a new vertex $v'$ to $G$ and joining $v'$ to every vertex in $G$ incident with $v$. They prove that the graphs obtained from the wheel $W_n$, $n \geq 5$, by duplicating any rim vertex is 3-equitable and the graphs obtained from the wheel $W_n$ by duplicating the center is 3-equitable when $n$ is even and not 3-equitable when $n$ is odd and at least 5. They also show that the graphs obtained from the wheel $W_n$, $n \neq 5$, by duplicating every vertex is 3-equitable.

Bhut-Nayak and Telang have shown that crowns $C_n \odot K_1$, are $k$-equitable for $k = n, \ldots, 2n-1$ [231] and $C_n \odot K_1$ is $k$-equitable for all $n$ when $k = 2, 3, 4, 5$, and 6 [232].

In [1005] Seoud and Abdel Maqsoud prove: a graph with $n$ vertices and $q$ edges in which every vertex has odd degree is not 3-equitable if $n \equiv 0 \pmod 3$ and $q \equiv 3 \pmod 6$; all fans except $P_2 + \overline{K_1}$ are 3-equitable; all double fans $P_n + \overline{K_2}$ except $P_4 + \overline{K_2}$ are 3-equitable; $P_n^2$ is 3-equitable for all $n$ except 3; $K_{1,1,n}$ is 3-equitable if and only if $n \equiv 0$ or $2 \pmod 3$; $K_{1,2,n}$, $n \geq 2$, is 3-equitable if and only if $n \equiv 2 \pmod 3$; $K_{m,n}$, $3 \leq m \leq n$, is 3-equitable if and only if $(m,n) = (4,4)$; and $K_{1,m,n}$, $3 \leq m \leq n$, is 3-equitable if and only if $(m,n) = (3,4)$.

Bapat and Limaye [185] have shown the following graphs are 3-equitable: helms $H_n$, $n \geq 4$; flowers (see §2.2 for the definition); the one-point union of any number of helms; the one-point union of any number of copies of $K_4$; $K_4$-snakes (see §2.2 for the definition); $C_t$-snakes where $t = 4$ or 6; $C_5$-snakes where the number of blocks is not congruent to 3 modulo 6. A *multiple shell* $\mathrm{MS}\{n_1^{t_1}, \ldots, n_r^{t_r}\}$ is a graph formed by $t_i$ shells each of order $n_i$, $1 \leq i \leq r$, that have a common apex. Bapat and Limaye [186] show that every multiple shell is 3-equitable and Chitre and Limaye [321] show that every multiple shell is 5-equitable.

Szaniszló [1170] has proved the following: $P_n$ is $k$-equitable for all $k$; $K_n$ is 2-equitable if and only if $n = 1, 2$, or 3; $K_n$ is not $k$-equitable for $3 \leq k < n$; $S_n$ is $k$-equitable for all $k$; $K_{2,n}$ is $k$-equitable if and only if $n \equiv k-1 \pmod k$, or $n \equiv 0, 1, 2, \ldots, \lfloor k/2 \rfloor - 1 \pmod k$, or $n = \lfloor k/2 \rfloor$ and $k$ is odd. She also proves that $C_n$ is $k$-equitable if and only if $k$ meets all of the following

conditions: $n \neq k$; if $k \equiv 2, 3 \pmod 4$, then $n \neq k-1$ and $n \not\equiv k \pmod{2k}$.

Vickrey [1201] has determined the $k$-equitability of complete multipartite graphs. He shows that for $m \geq 3$ and $k \geq 3$, $K_{m,n}$ is $k$-equitable if and only if $K_{m,n}$ is one of the following graphs: $K_{4,4}$ for $k = 3$; $K_{3,k-1}$ for all $k$; or $K_{m,n}$ for $k > mn$. He also shows that when $k$ is less than or equal to the number of edges in the graph and at least 3, the only complete multipartite graphs that are $k$-equitable are $K_{kn+k-1,2,1}$ and $K_{kn+k-1,1,1}$. Partial results on the $k$-equitability of $K_{m,n}$ were obtained by Krussel [684].

In [819] Lopez, Muntaner-Batle, and Rius-Font prove that if $n$ is an odd integer and $F$ is optimal $k$-equitable for all proper divisors $k$ of $|E(F)|$, then $nF$ is optimal $k$-equitable for all proper divisors $k$ of $|E(F)|$. They also prove that if $m-1$ and $n$ are odd, then then $nC_m$ is optimal $k$-equitable for all proper divisors $k$ of $|E(F)|$.

As a corollary of the result of Cairnie and Edwards [287] on the computational complexity of cordially labeling graphs it follows that the problem of finding $k$-equitable labelings of graphs is NP-complete as well.

Seoud and Abdel Maqsoud [1006] call a graph *$k$-balanced* if the vertices can be labeled from $\{0, 1, \ldots, k-1\}$ so that the number of edges labeled $i$ and the number of edges labeled $j$ induced by the absolute value of the differences of the vertex labels differ by at most 1. They prove that $P_n^2$ is 3-balanced if and only if $n = 2, 3, 4$, or 6; for $k \geq 4$, $P_n^2$ is not $k$-balanced if $k \leq n-2$ or $n+1 \leq k \leq 2n-3$; for $k \geq 4$, $P_n^2$ is $k$-balanced if $k \geq 2n-2$; for $k, m, n \geq 3$, $K_{m,n}$ is $k$-balanced if and only if $k \geq mn$; for $m \leq n$, $K_{1,m,n}$ is $k$-balanced if and only if $(i)$ $m = 1$, $n = 1$ or 2, and $k = 3$; $(ii)$ $m = 1$ and $k = n+1$ or $n+2$; or $(iii)$ $k \geq (m+1)(n+1)$.

Bloom has used the term $k$-equitable to describe another kind of labeling (see [1261] and [1262]). He calls a graph *$k$-equitable* if the edge labels induced by the absolute value of the difference of the vertex labels have the property that every edge label occurs exactly $k$ times. Bloom calls a graph of order $n$ *minimally $k$-equitable* if the vertex labels are 1, 2,..., $n$ and it is $k$-equitable. Both Bloom and Wojciechowski [1261], [1262] proved that $C_n$ is minimally $k$-equitable if and only if $k$ is a proper divisor of $n$. Barrientos and Hevia [201] proved that if $G$ is $k$-equitable of size $q = kw$ (in the sense of Bloom), then $\delta(G) \leq w$ and $\Delta(G) \leq 2w$. Barrientos, Dejter, and Hevia [200] have shown that forests of even size are 2-equitable. They also prove that for $k = 3$ or $k = 4$ a forest of size $kw$ is $k$-equitable if and only if its maximum degree is at most $2w$ and that if 3 divides $mn+1$, then the double star $S_{m,n}$ is 3-equitable if and only if $q/3 \leq m \leq \lfloor (q-1)/2 \rfloor$. ($S_{m,n}$ is $P_2$ with $m$ pendant edges attached at one end and $n$ pendant edges attached at the other end.) They discuss the $k$-equitability of forests for $k \geq 5$ and characterize all caterpillars of diameter 2 that are $k$-equitable for all possible values of $k$. Acharya and Bhat-Nayak [33] have shown that coronas of the form $C_{2n} \odot K_1$ are minimally 4-equitable. In [188] Barrientos proves that the one-point union of a cycle and a path (dragon) and the disjoint union of a cycle and a path are $k$-equitable for all $k$ that divide the size of the graph. Barrientos and Havia [201] have shown the following: $C_n \times K_2$ is 2-equitable when $n$ is even; books $B_n$ ($n \geq 3$) are 2-equitable when $n$ is odd; the vertex union of $k$-equitable graphs is $k$-equitable; and wheels $W_n$ are 2-equitable when $n \not\equiv 3 \pmod 4$. They conjecture that $W_n$ is 2-equitable when $n \equiv 3 \pmod 4$ except when $n = 3$. Their 2-equitable labelings of $C_n \times K_2$ and the $n$-cube utilized graceful labelings of those graphs.

M. Acharya and Bhat-Nayak [34] have proved the following: the crowns $C_{2n} \odot K_1$ are minimally 2-equitable, minimally $2n$-equitable, minimally 4-equitable, and minimally $n$-equitable; the crowns $C_{3n} \odot K_1$ are minimally 3-equitable, minimally $3n$-equitable, minimally $n$-equitable, and minimally 6-equitable; the crowns $C_{5n} \odot K_1$ are minimally 5-equitable, minimally $5n$-

equitable, minimally $n$-equitable, and minimally 10-equitable; the crowns $C_{2n+1} \odot K_1$ are minimally $(2n+1)$-equitable; and the graphs $P_{kn+1}$ are $k$-equitable.

In [190] Barrientos calls a $k$-equitable labeling *optimal* if the vertex labels are consecutive integers and *complete* if the induced edge labels are $1, 2, \ldots, w$ where $w$ is the number of distinct edge labels. Note that a graceful labeling is a complete 1-equitable labeling. Barrientos proves that $C_m \odot nK_1$ (that is, an $m$-cycle with $n$ pendant edges attached at each vertex) is optimal 2-equitable when $m$ is even; $C_3 \odot nK_1$ is complete 2-equitable when $n$ is odd; and that $C_3 \odot nK_1$ is complete 3-equitable for all $n$. He also shows that $C_n \odot K_1$ is $k$-equitable for every proper divisor $k$ of the size $2n$. Barrientos and Havia [201] have shown that the $n$-cube ($n \geq 2$) has a complete 2-equitable labeling and that $K_{m,n}$ has a complete 2-equitable labeling when $m$ or $n$ is even. They conjecture that every tree of even size has an optimal 2-equitable labeling.

### 3.9 Hamming-graceful Labelings

Mollard, Payan, and Shixin [879] introduced a generalization of graceful graphs called Hamming-graceful. A graph $G = (V, E)$ is called *Hamming-graceful* if there exists an injective labeling $g$ from $V$ to the set of binary $|E|$-tuples such that $\{d(g(v), g(u))|\ uv \in E\} = \{1, 2, \ldots, |E|\}$ where $d$ is the Hamming distance. Shixin and Yu [1084] have shown that all graceful graphs are Hamming-graceful; all trees are Hamming-graceful; $C_n$ is Hamming-graceful if and only if $n \equiv 0$ or 3 (mod 4); if $K_n$ is Hamming-graceful, then $n$ has the form $k^2$ or $k^2 + 2$; and $K_n$ is Hamming-graceful for $n = 2, 3, 4, 6, 9, 11, 16,$ and 18. They conjecture that $K_n$ is Hamming-graceful for $n$ of the forms $k^2$ and $k^2 + 2$ for $k \geq 5$.

## 4 Variations of Harmonious Labelings

### 4.1 Sequential and Strongly $c$-harmonious Labelings

Chang, Hsu, and Rogers [295] and Grace [480], [481] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling $f$ of a graph $G$ with $q$ vertices to be *strongly $c$-harmonious* if the vertex labels are from $\{0, 1, \ldots, q-1\}$ and the edge labels induced by $f(x) + f(y)$ for each edge $xy$ are $c, \ldots, c + q - 1$. Grace called such a labeling *sequential*. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices whereas Grace allows the vertex labels to range from 0 to $q$ with no vertex label being used twice. For graphs other than trees, we use the term $k$-sequential labelings interchangeably with strongly $k$-harmonious labelings. By taking the edge labels of a sequentially labeled graph with $q$ edges modulo $q$, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [481] proved that caterpillars, caterpillars with a pendant edge, odd cycles with zero or more pendant edges, trees with $\alpha$-labelings, wheels $W_{2n+1}$, and $P_n^2$ are sequential. Liu and Zhang [811] finished off the crowns $C_{2n} \odot K_1$. (The case $C_{2n+1} \odot K_1$ was a special case of Grace's results. Liu [808] proved crowns are harmonious.) Bu [258] also proved that crowns are sequential as are all even cycles with $m$ pendant edges attached at each vertex. Figueroa-Centeno, Ichishima, and Muntaner-Batle [414] proved that all cycles with $m$ pendant edges attached at each vertex are sequential. Wu [1266] has shown that caterpillars with $m$ pendant edges attached at each vertex are sequential.

Singh has proved the following: $C_n \odot K_2$ is sequential for all odd $n > 1$ [1091]; $C_n \odot P_3$ is sequential for all odd $n$ [1092]; $K_2 \odot C_n$ (each vertex of the cycle is joined by edges to the end

points of a copy of $K_2$) is sequential for all odd $n$ [1092]; helms $H_n$ are sequential when $n$ is even [1092]; and $K_{1,n}+K_2$, $K_{1,n}+\overline{K}_2$, and ladders are sequential [1094]. Santhosh [991] has shown that $C_n \odot P_4$ is sequential for all odd $n \geq 3$. Both Grace [480] and Reid (see [445]) have found sequential labelings for the books $B_{2n}$. Jungreis and Reid [621] have shown the following graphs are sequential: $P_m \times P_n \quad (m,n) \neq (2,2)$; $C_{4m} \times P_n \quad (m,n) \neq (1,2)$; $C_{4m+2} \times P_{2n}$; $C_{2m+1} \times P_n$; and $C_4 \times C_{2n} \quad (n>1)$. The graphs $C_{4m+2} \times C_{2n+1}$ and $C_{2m+1} \times C_{2n+1}$ fail to satisfy a necessary parity condition given by Graham and Sloane [484]. The remaining cases of $C_m \times P_n$ and $C_m \times C_n$ are open. Gallian, Prout, and Winters [446] proved that all graphs $C_n \times P_2$ with a vertex or an edge deleted are sequential.

Gnanajothi [471, pp. 68–78] has shown the following graphs are sequential: $K_{1,m,n}$; $mC_n$, the disjoint union of $m$ copies of $C_n$ if and only if $m$ and $n$ are odd; books with triangular pages or pentagonal pages; and books of the form $B_{4n+1}$, thereby answering a question and proving a conjecture of Gallian and Jungreis [445]. Sun [1149] has also proved that $B_n$ is sequential if and only if $n \not\equiv 3 \pmod 4$. Ichishima and Oshima [574] pose determining whether or not $mK_{s,t}$ is sequential as a problem.

Yuan and Zhu [1316] have shown that $mC_n$ is sequential when $m$ and $n$ are odd. Although Graham and Sloane [484] proved that the Möbius ladder $M_3$ is not harmonious, Gallian [441] established that all other Möbius ladders are sequential (see §2.3 for the definition of Möbius ladder). Chung, Hsu, and Rogers [295] have shown that $K_{m,n}+K_1$, which includes $S_m+K_1$, is sequential. Seoud and Youssef [1018] proved that if $G$ is sequential and has the same number of edges as vertices, then $G+\overline{K_n}$ is sequential for all $n$. Recall that $\Theta(C_m)^n$ denotes the book with $n$ $m$-polygonal pages. Lu [826] proved that $\Theta(C_{2m+1})^{2n}$ is $2mn$-sequential for all $n$ and $m=1,2,3,4$ and $\Theta(C_m)^2$ is $(m-2)$-sequential if $m \geq 3$ and $m \equiv 2,3,4,7 \pmod 8$.

Zhou and Yuan [1329] have shown that for every $k$-sequential graph $G$ with $p$ vertices and $q$ edges and any positive integer $m$ the graph $(G+\overline{K_m})+\overline{K_n}$ is also $k$-sequential when $q-p+1 \leq m \leq q-p+k$. Zhou [1328] has shown that the analogous results hold for strongly $k$-harmonious and strongly $k$-elegant graphs. Zhou and Yuan [1329] have shown that for every $k$-sequential graph $G$ with $p$ vertices and $q$ edges and any positive integer $m$ the graph $(G+\overline{K_m})+\overline{K_n}$ is $k$-sequential when $q-p+1 \leq m \leq q-p+k$.

Shee [743] proved that every graph is a subgraph of a sequential graph. Acharya, Germina, Princy, and Rao [23] prove that every connected graph can be embedded in a strongly $c$-harmonious graph for some $c$. Lu [825] provides three techniques for constructing larger sequential graphs from some smaller one: an attaching construction, an adjoining construction, and the join of two graphs. Using these, he obtains various families of sequential or strongly $k$-indexable graphs.

Youssef [1308] observed that a strongly $c$-harmonious graph with $q$ edges is $k$-cordial for all $k \geq q$ and a strongly $k$-indexable graph is $k$-cordial for every $k$. The converse of this latter result is not true.

In [571] Ichishima and Oshima show that the hypercube $Q_n$ $(n \geq 2)$ is sequential if and only if $n \geq 4$. They also introduce a special kind of sequential labeling of a graph $G$ with size $2t+s$ by defining a sequential labeling $f$ to be a *partitional labeling* if $G$ is bipartite with partite sets $X$ and $Y$ of the same cardinality $s$ such that $f(x) \leq t+s-1$ for all $x \in X$ and $f(y) \geq t-s$ for all $y \in Y$, and there is a positive integer $m$ such that the induced edge labels are partitioned into three sets $[m, m+t-1]$, $[m+t, m+t+s-1]$, and $[m+t+s, m+2t+s-1]$ with the properties that there is an involution $\pi$, which is an automorphism of $G$ such that $\pi$ exchanges $X$ and $Y$, $x\pi(x) \in E(G)$ for all $x \in X$, and $\{f(x)+f(\pi(x)) |\ x \in X\} = [m+t, m+t+s-1]$. They prove if

$G$ has a partitional labeling, then $G \times Q_n$ has a partitional labeling for every nonnegative integer $n$. Using this together with existing results and the fact that every graph that has a partitional labeling is sequential, harmonious, and felicitous (see §4.5) they show that the following graphs are partitional, sequential, harmonious, and felicitous: for $n \geq 4$, hypercubes $Q_n$; generalized books $S_{2m} \times Q_n$; and generalized ladders $P_{2m+1} \times Q_n$.

In [572] Ichishma and Oshima proved the following: if $G$ is a partitional graph, then $G \times K_2$ is partitional, sequential, harmonious and felicitous; if $G$ is a connected bipartite graph with partite sets of distinct odd order such that in each partite set each vertex has the same degree, then $G \times K_2$ is not partitional; for every positive integer $m$, the book $B_m$ is partitional if and only if $m$ is even; the graph $B_{2m} \times Q_n$ is partitional if and only if $(m, n) \neq (1, 1)$; the graph $K_{m,2} \times Q_n$ is partitional if and only if $(m, n) \neq (2, 1)$; for every positive integer $n$, the graph $K_{m,3} \times Q_n$ is partitional when $m = 4, 8, 12$, or 16. As open problems they ask which $m$ and $n$ is $K_{m,n} \times K_2$ partitional and for which $l, m$ and $n$ is $K_{l,m} \times Q_n$ partitional?

Ichishma and Oshima [572] also investigated the relationship between partitional graphs and strongly graceful graphs (see §3.1 for the definition) and partitional graphs and strongly felicitous graphs (see §4.5 for the definition). They proved the following. If $G$ is a partitional graph, then $G \times K_2$ is partitional, sequential, harmonious and felicitous. Assume that $G$ is a partitional graph of size $2t+s$ with partite sets $X$ and $Y$ of the same cardinality $s$, and let $f$ be a partitional labeling of $G$ such that $\lambda_1 = \max\{f(x) : x \in X\}$ and $\lambda_2 = \max\{f(y) : y \in Y\}$. If $\lambda_1 + 1 = m + 2t + s - \lambda_2$, where $m = \min\{f(x) + f(y) : xy \in E(G)\} = \min\{f(y) : y \in Y\}$, then $G$ has a strong $\alpha$-valuation. Assume that $G$ is a partitional graph of size $2t + s$ with partite sets $X$ and $Y$ of the same cardinality $s$, and let $f$ be a partitional labeling of $G$ such that $\lambda_1 = \max\{f(x) : x \in X\}$ and $\lambda_2 = \max\{f(y) : y \in Y\}$. If $\lambda_1 + 1 = m + 2t + s - \lambda_2$, where $m = \min\{f(x) + f(y) : xy \in E(G)\} = \min\{f(y) : y \in Y\}$, then $G$ is strongly felicitous. Assume that $G$ is a partitional graph of size $2t + s$ with partite sets $X$ and $Y$ of the same cardinality $s$, and let $f$ be a partitional labeling of $G$ such that $\mu_1 = f(x_1) = \min\{f(x) : x \in X\}$ and $\mu_2 = f(y_1) = \min\{f(y) : y \in Y\}$. If $t + s = m + 1$ and $\mu_1 + \mu_2 = m$, where $m = \min\{f(x) + f(y) : xy \in E(G)\}$ and $x_1y_1 \in E(G)$, then $G$ has a strong $\alpha$-valuation and strongly felicitous labeling.

Singh and Varkey [1097] call a graph with $q$ edges *odd sequential* if the vertices can be labeled with distinct integers from the set $\{0, 1, 2, \ldots, q\}$ or, in the case of a tree, from the set $\{0, 1, 2, \ldots, 2q-1\}$, such that the edge labels induced by addition of the labels of the endpoints take on the values $\{1, 3, 5, \ldots, 2q-1\}$. They prove that combs, grids, stars, and rooted trees of level 2 are odd sequential whereas odd cycles are not. Singh and Varkey call a graph $G$ *bisequential* if both $G$ and its line graph have a sequential labeling. They prove paths and cycles are bisequential.

Among the strongly 1-harmonious (also called *strongly harmonious*) graphs are: fans $F_n$ with $n \geq 2$ [295]; wheels $W_n$ with $n \not\equiv 2 \pmod 3$ [295]; $K_{m,n} + K_1$ [295]; French windmills $K_4^{(t)}$ [559], [627]; the friendship graphs $C_3^{(n)}$ if and only if $n \equiv 0$ or $1 \pmod 4$ [559], [627], [1283]; $C_{4k}^{(t)}$ [1150]; and helms [952].

Seoud, Diab, and Elsakhawi [1010] have shown that the following graphs are strongly harmonious: $K_{m,n}$ with an edge joining two vertices in the same partite set; $K_{1,m,n}$; the composition $P_n[P_2]$ (see §2.3 for the definition); $B(3, 2, m)$ and $B(4, 3, m)$ for all $m$ (see §2.4 for the notation); $P_n^2$ $(n \geq 3)$; and $P_n^3$ $(n \geq 3)$. Seoud et al. [1010] have also proved: $B_{2n}$ is strongly $2n$-harmonious; $P_n$ is strongly $\lfloor n/2 \rfloor$-harmonious; ladders $L_{2k+1}$ are strongly $(k+1)$-harmonious; and that if $G$ is strongly $c$-harmonious and has an equal number of vertices and edges, then $G + \overline{K_n}$ is also

strongly $c$-harmonious.

Sethuraman and Selvaraju [1048] have proved that the graph obtained by joining two complete bipartite graphs at one edge is graceful and strongly harmonious. They ask whether these results extend to any number of complete bipartite graphs.

For a graph $G(V, E)$ Gayathri and Hemalatha [458] define an *even sequential harmonious labeling* $f$ of $G$ as an injection from $V$ to $\{0, 1, 2, \ldots, 2|E|\}$ with the property that the induced mapping $f^+$ from $E$ to $\{2, 4, 6, \ldots, 2|E|\}$ defined by $f^+(uv) = f(u) + f(v)$ when $f(u) + f(v)$ is even, and $f^+(uv) = f(u) + f(v) + 1$ when $f(u) + f(v)$ is odd, is an injection. They prove the following have even sequential harmonious labelings (all cases are the nontrivial ones): $P_n, P_n^+, C_n (n \geq 3)$, triangular snakes, quadrilateral snakes, Möbius ladders, $P_m \times P_n$ $(m \geq 2, n \geq 2)$, $K_{m,n}$; crowns $C_m \odot K_1$, graphs obtained by joining the centers of two copies of $K_{1,n}$ by a path; banana trees (see §2.1), $P_n^2$, closed helms (see §2.2), $C_3 \odot nK_1 (n \geq 2)$; $D \odot K_{1,n}$ where $D$ is a dragon (see §2.2); $\langle K_{1,n} : m \rangle$ $(m, n \geq 2)$ (see §4.5); the wreath product $P_n * \overline{K_2}$ $(n \geq 2)$ (see §4.5); combs $P_n \odot K_1$; the one-point union of the end point of a path to a vertex of a cycle (tadpole); the one-point union of the end point of a tadpole and the center of a star; the graphs $PC_n$ obtained from $C_n = v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the cords $v_1 v_{n-1}, v_2 v_{n-2}, \ldots, v_{(n-2)/2}, v_{(n+2)/2}$ when $n$ is even and $v_1 v_{n-1}, v_2 v_{n-2}, \ldots, v_{(n-3)/2}, v_{(n+3)/2}$ when $n$ is odd (that is, cycles with a full set of cords); $P_m \cdot nK_1$; the one-point union of a vertex of a cycle and the center of a star; graphs obtained by joining the centers of two stars with an edge; graphs obtained by joining two disjoint cycles with an edge (dumbbells); graphs consisting of two even cycles of the same order sharing a common vertex with an arbitrary number of pendant edges attached at the common vertex (butterflies).

## 4.2 $(k, d)$-arithmetic Labelings

Acharya and Hegde [27] have generalized sequential labelings as follows. Let $G$ be a graph with $q$ edges and let $k$ and $d$ be positive integers. A labeling $f$ of $G$ is said to be $(k, d)$-*arithmetic* if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x) + f(y)$ for each edge $xy$ are $k, k + d, k + 2d, \ldots, k + (q - 1)d$. They obtained a number of necessary conditions for various kinds of graphs to have a $(k, d)$-arithmetic labeling. The case where $k = 1$ and $d = 1$ was called *additively graceful* by Hegde [521]. Hegde [521] showed: $K_n$ is additively graceful if and only if $n = 2, 3$, or 4; every additively graceful graph except $K_2$ or $K_{1,2}$ contains a triangle; and a unicyclic graph is additively graceful if and only if it is a 3-cycle or a 3-cycle with a single pendant edge attached. Jinnah and Singh [612] noted that $P_n^2$ is additively graceful. Hegde [522] proved that if $G$ is strongly $k$-indexable, then $G$ and $G + \overline{K_n}$ are $(kd, d)$-arithmetic. Acharya and Hegde [29] proved that $K_n$ is $(k, d)$-arithmetic if and only if $n \geq 5$ (see also [264]). They also proved that a graph with an $\alpha$-labeling is a $(k, d)$-arithmetic for all $k$ and $d$. Bu and Shi [264] proved that $K_{m,n}$ is $(k, d)$-arithmetic when $k$ is not of the form $id$ for $1 \leq i \leq n - 1$. For all $d \geq 1$ and all $r \geq 0$, Acharya and Hegde [27] showed the following: $K_{m,n,1}$ is $(d + 2r, d)$-arithmetic; $C_{4t+1}$ is $(2dt + 2r, d)$-arithmetic; $C_{4t+2}$ is not $(k, d)$-arithmetic for any values of $k$ and $d$; $C_{4t+3}$ is $((2t + 1)d + 2r, d)$-arithmetic; $W_{4t+2}$ is $(2dt + 2r, d)$-arithmetic; and $W_{4t}$ is $((2t + 1)d + 2r, d)$-arithmetic. They conjecture that $C_{4t+1}$ is $(2dt + 2r, d)$-arithmetic for some $r$ and that $C_{4t+3}$ is $(2dt + d + 2r, d)$-arithmetic for some $r$. Hegde and Shetty [537] proved the following: the generalized web $W(t, n)$ (see §2.2 for the definition) is $((n - 1)d/2, d)$-arithmetic and $((3n - 1)d/2, d)$-arithmetic for odd $n$; the join of the generalized web $W(t, n)$ with the center removed and $\overline{K_p}$ where $n$ is odd is $((n - 1)d/2, d)$-arithmetic; every $T_p$-tree (see §3.2

for the definition) with $q$ edges and every tree obtained by subdividing every edge of a $T_p$-tree exactly once is $(k+(q-1)d, d)$-arithmetic for all $k$ and $d$. Lu, Pan, and Li [828] proved that $K_{1,m} \cup K_{p,q}$ is $(k,d)$-arithmetic when $k > (q-1)d+1$ and $d > 1$.

Yu [1312] proved that a necessary condition for $C_{4t+1}$ to be $(k,d)$-arithmetic is that $k = 2dt+r$ for some $r \geq 0$ and a necessary condition for $C_{4t+3}$ to be $(k,d)$-arithmetic is that $k = (2t+1)d+2r$ for some $r \geq 0$. These conditions were conjectured by Acharya and Hegde [27]. Singh proved that the graph obtained by subdividing every edge of the ladder $L_n$ is $(5,2)$-arithmetic [1090] and that the ladder $L_n$ is $(n,1)$-arithmetic [1093]. He also proves that $P_m \times C_n$ is $((n-1)/2, 1)$-arithmetic when $n$ is odd [1093]. Acharya, Germina, and Anandavally [21] proved that the subdivision graph of the ladder $L_n$ is $(k,d)$-arithmetic if either $d$ does not divide $k$ or $k = rd$ for some $r \geq 2n$ and that $P_m \times P_n$ and the subdivision graph of the ladder $L_n$ are $(k,k)$-arithmetic if and only if $k$ is at least 3. Lu, Pan, and Li [828] proved that $S_m \cup K_{p,q}$ is $(k,d)$-arithmetic when $k > (q-1)d+1$ and $d > 1$.

A graph is called *arithmetic* if it is $(k,d)$-arithmetic for some $k$ and $d$. Singh and Vilfred [1099] showed that various classes of trees are arithmetic. Singh [1093] has proved that the union of an arithmetic graph and an arithmetic bipartite graph is arithmetic. He conjectures that the union of arithmetic graphs is arithmetic. He provides an example to show that the converse is not true.

Germina and Anandavally [465] investigated embedding of graphs in arithmetic graphs. They proved: every graph can be embedded as an induced subgraph of an arithmetic graph; every bipartite graph can be embedded in a $(k,d)$-arithmetic graph for all $k$ and $d$ such that $d$ does not divide $k$; and any graph containing an odd cycle cannot be embedded as an induced subgraph of a connected $(k,d)$-arithmetic with $k < d$.

## 4.3 $(k,d)$-indexable Labelings

Acharya and Hegde [27] call a graph with $p$ vertices and $q$ edges $(k,d)$-*indexable* if there is an injective function from $V$ to $\{0,1,2,\ldots,p-1\}$ such that the set of edge labels induced by adding the vertex labels is a subset of $\{k, k+d, k+2d, \ldots, k+q(d-1)\}$. When the set of edges is $\{k, k+d, k+2d, \ldots, k+q(d-1)\}$ the graph is said to be *strongly* $(k,d)$-*indexable*. A $(k,1)$-graph is more simply called $k$-*indexable* and strongly 1-indexable graphs are simply called *strongly indexable*. Notice that strongly indexable graphs are a stronger form of sequential graphs and for trees and unicyclic graphs the notions of sequential labelings and strongly $k$-indexable labelings coincide. Hegde and Shetty [542] have shown that the notions of $(1,1)$-strongly indexable graphs and super edge-magic total labelings (see §5.2) are equivalent.

Zhou [1328] has shown that for every $k$-indexable graph $G$ with $p$ vertices and $q$ edges the graph $(G + \overline{K_{q-p+k}}) + \overline{K_1}$ is strongly $k$-indexable. Acharaya and Hegde prove that the only nontrivial regular graphs that are strongly indexable are $K_2, K_3$, and $K_2 \times K_3$, and that every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. Acharya and Hegde [27] call a graph with $p$ vertices *indexable* if there is an injective labeling of the vertices with labels from $\{0,1,2,\ldots,p-1\}$ such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [75] who also proved that all trees are indexable. Bu and Shi [265] also proved that all trees are indexable and that all unicyclic graphs with the cycle $C_3$ are indexable. Hegde [522] has shown the following: every graph can be embedded as an induced subgraph of an indexable graph; if a connected graph

with $p$ vertices and $q$ edges ($q \geq 2$) is $(k,d)$-indexable, then $d \leq 2$; $P_m \times P_n$ is indexable for all $m$ and $n$; if $G$ is a connected $(1,2)$-indexable graph, then $G$ is a tree; the minimum degree of any $(k,1)$-indexable graph with at least two vertices is at most 3; a caterpillar with partite sets of orders $a$ and $b$ is strongly $(1,2)$-indexable if and only if $|a-b| \leq 1$; in a connected strongly $k$-indexable graph with $p$ vertices and $q$ edges, $k \leq p-1$; and if a graph with $p$ vertices and $q$ edges is $(k,d)$-indexable, then $q \leq (2p-3-k+d)/d$. As a corollary of the latter, it follows that $K_n$ ($n \geq 4$) and wheels are not $(k,d)$-indexable.

Lee and Lee [697] provide a way to construct a $(k,d)$-strongly indexable graph from two given $(k,d)$-strongly indexable graphs. Lee and Lo [721] show that every given (1,2)-strongly indexable spider can extend to an (1,2)-strongly indexable spider with arbitrarily many legs.

Hegde and Shetty [541] also prove that if $G$ is strongly $k$-indexable Eulerian graph with $q$ edges then $q \equiv 0,3 \pmod 4$ if $k$ is even and $q \equiv 0,1 \pmod 4$ if $k$ is odd. They further showed how strongly $k$-indexable graphs can be used to construct polygons of equal internal angles with sides of different lengths.

Germina [462] has proved the following: fans $P_n + K_1$ are strongly indexable if and only if $n = 1,2,3,4,5,6$; $P_n + K_2$ is strongly indexable if and only if $n \leq 2$; the only strongly indexable complete $m$-partite graphs are $K_{1,n}$ and $K_{1,1,n}$; ladders $P_n \times P_2$ are $\lceil \frac{n}{2} \rceil$-strongly indexable, if $n$ is odd; $K_n \times P_k$ is a strongly indexable if and only if $n = 3$; $C_m \times P_n$ is 2-strongly indexable if $m$ is odd and $n \geq 2$; $K_{1,n} + K_i$ is not strongly indexable for $n \geq 2$; for $G_i \cong K_{1,n}$, $1 \leq i \leq n$, the sequential join $G \cong (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$ is strongly indexable if and only if, either $i = n = 1$ or $i = 2$ and $n = 1$ or $i = 1, n = 3$; $P_1 \cup P_n$ is strongly indexable if and only if $n \leq 3$; $P_2 \cup P_n$ is not strongly indexable; $P_2 \cup P_n$ is $\lceil \frac{n+3}{2} \rceil$-strongly indexable; $mC_n$ is $k$-strongly indexable if and only if $m$ and $n$ are odd; $K_{1,n} \cup K_{1,n+1}$ is strongly indexable; and $mK_{1,n}$ is $\lceil \frac{3m-1}{2} \rceil$-strongly indexable when $m$ is odd.

Acharya and Germina [18] proved that every graph can be embedded in a strongly indexable graph and gave an algorithmic characterization of strongly indexable unicyclic graphs. In [19] they provide necessary conditions for an Eulerian graph to be strongly $k$-indexable and investigate strongly indexable $(p,q)$-graphs for which $q = 2p-3$.

Hegde and Shetty [537] proved that for $n$ odd the generalized web graph $W(t,n)$ with the center removed is strongly $(n-1)/2$-indexable. Hegde and Shetty [542] define a *level joined planar grid* as follows. Let $u$ be a vertex of $P_m \times P_n$ of degree 2. For every pair of distinct vertices $v$ and $w$ that do not have degree 4, introduce an edge between $v$ and $w$ provided that the distance from $u$ to $v$ equals the distance from $u$ to $w$. They prove that every level joined planar grid is strongly indexable. For any sequence of positive integers $(a_1, a_2, \ldots, a_n)$ Lee and Lee [712] show how to associate a strongly indexible $(1,1)$-graph. As a corollary, they obtain the aforementioned result Hegde and Shetty on level joined planar grids.

Section 5.2 of this survey includes a discussion of a labeling method called super edge-magic. In 2002 Hegde and Shetty [542] showed that a graph has a strongly $k$-indexable labeling if and only if it has a super edge-magic labeling.

## 4.4 Elegant Labelings

In 1981 Chang, Hsu, and Rogers [295] defined an *elegant labeling* $f$ of a graph $G$ with $q$ edges as an injective function from the vertices of $G$ to the set $\{0,1,\ldots,q\}$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{(q+1)}$ the resulting edge labels are distinct and nonzero. Note that in contrast to the definition of a harmonious labeling, for an elegant labeling

it is not necessary to make an exception for trees. Whereas the cycle $C_n$ is harmonious if and only if $n$ is odd, Chang et al. [295] proved that $C_n$ is elegant when $n \equiv 0$ or 3 (mod 4) and not elegant when $n \equiv 1 \pmod 4$. Chang et al. further showed that all fans are elegant and the paths $P_n$ are elegant for $n \not\equiv 0 \pmod 4$. Cahit [273] then showed that $P_4$ is the only path that is not elegant. Balakrishnan, Selvam, and Yegnanarayanan [181] have proved numerous graphs are elegant. Among them are $K_{m,n}$ and the $m$th-subdivision graph of $K_{1,2n}$ for all $m$. They prove that the bistar $B_{n,n}$ ($K_2$ with $n$ pendant edges at each endpoint) is elegant if and only if $n$ is even. They also prove that every simple graph is a subgraph of an elegant graph and that several families of graphs are not elegant. Deb and Limaye [348] have shown that triangular snakes (see §2.2 for the definition) are elegant if and only if the number of triangles is not equal to 3 (mod 4). In the case where the number of triangles is 3 (mod 4) they show the triangular snakes satisfy a weaker condition they call *semi-elegant* whereby the edge label 0 is permitted. In [349] Deb and Limaye define a graph $G$ with $q$ edges to be *near-elegant* if there is an injective function $f$ from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{(q+1)}$ the resulting edge labels are distinct and not equal to $q$. Thus, in a near-elegant labeling, instead of 0 being the missing value in the edge labels, $q$ is the missing value. Deb and Limaye show that triangular snakes where the number of triangles is 3 (mod 4) are near-elegant. For any positive integers $\alpha \le \beta \le \gamma$ where $\beta$ is at least 2, the *theta graph* $\theta_{\alpha,\beta,\gamma}$ consists of three edge disjoint paths of lengths $\alpha, \beta$, and $\gamma$ having the same end points. Deb and Limaye [349] provide elegant and near-elegant labelings for some theta graphs where $\alpha = 1, 2$, or 3. Seoud and Elsakhawi [1011] have proved that the following graphs are elegant: $K_{1,m,n}$; $K_{1,1,m,n}$; $K_2 + \overline{K_m}$; $K_3 + \overline{K_m}$; and $K_{m,n}$ with an edge joining two vertices of the same partite set. Sethuraman and Elumalai [1031] have proved that for every graph $G$ with $p$ vertices and $q$ edges the graph $G + K_1 + \overline{K_m}$ is graceful when $m \ge 2^p - p - q$.

Sethuraman and Elumalai [1031] proved that every graph is a vertex induced subgraph of a elegant graph and present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a strong form of elegant labeling. Acharya, Germina, Princy, and Rao [23] prove that every $(p, q)$-graph $G$ can be embedded in a connected elegant graph $H$. The construction is done in such a way that if $G$ is planar and elegant (harmonious), then so is $H$.

In [1030] Sethuraman and Elumalai define a graph $H$ to be a $K_{1,m}$-*star extension* of a graph $G$ with $p$ vertices and $q$ edges at a vertex $v$ of $G$ where $m > p - 1 - deg(v)$ if $H$ is obtained from $G$ by merging the center of the star $K_{1,m}$ with $v$ and merging $p - 1 - deg(v)$ pendent vertices of $K_{1,m}$ with the $p - 1 - deg(v)$ nonadjacent vertices of $v$ in $G$. They prove that for every graph $G$ with $p$ vertices and $q$ edges and for every vertex $v$ of $G$ and every $m \ge 2^{p-1} - 1 - q$, there is a $K_{1,m}$-star extension of $G$ that is both graceful and harmonious. In the case where $m \ge 2^{p-1} - q$, they show that $G$ has a $K_{1,m}$-star extension that is elegant. Sethuraman and Selvaraju [1049] have shown that certain cases of the union of any number of copies of $K_4$ with one or more edges deleted and one edge in common are elegant.

Gallian extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let $G$ be a graph with $q$ edges and $H$ a finite Abelian group (under addition) of order $q$. Define $G$ to be $H$-*harmonious* if there is an injection $f$ from the vertices of $G$ to $H$ such that when each edge $xy$ is assigned the label $f(x) + f(y)$ the resulting edge labels are distinct. When $G$ is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley, and Jungreis [206] have shown that if $H$ is a finite Abelian group of order $n > 1$ then $C_n$ is $H$-harmonious if and only if $H$ has a non-cyclic or trivial Sylow 2-subgroup and $H$ is not of the form $Z_2 \times Z_2 \times \cdots \times Z_2$.

Thus, for example, $C_{12}$ is not $Z_{12}$-harmonious but is $(Z_2 \times Z_2 \times Z_3)$-harmonious. Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let $G$ be a graph with $q$ edges and $H$ a finite Abelian group (under addition) with $q+1$ elements. We say $G$ is *$H$-elegant* if there is an injection $f$ from the vertices of $G$ to $H$ such that when each edge $xy$ is assigned the label $f(x)+f(y)$ the resulting set of edge labels is the non-identity elements of $H$. Beals et al. [206] proved that if $H$ is a finite Abelian group of order $n$ with $n \neq 1$ and $n \neq 3$, then $C_{n-1}$ is $H$-elegant using only the non-identity elements of $H$ as vertex labels if and only if $H$ has either a non-cyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang, Hsu, and Rogers [295] by showing that $C_n$ is elegant when $n \equiv 2 \pmod 4$. Mollard and Payan [878] also proved that $C_n$ is elegant when $n \equiv 2 \pmod 4$ and gave another proof that $P_n$ is elegant when $n \neq 4$.

For a graph $G(V,E)$ and an Abelian group $H$ Valentin [1198] defines a *polychrome* labeling of $G$ by $H$ to be a bijection $f$ from $V$ to $H$ such that the edge labels induced by $f(uv) = f(v)+f(u)$ are distinct. Valentin investigates the existence of polychrome labelings for paths and cycles for various Abelian groups.

## 4.5 Felicitous Labelings

Another generalization of harmonious labelings are felicitous labelings. An injective function $f$ from the vertices of a graph $G$ with $q$ edges to the set $\{0,1,\ldots,q\}$ is called *felicitous* if the edge labels induced by $f(x)+f(y) \pmod q$ for each edge $xy$ are distinct. (Recall a harmonious labeling only allows the vertex labels $0,1,\ldots,q-1$.) This definition first appeared in a paper by Lee, Schmeichel, and Shee in [743] and is attributed to E. Choo. Acharya, Germina, Princy, and Rao [23] observed that every harmonious labeling of a graph is also a felicitous labeling of the graph. Balakrishnan and Kumar [178] proved the conjecture of Lee, Schmeichel, and Shee [743] that every graph is a subgraph of a felicitous graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitous are: $C_n$ except when $n \equiv 2 \pmod 4$ [743]; $K_{m,n}$ when $m,n > 1$ [743]; $P_2 \cup C_{2n+1}$ [743]; $P_2 \cup C_{2n}$ [1175]; $P_3 \cup C_{2n+1}$ [743]; $S_m \cup C_{2n+1}$ [743]; $K_n$ if and only if $n \leq 4$ [1032]; $P_n + \overline{K_m}$ [1032]; the friendship graph $C_3^{(n)}$ for $n$ odd [743]; $P_n \cup C_3$ [1058]; $P_n \cup C_{n+3}$ [1175]; and the one-point union of an odd cycle and a caterpillar [1058]. Shee [1054] conjectured that $P_m \cup C_n$ is felicitous when $n > 2$ and $m > 3$. Lee, Schmeichel, and Shee [743] ask for which $m$ and $n$ is the one-point union of $n$ copies of $C_m$ felicitous. They showed that in the case where $mn$ is twice an odd integer the graph is not felicitous. In contrast to the situation for felicitous labelings, we remark that $C_{4k}$ and $K_{m,n}$ where $m,n > 1$ are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious. Lee, Schmeichel, and Shee [743] conjectured that the $n$-cube is felicitous. This conjecture was proved by Figueroa-Centeno and Ichishima in 2001 [409].

Balakrishnan, Selvam, and Yegnanarayanan [180] obtained numerous results on felicitous labelings. The *wreath product*, $G * H$, of graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ and $(g_1,h_1)$ is adjacent to $(g_2,h_2)$ whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. They define $H_{n,n}$ as the graph with vertex set $\{u_1,\ldots,u_n;v_1,\ldots,v_n\}$ and edge set $\{u_iv_j|\ 1 \leq i \leq j \leq n\}$. They let $\langle K_{1,n} : m\rangle$ denote the graph obtained by taking $m$ disjoint copies of $K_{1,n}$, and joining a new vertex to the centers of the $m$ copies of $K_{1,n}$. They prove the following are felicitous: $H_{n,n}$; $P_n * \overline{K_2}$; $\langle K_{1,m} : m\rangle$; $\langle K_{1,2} : m\rangle$ when $m \not\equiv 0 \pmod 3$, or $m \equiv 3 \pmod 6$, or $m \equiv 6 \pmod{12}$; $\langle K_{1,2n} : m\rangle$ for all $m$ and $n \geq 2$; $\langle K_{1,2t+1} : 2n+1\rangle$ when $n \geq t$; $P_n^k$ when $k = n-1$ and $n \not\equiv 2$

(mod 4), or $k = 2t$ and $n \geq 3$ and $k < n - 1$; the join of a star and $\overline{K_n}$; and graphs obtained by joining two end vertices or two central vertices of stars with an edge. Yegnanarayanan [1298] conjectures that the graphs obtained from an even cycle by attaching $n$ new vertices to each vertex of the cycle is felicitous. This conjecture was verified by Figueroa-Centeno, Ichishima, and Muntaner-Batle in [414]. In [1045] Sethuraman and Selvaraju [1049] have shown that certain cases of the union of any number of copies of $K_4$ with 3 edges deleted and one edge in common are felicitous. Sethuraman and Selvaraju [1045] present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a felicitous labeling. Krisha and Dulawat [682] give algorithms for finding graceful, harmonious, sequential, felicitous, and antimagic (see §5.7) labelings of paths.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [415] define a felicitous graph to be *strongly felicitous* if there exists an integer $k$ so that for every edge $uv$, $\min\{f(u), f(v)\}$ $\leq k < \max\{f(u), f(v)\}$. For a graph with $p$ vertices and $q$ edges with $q \geq p - 1$ they show that $G$ is strongly felicitous if and only if $G$ has an $\alpha$-labeling (see §3.1). They also show that for graphs $G_1$ and $G_2$ with strongly felicitous labelings $f_1$ and $f_2$ the graph obtained from $G_1$ and $G_2$ by identifying the vertices $u$ and $v$ such that $f_1(u) = 0 = f_2(v)$ is strongly felicitous and that the one-point union of two copies of $C_m$ where $m \geq 4$ and $m$ is even is strongly felicitous. As a corollary they have that the one-point union of $n$ copies of $C_m$ where $m$ is even and at least 4 and $n \equiv 2 \pmod 4$ is felicitous. They conjecture that the one-point union of $n$ copies of $C_m$ is felicitous if and only if $mn \equiv 0, 1$, or 3 (mod 4). In [419] Figueroa-Centeno, Ichishima, and Muntaner-Batle prove that $2C_n$ is strongly felicitous if and only if $n$ is even and at least 4. They conjecture [419] that $mC_n$ is felicitous if and only if $mn \not\equiv 2 \pmod 4$ and that $C_m \cup C_n$ is felicitous if and only if $m + n \not\equiv 2 \pmod 4$.

As consequences of their results about super edge-magic labelings (see §5.2) Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [419] have the following corollaries: if $m$ and $n$ are odd with $m \geq 1$ and $n \geq 3$, then $mC_n$ is felicitous; $3C_n$ is felicitous if and only if $n \not\equiv 2 \pmod 4$; and $C_5 \cup P_n$ is felicitous for all $n$.

Chang, Hsu, and Rogers [295] have given a sequential counterpart to felicitous labelings. They call a graph with $q$ edges *strongly $c$-elegant* if the vertex labels are from $\{0, 1, \ldots, q\}$ and the edge labels induced by addition are $\{c, c+1, \ldots, c+q-1\}$. (A strongly 1-elegant labeling has also been called a *consecutive* labeling.) Notice that every strongly $c$-elegant graph is felicitous and that strongly $c$-elegant is the same as $(c, 1)$-arithmetic in the case where the vertex labels are from $\{0, 1, \ldots, q\}$. Chang et al. [295] have shown: $K_n$ is strongly 1-elegant if and only if $n = 2, 3, 4$; $C_n$ is strongly 1-elegant if and only if $n = 3$; and a bipartite graph is strongly 1-elegant if and only if it is a star. Shee [1055] has proved that $K_{m,n}$ is strongly $c$-elegant for a particular value of $c$ and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

Seoud and Elsakhawi [1013] have shown: $K_{m,n}$ $(m \leq n)$ with an edge joining two vertices of the same partite set is strongly $c$-elegant for $c = 1, 3, 5, \ldots, 2n + 2$; $K_{1,m,n}$ is strongly $c$-elegant for $c = 1, 3, 5, \ldots, 2m$ when $m = n$, and for $c = 1, 3, 5, \ldots, m + n + 1$ when $m \neq n$; $K_{1,1,m,m}$ is strongly $c$-elegant for $c = 1, 3, 5, \ldots, 2m + 1$; $P_n + \overline{K_m}$ is strongly $\lfloor n/2 \rfloor$-elegant; $C_m + \overline{K_n}$ is strongly $c$-elegant for odd $m$ and all $n$ for $c = (m-1)/2, (m-1)/2 + 2, \ldots, 2m$ when $(m-1)/2$ is even and for $c = (m-1)/2, (m-1)/2 + 2, \ldots, 2m - (m-1)/2$ when $(m-1)/2$ is odd; ladders $L_{2k+1}$ $(k > 1)$ are strongly $(k+1)$-elegant; and $B(3, 2, m)$ and $B(4, 3, m)$ (see §2.4 for notation) are strongly 1-elegant and strongly 3-elegant for all $m$; the composition $P_n[P_2]$ (see §2.3 for the definition) is strongly $c$-elegant for $c = 1, 3, 5, \ldots, 5n - 6$ when $n$ is odd and for

$c = 1, 3, 5, \ldots, 5n-5$ when $n$ is even; $P_n$ is strongly $\lfloor n/2 \rfloor$-elegant; $P_n^2$ is strongly $c$-elegant for $c = 1, 3, 5, \ldots, q$ where $q$ is the number of edges of $P_n^2$; and $P_n^3$ $(n > 3)$ is strongly $c$-elegant for $c = 1, 3, 5, \ldots, 6k-1$ when $n = 4k$; $c = 1, 3, 5, \ldots, 6k+1$ when $n = 4k+1$; $c = 1, 3, 5, \ldots, 6k+3$ when $n = 4k+2$; $c = 1, 3, 5, \ldots, 6k+5$ when $n = 4k+3$.

### 4.6 Odd-harmonious Labelings

A function $f$ is said to be an *odd-harmonious* labeling of a $(p, q)$-graph $G$ if $f$ is an injection from the vertices of $G$ to the integers from 0 to $2q-1$ such that the induced mapping $f^*$ from the edges of $G$ to the odd integers between 1 to $2q-1$ is a bijection. A function $f$ is said to be an *strongly odd-harmonious* labeling of a $(p, q)$-graph $G$ if $f$ is an injection from the vertices of $G$ to the integers from 0 to $q$ such that the induced mapping $f^*$ from the edges of $G$ to the odd integers between 1 to $2q-1$ is a bijection. Liang and Bai [787] have show the following: odd-harmonious graphs are bipartite; if a $(p, q)$-graph is odd-harmonious, then $2\sqrt{q} \leq p \leq 2q-1$; if a $(p, q)$-graph with degree sequence $(d_1, d_2, \ldots, d_p)$ is odd-harmonious, then $\gcd(d_1, d_2, \ldots, d_p)$ divides $q^2$; $P_n (n > 1)$ is odd-harmonious and strongly odd-harmonious; $C_n$ is odd-harmonious if and only if $n \equiv 0 \bmod 4$; $K_n$ is odd-harmonious if and only if $n = 2$; $K_{n_1,n_2,\ldots,n_k}$ is odd-harmonious if and only if $k = 2$; $K_n^t$ is odd-harmonious if and only if $n = 2$; $P_m \times P_n$ is odd-harmonious; the tadpole graph obtained by identifying the endpoint of a path with a vertex of a cycle odd-harmonious if $n \equiv 0 \bmod 4$; the graph obtained by appending two or more pendant edges to each vertex of $C_{4n}$ is odd-harmonious; the graph obtained by subdividing every edge of the cycle of a wheel (gear graphs) is odd-harmonious; the graph obtained by appending an edge to each vertex of a strongly odd-harmonious graph is odd-harmonious; and caterpillars and lobsters are odd-harmonious. They conjecture that every tree is odd-harmonious.

## 5 Magic-type Labelings

### 5.1 Magic Labelings

Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedláček [998] in 1963. Responding to a problem raised by Sedláček, Stewart [1134] and [1135] studied various ways to label the edges of a graph in the mid 1960s. Stewart calls a connected graph *semi-magic* if there is a labeling of the edges with integers such that for each vertex $v$ the sum of the labels of all edges incident with $v$ is the same for all $v$. (Berge [213] used the term "regularisable" for this notion.) A semi-magic labeling where the edges are labeled with distinct positive integers is called a *magic* labeling. Stewart calls a magic labeling *supermagic* if the set of edge labels consists of consecutive positive integers. The classic concept of an $n \times n$ magic square in number theory corresponds to a supermagic labeling of $K_{n,n}$. Stewart [1134] proved the following: $K_n$ is magic for $n = 2$ and all $n \geq 5$; $K_{n,n}$ is magic for all $n \geq 3$; fans $F_n$ are magic if and only if $n$ is odd and $n \geq 3$; wheels $W_n$ are magic for $n \geq 4$; and $W_n$ with one spoke deleted is magic for $n = 4$ and for $n \geq 6$. Stewart [1134] also proved that $K_{m,n}$ is semi-magic if and only if $m = n$. In [1135] Stewart proved that $K_n$ is supermagic for $n \geq 5$ if and only if $n > 5$ and $n \not\equiv 0 \pmod 4$. Sedláček [999] showed that Möbius ladders $M_n$ (see §2.3 for the definition) are supermagic when $n \geq 3$ and $n$ is odd and that $C_n \times P_2$ is magic, but not supermagic, when $n \geq 4$ and $n$ is even. Shiu, Lam, and Lee [1069] have proved: the composition of $C_m$ and $\overline{K}_n$ (see §2.3 for the definition) is supermagic when $m \geq 3$ and $n \geq 2$;

the complete $m$-partite graph $K_{n,n,\ldots,n}$ is supermagic when $n \geq 3$, $m > 5$ and $m \not\equiv 0 \pmod 4$; and if $G$ is an $r$-regular supermagic graph, then so is the composition of $G$ and $\overline{K}_n$ for $n \geq 3$. Ho and Lee [548] showed that the composition of $K_m$ and $\overline{K_n}$ is supermagic for $m = 3$ or 5 and $n = 2$ or $n$ odd. Bača, Holländer, and Lih [132] have found two families of 4-regular supermagic graphs. Shiu, Lam, and Cheng [1066] proved that for $n \geq 2$, $mK_{n,n}$ is supermagic if and only if $n$ is even or both $m$ and $n$ are odd. Ivančo [577] gave a characterization of all supermagic regular complete multipartite graphs. He proved that $Q_n$ is supermagic if and only if $n = 1$ or $n$ is even and greater than 2 and that $C_n \times C_n$ and $C_{2m} \times C_{2n}$ are supermagic. He conjectures that $C_m \times C_n$ is supermagic for all $m$ and $n$. Trenklér [1181] has proved that a connected magic graph with $p$ vertices and $q$ edges other than $P_2$ exits if and only if $5p/4 < q \leq p(p-1)/2$. In [1151] Sun, Guan, and Lee give an efficient algorithm for finding a magic labeling of a graph. In [1255] Wen, Lee, and Sun show how to construct a supermagic multigraph from a given graph $G$ by adding extra edges to $G$.

In [678] Kovář provides a general technique for constructing supermagic labelings of copies of certain kinds of regular supermagic graphs. In particular, he proves: if $G$ is a supermagic $r$-regular graph ($r \geq 3$) with a proper edge $r$ coloring, then $nG$ is supermagic when $r$ is even and supermagic when $r$ and $n$ are odd; if $G$ is a supermagic $r$-regular graph with $m$ vertices and has a proper edge $r$ coloring and $H$ is a supermagic $s$-regular graph with $n$ vertices and has a proper edge $s$ coloring, then $G \times H$ is supermagic when $r$ is even or $n$ is odd and is supermagic when $s$ or $m$ is odd.

In [369] Drajnová, Ivančo, and Semaničová proved that the maximal number of edges in a supermagic graph of order $n$ is 8 for $n = 5$ and $\frac{n(n-1)}{2}$ for $6 \leq n \not\equiv 0 \pmod 4$, and $\frac{n(n-1)}{2} - 1$ for $8 \leq n \equiv 0 \pmod 4$. They also establish some bounds for the minimal number of edges in a supermagic graph of order $n$. Ivančo, and Semaničová [584] proved that every 3-regular triangle-free supermagic graph has an edge such that the graph obtained by contracting that edge is also supermagic and the graph obtained by contracting one of the edges joining the two $n$-cycles of $C_n \times K_2$ ($n \geq 3$) is supermagic.

Sedláček [999] proves that graphs obtained from an odd cycle with consecutive vertices $u_1, u_2, \ldots, u_m, u_{m+1}, v_m, \ldots, v_1$ ($m \geq 2$) by joining each $u_i$ to $v_i$ and $v_{i+1}$ and $u_1$ to $v_{m+1}$, $u_m$ to $v_1$ and $v_1$ to $v_{m+1}$ are magic. Trenklér and Vetchý [1184] have shown that if $G$ has order at least 5, then $G^n$ is magic for all $n \geq 3$ and $G^2$ is magic if and only if $G$ is not $P_5$ and $G$ does not have a 1-factor whose every edge is incident with an end-vertex of $G$. Avadayappan, Jeyanthi, and Vasuki [77] have shown that $k$-sequential trees are magic (see §4.1 for the definition). Seoud and Abdel Maqsoud [1005] proved that $K_{1,m,n}$ is magic for all $m$ and $n$ and that $P_n^2$ is magic for all $n$. However, Serverino has reported that $P_n^2$ is not magic for $n = 2, 3$, and 5 [467]. Jeurissan [591] characterized magic connected bipartite graphs. Ivančo [578] proved that bipartite graphs with $p \geq 8$ vertices, equal sized partite sets, and minimum degree greater than $p$ are magic. Bača [98] characterizes the structure of magic graphs that are formed by adding edges to a bipartite graph and proves that a regular connected magic graph of degree at least 3 remains magic if an arbitrary edge is deleted.

Ivančo [579] proved: the complement of a $d$-regular bipartite graph of order $8k$ is supermagic if and only if $d$ is odd; the complement of a $d$-regular bipartite graph of order $2n$ where $n$ is odd and $d$ is even is supermagic if and only if $(n, d) \neq (3, 2)$; if $G_1$ and $G_2$ are disjoint $d$-regular Hamiltonian graphs of odd order and $d \geq 4$ and even, then the join $G_1 \oplus G_2$ is supermagic; and if $G_1$ is $d$-regular Hamiltonian graph of odd order $n$, $G_2$ is $d-2$-regular Hamiltonian graph of order $n$ and $4 \leq d \equiv 0 \pmod 4$, then the join $G_1 \oplus G_2$ is supermagic.

A *prime-magic labeling* is a magic labeling for which every label is a prime. Sedláček [999] proved that the smallest magic constant for prime-magic labeling of $K_{3,3}$ is 53 while Bača and Holländer [128] showed that the smallest magic constant for a prime-magic labeling of $K_{4,4}$ is 114. Letting $\sigma_n$ be the smallest natural number such that $n\sigma_n$ is equal to the sum of $n^2$ distinct prime numbers we have that the smallest magic constant for a prime-magic labeling of $K_{n,n}$ is $\sigma_n$. Bača and Hollaänder [128] conjecture that for $n \geq 5$, $K_{n,n}$ has a prime-magic labeling with magic constant $\sigma_n$. They proved the conjecture for $5 \leq n \leq 17$ and confirm the conjecture for $n = 5$, 6 and 7.

Characterizations of regular magic graphs were given by Doob [368] and necessary and sufficient conditions for a graph to be magic were given in [591], [609], and [357]. Some sufficient conditions for a graph to be magic are given in [366], [1180], and [884]. The notion of magic graphs was generalized in [367] and [990].

Let $m, n, a_1, a_2, \ldots, a_m$ be positive integers where $1 \leq a_i \leq \lfloor n/2 \rfloor$ and the $a_i$ are distinct. The *circulant graph* $C_n(a_1, a_2, \ldots, a_m)$ is the graph with vertex set $\{v_1, v_2, \ldots, v_m\}$ and edge set $\{v_i v_{i+a_j} \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$ where addition of indices is done modulo $n$. In [1003] Semaničová characterizes magic circulant graphs and 3-regular supermagic circulant graphs. In particular, if $G = C_n(a_1, a_2, \ldots, a_m)$ has degree $r$ at least 3 and $d = \gcd(a_1, n/2)$ then $G$ is magic if and only if $r = 3$ and $n/d \equiv 2 \pmod 4$, $a_1/d \equiv 1 \pmod 2$, or $r \geq 4$ (a necessary condition for $C_n(a_1, a_2, \ldots, a_m)$ to be 3-regular is that $n$ is even). In the 3-regular case, $C_n(a_1, n/2)$ is supermagic if and only $n/d \equiv 2 \pmod 4$, $a_1/d \equiv 1 \pmod 2$ and $d \equiv 1 \pmod 2$. Semaničová also notes that a bipartite graph that is decomposable into an even number of Hamilton cycles is supermagic. As a corollary she obtains that $C_n(a_1, a_2, \ldots, a_{2k})$ is supermagic in the case that $n$ is even, every $a_i$ is odd, and $\gcd(a_{2j-1}, a_{2j}, n) = 1$ for $i = 1, 2, \ldots, 2k$ and $j = 1, 2, \ldots, k$.

Ivančo, Kovář, and Semaničová-Feňovčková [581] characterize all pairs $n$ and $r$ for which an $r$-regular supermagic graph of order $n$ exists. They prove that for positive integers $r$ and $n$ with $n \geq r+1$ there exists an $r$-regular supermagic graph of order $n$ if and only if one of the following statements holds: $r = 1$ and $n = 2$; $3 \leq r \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 4$; and $4 \leq r \equiv 0 \pmod 2$ and $n > 5$. The proof of the main result is based on finding supermagic labelings of circulant graphs. The authors construct supermagic labelings of several circulant graphs.

In [577] Ivančo completely determines the supermagic graphs that are the disjoint unions of complete $k$-partite graphs where every partite set has the same order.

Trenklér [1182] extended the definition of supermagic graphs to include hypergraphs and proved that the complete $k$-uniform $n$-partite hypergraph is supermagic if $n \neq 2$ or 6 and $k \geq 2$ (see also [1183]).

For connected graphs of size at least 5, Ivančo, Lastivkova, and Semaničová [583] provide a forbidden subgraph characterization of the line graphs that can be magic. As a corollary they obtain that the line graph of every connected graph with minimum degree at least 3 is magic. They also prove that the line graph of every bipartite regular graph of degree at least 3 is supermagic.

In 1976 Sedláček [999] defined a connected graph with at least two edges to be *pseudo-magic* if there exists a real-valued function on the edges with the property that distinct edges have distinct values and the sum of the values assigned to all the edges incident to any vertex is the same for all vertices. Sedláček proved that when $n \geq 4$ and $n$ is even, the Möbius ladder $M_n$ is not pseudo-magic and when $m \geq 3$ and $m$ is odd, $C_m \times P_2$ is not pseudo-magic.

Kong, Lee, and Sun [667] used the term "magic labeling" for a labeling of the edges with nonnegative integers such that for each vertex $v$ the sum of the labels of all edges incident with

$v$ is the same for all $v$. In particular, the edge labels need not be distinct. They let $M(G)$ denote the set of all such labelings of $G$. For any $L$ in $M(G)$, they let $s(L) = \max\{L(e): e \text{ in } E\}$ and define the *magic strength* of $G$ as $m(G) = \min\{s(L): L \text{ in } M(G)\}$. To distinguish these notions from others with the same names and notation, which we will introduced in the next section for labelings from the set of vertices and edges, we call the Kong, Lee, and Sun version the *edge magic strength* and use $em(G)$ for $\min\{s(L): L \text{ in } M(G)\}$ instead of $m(G)$. Kong, Lee, and Sun [667] use $DS(k)$ to denote the graph obtained by taking two copies of $K_{1,k}$ and connecting the $k$ pairs of corresponding leafs. They show: for $k > 1$, $em(DS(k)) = k - 1$; $em(P_k + K_1) = 1$ for $k = 1$ or 2, $em(P_k + K_1) = k$ if $k$ is even and greater than 2, and 0 if $k$ is odd and greater than 1; for $k \geq 3$, $em(W(k)) = k/2$ if $k$ is even and $em(W(k)) = (k-1)/2$ if $k$ is odd; $em(P_2 \times P_2) = 1$, $em(P_2 \times P_n) = 2$ if $n > 3$, $em(P_m \times P_n) = 3$ if $m$ or $n$ is even and greater than 2; $em(C_3^{(n)}) = 1$ if $n = 1$ (Dutch windmill, – see §2.4), and $em(C_3^{(n)}) = 2n - 1$ if $n > 1$. They also prove that if $G$ and $H$ are magic graphs then $G \times H$ is magic and $em(G \times H) = \max\{em(G), em(H)\}$ and that every connected graph is an induced subgraph of a magic graph (see also [390] and [412]). They conjecture that almost all connected graphs are not magic. In [740] Lee, Saba, and Sun show that the edge magic strength of $P_n^k$ is 0 when $k$ and $n$ are both odd. Sun and Lee [1152] show that the Cartesian, conjunctive, normal, lexicographic, and disjunctive products of two magic graphs are magic and the sum of two magic graphs is magic. They also determine the edge magic strengths of the products and sums in terms of the edge magic strengths of the components graphs.

In [50] Akka and Warad define the *super magic strength* of a graph $G$, $\text{sms}(G)$ as the minimum of all magic constants $c(f)$ where the minimum is taken over all super magic labeling $f$ of $G$ if there exist at least one such super magic labeling. They determine the super magic strength of paths, cycles, wheels, stars, bistars, $P_n^2$, $< K_{1,n} : 2 >$ (the graph obtained by joining the centers of two copies of $K_{1,n}$ by a path of length 2), and $(2n+1)P_2$.

A *Halin graph* Halin ia a planar 3-connected graphs that consist of a tree and a cycle connecting the end vertices of the tree. Let $G$ be a $(p,q)$-graph in which the edges are labeled $k, k+1, \ldots, k+q-1$, where $k \geq 0$. In [757] Lee, Su, and Wang define a graph with $p$ vertices to be *$k$-edge-magic* for every vertex $v$ the sum of the labels of the incident edges at $v$ are constant modulo $p$. They investigate some classes of Halin graphs that are $k$-edge-magic.

S. M. Lee and colleagues [777] and [714] call a graph $G$ *$k$-magic* if there is a labeling from the edges of $G$ to the set $\{1, 2, \ldots, k-1\}$ such that for each vertex $v$ of $G$ the sum of all edges incident with $v$ is a constant independent of $v$. The set of all $k$ for which $G$ is $k$-magic is denoted by $\text{IM}(G)$ and called the *integer-magic spectrum* of $G$. In [777] Lee and Wong investigate the integer-magic spectrum of powers of paths. They prove: $\text{IM}(P_4^2)$ is $\{4, 6, 8, 10, \ldots\}$; for $n > 5$, $\text{IM}(P_n^2)$ is the set of all positive integers except 2; for all odd $d > 1$, $\text{IM}(P_{2d}^d)$ is the set of all positive integers except 1; $\text{IM}(P_4^3)$ is the set of all positive integers; for all odd $n \geq 5$, $\text{IM}(P_n^3)$ is the set of all positive integers except 1 and 2; and for all even $n \geq 6$, $\text{IM}(P_n^3)$ is the set of all positive integers except 2. For $k > 3$ they conjecture: $\text{IM}(P_n^k)$ is the set of all positive integers when $n = k+1$; the set of all positive integers except 1 and 2 when $n$ and $k$ are odd and $n \geq k$; the set of all positive integers except 1 and 2 when $n$ and $k$ are even and $k \geq n/2$; the set of all positive integers except 2 when $n$ is even and $k$ is odd and $n \geq k$; and the set of all positive integers except 2 when $n$ and $k$ are even and $k \leq n/2$. In [755] Lee, Su, and Wang showed that besides the natural numbers there are two types of the integer-magic spectra of honeycomb graphs.

In [714] Lee, Lee, Sun, and Wen investigated the integer-magic spectrum of various graphs

such as stars, double stars (trees obtained by joining the centers of two disjoint stars $K_{1,m}$ and $K_{1,n}$ with an edge), wheels, and fans. In [983] Salehi and Bennett report that a number of the results of Lee et al. are incorrect and provide a detailed accounting of these errors as well as determine the integer-magic spectra of caterpillars.

Lee, Lee, Sun, and Wen [714] use the notation $C_m@C_n$ to denote the graph obtained by starting with $C_m$ and attaching paths $P_n$ to $C_m$ by identifying the endpoints of the paths with each successive pairs of vertices of $C_m$. They prove that $\mathrm{IM}(C_m@C_n)$ is the set of all positive integers if $m$ or $n$ is even and $\mathrm{IM}(C_m@C_n)$ is the set of all even positive integers if $m$ and $n$ are odd.

Lee, Valdés, and Ho [764] investigate the integer magic spectrum for special kinds of trees. For a given tree $T$ they define the *double tree* $DT$ of $T$ as the graph obtained by creating a second copy $T^*$ of $T$ and joining each end vertex of $T$ to its corresponding vertex in $T^*$. They prove that for any tree $T$, $\mathrm{IM}(DT)$ contains every positive integer with the possible exception of 2 and $\mathrm{IM}(DT)$ contains all positive integers if and only if the degree of every vertex that is not an end vertex is even. For a given tree $T$ they define $ADT$, the *abbreviated double tree of* $T$, as the the graph obtained from $DT$ by identifying the end vertices of $T$ and $T^*$. They prove that for every tree $T$, $\mathrm{IM}(ADT)$ contains every positive integer with the possible exceptions of 1 and 2 and $\mathrm{IM}(ADT)$ contains all positive integers if and only if $T$ is a path.

Lee, Salehi, and Sun [742] have investigated the integer-magic spectra of trees with diameter at most four. Among their findings are: if $n \geq 3$ and the prime power factorization of $n-1 = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, then $\mathrm{IM}(K_{1,n}) = p_1\mathbb{N}\cup p_2\mathbb{N}\cup\cdots\cup p_k\mathbb{N}$ (here $p_i\mathbb{N}$ means all positive integer multiples of $p_i$); for $m, n \geq 3$, the double star $\mathrm{IM}(DS(m,m))$ (that is, stars $K_{m,1}$ and $K_{n,1}$ that have an edge in common) is the set of all natural numbers excluding all divisors of $m-2$ greater than 1; if the prime power factorization of $m-n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ and the prime power factorization of $n-2 = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, (the exponents are permitted to be 0) then $\mathrm{IM}(DS(m,n)) = A_1\cup A_2\cup\cdots\cup A_k$ where $A_i = p_i^{1+s_i}\mathbb{N}$ if $r_i > s_i \geq 0$ and $A_i = \emptyset$ if $s_i \geq r_i \geq 0$; for $m, n \geq 3$, $\mathrm{IM}(DS(m,n)) = \emptyset$ if and only if $m-n$ divides $n-2$; if $m, n \geq 3$ and $|m-n| = 1$, then $DS(m,n)$ is not magic. Lee and Salehi [741] give formulas for the integer-magic spectra of trees of diameter four but they are too complicated to include here.

For a graph $G(V,E)$ and a function $f$ from the $V$ to the positive integers, Salehi and Lee [986] define the *functional extension of* $G$ *by* $f$, as the graph $H$ with $V(H) = \cup\{u_i|\ u \in V(G)$ and $i = 1, 2, \ldots, f(u)\}$ and $E(H) = \cup\{u_iu_j|\ uv \in E(G), i = 1, 2, \ldots, f(u); j = 1, 2, \ldots, f(v)\}$. They determine the integer-magic spectra for $P_2, P_3$ and $P_4$.

More specialized results about the integer-magic spectra of amalgamations of stars and cycles are given by Lee and Salehi in [741].

Table 4 summarizes the state of knowledge about magic-type labelings. In the table
**SM** means semi-magic
**M** means magic
**SPM** means supermagic.

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Table 4: **Summary of Magic Labelings**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $K_n$ | M | if $n = 2,\ n \geq 5$ [1134] |
| | SPM | for $n \geq 5$ iff $n > 5$ |
| | | $n \not\equiv 0 \pmod 4$ [1135] |
| $K_{m,n}$ | SM | if $n \geq 3$ [1134] |
| $K_{n,n}$ | M | if $n \geq 3$ [1134] |
| fans $f_n$ | M | iff $n$ is odd, $n \geq 3$ [1134] |
| | not SM | if $n \geq 2$ [467] |
| wheels $W_n$ | M | if $n \geq 4$ [1134] |
| | SM | if $n = 5$ or 6 [467] |
| wheels with one spoke deleted | M | if $n = 4$, $n \geq 6$ [1134] |
| Möbius ladders $M_n$ | SPM | if $n \geq 3$, $n$ is odd [999] |
| $C_n \times P_2$ | not SPM | for $n \geq 4$, $n$ even [999] |
| $C_m[\overline{K}_n]$ | SPM | if $m \geq 3$, $n \geq 2$ [1069] |
| $K_{\underbrace{n, n, \ldots, n}_{p}}$ | SPM | $n \geq 3$, $p > 5$ and |
| | | $p \not\equiv 0 \pmod 4$ [1069] |
| composition of $r$-regular SPM graph and $\overline{K}_n$ | SPM | if $n \geq 3$ [1069] |
| $K_k[\overline{K}_n]$ | SPM | if $k = 3$ or 5, $n = 2$ or $n$ odd [548] |

Table 4: **Summary of Magic Labelings continued**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $mK_{n,n}$ | SPM | for $n \geq 2$ iff $n$ is even or both $n$ and $m$ are odd [1066] |
| $Q_n$ | SPM | iff $n = 1$ or $n > 2$ even [577] |
| $C_m \times C_n$ | SPM | $m = n$ or $m$ and $n$ are even [577] |
| $C_m \times C_n$ | SPM? | for all $m$ and $n$ [577] |
| connected $(p, q)$-graph other than $P_2$ | M | iff $5p/4 < q \leq p(p-1)/2$ [1181] |
| $G^i$ | M | $\|G\| \geq 5$, $i \geq 3$ [1184] |
| $G^2$ | M | $G \neq P_5$ and $G$ does not have a 1-factor whose every edge is incident with an end-vertex of $G$ [1184] |
| $K_{1,m,n}$ | M | for all $m$, $n$ [1005] |
| $P_n^2$ | M | for all $n$ except 2, 3, 5 [1005], [467] |
| $G \times H$ | M | iff $G$ and $H$ are magic [667] |

## 5.2 Edge-magic Total and Super Edge-magic Total Labelings

In 1970 Kotzig and Rosa [673] defined a *magic valuation* of a graph $G(V,E)$ as a bijection $f$ from $V \cup E$ to $\{1,2,\ldots,|V \cup E|\}$ such that for all edges $xy$, $f(x)+f(y)+f(xy)$ is constant (called the *magic constant*). This notion was rediscovered by Ringel and Lladó [963] in 1996 who called this labeling *edge-magic*. To distinguish between this usage from that of other kinds of labelings that use the word magic we will use the term *edge-magic total* labeling as introduced by Wallis [1228] in 2001. (We note that for 2-regular graphs a vertex-magic total labeling is an edge-magic total labeling and vice versa.) Inspired by Kotzig-Rosa notion, Enomoto, Lladó, Nakamigawa, and Ringel [390] called a graph $G(V,E)$ with an edge-magic total labeling that has the additional property that the vertex labels are 1 to $|V|$ *super edge-magic total* labeling. Kotzig and Rosa proved: $K_{m,n}$ has an edge-magic total labeling for all $m$ and $n$; $C_n$ has an edge-magic total labeling for all $n \geq 3$ (see also [472], [970], [216], and [390]); and the disjoint union of $n$ copies of $P_2$ has an edge-magic total labeling if and only if $n$ is odd. They further state that $K_n$ has an edge-magic total labeling if and only if $n = 1,2,3,5$ or 6 (see [674], [341], and [390]) and ask whether all trees have edge-magic total labelings. Wallis, Baskoro, Miller, and Slamin [1232] enumerate every edge-magic total labeling of complete graphs. They also prove that the following graphs are edge-magic total: paths, crowns, complete bipartite graphs, and cycles with a single edge attached to one vertex. Enomoto, Llado, Nakamigana, and Ringel [390] prove that all complete bipartite graphs are edge-magic total. They also show that wheels $W_n$ are not edge-magic total when $n \equiv 3 \pmod 4$ and conjectured that all other wheels are edge-magic total. This conjecture was proved when $n \equiv 0,1 \pmod 4$ by Phillips, Rees, and Wallis [933] and when $n \equiv 6 \pmod 8$ by Slamin, Bača, Lin, Miller, and Simanjuntak [1103]. Fukuchi [438] verified all cases of the conjecture independently of the work of others. Slamin et al. further show that all fans are edge-magic total. Ringel and Llado [963] prove that a graph with $p$ vertices and $q$ edges is not edge-magic total if $q$ is even and $p+q \equiv 2 \pmod 4$ and each vertex has odd degree. Ringel and Llado conjecture that trees are edge-magic total. In [165] Babujee and Rao show that the path with $n$ vertices has an edge-magic total labeling with magic constant $(5n+2)/2$ when $n$ is even and $(5n+1)/2$ when $n$ is odd. For stars with $n$ vertices they provide an edge-magic total labeling with magic constant $3n$. In [398] Eshghi and Azimi discuss a zero-one integer programming model for finding edge-magic total labelings of large graphs.

Santhosh [994] proved that for $n$ odd and at least 3, the crown $C_n \odot P_2$ has an edge-magic total labeling with magic constant $(27n+3)/2$ and for $n$ odd and at least 3, $C_n \odot P_3$ has an edge-magic total labeling with magic constant $(39n+3)/2$.

Ahmad, Baig, and Imran [47] define a *zig-zag triangle* as the graph obtained from the path $x_1, x_2, \ldots, x_n$ by adding $n$ new vertices $y_1, y_2, \ldots, y_n$ and new edges $y_1x_1, y_nx_{n-1}$; $x_iy_i$ for $1 \leq i \leq n$; $y_ix_{i-1}y_ix_{i+1}$ for $2 \leq i \leq n-1$. They define a graph $Cb_n$ as one obtained from the path $x_1, x_2, \ldots, x_n$ adding $n-1$ new vertices $y_1, y_2, \ldots, y_{n-1}$ and new edges $y_ix_{i+1}$ for $1 \leq i \leq n-1$. The graph $Cb_n^*$ is obtained from the $Cb_n$ by joining a new edge $x_1y_1$. They prove that zig-zag triangles, graphs that are the disjoint union of a star and a banana tree, certain disjoint unions of stars, and for $n \geq 4$, $Cb_n^* \cup CB_{n-1}$ are super edge-magic total.

Beardon [208] extended the notion of edge-magic total to countable infinite graphs $G(V,E)$ (that is, $V \cup E$ is countable). His main result is that a countably infinite tree that processes an infinite simple path has a bijective edge-magic total labeling using the integers as labels. He asks whether all countably infinite trees have an edge-magic total labeling with the integers as labels and whether the graph with the integers as vertices and an edge joining every two distinct

vertices has a bijective edge-magic total labeling using the integers.

Cavenagh, Combe, and Nelson [292] investigate edge-magic total labelings of countably infinite graphs with labels from a countable Abelian group $A$. Their main result is that if $G$ is a countable graph that has an infinite set of mutually disjoint edges and $A$ is isomorphic to a countable subgroup of the real numbers under addition then for any $k$ in $A$ there is an edge-magic labeling of $G$ with elements from $A$ that has magic constant $k$.

Balakrishnan and Kumar [178] proved that the join of $\overline{K_n}$ and two disjoint copies of $K_2$ is edge-magic total if and only if $n = 3$. Yegnanarayanan [1299] has proved the following graphs have edge-magic total labelings: $nP_3$ where $n$ is odd; $P_n + K_1; P_n \times C_3$ $(n \geq 2)$; the crown $C_n \odot K_1$; and $P_m \times C_3$ with $n$ pendant vertices attached to each vertex of the outermost $C_3$. He conjectures that for all $n$, $C_n \odot \overline{K_n}$, the $n$-cycle with $n$ pendant vertices attached at each vertex of the cycle, and $nP_3$ have edge-magic total labelings. In fact, Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [419] have proved the stronger statement that for all $n \geq 3$, the corona $C_n \odot \overline{K_m}$ admits an edge-magic labeling where the set of vertex labels is $\{1, 2, \ldots, |V|\}$. Yegnanarayanan [1299] also introduces several variations of edge-magic labelings and provides some results about them. Kotzig [1230] provides some necessary conditions for graphs with an even number of edges in which every vertex has odd degree to have an edge-magic total labeling. Craft and Tesar [341] proved that an $r$-regular graph with $r$ odd and $p \equiv 4 \pmod 8$ vertices can not be edge-magic total. Wallis [1228] proved that if $G$ is an edge-magic total $r$-regular graph with $p$ vertices and $q$ edges where $r = 2^t s + 1$ $(t > 0)$ and $q$ is even, then $2^{t+2}$ divides $p$.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [413] have proved the following graphs are edge-magic total: $P_4 \cup nK_2$ for $n$ odd; $P_3 \cup nK_2$; $P_5 \cup nK_2$; $nP_i$ for $n$ odd and $i = 3, 4, 5$; $2P_n$; $P_1 \cup P_2 \cup \cdots \cup P_n$; $mK_{1,n}$; $C_m \odot nK_1$; $K_1 \odot nK_2$ for $n$ even; $W_{2n}$; $K_2 \times \overline{K}_n$, $nK_3$ for $n$ odd (the case $nK_3$ for $n$ even and larger than 2 is done in [852]); binary trees, generalized Petersen graphs (see also [898]), ladders (see also [1257]), books, fans, and odd cycles with pendant edges attached to one vertex.

In [419] Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima, investigate super edge-magic labelings of graphs with two components. Among their results are: $C_3 \cup C_n$ is super edge-magic if and only if $n \geq 6$ and $n$ is even; $C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$ and $n$ is odd; $C_5 \cup C_n$ is super edge-magic if and only if $n \geq 4$ and $n$ is even; if $m$ is even with $m \geq 4$ and $n$ is odd with $n \geq m/2 + 2$, then $C_m \cup C_n$ is super edge-magic; for $m = 6, 8$, or 10, $C_m \cup C_n$ is super edge-magic if and only if $n \geq 3$ and $n$ is odd; $2C_n$ is strongly felicitous if and only if $n \geq 4$ and $n$ is even (the converse was proved by Lee, Schmeichel, and Shee in [743]); $C_3 \cup P_n$ is super edge-magic for $n \geq 6$; $C_4 \cup P_n$ is super edge-magic if and only if $n \neq 3$; $C_5 \cup P_n$ is super edge-magic for $n \geq 4$; if $m$ is even with $m \geq 4$ and $n \geq m/2 + 2$ then $C_m \cup P_n$ is super edge-magic; $P_m \cup P_n$ is super edge-magic if and only $(m, n) \neq (2, 2)$ or(3,3); and $P_m \cup P_n$ is edge-magic if and only $(m, n) \neq (2, 2)$.

Enomoto, Llado, Nakamigawa, and Ringel [390] conjecture that if $G$ is a graph of order $n + m$ that contains $K_n$, then $G$ is not edge-magic total for $n \gg m$. Wijaya and Baskoro [1257] proved that $P_m \times C_n$ is edge-magic total for odd $n$ at least 3. Ngurah and Baskoro [898] state that $P_2 \times C_n$ is not edge-magic total. Hegde and Shetty [533] have shown that every $T_p$-tree (see §4.4 for the definition) is edge-magic total. Ngurah, Simanjuntak, and Baskoro [905] show that certain subdivisions of the star $K_{1,3}$ have edge-magic total labelings. Wallis [1228] proves that a cycle with one pendent edge is edge-magic total. In [1228] Wallis poses a large number of research problems about edge-magic total graphs.

For $n \geq 3$, López, Muntaner-Batle, and Rius-Font [819] let $S_n$ denote the set of all super edge-

magic total 1-regular labeled digraphs of order $n$ where each vertex takes the name of the label that has been assigned to it. For $\pi \in S_n$. they define a generalization of generalized Petersen graphs that they denote by $GGP(n;\pi)$, which consists of an outer $n$-cycle $x_0, x_1, \ldots, x_{n-1}, x_0$, a set of $n$-spokes $x_i y_i$, $0 \leq i \leq n-1$, and $n$ inner edges defined by $y_i y_{\pi(i)}, i = 0, \ldots, n-1$. Notice that, for the permutation $\pi$ defined by $\pi(i) = i + k \pmod n$ we have $GGP(n;\pi) = P(n;k)$. They define a second generalization of generalized Petersen graphs, $GGP(n;\pi_2, \ldots, \pi_m)$, as the graphs with vertex sets $\cup_{j=1}^{m}\{x_i^j : i = 0, \ldots, n-1\}$, an outer $n$-cycle $x_0^1, x_1^1, \ldots, x_{n-1}^1, x_0^1$, and inner edges $x_i^{j-1}x_i^j$ and $x_i^j x_{\pi_j(i)}^j$, for $j = 2, \ldots, m$, and $i = 0, \ldots, n-1$. Notice that, $GGP(n;\pi_2, \ldots, \pi_m) = P_m \times C_n$, when $\pi_j(i) = i + 1 \pmod n$ for every $j = 2, \ldots, m$. Among their results are the Petersen graphs are super edge-magic total; for each $m$ with $1 < l \leq m$ and $1 \leq k \leq 2$, the graph $GGP(5;\pi_2, \ldots, \pi_m)$, where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \sigma_k$, is super edge-magic total; for each $1 \leq k \leq 2$, the graph $P(5n; k + 5r)$ where $r$ is the smallest integer such that $k + 5r = 1 \pmod n$ is super edge-magic total.

In 1996 Erdős asked for $M(n)$, the maximum number of edges that an edge-magic total graph of order $n$ can have (see [341]). In 1999 Craft and Tesar [341] gave the bound $\lfloor n^2/4 \rfloor \leq M(n) \leq \lfloor n(n-1)/2 \rfloor$. For large $n$ this was improved by Pikhurko [936] in 2006 to $2n^2/7 + O(n) \leq M(n) \leq (0.489 + \cdots + o(1)n^2)$.

Enomoto, Lladó, Nakamigawa, and Muntaner-Batle [390] proved that a super edge-magic total graph $G(V, E)$ with $|V| \geq 4$ and with girth at least 4 has at most $2|V| - 5$ edges. They prove this bound is tight for graphs with girth 4 and 5 in [390] and [570].

In his Ph.D. thesis, Barrientos [192] introduced the following notion. Let $L_1, L_2, \ldots, L_h$ be ordered paths in the grid $P_r \times P_t$ that are maximal straight segments such that the end vertex of $L_i$ is the beginning vertex of $L_{i+1}$ for $i = 1, 2, \ldots h-1$. Suppose for some $i$ with $1 < i < h$ we have $V(L_i) = \{u_0, v_0\}$ where $u_0$ is the end vertex of $L_{i-1}$ and the beginning vertex of $L_i$ and $v_0$ is the end vertex of $L_1$ and the beginning vertex of $L_{i+1}$. Let $u \in V(L_{i-1}) - \{u_0\}$ and $v \in V(L_{i+1}) - \{v_0\}$. The replacement of the edge $u_0 v_0$ by a new edge $uv$ is called an *elementary transformation* of the path $P_n$. A tree is called a *path-like tree* if it can be obtained from $P_n$ by a sequence of elementary transformations on an embedding of $P_n$ in a 2-dimensional grid. In [144] Bača, Lin, and Muntaner-Batle proved that if $T_1, T_2, \ldots, T_m$ are path-like trees each of order $n \geq 4$ where $m$ is odd and at least 3, then $T_1 \cup T_2, \cup \cdots \cup T_m$ has a super edge-magic labeling. In [143] Bača, Lin, Muntaner-Batle and Rius-Font proved that the number of such trees grows at least exponentially with $m$. As an open problem Bača, Lin, Muntaner-Batle and Rius-Font ask if graphs of the form $T_1 \cup T_2 \cup \cdots \cup T_m$ where $T_1, T_2, \ldots, T_m$ are path-like trees each of order $n \geq 2$ and $m$ is even have a super edge-magic labeling. In [192] Barrientos proved that all path-like trees admit an $\alpha$-valuation. Using Barrientos's result, it is very easy to obtain that all path-like trees are a special kind of super edge-magic by using a super edge-magic labeling of the path $P_n$, and hence they are also super edge-magic. Furthermore in [7] Figueroa-Centeno at al. proved that if a tree is super edge-magic, then it is also harmonious. Therefore all path-like trees are also harmonious. In [817] López, Muntaner-Batle, and Rius-Font also use a variation of the Kronecker product of matrices in order to obtain lower bounds for the number of non isomorphic super edge-magic labeling of some types of path-like trees. As a corollary they obtain lower bounds for the number of harmonious labeling of the same type of trees.

Let $G = (V, E)$ be a $(p, q)$-linear forest. In [143] Bača, Lin, Muntaner-Batle, and Rius-Font call a labeling $f$ a *strong super edge-magic* labeling of $G$ and $G$ a *strong super edge-magic* graph if $f : V \cup E \to \{1, 2, \ldots, p+q\}$ with the extra property that if $uv \in E, u', v' \in V(G)$ and $d_G(u, u') = d_G(v, v') < +\infty$, then we have that $f(u) + f(v) = f(u') + f(v')$. In [49] Ahmad,

López, Muntaner-Batle, and Rius-Font define the concept of strong super edge-magic labeling of a graph with respect to a linear forest as follows. Let $G = (V, E)$ be a $(p, q)$-graph and let $F$ be any linear forest contained in $G$. A *strong super edge-magic labeling of $G$ with respect to $F$* is a super edge-magic labeling $f$ of $G$ with the extra property with if $uv \in E(F), u', v' \in V(F)$ and $d_F(u, u') = d_F(v, v') < +\infty$ then we have that $f(u) + f(v) = f(u') + f(v')$. If a graph $G$ admits a strong super edge-magic labeling with respect to some linear forest $F$, they say that $G$ is a *strong super edge-magic graph with respect to $F$*. They prove that if $m$ is odd and $G$ is an acyclic graph which is strong super edge-magic with respect to a linear forest $F$, then $mG$ is strong super edge-magic with respect to $F_1 \cup F_2 \cup \ldots \cup F_m$, where $F_i \simeq F$ for $i = 1, 2, \ldots, m$ and every regular caterpillar is strong super edge-magic with respect to its spine.

Noting that for a super edge-magic graph $G$ with $p$ vertices and $q$ edges, the magic constant $k$ is given by the formula: $k = (\sum_{u \in V} \deg(u) f(u) + \sum_{i=p+1}^{p+q} i)/q$, López, Muntaner-Batle and Rius-Font [818] define the set

$$S_G = \left\{ \frac{\sum_{u \in V} \deg(u) g(u) + \sum_{i=p+1}^{p+q} i}{q} : \text{ the function } g : V \to \{i\}_{i=1}^{p} \text{ is bijective} \right\}.$$

If $\lceil \min S_G \rceil \leq \lfloor \max S_G \rfloor$ then the *super edge-magic interval* of $G$ $I_G$ is the set

$$I_G = [\lceil \min S_G \rceil \lfloor \max S_G \rfloor] \cap \mathbb{N}.$$

The *super edge-magic set* $\sigma_G$ of $G$ is

$$\sigma_G = \{k \in I_G : \text{ there exists a super edge-magic labeling of } G \text{ with valence } k\}.$$

They call a graph $G$ *perfect super edge-magic if* $I_G = \sigma_G$. They show that the family of paths $P_n$ is a family of perfect super edge-magic graphs with $|I_{P_n}| = 1$ 1 if $n$ is even and $|I_{P_n}| = 2$ if $n$ is odd and raise the question of whether there is an infinite family $\mathfrak{F} = \{F_1, F_2, \ldots\}$ of graphs such that each member of the family is perfect super edge-magic and $\lim_{i \to +\infty} |I_{F_i}| = +\infty$. They show that graphs $G \cong C_{p^k} \bigodot \overline{K_n}$ where $p > 2$ is a prime is such a family.

McSorley and Trono [856] define a relaxed version of edge-magic total labelings of a graph as follows. An *edge-magic injection* $\mu$ of a graph $G$ is an injection $\mu$ from the set of vertices and edges of $G$ to the natural numbers such that for every edge $uv$ the sum $\mu(u) + \mu(v) + \mu(uv)$ is some constant $k_\mu$. They investigate $\kappa(G)$, the smallest $k_\mu$ among all edge-magic injections of a graph $G$. They determine $\kappa(G)$ in the cases that $G$ is $K_2, K_3, K_5, K_6$ (recall that these are the only complete graphs that have edge-magic total labelings), a path, a cycle, or certain types of trees. They also show that every graph has an edge-magic injection and give bounds for $\kappa(K_n)$.

Avadayappan, Vasuki, and Jeyanthi [78] define the *edge-magic total strength* of a graph $G$ as the minimum of all constants over all edge-magic total labelings of $G$. We denote this by $emt(G)$. They use the notation $< K_{1,n} : 2 >$ for the tree obtained from the bistar $B_{n,n}$ (the graph obtained by joining the center vertices of two copies of $K_{1,n}$ with an edge) by subdividing the edge joining the two stars. They prove: $emt(P_{2n}) = 5n+1$; $emt(P_{2n+1}) = 5n+3$; $emt(< K_{1,n} : 2 >) = 4n+9$; $emt(B_{n,n}) = 5n+6$; $emt((2n+1)P_2) = 9n+6$; $emt(C_{2n+1}) = 5n+4$; $emt(C_{2n}) = 5n+2$; $emt(K_{1,n}) = 2n+4$; $emt(P_n^2) = 3n$; and $emt(K_{n,m}) \leq (m+2)(n+1)$ where $n \leq m$. Using an analogous definition for super edge-magic total strength, Swaninathan and Jeyanthi [1166], [1166], [1167] provide results about the super edge-magic strength of trees, fire crackers, unicyclic graphs, and generalized theta graphs. Ngurah, Simanjuntak, and Baskoro [905] show that

certain subdivisions of the star $K_{1,3}$ have super edge-magic total labelings. In [390] Enomoto, Lladó, Nakamigawa and Ringel conjectured that all trees have a super edge-magic total labeling. Ichishima, Muntaner-Batle, and Rius-Font [569] have shown that any tree of order $p$ is contained in a tree of order at most $2p-3$ that has a super edge-magic total labeling.

In [143] Bača, Lin, Muntaner-Batle, and Rius-Font call a super edge-magic labeling $f$ of a linear forest $G$ of order $p$ and size $q$ satisfying $f: V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ with the additional property that if $uv \in E(G)$, $u'v' \notin E(G)$ and $d_G(u,u') = d_G(v,v') < \infty$, then $f(u)+f(v) = f(u')+f(v')$ a *strong super edge-magic labeling* of $G$. They use a generalization of the Kronecker product of matrices introduced by Figueroa-Centeno, Ichishima, Muntaner-Batle, and Rius-Font [421] to obtain an exponential lower bound for the number of non-isomorphic strong super edge-magic labelings of the graph $mP_n$, for $m$ odd and any $n$, starting from the strong super edge-magic labeling of $P_n$. They prove that the number of non-isomorphic strong super edge-magic labelings of the graph $mP_n$, $n \geq 4$, is at least $\frac{5}{2}2^{\lfloor \frac{m}{2} \rfloor}+1$ where $m \geq 3$ is an odd positive integer. This result allows them to generate an exponential number of non-isomorphic super edge-magic labelings of the forest $F \cong \bigcup_{j=1}^{m} T_j$, where each $T_j$ is a path-like tree of order $n$ and $m$ is an odd integer.

López, Muntaner-Batle, and Rius-Font [816] introduced a generalization of super edge-magic graphs called *super edge-magic models* and prove some results about them.

Yegnanarayanan and Vaidhyanathan [1300] use the term *nice (1,1) edge-magic labeling* for a super edge-magic total labeling. They prove: a super edge-magic total labeling $f$ of a $(p,q)$-graph $G$ satisfies $2\sum_{v \in V(G)} f(v)deg(v) \equiv 0 \bmod q$; if $G$ is $(p,q)$ $r$-regular graph $(r>1)$ with a super edge-magic total labeling then $q$ is odd and the magic constant is $(4p+q+3)/2$; every super edge-magic total labeling has at least two vertices of degree less than 4; fans $P_n + K_1$ are edge-magic total for all $n$ and super edge-magic total if and only if is is at most 6; books $B_n$ are edge-magic total for all $n$; a super edge-magic total $(p,q)$-graph with $q \geq p$ is sequential; a super edge-magic total tree is sequential; and a super edge-magic total tree is cordial.

Hegde and Shetty [539] (see also [538]) define the *maximum magic strength* of a graph $G$ as the maximum magic constant over all edge-magic total labelings of $G$. We use $eMt(G)$ to denote the maximum magic strength of $G$. Hegde and Shetty call a graph $G$ with $p$ vertices *strong magic* if $eMt(G) = emt(G)$; *ideal magic* if $1 \leq eMt(G) - emt(G) \leq p$; and *weak magic* if $eMt(G) - emt(G) > p$. They prove that for an edge-magic total graph $G$ with $p$ vertices and $q$ edges, $eMt(G) = 3(p+q+1) - emt(G)$. Using this result they obtain: $P_n$ is ideal magic for $n > 2$; $K_{1,1}$ is strong magic; $K_{1,2}$ and $K_{1,3}$ are ideal magic; and $K_{1,n}$ is weak magic for $n > 3$; $B_{n,n}$ is ideal magic; $(2n+1)P_2$ is strong magic; cycles are ideal magic; and the generalized web $W(t,3)$ (see §2.2 for the definition) with the central vertex deleted is weak magic.

Santhosh [994] has shown that for $n$ odd and at least 3, $eMt(C_n \odot P_2) = (27n+3)/2$ and for $n$ odd and at least 3, $(39n+3)/2 \leq eMt(C_n \odot P_2) \leq (40n+3)/2$. Moreover, he proved that for $n$ odd and at least 3 both $C_n \odot P_2$ and $C_n \odot P_3$ are weak magic. In [323] Chopra and Lee provide an number of families of super edge-magic graphs that are weak magic.

In [886] Murugan introduces the notions of *almost-magic labeling, relaxed-magic labeling, almost-magic strength*, and *relaxed-magic strength* of a graph. He determines the magic strength of Huffman trees and twigs of odd order and the almost-magic strength of $nP_2$ ($n$ is even) and twigs of even order. Also, he obtains a bound on the magic strength of the path-union $P_n(m)$ and on the relaxed-magic strength of $kS_n$ and $kP_n$.

Enomoto, Llado, Nakamigawa, and Ringel [390] call an edge-magic total labeling *super edge-magic* if the set of vertex labels is $\{1, 2, \ldots, |V|\}$ (Wallis [1228] calls these labelings *strongly edge-*

*magic*). They prove the following: $C_n$ is super edge-magic if and only if $n$ is odd; caterpillars are super edge-magic; $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$; and $K_n$ is super edge-magic if and only if $n = 1, 2$, or 3. They also prove that if a graph with $p$ vertices and $q$ edges is super edge-magic then, $q \leq 2p - 3$. In [838] MacDougall and Wallis study super edge-magic $(p, q)$-graphs where $q = 2p - 3$. Enomoto et al. [390] conjecture that every tree is super edge-magic. Lee and Shan [751] have verified this conjecture for trees with up to 17 vertices with a computer. Fukuchi, and Oshima, [440] have shown that if $T$ is a tree of order $n \geq 2$ such that $T$ has diameter greater than or equal to $n - 5$, then $T$ has a super edge-magic labeling.

Various classes of banana trees that have super edge-magic total labelings have been found by Swaminathan and Jeyanthi [1166] and Hussain, Baskoro, and Slamin [567]. In [43] Ahmad, Ali, and Baskoro [43] investigate the existence of super edge-magic labelings of subdivisions of banana trees and disjoint unions of banana trees. The pose three open problems.

Kotzig and Rosa's ([673] and [674]) proof that $nK_2$ is edge-magic total when $n$ is odd actually shows that it is super edge-magic. Kotzig and Rosa also prove that every caterpillar is super-edge magic. Figueroa-Centeno, Ichishima, and Muntaner-Batle prove the following: if $G$ is a bipartite or tripartite (super) edge-magic graph, then $nG$ is (super) edge-magic when $n$ is odd [416]; if $m$ is a multiple of $n+1$, then $K_{1,m} \cup K_{1,n}$ is super edge-magic [416]; $K_{1,2} \cup K_{1,n}$ is super edge-magic if and only if $n$ is a multiple of 3; $K_{1,m} \cup K_{1,n}$ is edge-magic if and only if $mn$ is even [416]; $K_{1,3} \cup K_{1,n}$ is super edge-magic if and only if $n$ is a multiple of 4 [416]; $P_m \cup K_{1,n}$ is super edge-magic when $m \geq 4$ [416]; $2P_n$ is super edge-magic if and only if $n$ is not 2 or 3; $K_{1,m} \cup 2nK_2$ is super edge-magic for all $m$ and $n$ [416]; $C_3 \cup C_n$ is super edge-magic if and only if $n \geq 6$ and $n$ is even [419] (see also [486]); $C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$ and $n$ is odd [419] (see also [486]); $C_5 \cup C_n$ is super edge-magic if and only if $n \geq 4$ and $n$ is even [419]; if $m$ is even and at least 6 and $n$ is odd and satisfies $n \geq m/2 + 2$, then $C_m \cup C_n$ is super edge-magic [419]; $C_4 \cup P_n$ is super edge-magic if and only if $n \neq 3$ [419]; $C_5 \cup P_n$ is super edge-magic if $n \geq 4$ [419]; if $m$ is even and at least 6 and $n \geq m/2 + 2$, then $C_m \cup P_n$ is super edge-magic [419]; and $P_m \cup P_n$ is super edge-magic if and only if $(m, n) \neq (2, 2)$ or (3,3) [419]. They [416] conjecture that $K_{1,m} \cup K_{1,n}$ is super edge-magic only when $m$ is a multiple of $n + 1$ and they prove that if $G$ is a super edge-magic graph with $p$ vertices and $q$ edges with $p \geq 4$ and $q \geq 2p - 4$, then $G$ contains triangles. In [419] Figueroa-Centeno et al. conjecture that $C_m \cup C_n$ is super edge-magic if and only if $m + n \geq 9$ and $m + n$ is odd. In [439] Fukuchi and Oshima describe a construction of super-edge-magic labelings of some families of trees with diameter 4. Salman, Ngurah, and Izzati [988] use $S_n^m$ $(n \geq 3)$ to denote the graph obtained by inserting $m$ vertices in every edge of the star $S_n$. They prove that $S_n^m$ is super edge-magic when $m = 1$ or 2.

Muntaner-Batle calls a bipartite graph with partite sets $V_1$ and $V_2$ *special super edge-magic* if is has a super edge-magic total labeling $f$ with the property that $f(V_1) = \{1, 2, \ldots, |V_1|\}$. He proves that a tree has a special super edge-magic labeling if and only if it has an $\alpha$-labeling (see §3.1 for the definition). Figueroa-Centeno, Ichishima, Muntaner-Batle, and Rius-Font [421] use matrices to generate edge-magic total labeling and define the concept of super edge-magic total labelings for digraphs. They prove that if $G$ is a graph with a super edge-magic total labeling then for every natural number $d$ there exists a natural number $k$ such that $G$ has a $(k, d)$-arithmetic labeling (see §4.2 for the definition). In [697] Lee and Lee prove that a graph is super edge-magic if and only if it is $(k, 1)$-strongly indexable (see §4.3 for the definition of $(k, d)$-strongly indexable graphs). They also provide a way to construct $(k, d)$-strongly indexable graphs from two given $(k, d)$-strongly indexable graphs. This allows them to obtain several existing results about super

edge-magic graphs as special cases of their constructions. Acharya and Germina [18] proved that the class of strongly indexable graphs is a proper subclass of super edge-magic graphs.

In [568] Ichishima, López, Muntaner-Batle and Rius-Font show how one can use the product $\otimes_h$ of super edge-magic 1-regular labeled digraphs and digraphs with harmonious, or sequential labelings to create new undirected graphs that have harmonious, sequential labelings or partitional labelings (see §4.1 for the definition). They define the product $\otimes_h$ as follows. Let $\overrightarrow{D} = (V, E)$ be a digraph with adjacency matrix $A(\overrightarrow{D}) = (a_{ij})$ and let $\Gamma = \{F_i\}_{i=1}^m$ be a family of $m$ digraphs all with the same set of vertices $V'$. Assume that $h : E \longrightarrow \Gamma$ is any function that assigns elements of $\Gamma$ to the arcs of $D$. Then the digraph $\overrightarrow{D} \otimes_h \Gamma$ is defined by $V(D \otimes_h \Gamma) = V \times V'$ and $((a_1, b_1), (a_2, b_2)) \in E(D \otimes_h \Gamma) \Longleftrightarrow [(a_1, a_2) \in E(D) \wedge (b_1, b_2) \in E(h(a_1, a_2))]$. An alternative way of defining the same product is through adjacency matrices, since one can obtain the adjacency matrix of $\overrightarrow{D} \otimes_h \Gamma$ as follows: if $a_{ij} = 0$ then $a_{ij}$ is multiplied by the $p' \times p'$ 0-square matrix, where $p' = |V'|$. If $a_{ij} = 1$ then $a_{ij}$ is multiplied by $A(h(i, j))$ where $A(h(i, j))$ is the adjacency matrix of the digraph $h(i, j)$. They prove the following. Let $\overrightarrow{D} = (V, E)$ be a harmonious $(p, q)$-digraph with $p \leq q$ and let $h$ be any function from $E$ to the set of all super edge-magic 1-regular labeled digraphs of order $n$, which we denote by $S_n$. Then the undirected graph $und(\overrightarrow{D} \otimes_h S_n)$ is harmonious. Let $\overrightarrow{D} = (V, E)$ be a sequential digraph and let $h : E \longrightarrow S_n$ be any function. Then $und(\overrightarrow{D} \otimes_h S_n)$ is sequential. Let $D$ be a partitional graph and let $h : E \longrightarrow S_n$ be any function, where $\overrightarrow{D} = (V, E)$ is the digraph obtained by orienting all edges from one stable set to the other one. Then

In [820] López, Muntaner-Batle and Rius-Font introduce the concept of $\{H_i\}_{i\in I}$-*super edge-magic decomposable* as follows: Let $G = (V, E)$ be any graph and let $\{H_i\}_{i\in I}$ be a set of graphs such that $G = \oplus_{i\in I} H_i$ (that is to say $G$ decomposes into the graphs in the set $\{H_i\}_{i\in I}$). Then we say that $G$ is $\{H_i\}_{i\in I}$-super edge-magic decomposable if there is a bijection $\beta : V \to [1, |V|]$ such that for each $i \in I$ the subgraph $H_i$ meets the following two requirements: (i) $\beta(V(H_i)) = [1, |V(H_i)|]$ and (ii) $\{\beta(a) + \beta(b) : \ ab \in E(H_i)\}$ is a set of consecutive integers. Such function $\beta$ is called an $\{H_i\}_{i\in I}$-*super edge-magic labeling* of $G$. When $H_i = H$ for every $i \in I$ we just use the notation $H$-super edge-magic decomposable labeling.

Among their results are the following. Let $G = (V, E)$ be a $(p, q)$-graph which is $\{H_1, H_2\}$-super edge-magic decomposable for a pair of graphs $H_1$ and $H_2$. Then $G$ is super edge-bimagic; Let $n$ be an even integer. Then the cycle $C_n$ is $(n/2)K_2$-super edge-magic decomposable if and only if $n \equiv 2 \pmod 4$. Let $n$ be odd. Then for any super edge-magic tree $T$ there exists a bipartite connected graph $G = G(T, n)$ such that $G$ is $(nT)$-super edge-magic decomposable. Let $G$ be a $\{H_i\}_{i\in I}$-super edge magic decomposable graph, where $H_i$ is an acyclic digraph for each $i \in I$. Assume that $\overrightarrow{G}$ is any orientation of $G$ and $h : E(\overrightarrow{G}) \to S_p$ is any function. Then $\text{und}(\overrightarrow{G} \otimes_h S_p)$ is $\{pH_i\}_{i\in I}$-super edge magic decomposable.

As a corollary of the last result they have that if $G$ is a 2-regular, (1-factor)-super edge-magic decomposable graph and $\overrightarrow{G}$ is any orientation of $G$ and $h : E(\overrightarrow{G}) \to S_p$ is any function, then $\text{und}(\overrightarrow{G} \otimes_h S_p)$ is a 2-regular, (1-factor)-super edge-magic decomposable graph. Moreover, if we denote the 1-factor of $G$ by $F$ then $pF$ is the 1-factor of $\text{und}(\overrightarrow{G} \otimes_h S_p)$.

They pose the following two open questions: Fix $p \in \mathbb{N}$. Find the maximum $r \in \mathbb{N}$ such that there is a $r$-regular graph of order $p$ which is $(p/2)K_2$-super edge-magic decomposable. Characterize the set of 2-regular graphs of order $n$, $n \equiv 2 \pmod 4$, such that each component has even order and admits an $(n/2)K_2$-super edge-magic decomposition.

In connection to open question 1 they prove: For all $r \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that there exists a $k$-regular bipartite graph $B(n)$, with $k > r$ and $|V(B(n))| = 2 \cdot 3^n$, such that $B(n)$ is

$(3^n K_2)$-super edge-magic decomposable.

Avadayappan, Jeyanthi, and Vasuki [77] define the *super magic strength* of a graph $G$ as $sm(G) = \min\{s(L)\}$ where $L$ runs over all super edge-magic labelings of $G$. They use the notation $< K_{1,n} : 2 >$ for the tree obtained from the bistar $B_{n,n}$ (the graph obtained by joining the center vertices of two copies of $K_{1,n}$ with an edge) by subdividing the edge joining the two stars. They prove: $sm(P_{2n}) = 5n + 1$; $sm(P_{2n+1}) = 5n + 3$; $sm(< K_{1,n} : 2 >) = 4n + 9$; $sm(B_{n,n}) = 5n + 6$; $sm((2n + 1)P_2) = 9n + 6$; $sm(C_{2n+1}) = 5n + 4$; $emt(C_{2n}) = 5n + 2$; $sm(K_{1,n}) = 2n + 4$; and $sm(P_n^2) = 3n$. Note that in each case the super magic strength of the graph is the same as its magic strength.

Santhosh and Singh [993] proved that $C_n \odot P_2$ and $C_n \odot P_3$ are super edge-magic for all odd $n \geq 3$ and prove for odd $n \geq 3$, $sm(C_n \odot P_2) = (15n + 3)/2$ and $(20n + 3) \leq sm(C_n \odot P_3) \leq (21n + 3)/2$.

In his Ph.D. thesis [487] Gray proves that $C_3 \cup C_n$ is super edge-magic if and only if $n \geq 6$ and $C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$. His computer search shows that $C_5 \cup 2C_3$ does not have a super edge-magic labeling.

In [1228] Wallis posed the problem of investigating the edge-magic properties of $C_n$ with the path of length $t$ attached to one vertex. Kim and Park [647] call such a graph an $(n, t)$-*kite*. They prove that an $(n, 1)$-kite is super edge-magic if and only if $n$ is odd and an $(n, 3)$-kite is super edge-magic if and only if $n$ is odd and at least 5. Park, Choi, and Bae [922] show that $(n, 2)$-kite is super edge-magic if and only if $n$ is even. Wallis [1228] also posed the problem of determining when $K_2 \cup C_n$ is super edge-magic. In [922] and [647] Park et al. prove that $K_2 \cup C_n$ is super edge-magic if and only if $n$ is even. Kim and Park [647] show that the graph obtained by attaching a pendant edge to a vertex of degree one of a star is super-edge magic and that a super edge-magic graph with edge magic constant $k$ and $q$ edges satisfies $q \leq 2k/3 - 3$.

Lee and Kong [710] use $\mathrm{St}(a_1, a_2, \ldots, a_n)$ to denote the disjoint union of the $n$ stars $\mathrm{St}(a_1)$, $\mathrm{St}(a_2)$, $\ldots$, $\mathrm{St}(a_n)$. They prove the following graphs are super edge-magic: $\mathrm{St}(m, n)$ where $n \equiv 0 \mod(m + 1)$; $\mathrm{St}(1, 1, n)$; $\mathrm{St}(1, 2, n)$; $\mathrm{St}(1, n, n)$; $\mathrm{St}(2, 2, n)$; $\mathrm{St}(2, 3, n)$; $\mathrm{St}(1, 1, 2, n)$ $(n \geq 2)$; $\mathrm{St}(1, 1, 3, n)$; $\mathrm{St}(1, 2, 2, n)$; and $\mathrm{St}(2, 2, 2, n)$. They conjecture that $\mathrm{St}(a_1, a_2, \ldots, a_n)$ is super edge-magic when $n > 1$ is odd.

In [837] MacDougall and Wallis investigate the existence of super edge-magic labelings of cycles with a chord. They use $C_v^t$ to denote the graph obtained from $C_v$ by joining two vertices that are distance $t$ apart in $C_v$. They prove: $C_{4m+1}^t$ $(m \geq 3)$ has a super edge-magic labeling for every $t$ except $4m - 4$ and $4m - 8$; $C_{4m}^t$ $(m \geq 3)$ has a super edge-magic labeling when $t \equiv 2 \mod 4$; and that $C_{4m+2}^t$ $(m > 1)$ has a super edge-magic labeling for all odd $t$ other than 5, and for $t = 2$ and 6. They pose the problem of what values of $t$ does $C_{2n}^t$ have a super edge-magic labeling.

Enomoto, Masuda, and Nakamigawa [391] have proved that every graph can be embedded in a connected super edge-magic graph as an induced subgraph. Slamin, Bača, Lin, Miller, Simanjuntak [1103] proved that the friendship graph consisting of $n$ triangles is super edge-magic if and only if $n$ is 3, 4, 5 or 7. Fukuchi proved [437] the generalized Petersen graph $P(n, 2)$ (see §2.7 for the definition) is super edge-magic if $n$ is odd and at least 3 while Xu, Yang, Xi, Haque, and Shen [1281] showed that $P(n, 3)$ is super edge-magic for odd $n$ is odd and at least 5. Baskoro and Ngurah [204] showed that $nP_3$ is super edge-magic for $n \geq 4$ and $n$ even.

Hegde and Shetty [542] showed that a graph is super edge-magic if and only if it is strongly $k$-indexable (see §4.1 for the definition). Figueroa-Centeno, Ichishima, and Muntaner-Batle [412] proved that a graph is super edge-magic if and only if it is strongly 1-harmonious and that every

super edge-magic graph is cordial. They also proved that $P_n^2$ and $K_2 \times C_{2n+1}$ are super edge-magic. In [413] Figueroa-Centeno et al. show that the following graphs are super edge-magic: $P_3 \cup kP_2$ for all $k$; $kP_n$ when $k$ is odd; $k(P_2 \cup P_n)$ when $k$ is odd and $n = 3$ or $n = 4$; and fans $F_n$ if and only if $n \leq 6$. They conjecture that $kP_2$ is not super edge-magic when $k$ is even. This conjecture has been proved by Z. Chen [314] who showed that $kP_2$ is super edge-magic if and only if $k$ is odd. Figueroa-Centeno et al. proved that the book $B_n$ is not super edge-magic when $n \equiv 1, 3, 7 \pmod 8$ and when $n = 4$. They proved that $B_n$ is super edge-magic for $n = 2$ and 5 and conjectured that for every $n \geq 5$, $B_n$ is super edge-magic if and only if $n$ is even or $n \equiv 5 \pmod 8$. Yuansheng, Yue, Xirong, and Xinhong [1317] proved this conjecture for the case that $n$ is even.

They prove that every tree with an $\alpha$-labeling is super edge-magic. Yokomura (see [390]) has shown that $P_{2m+1} \times P_2$ and $C_{2m+1} \times P_m$ are super edge-magic (see also [412]). In [414], Figueroa-Centeno et al. proved that if $G$ is a (super) edge-magic 2-regular graph, then $G \odot \overline{K}_n$ is (super) edge-magic and that $C_m \odot \overline{K}_n$ is super edge-magic. Fukuchi [436] shows how to recursively create super edge-magic trees from certain kinds of existing super edge-magic trees. Ngurah, Baskoro, and Simanjuntak [902] provide a method for constructing new (super) edge-magic graphs from existing ones. One of their results is that if $G$ has an edge-magic total labeling and $G$ has order $p$ and size $p$ or $p - 1$, then $G \odot nK_1$ has an edge-magic total labeling.

Lee and Lee [713] investigate the existence of total edge-magic labelings and super edge-magic labelings of unicylic graphs. They obtain a variety of positive and negative results and conjecture that all unicyclic are edge-magic total.

Shiu and Lee [1072] investigated edge labelings of multigraphs. Given a multigraph $G$ with $q$ edges they call a bijection from the set of edges of $G$ to $\{1, 2, \ldots, q\}$ with the property that for each vertex $v$ the sum of all edge labels incident to $v$ is a constant independent of $v$ a *supermagic* labeling of $G$. They use $K_2[n]$ to denote the multigraph consisting of $n$ edges joining 2 vertices and $mK_2[n]$ to denote the disjoint union of $m$ copies of $K_2[n]$. They prove that for $m$ and $n$ at least 2, $mK_2[n]$ is supermagic if and only if $n$ is even or if both $m$ and $n$ are odd.

In 1970 Kotzig and Rosa [673] defined the *edge-magic deficiency*, $\mu(G)$, of a graph $G$ as the minimum $n$ such that $G \cup nK_1$ is edge-magic total. If no such $n$ exists they define $\mu(G) = \infty$. In 1999 Figueroa-Centeno, Ichishima, and Muntaner-Batle [418] extended this notion to *super edge-magic deficiency*, $\mu_s(G)$, is the analogous way. They prove the following: $\mu_s(nK_2) = \mu(nK_2) = n - 1 \pmod 2$; $\mu_s(C_n) = 0$ if $n$ is odd; $\mu_s(C_n) = 1$ if $n \equiv 0 \pmod 4$; $\mu_s(C_n) = \infty$ if $n \equiv 2 \pmod 4$; $\mu_s(K_n) = \infty$ if and only if $n \geq 5$; $\mu_s(K_{m,n}) \leq (m-1)(n-1)$; $\mu_s(K_{2,n}) = n - 1$; and $\mu_s(F)$ is finite for all forests $F$. They also prove that if a graph $G$ has $q$ edges with $q/2$ odd, and every vertex is even, then $\mu_s(G) = \infty$ and conjecture that $\mu_s(K_{m,n}) \leq (m-1)(n-1)$. This conjecture was proved for $m = 3, 4$, and 5 by Hegde, Shetty, and Shankaran [543] using the notion of strongly $k$-indexable labelings.

In [171] Baig, Ahmad, Baskoro, and Simanjuntak provide an upper bound for the super edge-magic deficiency of a forest formed by paths, stars, combs, banana trees, and subdivisions of $K_{1,3}$. Baig, Baskoro, and Semaničová-Feňovčíková [172] investigate the super edge-magic deficiency of forests consisting of stars. Among their results are: a forest consisting of $k \geq 3$ stars has super edge-magic deficiency at most $k-2$; for every positive integer $n$ a forest consisting of 4 stars with exactly 1, $n$, $n$, and $n + 2$ leaves has a super edge-magic total labeling; for every positive integer $n$ a forest consisting of 4 stars with exactly 1, $n + 5, 2n + 6$, and $n + 1$ leaves has a super edge-magic total labeling; and for every positive integers $n$ and $k$ a forest consisting of $k$ identical stars has super edge-magic deficiency at most 1 when $k$ is even and deficiency 0

when $k$ is odd.

In [417] Figueroa-Centeno, Ichishima, and Muntaner-Batle proved that $\mu_s(P_m \cup K_{1,n}) = 1$ if $m = 2$ and $n$ is odd, or $m = 3$ and $n$ is not congruent to 0 mod 3, whereas in all other cases $\mu_s(P_m \cup K_{1,n}) = 0$. They also proved that $\mu_s(2K_{1,n}) = 1$ when $n$ is odd and $\mu_s(2K_{1,n}) \leq 1$ when $n$ is even. They conjecture that $\mu_s(2K_{1,n}) = 1$ in all cases. Other results in [417] are: $\mu_s(P_m \cup P_n) = 1$ when $(m, n) = (2, 2)$ or $(3, 3)$ and $\mu_s(P_m \cup P_n) = 0$ in all other cases; $\mu_s(K_{1,m} \cup K_{1,n}) = 0$ when $mn$ is even and $\mu_s(K_{1,m} \cup K_{1,n}) = 1$ when $mn$ is odd; $\mu(P_m \cup K_{1,n}) = 1$ when $m = 2$ and $n$ is odd and $\mu(P_m \cup K_{1,n}) = 0$ in all other cases; $\mu(P_m \cup P_n) = 1$ when $(m, n) = (2, 2)$ and $\mu(P_m \cup P_n) = 0$ in all other cases; $\mu_s(2C_n) = 1$ when $n$ is even and $\infty$ when $n$ is odd; $\mu_s(3C_n) = 0$ when $n$ is odd; $\mu_s(3C_n) = 1$ when $n \equiv 0 \pmod 4$; $\mu_s(3C_n) = \infty$ when $n \equiv 2 \pmod 4$; and $\mu_s(4C_n) = 1$ when $n \equiv 0 \pmod 4$. They conjecture the following: $\mu_s(mC_n) = 0$ when $mn$ is odd; $\mu_s(mC_n) = 1$ when $mn \equiv 0 \pmod 4$; $\mu_s(mC_n) = \infty$ when $mn \equiv 2 \pmod 4$; $\mu_s(2K_{1,n}) = 1$; and if $F$ is a forest with two components, then $\mu(F) \leq 1$ and $\mu_s(F) \leq 1$. Santhosh and Singh [992] proved: for $n$ odd at least 3, $\mu_s(K_2 \odot C_n) \leq (n-3)/2$; for $n > 1$, $1 \leq \mu_s(P_n[P_2]) = \lceil (n-1)/2 \rceil$; and for $n \geq 1$, $1 \leq \mu_s(P_n \times K_4) \leq n$.

Ichishima and Oshima [574] prove the following: if a graph $G(V, E)$ has an $\alpha$-labeling and no isolated vertices, then $\mu_s(G) \leq |E| - |V| + 1$; if a graph $G(V, E)$ has an $\alpha$-labeling, is not sequential, and has no isolated vertices, then $\mu_s(G) = |E| - |V| + 1$; and, if $m$ is even, then $\mu_s(mK_{1,n}) \leq 1$. As corollaries of the last result they have: $\mu_s(2K_{1,n}) = 1$; when $m \equiv 2 \pmod 4$ and $n$ is odd, $\mu_s(mK_{1,n}) = 1$; $\mu_s(mK_{1,3}) = 0$ when $m \equiv 4 \pmod 8$ or $m$ is odd; $\mu_s(mK_{1,3}) = 1$ when $m \equiv 2 \pmod 4$; $\mu_s(mK_{2,2}) = 1$; for $n \geq 4$, $(n-4)2^{n-2} + 3 \leq \mu_s(Q_n) \leq (n-2)2^{n-1} - 4$; and for $s \geq 2$ and $t \geq 2$, $\mu_s(mK_{s,t}) \leq m(st - s - t) + 1$. They conjecture that for $s \geq 2$ and $t \geq 2$, $\mu_s(mK_{s,t}) = m(st - s - t) + 1$ and pose as a problem determining the exact value of $\mu_s(Q_n)$.

Ichishima and Oshima [573] determined the super edge-magic deficiency of graphs of the form $C_m \cup C_n$ for $m$ and $n$ even and for arbitrary $n$ when $m = 3, 4, 5$, and 7. They state a conjecture for the super edge-magic deficiency of $C_m \cup C_n$ in the general case.

A *block* of a graph is a maximal subgraph with no cut-vertex. The *block-cut-vertex graph* of a graph $G$ is a graph $H$ whose vertices are the blocks and cut-vertices in $G$; two vertices are adjacent in $H$ if and only if one vertex is a block in $G$ and the other is a cut-vertex in $G$ belonging to the block. A *chain graph* is a graph with blocks $B_1, B_2, B_3, \ldots, B_k$ such that for every $i, B_i$ and $B_{i+1}$ have a common vertex in such a way that the block-cut-vertex graph is a path. The chain graph with $k$ blocks where each block is identical and isomorphic to the complete graph $K_n$ is called the *$kK_n$-path*.

Ngurah, Baskoro, and Simanjuntak [901] investigate the exact values of $\mu_s(kK_n\text{-path})$ when $n = 2$ or 4 for all values of $k$ and when $n = 3$ for $k \equiv 0, 1, 2 \pmod 4$, and give an upper bound for $k \equiv 3 \pmod 4$. They determine the exact super edge-magic deficiencies for fans, double fans, wheels of small order and provide upper and lower bounds for the general case as well as bounds for some complete partite graphs. They also include some open problems. Lee and Wang [769] show that various chain graphs with blocks that are complete graphs are super edge-magic.

Figueroa-Centeno and Ichishima [410] introduce the notion of the *sequential number* $\sigma(G)$ of a graph $G$ without isolated vertices to be either the smallest positive integer $n$ for which it is possible to label the vertices of $G$ with distinct elements from the set $\{0, 1, \ldots, n\}$ in such a way that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels are $|E(G)|$ consecutive integers or $+\infty$ if there exists no such integer $n$. They prove that $\sigma(G) = \mu_s(G) + |V(G)| - 1$

for any graph $G$ without isolated vertices, and $\sigma(K_{m,n)} = mn$, which settles the conjecture of Figueroa-Centeno, Ichishima, and Muntaner-Batle [418] that $\mu_s(K_m, n) = (m-1)(n-1)$.

Z. Chen [314] has proved: the join of $K_1$ with any subgraph of a star is super edge-magic; the join of two nontrivial graphs is super edge-magic if and only if at least one of them has exactly two vertices and their union has exactly one edge; and if a $k$-regular graph is super edge-magic, then $k \leq 3$. Chen also obtained the following: there is a connected super edge-magic graph with $p$ vertices and $q$ edges if and only if $p-1 \leq q \leq 2p-3$; there is a connected 3-regular super edge-magic graph with $p$ vertices if and only if $p \equiv 2 \pmod 4$; and if $G$ is a $k$-regular edge-magic total graph with $p$ vertices and $q$ edges then $(p+q)(1+p+q) \equiv 0 \pmod{2d}$ where $d = \gcd(k-1, q)$. As a corollary of the last result, Chen observes that $nK_2 + nK_2$ is not edge-magic total.

Another labeling that has been called "edge-magic" was introduced by Lee, Seah, and Tan in 1992 [749]. They defined a graph $G = (V, E)$ to be *edge-magic* if there exists a bijection $f: E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $f^+: V \to N$ defined by $f^+(u) = \sum_{(u,v)\in E} f(u, v) \pmod{|V|}$ is a constant map. Lee (see [737]) conjectured that a cubic graph with $p$ vertices is edge-magic if and only if $p \equiv 2 \pmod 4$. Lee, Pigg, and Cox [737] verified this conjecture for prisms and several other classes of cubic graphs. They also show that $C_n \times K_2$ is edge-magic if and only if $n$ is odd. Shiu and Lee [1072] showed that the conjecture is not true for multigraphs and disconnected graphs. In [1072] Lee's conjecture was modified by restricting it to simple connected cubic graphs. A computer search by Lee, Wang, and Wen [772] showed that the new conjecture was false for a graph of order 10. Lee, Seah, and Tan [749] establish that a necessary condition for a multigraph with $p$ vertices and $q$ edges to be edge-magic is that $p$ divides $q(q+1)$ and they exhibit several new classes of cubic edge-magic graphs. They also proved: $K_{n,n}$ $(n \geq 3)$ is edge-magic and $K_n$ is edge-magic for $n \equiv 1, 2 \pmod 4$ and for $n \equiv 3 \pmod 4$ $(n \geq 7)$. Lee, Seah, and Tan further proved that following graphs are not edge-magic: all trees except $P_2$; all unicyclic graphs; and $K_n$ where $n \equiv 0 \pmod 4$. Schaffer and Lee [997] have proved that $C_m \times C_n$ is always edge-magic. Lee, Tong, and Seah [763] have conjectured that the total graph of a $(p, p)$-graph is edge-magic if and only if $p$ is odd. They prove this conjecture for cycles. Lee, Kitagaki, Young, and Kocay [709] proved that a maximal outerplanar graph with $p$ vertices is edge-magic if and only if $p = 6$. Shiu [1062] used matrices with special properties to prove that the composition of $P_n$ with $\overline{K_n}$ and the composition of $P_n$ with $\overline{K_{kn}}$ where $kn$ is odd and $n$ is at least 3 have edge-magic labelings.

Chopra, Dios, and Lee [322] investigated the edge-magicness of joins of graphs. Among their results are: $K_{2,m}$ is edge-magic if and only if $m = 4$ or 10; the only possible edge-magic graphs of the form $K_{3,m}$ are those with $m = 3, 5, 6, 15, 33$, and 69; for any fixed $m$ there are only finitely many $n$ such that $K_{m,n}$ is edge-magic; for any fixed $m$ there are only finitely many trees $T$ such that $T + \overline{K_m}$ is edge-magic; and wheels are not edge-magic.

For any graph $G$ and any positive integer $k$ the graph $G[k]$, called the *k-fold* $G$, is the hypergraph obtained from $G$ by replacing each edge of $G$ with $k$ parallel edges. Lee, Seah, and Tan [749] proved that for any graph $G$ with $p$ vertices, $G[2p]$ is edge-magic and, if $p$ is odd, $G[p]$ is edge-magic. Shiu, Lam, and Lee [1070] show that if $G$ is an $(n+1, n)$-multigraph, then $G$ is edge-magic if and only if $n$ is odd and $G$ is isomorphic to the disjoint union of $K_2$ and $(n-1)/2$ copies of $K_2[2]$. They also prove that if $G$ is a $(2m+1, 2m)$-multigraph and $k \geq 2$, then $G[k]$ is edge-magic if and only if $2m+1$ divides $k(k-1)$. For a $(2m, 2m-1)$-multigraph $G$ and $k$ at least 2, they show that $G[k]$ is edge-magic if $4m$ divides $(2m-1)k((2m-1)k+1)$ or if $4m$ divides $(2m+k-1)k$. In [1068] Shiu, Lam, and Lee characterize the $(p, p)$-multigraphs

that are edge-magic as $mK_2[2]$ or the disjoint union of $mK_2[2]$ and two particular multigraphs or the disjoint union of $K_2$, $mK_2[2]$, and four particular multigraphs. They also show for every $(2m+1, 2m+1)$-multigraph $G$, $G[k]$ is edge-magic for all $k$ at least 2. Lee, Seah, and Tan [749] prove that the multigraph $C_n[k]$ is edge-magic for $k \geq 2$.

Tables 5 and 6 summarize what is known about edge-magic total labelings and super edge-magic total labelings. We use **SEM** to indicate the graphs have super edge-magic total labelings and **EMT** to indicate the graphs have edge-magic total labelings. A question mark following SEM or EMT indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Table 5: **Summary of Edge-magic Total Labelings**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $P_n$ | EMT | [1232] |
| trees | EMT? | [674], [963] |
| $C_n$ | EMT | for $n \geq 3$ [673], [472], [970], [216] |
| $K_n$ | EMT | iff $n = 1, 2, 3, 4, 5,$ or 6 [674], [341], [390]<br>enumeration of all EMT of $K_n$ [1232] |
| $K_{m,n}$ | EMT | for all $m$ and $n$ [673] |
| crowns $C_n \odot K_1$ | EMT | [1232] |
| $K_{m,n}$ | EMT | [1232] |
| $C_n$ with a single edge attached to one vertex | EMT | [1232] |
| wheels $W_n$ | EMT | iff $n \not\equiv 3 \pmod 4$ [390],[437] |
| fans | EMT | [1103], [412], [413] |
| $(p, q)$-graph | not EMT | if $q$ even and $p + q \equiv 2 \pmod 4$ [963] |

Table 5: **Summary of Edge-magic Total continued**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $nP_2$ | EMT | iff $n$ odd [673] |
| $P_n + K_1$ | EMT | [1299] |
| $P_n \times C_3$ | EMT | $n \geq 2$ [1299] |
| crown $C_n \odot K_1$ | EMT | [1299] |
| $r$-regular graph | not EMT | $r$ odd and $p \equiv 4 \pmod 8$ [341] |
| $P_3 \cup nK_2$ and $P_5 \cup nK_2$ | EMT | [412], [413] |
| $P_4 \cup nK_2$ | EMT | $n$ odd [412], [413] |
| $nP_i$ | EMT | $n$ odd, $i = 3, 4, 5$ [1299] [412],[413] |
| $nP_3$ | EMT? | [1299] |
| $2P_n$ | EMT | [412], [413] |
| $P_1 \cup P_2 \cup \cdots \cup P_n$ | EMT | [412], [413] |
| $mK_{1,n}$ | EMT | [412], [413] |
| $C_m \odot \overline{K}_n$ | EMT | [412], [413] |
| unicylic graphs | EMT? | [713] |

Table 5: **Summary of Edge-magic Total continued**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $K_1 \odot nK_2$ | EMT | $n$ even [412], [413] |
| $K_2 \times \overline{K}_n$ | EMT | [412], [413] |
| $nK_3$ | EMT | iff $n \neq 2$ odd [412], [413], [852] |
| binary trees | EMT | [412], [413] |
| $P(m,n)$ (generalized Petersen graph see §2.7) | EMT | [412], [413], [898] |
| ladders | EMT | [412], [413] |
| books | EMT | [412], [413] |
| odd cycle with pendant edges attached to one vertex | EMT | [412], [413] |
| $P_m \times C_n$ | EMT | $n$ odd $n \geq 3$ [1257] |
| $P_m \times P_2$ | EMT | $m$ odd $m \geq 3$ [1257] |
| $P_2 \times C_n$ | not EMT | [898] |
| $K_{1,m} \cup K_{1,n}$ | EMT | iff $mn$ is even [416] |
| $G \odot \overline{K}_n$ | EMT | if $G$ is EMT 2-regular graph [414] |

Table 6: **Summary of Super Edge-magic Labelings**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $C_n$ | SEM | iff $n$ is odd [390] |
| caterpillars | SEM | [390], [673], [674] |
| $K_{m,n}$ | SEM | iff $m = 1$ or $n = 1$ [390] |
| $K_n$ | SEM | iff $n = 1, 2$ or 3 [390] |
| trees | SEM? | [390] |
| $nK_2$ | SEM | iff $n$ odd [314] |
| $nG$ | SEM | if $G$ is a bipartite or tripartite SEM graph and $n$ odd [416] |
| $K_{1,m} \cup K_{1,n}$ | SEM | if $m$ is a multiple of $n + 1$ [416] |
| $K_{1,m} \cup K_{1,n}$ | SEM? | iff $m$ is a multiple of $n + 1$ [416] |
| $K_{1,2} \cup K_{1,n}$ | SEM | iff $n$ is a multiple of 3 [416] |
| $K_{1,3} \cup K_{1,n}$ | SEM | iff $n$ is a multiple of 4 [416] |
| $P_m \cup K_{1,n}$ | SEM | if $m \geq 4$ is even [416] |
| $2P_n$ | SEM | iff $n$ is not 2 or 3 [416] |
| $2P_{4n}$ | SEM | for all $n$ [416] |
| $K_{1,m} \cup 2nK_{1,2}$ | SEM | for all $m$ and $n$ [416] |

Table 6: **Summary of Super Edge-magic Labelings continued**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $C_3 \cup C_n$ | SEM | iff $n \geq 6$ even [419], [486] |
| $C_4 \cup C_n$ | SEM | iff $n \geq 5$ odd [419], [486] |
| $C_5 \cup C_n$ | SEM | iff $n \geq 4$ even [419] |
| $C_m \cup C_n$ | SEM | if $m \geq 6$ even and $n$ odd $n \geq m/2 + 2$ [419] |
| $C_m \cup C_n$ | SEM? | iff $m + n \geq 9$ and $m + n$ odd [419] |
| $C_4 \cup P_n$ | SEM | iff $n \neq 3$ [419] |
| $C_5 \cup P_n$ | SEM | if $n \neq 4$ [419] |
| $C_m \cup P_n$ | SEM | if $m \geq 6$ even and $n \geq m/2 + 2$ [419] |
| $P_m \cup P_n$ | SEM | iff $(m, n) \neq (2, 2)$ or $(3, 3)$ [419] |
| corona $C_n \odot \overline{K}_m$ | SEM | $n \geq 3$ [419] |
| $St(m, n)$ | SEM | $n \equiv 0 \pmod{m+1}$ [710] |
| $St(1, k, n)$ | SEM | $k = 1, 2$ or $n$ [710] |
| $St(2, k, n)$ | SEM | $k = 2, 3$ [710] |
| $St(1, 1, k, n)$ | SEM | $k = 2, 3$ [710] |
| $St(k, 2, 2, n)$ | SEM | $k = 1, 2$ [710] |
| $St(a_1, \ldots, a_n)$ | SEM? | for $n > 1$ odd [710] |

Table 6: **Summary of Super Edge-magic Labelings continued**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $C_{4m}^t$ | SEM | [837] |
| $C_{4m+1}^t$ | SEM | [837] |
| friendship graph of $n$ triangles | SEM | iff $n = 3, 4, 5,$ or 7 [1103] |
| generalized Petersen graph $P(n, 2)$ (see §2.7) | SEM | if $n \geq 3$ odd [436] |
| $nP_3$ | SEM | if $n \geq 4$ even [204] |
| $P_n^2$ | SEM | [412] |
| $K_2 \times C_{2n+1}$ | SEM | [412] |
| $P_3 \cup kP_2$ | SEM | for all $k$ [413] |
| $kP_n$ | SEM | if $k$ is odd [413] |
| $k(P_2 \cup P_n)$ | SEM | if $k$ is odd and $n = 3, 4$ [413] |
| fans $F_n$ | SEM | iff $n \leq 6$ [413] |
| books $B_n$ | SEM | if $n$ even [1317] |
| books $B_n$ | SEM? | if $n$ even or $n \equiv 5 \pmod 8$[413] |
| trees with $\alpha$-labelings | SEM | [413] |

Table 6: **Summary of Super Edge-magic Labelings continued**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $P_{2m+1} \times P_2$ | SEM | [390], [412] |
| $C_{2m+1} \times P_m$ | SEM | [390], [412] |
| $G \odot \overline{K}_n$ | SEM | if $G$ is SEM 2-regular graph [414] |
| $C_m \odot \overline{K}_n$ | SEM | [414] |
| join of $K_1$ with any subgraph of a star | SEM | [314] |
| if $G$ is $k$-regular SEM graph | | then $k \leq 3$ [314] |
| $G$ is connected $(p, q)$-graph | SEM | $G$ exists iff $p - 1 \leq q \leq 2p - 3$ [314] |
| $G$ is connected 3-regular graph on $p$ vertices | SEM | iff $p \equiv 2 \pmod 4$ [314] |
| $nK_2 + nK_2$ | not SEM | [314] |

## 5.3 Vertex-magic Total Labelings

MacDougall, Miller, Slamin, and Wallis [834] introduced the notion of a vertex-magic total labeling in 1999. For a graph $G(V,E)$ an injective mapping $f$ from $V \cup E$ to the set $\{1,2,\ldots,|V|+|E|\}$ is a *vertex-magic total labeling* if there is a constant $k$, called the *magic constant*, such that for every vertex $v$, $f(v)+\sum f(vu)=k$ where the sum is over all vertices $u$ adjacent to $v$ (some authors use the term "vertex-magic" for this concept). They prove that the following graphs have vertex-magic total labelings: $C_n$; $P_n$ $(n>2)$; $K_{m,m}$ $(m>1)$; $K_{m,m}-e$ $(m>2)$; and $K_n$ for $n$ odd. They also prove that when $n>m+1$, $K_{m,n}$ does not have a vertex-magic total labeling. They conjectured that $K_{m,m+1}$ has a vertex-magic total labeling for all $m$ and that $K_n$ has vertex-magic total labeling for all $n \geq 3$. The latter conjecture was proved by Lin and Miller [794] for the case that $n$ is divisible by 4 while the remaining cases were done by MacDougall, Miller, Slamin, and Wallis [834]. McQuillan [851] provided many vertex-magic total labelings for cycles $C_{nk}$ for $k \geq 3$ and odd $n \geq 3$ using given vertex-magic labelings for $C_k$. Gray, MacDougall, and Wallis [496] then gave a simpler proof that all complete graphs are vertex-magic total. Krishnappa, Kothapalli, and Venkaiah [665] gave another proof that all complete graphs are vertex-magic total.

In [834] MacDougall, Miller, Slamin, and Wallis conjectured that for $n \geq 5$, $K_n$ has a vertex-magic total labeling with magic constant $h$ if and only if $h$ is an integer satisfying $n^3+3n \leq 4h \leq n^3+2n^2+n$. In [853] McQuillan and Smith proved that this conjecture is true when $n$ is odd. Armstrong and McQuillan [74] proved that if $n \equiv 2 \pmod 4$ $(n \geq 6)$ then $K_n$ has a vertex-magic total labeling with magic constant $h$ for each integer $h$ satisfying $n^3+6n \leq 4h \leq n^3+2n^2-2n$. If, in addition, $n \equiv 2 \pmod 8$, then $K_n$ has a vertex-magic total labeling with magic constant $h$ for each integer $h$ satisfying $n^3+4n \leq 4h \leq n^3+2n^2$. They further showed that for each odd integer $n \geq 5$, $2K_n$ has a vertex-magic total labeling with magic constant $h$ for each integer $h$ such that $n^3+5n \leq 2h \leq n^3+2n^2-3n$. If, in addition, $n \equiv 1 (\mathrm{mod}\ 4)$, then $2K_n$ has a vertex-magic total labeling with magic constant $h$ for each integer $h$ such that $n^3+3n \leq 2h \leq n^3+2n^2-n$.

In [852] McQuillan and McQuillan investigate the existence of vertex-magic labelings of $nC_3$. They prove: for every even integer $n \geq 4$, $nC_3$ is vertex-magic (and therefore also edge-magic); for each even integer $n \geq 6$, $nC_3$ has vertex-magic total labelings with at least $2n-2$ different magic constants; if $n \equiv 2 \bmod 4$, two extra vertex-magic total labelings with the highest possible and lowest possible magic constants exist; if $n=2\cdot 3^k$, $k>1$, $nC_3$ has a vertex-magic total labeling with magic constant $k$ if and only if $(1/2)(15n+4) \leq k \leq (1/2)(21n+2)$; if $n$ is odd, there are vertex-magic total labelings for $nC_3$ with $n+1$ different magic constants. In [850] McQuillan provides a technique for constructing vertex-magic total labelings of 2-regular graphs. In particular, if $m$ is an odd positive integer, $G=C_{n_1}\cup C_{n_2}\cup\cdots\cup C_{n_k}$ has a vertex-magic total labeling, and $J$ is any subset of $I=\{1,2,\ldots,k\}$ then $(\cup_{i\in J}\ mC_{n_i})\cup(\cup_{i\in I-J}\ mC_{n_i})$ has a vertex-magic total labeling.

Lin and Miller [794] have shown that $K_{m,m}$ is vertex-magic total for all $m>1$ and that $K_n$ is vertex-magic total for all $n \equiv 0 \pmod 4$. Phillips, Rees, and Wallis [934] generalized the Lin and Miller result by proving that $K_{m,n}$ is vertex-magic total if and only if $m$ and $n$ differ by at most 1. Cattell [290] has shown that a necessary condition for a graph of the form $H+\overline{K_n}$ to be vertex-magic total is that the number of vertices of $H$ is at least $n-1$. As a corollary he gets that a necessary condition for $K_{m_1,m_2,\ldots,m_r,n}$ where $n$ is the largest size of any partite set to be vertex-magic total is that $m_1+m_2+\cdots+m_r \geq n$. He poses as an open question whether graphs that meet the conditions of the theorem are vertex-magic total. Cattell also proves that $K_{1,n,n}$

has a vertex-magic total labeling when $n$ is odd and $K_{2,n,n}$ has a vertex-magic total labeling when $n \equiv 3 \pmod 4$.

Miller, Bača, and MacDougall [865] have proved that the generalized Petersen graphs $P(n,k)$ (see §2.7) for the definition) are vertex-magic total when $n$ is even and $k \leq n/2 - 1$. They conjecture that all $P(n,k)$ are vertex-magic total when $k \leq (n-1)/2$ and all prisms $C_n \times P_2$ are vertex-magic total. Bača, Miller, and Slamin [154] proved the first of these conjectures (see also [1104] for partial results) while Slamin and Miller prove the second. Slamin, Prihandoko, Setiawan, Rosita and Shaleh [1105] constructed vertex-magic total labelings for the disjoint union of two copies of $P(n,k)$ and Silaban, Parestu, Herawati, Sugeng, and Slamin [1085] extended this to any number of copies of $P(n,k)$. More generally, they proved that for $n_j \geq 3$ and $1 \leq k_j \leq \lfloor (n_j - 1)/2 \rfloor$, the union $P(n_1,k_1) \cup P(n_2,k_2) \cup \cdots \cup P(n_t,k_t)$ has a vertex-magic total labeling with vertex magic constant $10(n_1 + n_2 + \cdots + n_t) + 2$. In the same article Silaban et al. define the union of $t$ special circulant graphs $\cup_{j=1}^{t} C_n(1,m_j)$ as the graph with vertex set $\{v_i^j |\ 0 \leq i \leq n-1, 1 \leq j \leq t\}$ and edge set $\{v_i^j v_{i+1}^j |\ 0 \leq i \leq n-1, 1 \leq j \leq t\} \cup \{v_i^j v_{i+m_j}^j |\ 0 \leq i \leq n-1, 1 \leq j \leq t\}$. They prove that for odd $n$ at least 5 and $m_j \in \{2,3,\ldots,(n-1)/2\}$, the disjoint union $\cup_{j=1}^{t} C_n(1,m_j)$ has a vertex-magic total labeling with constant $8tn + (n - 10/2 + 3$.

MacDougall et al. ([834], [836] and [494]) have shown: $W_n$ has a vertex-magic total labeling if and only if $n \leq 11$; fans $F_n$ have a vertex-magic total labelings if and only if $n \leq 10$; friendship graphs have vertex-magic total labelings if and only if the number of triangles is at most 3; $K_{m,n}$ $(m > 1)$ has a vertex-magic total labeling if and only if $m$ and $n$ differ by at most 1. Wallis [1228] proved: if $G$ and $H$ have the same order and $G \cup H$ is vertex-magic total then so is $G + H$; if the disjoint union of stars is vertex-magic total, then the average size of the stars is less than 3; if a tree has $n$ internal vertices and more than $2n$ leaves then it does not have a vertex-magic total labeling. Wallis [1229] has shown that if $G$ is a regular graph of even degree that has a vertex-magic total labeling then the graph consisting of an odd number of copies of $G$ is vertex-magic total. He also proved that if $G$ is a regular graph of odd degree (not $K_1$) that has a vertex-magic total labeling then the graph consisting of any number of copies of $G$ is vertex-magic total.

Gray, MacDougall, McSorley, and Wallis [495] investigated vertex-magic total labelings of forests. They provide sufficient conditions for the nonexistence of a vertex-magic total labeling of forests based on the maximum degree and the number of internal vertices, and leaves or the number of components. They also use Skolem sequences to prove a star forest with each component a $K_{1,2}$ has a vertex-magic total labeling.

Recall a helm $H_n$ is obtained from a wheel $W_n$ by attaching a pendant edge at each vertex of the $n$-cycle of the wheel. A *generalized helm* $H(n,t)$ is a graph obtained from a wheel $W_n$ by attaching a path on $t$ vertices at each vertex of the $n$-cycle. A *generalized web* $\mathrm{W}(n,t)$ is a graph obtained from a generalized helm $H(n,t)$ by joining the corresponding vertices of each path to form an $n$-cycle. Thus $\mathrm{W}(n,t)$ has $(t+1)n + 1$ vertices and $2(t+1)n$ edges. A *generalized Jahangir graph* $J_{k,s}$ is a graph on $ks + 1$ vertices consisting of a cycle $C_{ks}$ and one additional vertex that is adjacent to $k$ vertices of $C_{ks}$ at distance $s$ to each other on $C_{ks}$. Rahim, Tomescu, and Slamin [951] prove: $H_n$ has no vertex-magic total labeling for any $n \geq 3$; $\mathrm{W}(n,t)$ has a vertex-magic total labeling for $n = 3$ or $n = 4$ and $t = 1$, but it is not vertex-magic total for $n \geq 17t + 12$ and $t \geq 0$; and $J_{n,t+1}$ is vertex-magic total for $n = 3$ and $t = 1$, but it does not have this property for $n \geq 7t + 11$ and $t \geq 1$. Recall a flower is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm. Ahmad and Tomescu [48] proved that flower graph is vertex-magic if and only if the underlying cycle is $C_3$.

Fronček, Kovář, and Kovářová [425] proved that $C_n \times C_{2m+1}$ and $K_5 \times C_{2n+1}$ are vertex-magic total. Kovář [676] furthermore proved some general results about products of certain regular vertex-magic total graphs. In particular, if $G$ is a $(2r+1)$-regular vertex-magic total graph that can be factored into an $(r+1)$-regular graph and an $r$-regular graph, then $G \times K_5$ and $G \times C_n$ for $n$ even are vertex-magic total. He also proved that if $G$ an $r$-regular vertex-magic total graph and $H$ is a $2s$-regular supermagic graph that can be factored into two $s$-regular factors, then their Cartesian product $G \times H$ is vertex-magic total if either $r$ is odd, or $r$ is even and $|H|$ is odd.

Beardon [207] has shown that a necessary condition for a graph with $c$ components, $p$ vertices, $q$ edges and a vertex of degree $d$ to be vertex-magic total is $(d+2)^2 \leq (7q^2+(6c+5)q+c^2+3c)/p$. When the graph is connected this reduces to $(d+2)^2 \leq (7q^2+11q+4)/p$. As a corollary, the following are not vertex-magic total: wheels $W_n$ when $n \geq 12$; fans $F_n$ when $n \geq 11$; and friendship graphs $C_3^{(n)}$ when $n \geq 4$.

MacDougall has conjectured (see [677]) that every $r$-regular ($r > 1$) graph with the exception of $2K_3$ has a vertex-magic total labeling. As a corollary of a general result Kovář [677] has shown that every $2r$-regular graph with an odd number of vertices and a Hamiltonian cycle has a vertex-magic total labeling.

Beardon [209] has investigated how vertices of small degree effect vertex-magic total labelings. Let $G(p,q)$ be a graph with a vertex-magic total labeling with magic constant $k$ and let $d_0$ be the minimum degree of any vertex. He proves $k \leq (1+d_0)(p+q-d_0/2)$ and $q < (1+d_0)q$. He also shows that if $G(p,q)$ is a vertex-magic graph with a vertex of degree one and $t$ is the number of vertices of degree at least two, then $t > q/3 \geq (p-1)/3$. Beardon [209] has shown that the graph obtained by attaching a pendant edge to $K_n$ is vertex-magic total if and only if $n = 2, 3$, or 4.

Meissner and Zwierzyński [858] used finding vertex-magic total labelings of graphs as a way to compare the efficiency of parallel execution of a program versus sequential processing.

MacDougall, Miller, and Sugeng [835] define a *super vertex-magic total labeling* of a graph $G(V,E)$ as a vertex-magic total labeling $f$ of $G$ with the additional property that $f(V) = \{1, 2, \ldots, |V|\}$ and $f(E) = \{|V|+1, |V|+2, \ldots, |V|+|E|\}$ (some authors use the term "super vertex-magic" for this concept). They show that a $(p,q)$-graph that has a super vertex-magic total labeling with magic constant $k$ satisfies the following conditions: $k = (p+q)(p+q+1)/v - (v+1)/2$; $k \geq (41p+21)/18$; if $G$ is connected, $k \geq (7p-5)/2$; $p$ divides $q(q+1)$ if $p$ is odd, and $p$ divides $2q(q+1)$ if $p$ is even; if $G$ has even order either $p \equiv 0 \pmod 8$ and $q \equiv 0$ or $3 \pmod 4$ or $p \equiv 4 \pmod 8$ and $q \equiv 1$ or $2 \pmod 4$; if $G$ is $r$-regular and $p$ and $r$ have opposite parity then $p \equiv 0 \pmod 8$ implies $q \equiv 0 \pmod 4$ and $p \equiv 4 \pmod 8$ implies $q \equiv 2 \pmod 4$. They also show: $C_n$ has a super vertex-magic total labeling if and only if $n$ is odd; and no wheel, ladder, fan, friendship graph, complete bipartite graph or graph with a vertex of degree 1 has a super vertex-magic total labeling. They conjecture that no tree has a super vertex-magic total labeling and that $K_{4n}$ has a super vertex-magic total labeling when $n > 1$. The latter conjecture was proved by Gómez in [476]. In [477] Gómez proved that if $G$ is a $d$-regular graph that has a vertex-magic total labeling and $k$ is a positive integer such that $(k-1)(d+1)$ is even, then $kG$ has a super vertex-magic total labeling. As a corollary, we have that if $n$ and $k$ are odd or if $n \equiv 0 \pmod 4$ and $n > 4$, then $kK_n$ has a super vertex-magic total labeling. Gómez also shows how graphs with super vertex-magic total labeling can be constructed from a given graph $G$ with super vertex-magic total labeling by adding edges to $G$ in various ways.

Swaminathan and Jeyanthi [1164] prove the following graphs are super vertex-magic total:

$P_n$ if and only if $n$ is odd and $n \geq 3$; $C_n$ if and only if $n$ is odd; the star graph if and only if it is $P_2$; and $mC_n$ if and only if $m$ and $n$ are odd. In [1165] they prove the following: no super vertex-magic total graph has two or more isolated vertices or an isolated edge; a tree with $n$ internal edges and $tn$ leaves is not super vertex-magic total if $t > (n+1)/n$; if $\Delta$ is the largest degree of any vertex in a tree $T$ with $p$ vertices and $\Delta > (-3+\sqrt{1+16p})/2$, then $T$ is not super vertex-magic total; the graph obtained from a comb by appending a pendant edge to each vertex of degree 2 is super vertex-magic total; the graph obtained by attaching a path with $t$ edges to a vertex of an $n$-cycle is super vertex-magic total if and only if $n+t$ is odd. Ali, Bača, and Bashir [53] proved that $mP_3$ and $mP_4$ have no super vertex-magic total labeling

For $n > 1$ and distinct odd integers $x, y$ and $z$ in [1,$n-1$] Javaid, Ismail, and Salman [588] define the *chordal ring* of order $n$ $CR_n(x,y,z)$, as the graph with vertex set $Z_n$, the additive group of integers modulo $n$, and edges $(i, i+x), (i, i+y), (i, i+z)$ for all even $i$. They prove that $CR_n(1,3,n-1)$ has a super vertex-magic total labeling when $n \equiv 0 \bmod 4$ and $n \geq 8$ and conjecture that for an odd integer $\Delta$, $3 \leq \Delta \leq n-3, n \equiv 0 \bmod 4$, $CR_n(1,\Delta,n-1)$ has a super vertex-magic total labeling with magic constant $23n/4+2$.

The *Knödel graphs* $W_{\Delta,n}$ with $n$ even and degree $\Delta$, where $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ have vertices pairs $(i,j)$ with $i = 1,2$ and $0 \leq j \leq n/2-1$ where for every $0 \leq j \leq n/2-1$ and there is an edge between vertex $(1,j)$ and every vertex $(2,(j+2^k-1) \bmod n/2)$, for $k = 0,1,\ldots,\Delta-1$. Xi, Yang, Mominul, and Wong [1270] have shown that $W_{3,n}$ is super vertex-magic total when $n \equiv 0 \bmod 4$.

Balbuena, Barker, Das, Lin, Miller, Ryan, and Slamin [173] call a vertex-magic total labeling of $G(V,E)$ a *strongly vertex-magic total labeling* if the vertex labels are $\{1,2,\ldots,|V|\}$. They prove: the minimum degree of a strongly vertex-magic total graph is at least 2; for a strongly vertex-magic total graph $G$ with $n$ vertices and $e$ edges, if $2e \geq \sqrt{10n^2-6n+1}$ then the minimum degree of $G$ is at least 3; and for a strongly vertex-magic total graph $G$ with $n$ vertices and $e$ edges if $2e < \sqrt{10n^2-6n+1}$ then the minimum degree of $G$ is at most 6. They also provide strongly vertex-magic total labelings for certain families of circulant graphs. In [850] McQuillan provides a technique for constructing vertex-magic total labelings of 2-regular graphs. In particular, if $m$ is an odd positive integer, $G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_k}$ has a strongly vertex-magic total labeling, and $J$ is any subset of $I = \{1,2,\ldots,k\}$ then $(\cup_{i \in J}\ mC_{n_i}) \cup (\cup_{i \in I-J}\ mC_{n_i})$ has a strongly vertex-magic total labeling.

Gray [487] proved that if $G$ is a graph with a spanning subgraph $H$ that possesses a strongly vertex-magic total labeling and $G - E(H)$ is even regular, then $G$ also possesses a strongly vertex-magic total labeling. As a corollary one has that regular Hamiltonian graphs of odd order have a strongly vertex-magic total labelings.

In a series of papers Gray and MacDougall expand on McQuillan's technique to obtain a variety of results. In [490] Gray and MacDougall show that for any $r \geq 4$, every $r$-regular graph of odd order at most 17 has a strong vertex-magic total labeling. They also show that several large classes of $r$-regular graphs of even order, including some Hamiltonian graphs, have vertex-magic total labelings. They conjecture that every 2-regular graph of odd order possesses a strong vertex-magic total labeling if and only if it is not of the form $(2t-1)C_3 \cup C_4$ or $2tC_3 \cup C_5$. They include five open problems.

In [491] Gray and MacDougall introduce a procedure called a *mutation* that transforms one vertex-magic totaling labeling into another one by swapping sets of edges among vertices that may result in different labeling of the same graph or a labeling of a different graph. Among their results are: a description of all possible mutations of a labeling of the path and the cycle; for

all $n \geq 2$ and all $i$ from 1 to $n-1$ the graphs obtained by identifying an end points of paths of lengths $i, i+1$, and $2n-2i-1$ have a vertex-magic total labeling; for odd $n$, the graph obtained by attaching a path of length $n-m$ to an $m$ cycle, (such graphs are called $(m; n-m)$-*kites* ) have strong vertex-magic total labelings for $m = 3, \ldots, n-2$; $C_{2n+1} \cup C_{4n+4}$ and $3C_{2n+1}$ have a strong vertex-magic total labeling; and for $n \geq 2$, $C_{4n} \cup C_{6n-1}$ has a strong vertex-magic total labeling. They conclude with three open problems.

Gray and MacDougall [492] show how to construct vertex-magic total labelings for several families of non-regular graphs, including the disjoint union of two other graphs already possessing vertex-magic total labelings. They prove that if $G$ is a $d$-regular graph of order $v$ and $H$ a $t$-regular graph of order $u$ with each having a strong vertex magic total labeling and $vd^2 + 2d + 2v + 2u = 2tvd + 2t + ut^2$ then $G \cup H$ possesses a strong vertex-magic total labeling. They also provide bounds on the minimum degree of a graph with a vertex-magic total labeling.

In [493] Gray and MacDougall establish the existence of vertex-magic total labelings for several infinite classes of regular graphs. Their method enables them to begin with any even-regular graph and construct a cubic graph possessing a vertex-magic total labeling that produces strong vertex-magic total labelings for many even order regular graphs. The construction also extends to certain families of non-regular graphs.

Rahim and Slamin [949] give the bounds for the number of vertices for Jahangir graphs, helms, webs, flower graphs and sunflower graphs when the graphs considered are not vertex-magic total.

Thirusangu, Nagar, and Rajeswari [1177] show that certain Cayley digraphs of cyclic groups have vertex-magic total labelings.

Balbuena, Barker, Lin, Miller, and Sugeng [175] call vertex-magic total labeling an *a-vertex consecutive magic* labeling if the vertex labels are $\{a, a+1, \ldots, a+|V|\}$. For an $a$-vertex consecutive magic labeling of a graph $G$ with $p$ vertices and $q$ edges they prove: if $G$ has one isolated vertex, then $a = q$ and $(p-1)^2 + p^2 = (2q+1)^2$; if $q = p-1$, then $p$ is odd and $a = p-1$; if $p = q$, then $p$ is odd and if $G$ has minimum degree 1, then $a = (p+1)/2$ or $a = p$; if $G$ is 2-regular, then $p$ is odd and $a = 0$ or $p$; and if $G$ is $r$-regular, then $p$ and $r$ have opposite parities. They also define an *b-edge consecutive magic* labeling analogously and state some results for these labelings.

Wood [1263] generalizes vertex-magic total and edge-magic total labelings by requiring only that the labels be positive integers rather than consecutive positive integers. He gives upper bounds for the minimum values of the magic constant and the largest label for complete graphs, forests, and arbitrary graphs.

Exoo, Ling, McSorley, Phillips, and Wallis [403] call a function $\lambda$ a *totally magic labeling* of a graph $G$ if $\lambda$ is both an edge-magic total and a vertex-magic total labeling of $G$. A graph with such a labeling is called *totally magic*. Among their results are: $P_3$ is the only connected totally magic graph that has a vertex of degree 1; the only totally magic graphs with a component $K_1$ are $K_1$ and $K_1 \cup P_3$; the only totally magic complete graphs are $K_1$ and $K_3$; the only totally magic complete bipartite graph is $K_{1,2}$; $nK_3$ is totally magic if and only if $n$ is odd; $P_3 \cup nK_3$ is totally magic if and only if $n$ is even. In [1231] Wallis asks: Is the graph $K_{1,m} \cup nK_3$ ever totally magic? That question was answered by Calhoun, Ferland, Lister, and Polhill [286] who proved that if $K_{1,m} \cup nK_3$ is totally magic then $m = 2$ and $K_{1,2} \cup nK_3$ is totally magic if and only if $n$ is even.

McSorley and Wallis [855] examine the possible totally magic labelings of a union of an odd number of triangles and determine the spectrum of possible values for the sum of the label on

a vertex and the labels on its incident edges and the sum of an edge label and the labels of the endpoints of the edge for all known totally magic graphs.

Gray and MacDougall [488] define an *order $n$ sparse semi-magic square* to be an $n \times n$ array containing the entries $1, 2, \ldots, m$ once (for some $m < n^2$), has its remaining entries equal to 0, and whose rows and columns have a constant sum of $k$. They prove some basic properties of such squares and provide constructions for several infinite families of squares, including squares of all orders $n \geq 3$. Moreover, they show how such arrays can be used to construct vertex-magic total labelings for certain families of graphs.

In Tables 7, 8 and 9
**VMT** means vertex-magic total labeling
**SVMT** means super vertex magic total
**TM** means totally magic labeling.

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2007.

Table 7: **Summary of Vertex-magic Total Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $C_n$ | VMT | [834] |
| $P_n$ | VMT | $n > 2$ [834] |
| $K_{m,m} - e$ | VMT | $m > 2$ [834] |
| $K_{m,n}$ | VMT | iff $\|m - n\| \leq 1$ [934],[834],[836] |
| $K_n$ | VMT | for $n$ odd [834]<br>for $n \equiv 2 \pmod 4$,$n > 2$ [794] |
| $nK_3$ | VMT | iff $n \neq 2$ [412], [413], [852] |
| $mK_n$ | VMT | $m \geq 1,\ n \geq 4$ [854] |
| Petersen $P(n, k)$ | VMT | [154] |
| prisms $C_n \times P_2$ | VMT | [1104] |
| $W_n$ | VMT | iff $n \leq 11$ [834],[836] |
| $F_n$ | VMT | iff $n \leq 10$ [834],[836] |

Table 7: **Summary of Vertex-magic Total Labelings continued**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| friendship graphs | VMT | iff # of triangles $\leq 3$ [834],[836] |
| $G + H$ | VMT | $\|V(G)\| = \|V(H)\|$ and $G \cup H$ is VMT [1228] |
| unions of stars | VMT | [1228] |
| tree with $n$ internal vertices and more than $2n$ leaves | not VMT | [1228] |
| $nG$ | VMT | $n$ odd, $G$ regular of even degree, VMT [1229] |
| | | $G$ is regular of odd degree, VMT, but not $K_1$ [1229] |
| $C_n \times C_{2m+1}$ | VMT | [425] |
| $K_5 \times C_{2n+1}$ | VMT | [425] |
| $G \times C_{2n}$ | VMT | $G$ $2r+1$-regular VMT [676] |
| $G \times K_5$ | VMT | $G$ $2r+1$-regular VMT [676] |
| $G \times H$ | VMT | $G$ $r$-regular VMT, $r$ odd or $r$ even and $\|H\|$ odd, $H$ $2s$-regular supermagic [676] |

Table 8: **Summary of Super Vertex-magic Total Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $P_n$ | SVMT | iff $n > 1$ is odd [1164] |
| $C_n$ | SVMT | iff $n$ is odd [1164] and [835] |
| $K_{1,n}$ | SVMT | iff $n = 1$ [1164] |
| $mC_n$ | SVMT | iff $m$ and $n$ are odd [1164] |
| $W_n$ | not SVMT | [835] |
| ladders | not SVMT | [835] |
| friendship graphs | not SVMT | [835] |
| $K_{m,n}$ | not SVMT | [835] |
| dragons (see §2.2) | SVMT | iff order is even [1165], [1165] |
| Knödel graphs $W_{3,n}$ | SVMT | $n \equiv 0 \pmod 4$ [1270] |
| graphs with minimum degree 1 | not SVMT | [835] |
| $K_{4n}$ | SVMT | $n > 1$ [476] |

Table 9: **Summary of Totally Magic Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $P_3$ | TM | the only connected TM graph with vertex of degree 1 [403] |
| $K_n$ | TM | iff $n = 1, 3$ [403] |
| $K_{m,n}$ | TM | iff $K_{m,n} = K_{1,2}$ [403] |
| $nK_3$ | TM | iff $n$ is odd [403] |
| $P_3 \cup nK_3$ | TM | iff $n$ is even [403] |
| $K_{1,m} \cup nK_3$ | TM | iff $m = 2$ and $n$ is even [286] |

## 5.4 Magic Labelings of Type $(a, b, c)$

A magic-type method for labeling the vertices, edges, and faces of a planar graph was introduced by Lih [791] in 1983. Lih defines a *magic labeling of type* (1,1,0) of a planar graph $G(V, E)$ as an injective function from $\{1, 2, \ldots, |V| + |E|\}$ to $V \cup E$ with the property that for each interior face the sum of the labels of the vertices and the edges surrounding that face is some fixed value. Similarly, Lih defines a *magic labeling of type* $(1, 1, 1)$ of a planar graph $G(V, E)$ with face set $F$ as an injective function from $\{1, 2, \ldots, |V| + |E| + |F|\}$ to $V \cup E \cup F$ with the property that for each interior face the sum of the labels of the face and the vertices and the edges surrounding that face is some fixed value. Lih calls a labeling involving the faces of a plane graph *consecutive* if for every integer $s$ the weights of all $s$-sided faces constitute a set of consecutive integers. Lih gave consecutive magic labelings of type $(1, 1, 0)$ for wheels, friendship graphs, prisms, and some members of the Platonic family. In [99] Bača shows that the cylinders $C_n \times P_m$ have magic labelings of type $(1, 1, 0)$ when $m \geq 2, n \geq 3, n \neq 4$. In [109] Bača proves that the generalized Petersen graph $P(n, k)$ (see §2.7 for the definition) has a consecutive magic labeling if and only if $n$ is even and at least 4 and $k \leq n/2 - 1$.

Bača gave magic labelings of type $(1, 1, 1)$ for fans [93], ladders [93], planar bipyramids (that is, 2-point suspensions of paths) [93], grids [102], hexagonal lattices [101], Möbius ladders [96], and $P_n \times P_3$ [97]. Kathiresan and Ganesan [636] show that the graph $P_{a,b}$ consisting of $b \geq 2$ internally disjoint paths of length $a \geq 2$ with common end points has a magic labeling of type $(1, 1, 1)$ when $b$ is odd, and when $a = 2$ and $b \equiv 0 \pmod 4$. They also show that $P_{a,b}$ has a consecutive labeling of type $(1, 1, 1)$ when $b$ is even and $a \neq 2$.

Bača [95], [94], [105], [103], [97], [104] and Bača and Holländer [129] gave magic labelings of type $(1, 1, 1)$ and type $(1, 1, 0)$ for certain classes of convex polytopes. Kathiresan and Gokulakrishnan [638] provided magic labelings of type $(1, 1, 1)$ for the families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces, and one external infinite face. Bača [100] also provides consecutive and magic labelings of type $(0, 1, 1)$ (that is, an injective function from $\{1, 2, \ldots, |E| + |F|\}$ to $E \cup F$ with the property that for each interior face the sum of the labels of the face and the edges surrounding that face is some fixed value) and a consecutive labeling of type $(1, 1, 1)$ for a kind of planar graph with hexagonal faces.

A *magic labeling of type* (1,0,0) of a planar graph $G$ with vertex set $V$ is an injective function from $\{1, 2, \ldots, |V|\}$ to $V$ with the property that for each interior face the sum of the labels of the vertices surrounding that face is some fixed value. Kathiresan, Muthuvel, and Nagasubbu [639] define a *lotus inside a circle* as the graph obtained from the cycle with consecutive vertices $a_1, a_2, \ldots, a_n$ and the star with central vertex $b_0$ and end vertices $b_1, b_2, \ldots, b_n$ by joining each $b_i$ to $a_i$ and $a_{i+1}$ $(a_{n+1} = a_1)$. They prove that these graphs $(n \geq 5)$ and subdivisions of ladders have consecutive labelings of type $(1, 0, 0)$. Devaraj [360] proves that graphs obtained by subdividing each edge of a ladder exactly the same number of times has a magic labeling of type $(1, 0, 0)$.

Bača, Baskoro, Jendroľ, and Miller [118] investigated various $d$-antimagic labelings for graphs in the shape of hexagonal honeycombs. They use $H_n^m$ to denote the honeycomb graph with $m$ rows, $n$ columns, and $mn$ 6-sided faces. They prove: for $n$ odd $H_n^m$, has a 0-antimagic vertex labeling and a 2-antimagic edge labeling; if $n$ is odd and $mn > 1$, $H_n^m$ has a 1-antimagic face labeling; for $n$ odd and $mn > 1$, $H_n^m$ has $d$-antimagic labelings of type $(1, 1, 1)$ for $d = 1, 2, 3$, and 4.

In Table 10 we use following abbreviations

**M$(a,b,c)$** magic labeling of type $(a,b,c)$

**CM$(a,b,c)$** consecutive magic labeling of type $(a,b,c)$.

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Table 10: **Summary of Magic Labelings of Type** $(a, b, c)$

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $W_n$ | CM(1,1,0) | [791] |
| friendship graphs | CM(1,1,0) | [791] |
| prisms | CM(1,1,0) | [791] |
| cylinders $C_n \times P_m$ | M(1,1,0) | $m \geq 2$, $n \geq 3$, $n \neq 4$ [99] |
| fans $F_n$ | M(1,1,1) | [93] |
| ladders | M(1,1,1) | [93] |
| planar bipyramids (see §5.3) | M(1,1,1) | [93] |
| grids | M(1,1,1) | [102] |
| hexagonal lattices | M(1,1,1) | [101] |
| Möbius ladders | M(1,1,1) | [96] |
| $P_n \times P_3$ | M(1,1,1) | [97] |
| certain classes of<br>convex polytopes | M(1,1,1)<br>M(1,1,0) | [95], [105], [103], [97]<br>[104], [129] |
| certain classes of planar graphs<br>with hexagonal faces | M(0,1,1)<br>CM(0,1,1)<br>CM(1,1,1) | [100] |
| lotus inside a circle (see §5.3) | CM(1,0,0) | $n \geq 5$ [639] |
| subdivisions of ladders | M(1,0,0)<br>CM(1,0,0) | [360]<br>[639] |

## 5.5 Sigma Labelings/1-vertex magic labelings

In 1987 Vilfred [1213] (see also [1214]) defined a *sigma-labeling* of a graph $G$ with $n$ vertices as a bijection $f$ from the vertices of $G$ to $\{1, 2, \ldots, n\}$ such that there is a constant $k$ with the property that, at any vertex $v$ the sum $\sum f(u)$ taken over all neighbors $u$ of $v$ is $k$. In [1215] Vilfred and Jinnah give a number of necessary conditions for a graph to have a sigma labeling. One of them is that if $u$ and $v$ are vertices of a graph with a sigma labeling, then the order of the symmetric difference of $N(u)$ and $N(v)$ (neighborhoods of $u$ and $v$) is not 1 or 2. This condition rules out a large class of graphs as having sigma labelings. Rao, Singh, and Parameswaran [960] have shown $C_m \times C_n$ has a sigma labeling if and only if $m = n \equiv 2 \pmod 4$ and $K_m \times K_n,\ m \geq 2, n \geq 3$ does not have a sigma labeling. In [210] Benna gives necessary and sufficient condition for $K_{m,n}$ to be a sigma labeled graph and proves that if $G_1$ and $G_2$ are connected graphs with minimum degree 1 and at least three vertices, then $G_1 \times G_2$ does not have a sigma labeling. Rao, Sighn, and Parameswaran [32] prove that every graph is an induced subgraph of a regular graph that has a sigma labeling. As open problems, Rao [959] asks for a characterize 4-regular graphs that have sigma labelings and which graphs of the form $C_m \times C_n,\ m = n \equiv 2 \pmod 4$ have sigma labelings. Acharaya, Rao, Signh, and Parameswaran [31] proved $P_m \times C_n$ does not have a sigma labeling when $m$ is at least 3 and provide necessary and sufficient conditions for $K_{m,n}$ to have a sigma labeling.

The concept of sigma labeling was independently studied in 2003 by Miller, Rodger, and Simanjuntak in [869] under the name 1-*vertex magic vertex*. Among their results are: the only trees that have a 1-vertex magic labeling are $P_1$ and $P_3$; $C_n$ has a 1-vertex magic labeling if and only if $n = 4$; $K_n$ has a 1-vertex magic labeling if and only if $n = 1$; the wheel $W_n = C_n + P_1$ has a 1-vertex magic labeling if and only if $n = 4$; the complete graph $K_{n,n,\ldots,n}$ with $p$ partite sets has a 1-vertex magic labeling if and only if $n$ is even or both $n$ and $p$ are odd; an $r$-regular graph where $n$ is odd does not have a 1-vertex magic labeling; and $G \times \overline{K_{2n}}$ has a 1-vertex magic labeling for any regular graph $G$. They also give necessary and sufficient conditions for complete tripartite graphs to have a 1-vertex magic labeling.

In [1008] Seoud, Maqsoud, and Aldiban determined whether or not the following families of graphs have a 1-vertex magic vertex labeling: $K_n - \{e\}$; $K_n - \{2e\}$; $P_n^k$; $C_n^2$; $K_m \times C_n$; $C_m + P_n$; $C_m + C_n$; $P_m + P_n$; $K_{1,r,s}$; $K_{1,r,m,n}$; $K_{2,r,m,n}$; $K_{m,n} + P_k$; $K_{m,n} + C_k$; $C_m + \overline{K_n}$; $P_m + \overline{K_n}$; $P_m \times P_n$; $K_{m,n} \times P_k$; $K_m \times P_n$; the splitting graph of $K_{m,n}$; $K_n + G$; $K_m + \overline{K_n}$; $K_m + C_n$; $K_m + P_n$; $K_{m,n} + K_r$; $C_m \times P_n$; $C_m \times K_{1,n}$; $C_m \times K_{n,n}$; $C_m \times K_{n,n+1}$; $K_m \times K_{n,r}$; and $K_m \times K_n$. Typically, 1-vertex magic labelings exist only a few low parameter cases.

## 5.6 Other Types of Magic Labelings

In 2004 Babujee [80] and [81] introduced the notion of *bimagic* labeling in which there exist two constants $k_1$ and $k_2$ such that the sums involved in a specified type of magic labeling is $k_1$ or $k_2$. Thus a vertex-bimagic total labeling with bimagic constants $k_1$ and $k_2$ is the same as a vertex-magic total labeling except for each vertex $v$ the sum of the label of $v$ and all edges adjacent to $v$ may be $k_1$ or $k_2$. A bimagic labeling is of interest for graphs that do not have a magic labeling of a particular type. Bimagic labelings for which the number of sums equal to $k_1$ and the number of sums equal to $k_2$ differ by at most 1 are called *equitable*. When all sums except one are the same the labeling is called *almost magic*. Although the wheel $W_n$ does not have an edge-magic total labeling when when $n \equiv 3 \pmod 4$, Marr, Phillips and Wallis [845]

showed that these wheels have both equitable bimagic and almost magic labelings. They also show that whereas $nK_2$ has an edge-magic total labeling if and only if $n$ is odd, $nK_2$ has an edge-bimagic total labeling when $n$ is even and although even cycles do not have super edge-magic total labelings all cycles have super edge-bimagic total labelings. They conjecture that there is a constant $N$ such that $K_n$ has a edge-bimagic total labeling if and only if $n$ is at most $N$. They show that such an $N$ must be at least 8. They also prove that if $G$ has an edge-magic total labeling then $2G$ has an edge-bimagic total equitable labeling.

Babujee and Jagadesh [81], [86], [87], and [85] proved the following graphs have super edge bimagic labelings: cycles of length 3 with a nontrivial path attached; $P_3 \odot K_{1,n}$ $n$ even; $P_n + \overline{K_2}$ ($n$ odd);$P_2 + mK_1$ ($m \geq 2$); $2P_n$ ($n \geq 2$); the disjoint union of two stars; $3K_{1,n}$ ($n \geq 2$); $P_n \cup P_{n+1}$ ($n \geq 2$); $C_3 \cup K_{1,n}$;$P_n$; $K_{1,n}$; $K_{1,n,n}$; the graphs obtained by joining the centers of any two stars with an edge or a path of length 2; the graphs obtained by joining the centers of two copies of $K_{1,n}$ ($n \geq 3$) with a path of length 2 then joining the center one of copies of $K_{1,n}$ to the center of a third copy of $K_{1,n}$ with a path of length 2; combs $P_n \odot K_1$; cycles; wheels; fans; gears; $K_n$ if and only if $n \leq 5$.

In [819] López, Muntaner-Batle, and Rius-Font give a necessary condition for a complete graph to be edge bimagic in the case that the two constants have the same parity.

In [83] Babujee, Babitha, and Vishnupriya make the following definitions. For any natural number $a$, a graph $G(p,q)$ is said to be *$a$-additive super edge bimagic* if there exists a bijective function $f$ from $V(G) \cup E(G)$ to $\{a+1, a+2, \ldots, a+p+q\}$ such that for every edge $uv$, $f(u) + f(v) + f(uv) = k_1$ or $k_2$. For any natural number $a$, a graph $G(p,q)$ is said to be *$a$-multiplicative super edge bimagic* if there exists a bijective $f$ from $V(G) \cup E(G)$ to $\{a, 2a, \ldots, (p+q)a\}$ such that for every edge $uv$, $f(u) + f(v) + f(uv) = k_1$ or $k_2$. A graph $G(p,q)$ is said to be *super edge-odd bimagic* if there exists a bijection $f$ from $V(G) \cup E(G)$ to $\{1, 3, 5, \ldots, 2(p+q)-1\}$ such that for every edge $uv$ $f(u) + f(v) + f(uv) = k_1$ or $k_2$. If $f$ is a super edge bimagic labeling, then a function $g$ from $E(G)$ to $\{0,1\}$ with the property that for every edge $uv$, $g(uv) = 0$ if $f(u)+f(v)+f(uv) = k_1$ and $g(uv) = 1$ if $f(u)+f(v)+f(uv) = k_2$ is called a *super edge bimagic cordial labeling* if the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. They prove: super edge bimagic graphs are $a$-additive super edge bimagic; super edge bimagic graphs are $a$-multiplicative super edge bimagic; if $G$ is super edge-magic, then $G + K_1$ is super edge bimagic labeling; the union of two super edge magic graphs is super edge bimagic; and $P_n$, $C_{2n}$ and $K_{1,n}$ are super edge bimagic cordial.

For any nontrivial Abelian group $A$ under addition a graph $G$ is said to be *$A$-magic* if there exists a labeling $f$ of the edges of $G$ with the nonzero elements of $A$ such that the vertex labeling $f^+$ defined by $f^+(v) = \Sigma f(vu)$ over all edges $vu$ is a constant. In [1131] and [1132] Stanley noted that $Z$-magic graphs can be viewed in the more general context of linear homogeneous diophantine equations. Shiu, Lam, and Sun [1071] have shown the following: the union of two edge-disjoint $A$-magic graphs with the same vertex set is $A$-magic; the Cartesian product of two $A$-magic graphs is $A$-magic; the lexicographic product of two $A$-magic connected graphs is $A$-magic; for an Abelian group $A$ of even order a graph is $A$-magic if and only if the degrees of all of its vertices have the same parity; if $G$ and $H$ are connected and $A$-magic, $G$ composed with $H$ is $A$-magic; $K_{m,n}$ is $A$-magic when $m, n \geq 2$ and $A$ has order at least 4; $K_n$ with an edge deleted is $A$-magic when $n \geq 4$ and $A$ has order at least 4; all generalized theta graphs (§4.4 for the definition) are $A$-magic when $A$ has order at least 4; $C_n + \overline{K_m}$ is $A$-magic when $n \geq 3, m \geq 2$ and $A$ has order at least 2; wheels are $A$-magic when $A$ has order at least 4; flower graphs $C_m@C_n$ are $A$-magic when $m, n \geq 2$ and $A$ has order at least 4 ($C_m@C_n$ is obtained

from $C_n$ by joining the end points of a path of length $m-1$ to each pair of consecutive vertices of $C_n$).

In [739] Lee, Saba, Salehi, and Sun investigate graphs that are $A$-magic where $A = V_4 \approx Z_2 \oplus Z_2$ is the Klein four-group. Many of theorems are special cases of the results of Shiu, Lam, and Sun [1071] given in the previous paragraph. They also prove the following are $V_4$-magic: a tree if and only if every vertex has odd degree; the star $K_{1,n}$ if and only if $n$ is odd; $K_{m,n}$ for all $m, n \geq 2$; $K_n - e$ (edge deleted $K_n$) when $n > 3$; even cycles with $k$ pendent edges if and only if $k$ is even; odd cycles with $k$ pendent edges if and only if $k$ is odd; wheels; $C_n + \overline{K_2}$; generalized theta graphs; graphs that are copies of $C_n$ that share a common edge; and $G + \overline{K_2}$ whenever $G$ is $V_4$-magic.

Babujee and Shobana [92] prove that the following graphs have $Z_3$-magic labelings: $C_{2n}$; $K_n$ $(n \geq 4)$; $K_{m,2m}$ $(m \geq 3)$; ladders $P_n \times P_2$ $(n \geq 4)$; bistars $B_{3n-1,3n-1}$; and cyclic, dihedral and symmetric Cayley digraphs for certain generating sets. In [327] Chou and Lee investigate $Z_3$-magic graphs. They also show that every graph is an induced subgraph of an $A$-magic graph for any non-trivial Abelian group $A$. Thus it is impossible to find a Kuratowski type characterization of $A$-magic graphs. Low and Lee [821] have shown that if a graph is $A_1$-magic then it is $A_2$-magic for any subgroup $A_2$ of $A_1$ and for any nontrivial Abelian group $A$ every Eulerian graph of even size is $A$-magic. For a connected graph $G$, Low and Lee define $T(G)$ to be the graph obtained from $G$ by adding a disjoint $uv$ path of length 2 for every pair of adjacent vertices $u$ and $v$. They prove that for every finite nontrivial Abelian group $A$ the graphs $T(P_{2k})$ and $T(K_{1,2n+1})$ are $A$-magic. Shiu and Low [1077] show that $K_{k_1,k_2,\ldots,k_n}(k_i \geq 2)$ is $A$-magic, for all $A$ where $|A| \geq 3$. Lee, Salehi and Sun [742] have shown that for $m, n \geq 3$ the double star $DS(m,n)$ is $Z$-magic if and only if $m = n$.

In [688] Kwong and Lee call the set of all $k$ for which a graph is $Z_k$-magic the *integer-magic spectrum* of the graph. They investigate the integer-magic spectra of the coronas of some specific graphs including paths, cycles, complete graphs, and stars. Low and Sue [824] have obtained some results on the integer-magic spectra of tessellation graphs. Shiu and Low [1078] provide the integer-magic spectra of sun graphs. Chopra and Lee [325] determined the integer-magic spectra of all graphs consisting of any number of pairwise disjoint paths with common end vertices (that is, generalized theta graphs). Low and Lee [821] show that Eulerian graphs of even size are $A$-magic for every finite nontrivial Abelian group $A$ whereas Wen and Lee [1253] provide two families of Eularian graphs that are not $A$-magic for every finite nontrivial Abelian group $A$ and eight infinite families of Eulerian graphs of odd sizes that are $A$-magic for every finite nontrivial Abelian group $A$. Low and Lee [821] also prove that if $A$ is an Abelian group and $G$ and $H$ are $A$-magic, then so are $G \times H$ and the lexicographic product of $G$ and $H$. Low and Shiu [823] prove: $K_{1,n} \times K_{1,n}$ has a $Z_{n+1}$-magic labeling with magic constant 0; if $G \times H$ is $Z_2$-magic, then so are $G$ and $H$; if $G$ is $Z_m$-magic and $H$ is $Z_n$-magic, then the integer-magic spectra of $G \times H$ contains all common multiples of $m$ and $n$; if $n$ is even and $k_i \geq 3$ then the integer-magic spectra of $P_{k_1} \times P_{k_2} \times \cdots \times P_{k_n} = \{3, 4, 5, \ldots\}$. In [1080] Shiu and Low determine all positive integers $k$ for which fans and wheels have a $Z_k$-magic labeling with magic constant 0.

Shiu and Low [1079] have introduced the notion of ring-magic as follows. Given a commutative ring $R$ with unity, a graph $G$ is called *$R$-ring-magic* if there exists a labeling $f$ of the edges of $G$ with the nonzero elements of $R$ such that the vertex labeling $f^+$ defined by $f^+(v) = \Sigma f(vu)$ over all edges $vu$ and vertex labeling $f^\times$ defined by $f^\times(v) = \Pi f(vu)$ over all edges $vu$ are constant. They give some results about $R$-ring-magic graphs.

In [281] Cahit says that a graph $G(p,q)$ is *total magic cordial* (TMC) provided there is a

mapping $f$ from $V(G) \cup E(G)$ to $\{0,1\}$ such that $(f(a) + f(b) + f(ab))$ mod 2 is a constant modulo 2 for all edges $ab \in E(G)$ and $|f(0) - f(1)| \leq 1$ where $f(0)$ denotes the sum of the number of vertices labeled with 0 and the number of edges labeled with 0 and $f(1)$ denotes the sum of the number of vertices labeled with 1 and the number of edges labeled with 1. He says a graph $G$ is *total sequential cordial* (TSC) if there is a mapping $f$ from $V(G) \cup E(G)$ to $\{0,1\}$ such that for each edge $e = ab$ with $f(e) = |f(a) - f(b)|$ it is true that $|f(0) - f(1)| \leq 1$ where $f(0)$ denotes the sum of the number of vertices labeled with 0 and the number of edges labeled with 0 and $f(1)$ denotes the sum of the number of vertices labeled with 1 and the number of edges labeled with 1. He proves that the following graphs have a TMC labeling: $K_{m,n}$ $(m, n > 1)$, trees, cordial graphs, and $K_n$ if and only if $n = 2, 3, 5$, or 6. He also proves that the following graphs have a TSC labeling: trees; cycles; complete bipartite graphs; friendship graphs; cordial graphs; cubic graphs other than $K_4$; wheels $W_n$ $(n > 3)$; $K_{4k+1}$ if and only if $k \geq 1$ and $\sqrt{k}$ is an integer; $K_{4k+2}$ if and only if $\sqrt{4k+1}$ is an integer; $K_{4k}$ if and only if $\sqrt{4k+1}$ is an integer; and $K_{4k+3}$ if and only if $\sqrt{k+1}$ is an integer.

In 2001, Simanjuntak, Rodgers, and Miller [869] defined a *1-vertex magic vertex labeling* of $G(V, E)$ as a bijection from $V$ to $\{1, 2, \ldots, |V|\}$ with the property that there is a constant $k$ such that at any vertex $v$ the sum $\sum f(u)$ taken over all neighbors of $v$ is $k$. Among their results are: $H \times \overline{K}_{2k}$ has a 1-vertex-magic vertex labeling for any regular graph $H$; the symmetric complete multipartite graph with $p$ parts, each of which contains $n$ vertices, has a 1-vertex-magic vertex labeling if and only if whenever $n$ is odd, $p$ is also odd; $P_n$ has a 1-vertex-magic vertex labeling if and only if $n = 1$ or 3; $C_n$ has a 1-vertex-magic vertex labeling if and only if $n = 4$; $K_n$ has a 1-vertex-magic vertex labeling if and only if $n = 1$; $W_n$ has a 1-vertex-magic vertex labeling if and only if $n = 4$; a tree has a 1-vertex-magic vertex labeling if and only if it is $P_1$ or $P_3$; and $r$-regular graphs with $r$ odd do not have a 1-vertex-magic vertex labeling.

Miller, Rogers, and Simanjuntak [869] the complete $p$-partite $(p > 1)$ graph $K_{n,n,\ldots,n}$ $(n > 1)$ has a 1-vertex-magic vertex labeling if and only if either $n$ is even or $np$ is odd. Shafiq, Ali, Simanjuntak [1052] proved $mK_{n,n,\ldots,n}$ has a 1-vertex-magic vertex labeling if $n$ is even or $mnp$ is odd and $m \geq 1, n > 1$, and $p > 1$ and $mK_{n,n,\ldots,n}$ does not have a 1-vertex-magic vertex labeling if $np$ is odd, $p \equiv 3$ (mod 4, and $m$ is even.

Recall if $V(G) = \{v_1, v_2, \ldots, v_p\}$ is the vertex set of a graph $G$ and $H_1, H_2, \ldots, H_p$ are isomorphic copies of a graph $H$, then $G[H]$ is the graph obtained from $G$ by replacing each vertex $v_i$ of $G$ by $H_i$ and joining every vertex in $H_i$ to every neighbor of $v_i$. Shafiq, Ali, Simanjuntak [1052] proved if $G$ is an $r$-regular graph $(r \geq 1)$ then $G[C_n]$ has a 1-vertex-magic vertex labeling if and only if $n = 4$. They also prove that for $m \geq 1$ and $n > 1$, $mC_p[\overline{K_n}]$ has 1-vertex-magic vertex labeling if and only if either $n$ is even or $mnp$ is odd or $n$ is odd and $p \equiv 3$ (mod 4).

Balbuena, Barker, Lin, Miller, and Sugeng [182] call a vertex-magic total labeling of a graph $G(V, E)$ an *a-vertex consecutive magic labeling* if the vertex labels are $\{a+1, a+2, \ldots, a+|V|\}$ where $0 \leq a \leq |E|$. They prove: if a tree of order $n$ has an $a$-vertex consecutive magic labeling then $n$ is odd and $a = n - 1$; if $G$ has an $a$-vertex consecutive magic labeling with $n$ vertices and $e = n$ edges, then $n$ is odd and if $G$ has minimum degree 1, then $a = (n+1)/2$ or $a = n$; if $G$ has an $a$-vertex consecutive magic labeling with $n$ vertices and $e$ edges such that $2a \leq e$ and $2e \geq \sqrt{6}n - 1$, then the minimum degree of $G$ is at least 2; if a 2-regular graph of order $n$ has an $a$-vertex consecutive magic labeling, then $n$ is odd and $a = 0$ or $n$; and if a $r$-regular graph of order $n$ has an $a$-vertex consecutive magic labeling, then $n$ and $r$ have opposite parities.

Balbuena et al. also call a vertex-magic total labeling of a graph $G(V, E)$ a *b-edge consecutive*

*magic labeling* if the edge labels are $\{b+1, b+2, \ldots, b+|E|\}$ where $0 \le b \le |V|$. They prove: if $G$ has $n$ vertices and $e$ edges and has a $b$-edge consecutive magic labeling and one isolated vertex, then $b = 0$ and $(n-1)^2+n^2 = (2e+1)^2$; if a tree with odd order has a $b$-edge consecutive magic labeling then $b = 0$; if a tree with even order has a $b$-edge consecutive magic labeling then it is $P_4$; a graph with $n$ vertices and $e$ edges such that $e \ge 7n/4$ and $b \ge n/4$ and a $b$-edge consecutive magic labeling has minimum degree 2; if a 2-regular graph of order $n$ has a $b$-edge consecutive magic labeling, then $n$ is odd and $b = 0$ or $b = n$; and if a $r$-regular graph of order $n$ has an $b$-edge consecutive magic labeling, then $n$ and $r$ have opposite parities.

Sugeng and Miller [1140] prove: If $(V, E)$ has an $a$-vertex consecutive edge magic labeling, where $a \ne 0$ and $a \ne |E|$, then $G$ is disconnected; if $(V, E)$ has an $a$-vertex consecutive edge magic labeling, where $a \ne 0$ and $a \ne |E|$, then $G$ cannot be the union of three trees with more than one vertex each; for each nonnegative $a$ and each positive $n$, there is an $a$-vertex consecutive edge magic labeling with $n$ vertices; the union of $r$ stars and a set of $r-1$ isolated vertices has an $s$-vertex consecutive edge magic labeling, where $s$ is the minimum order of the stars; for every $b$ every caterpillar has a $b$-edge consecutive edge magic labeling; if a connected graph $G$ with $n$ vertices has a $b$-edge consecutive edge magic labeling where $1 \le b \le n-1$, then $G$ is a tree; the union of $r$ stars and a set of $r-1$ isolated vertices has an $r$-edge consecutive edge magic labeling.

Babujee, Vishnupriya, and Jagadesh [170] introduced a labeling called *a-vertex consecutive edge bimagic total* as a graph $G(V, E)$ for which there are two positive integers $k_1$ and $k_2$ and a bijection $f$ from $V \cup E$ to $\{1, 2, \ldots, |V| + |E|\}$ such that $f(u) + f(v) + f(uv) = k_1$ or $k_2$ for all edges $uv$ and $f(V) = \{a+1, a+2, \ldots, a+|V|\}$, $0 \le a \le |V|$. They proved the following graphs have such labelings: $P_n$, $K_{1,n}$, combs, bistars $B_{m,n}$, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, trees obtained by joining the centers of two stars with a path of length 2, trees obtained from $P_5$ by identifying the center of a copy $K_{1,n}$ with the two end vertices and the middle vertex. In [162] Babujee and Jagadesh proved that cycles, fans, wheels, and gear graphs have $a$-vertex consecutive edge bimagic total labelings. Babujee, Jagadesh, Vishnupriya [164] study the properties of $a$-vertex consecutive edge bimagic total labeling for $P_3 \odot K_{1,2n}$, $P_n + \overline{K_2}$ ($n$ is odd and $n \ge 3$), $(P_2 \cup mK_1) + \overline{K_2}$, $(P_2 + mK_1)$ $(m \ge 2)$, $C_n$, fans $P_n + K_1$, double fans $P_n + 2K_1$, and graphs obtained by appending a path of length at least 2 to a vertex of $C_3$. Babujee, Jagadesh [163] prove the following graphs have $a$-vertex consecutive edge bimagic total labelings: $2P_n$ $(n \ge 2)$, $P_n \cup P_n + 1 (n \ge 2)$, $K_{2,n}$, $C_n \odot K_1$, and that $C_3 \cup K_{1,n}$ an $a$-vertex consecutive edge bimagic labeling for $a = n + 3$

In 2005 Gutiérrez and Lladó [500] introduced the notion of an $H$-magic labeling of a graph, which generalizes the concept of a magic valuation. Let $H$ and $G = (V, E)$ be finite simple graphs with the property that every edge of $G$ belongs to at least one subgraph isomorphic to $H$. A bijection $f: V \cup E \to \{1, \ldots, |V| + |E|\}$ is an *$H$-magic labeling* of $G$ if there exists a positive integer $m(f)$, called the *magic sum*, such that for any subgraph $H'(V', E')$ of $G$ isomorphic to $H$, the sum

$$\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$$

is equal to the magic sum, $m(f)$. A graph is *$H$-magic* if it admits an $H$-magic labeling. If, in addition, the $H$-magic labeling $f$ has the property that $\{f(v)\}_{v \in V} = \{1, \ldots, |V|\}$, then the graph is *$H$-supermagic*. A $K_2$-magic labeling is also known as an edge-magic total labeling. Gutiérrez and Lladó investigate the cases where $G = K_n$ or $G = K_{m,n}$ and $H$ is a star or a path. Among their results are: a $d$-regular graph is not $K_{1,h}$ for any $1 < h < d$; $K_{n,n}$ is $K_{1,n}$-magic for all $n$;

$K_{n,n}$ is not $K_{1,n}$-supermagic for $n > 1$; for any integers $1 < r < s$, $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$; $P_n$ is $P_h$-supermagic for all $2 \leq h \leq n$; $K_n$ is not $P_h$-magic for any $2 < h \leq n$; $C_n$ is $P_h$-magic for any $2 \leq h < n$ such that $\gcd(n, h(h-1)) = 1$. They also show that by uniformly gluing copies of $H$ along edges of another graph $G$, one can construct connected $H$-magic graphs from a given 2-connected graph $H$ and an $H$-free supermagic graph $G$.

Lladó and Moragas [814] studied cycle-magic graphs. They proved: wheels $W_n$ are $C_3$-magic for odd $n$ at least 5; for $r \geq 3$ and $k \geq 2$ the windmill graphs $C_r^{(k)}$ (the one-point union of $k$ copies of $C_r$) are $C_r$-supermagic; and if $G$ is $C_4$-free supermagic graph of odd size, then $G \times K_2$ is $C_4$-supermagic. As corollaries of the latter result, they have that for $n$ odd, prisms $C_n \times K_2$ and books $K_{1,n} \times K_2$ are $C_4$-magic. They define a subdivided wheel $W_n(r,k)$ as the graph obtained from a wheel $W_n$ by replacing each radial edge $vv_i, 1 \leq i \leq n$ by a $vv_i$-path of size $r \geq 1$, and every external edge $v_iv_{i+1}$ by a $v_iv_{i+1}$-path of size $k \geq 1$. They prove that $W_n(r,k)$ is $C_{2r+k}$-magic for any odd $n \neq 2r/k + 1$ and that $W_n(r,1)$ is $C_{2r+1}$-supermagic. They also prove that the graph obtained by joining the end points of any number of internally disjoint paths of length $p \geq 2$ is $C_{2p}$-supermagic.

In [846] Maryati, Baskoro, and Salman provided $P_h$-(super)magic labelings of shrubs and banana trees. Ngurah, Salman, and Sudarsana [903] construct $C_h$-(super)magic labelings for some fans and ladders. For any connected graph $H$, Maryati, Salman, Baskoro, and Irawati [847] proved that the disjoint union of $k$ isomorphic copies of a connected graph $H$ is a $H$-supermagic graph if and only if $|V(H)| + |E(H)|$ is even or $k$ is odd.

[848] Maryati, Salman, Baskoro, Ryan, and Miller define a *shackle* as a graph obtained from non-trivial connected graphs $G_1, G_2, \ldots, G_k$ $(k \geq 2)$ such that $G_s$ and $G_t$ have no common vertex for every $s$ and $t$ in $[1,k]$ with $|s-t| \geq 2$, and for every $i$ in $[1,k-1]$, $G_i$ and $G_{i+1}$ share exactly one common vertex that are all distinct. They prove that shackles and amalgamations constructed from copies of a connected graph $H$ is $H$-supermagic. (Recall for finite collection of graph $G_1, G_2, \ldots, G_k$ with a fixed vertex $v_i$ from each $G_i$, an *amalgamation*, $\mathrm{Amal}G_i, v_i)$, is the graph obtained by identifying the $v_i$.)

Ngurah, Salman, and Susilowati [904] proved the following: chain graphs with identical blocks each isomorphic to $C_n$ are $C_n$-supermagic; fans are $C_3$-supermagic; ladders and books are $C_4$-supermagic; $K_{1,n} + K_1$ are $C_3$-supermagic; grids $P_m \times P_n$ are $C_4$-supermagic for $m \geq 3$ and $n = 3, 4$, and 5. They pose the case that $P_m \times P_n$ are $C_4$-supermagic for $n > 5$ as an open problem. They also have some results on $P_t$-(super)magic labelings of cycles.

Selvagopal and Jeyanthi proved: for any positive integer $n$, a the $k$-polygonal snake of length $n$ is $C_k$-supermagic [1002]; for $m \geq 2$, $n = 3$, or $n > 4$, $C_n \times P_m$ is $C_4$-supermagic [606]; s $P_2 \times P_n$ and $P_3 \times P_n$ are $C_4$-supermagic for all $n \geq 2$ [606]; the one-point union of any number of copies of a 2-connected $H$ is $H$-magic [604]; graphs obtained by taking copies $H_1, H_2, \ldots, H_n$ of a 2-connected graph $H$ and two distinct edges $e_i, e_i'$ from each $H_i$ and identifying $e_i'$ of $H_i$ with $e_{i+1}$ of $H_{i+1}$ where $|V(H)| \geq 4, |E(H)| \geq 4$ and $n$ is odd or both $n$ and $|V(H)| + |E(H)|$ are even are $H$-supermagic [604]. For simple graphs $H$ and $G$ the *$H$-supermagic strength of $G$* is the minimum constant value of all $H$-magic total labelings of $G$ for which the vertex labels are $\{1, 2, \ldots, |V|\}$. Jeyanthi and Selvagopal [605] found the $C_n$-supermagic strength of $n$-polygonal snakes of any length and the $H$-supermagic strength of a chain of an arbitrary 2-connected simple graph.

Let $H_1, H_2, \ldots, H_n$ be copies of a graph $H$. Let $u_i$ and $v_i$ be two distinct vertices of $H_i$ for $i = 1, 2, \ldots, n$. The *chain graph* $H_n$ of $H$ of length $n$ is the graph obtained by identifying the vertices $u_i$ and $v_{i+1}$ for $i = 1, 2, \ldots, n-1$. In [603] Jayanthi and Selvagopal show that a

chain graph of any 2-connected simple graph $H$ is $H$-supermagic and if $H$ is a 2-connected $(p, q)$ simple graph, then $H_n$ is $H$-supermagic if $p + q$ is even or $p + q + n$ is even.

The *antiprism* on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ (subscripts are taken modulo $n$). Jeyanthi, Selvagopal, and Sundaram [608] proved the following graphs are $C_3$-supermagic: antiprisms, fans, and graphs obtained from the ladders $P_2 \times P_n$ with the two paths $v_{1,1}, \ldots, v_{1,n}$ and $v_{2,1}, \ldots, v_{2,n}$ by adding the edges $v_{1,j}v_{2,j+1}$.

Jeyanthi and Selvagopal [607] show that for any 2-connected simple graph $H$ the edge amalogamation of a finite number of copies of $H$ is $H$-supermagic. They also show that the graph obtained by picking one endpoint $v_i$ from each of $k$ copies of $K_{1,k}$ then creating a new graph by joining each $v_i$ to a fixed new vertex $v$ is $K_{1,k}$-supermagic.

Vishnupriya, Manimekalai, and Babujee [1226] define a labeling $f$ of a graph $G(p, q)$ to be a *edge bimagic total labeling* if there exists a bijection $f$ from $V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that for each edge $e = (u, v) \in E(G)$ we have $f(u) + f(e) + f(v) = k_1$ or $k_2$, where $k_1$ and $k_2$ are two constants. They provide edge bimagic total labelings for $B_{m,n}$, $K_{1,n,n}$, and trees obtained from a path by appending an edge to one of the vertices adjacent to an endpoint of the path. An edge bimagic total labeling is $G(V, E)$ is called an *a-vertex consecutive edge bimagic total labeling* if the vertex labels are $\{a+1, a+2, \ldots, a+|V|\}$ where $0 \leq a \leq |E|$. Babujee and Jagadesh [89] prove the following graphs $a$-vertex consecutive edge-bimagic total labelings: the trees obtained from $K_{1,n}$ by adding a new pendent edge to each of the existing $n$ pendent vertices; the trees obtained by adding a pendent path of length 2 to each of the $n$ pendent vertices of $K_{1,n}$; the graphs obtained by joining the centers of two copies of identical stars by a path of length 2; and the trees obtained from a path by adding new pendent edges to one pendent vertex of the path. Babujee, Vishnupriya, and Jagadesh [170] proved the following graphs have such labelings: $P_n$, $K_{1,n}$, combs, bistars $B_{m,n}$, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, trees obtained by joining the centers of two stars with a path of length 2, trees obtained from $P_5$ by identifying the center of a copy $K_{1,n}$ with the two end vertices and the middle vertex. In [162] Babujee and Jagadesh proved that cycles, fans, wheels, and gear graphs have $a$-vertex consecutive edge bimagic total labelings. Babujee, Jagadesh, Vishnupriya [164] study the properties of $a$-vertex consecutive edge bimagic total labeling for $P_3 \odot K_{1,2n}$, $P_n + \overline{K_2}$ ($n$ is odd and $n \geq 3$), $(P_2 \cup mK_1) + \overline{K_2}$, $(P_2 + mK_1)$ $(m \geq 2), C_n$, fans $P_n + K_1$, double fans $P_n + 2K_1$, and graphs obtained by appending a path of length at least 2 to a vertex of $C_3$. J. Babujee, R. Jagadesh [163] prove the following graphs have $a$-vertex consecutive edge bimagic total labelings: $2P_n$ $(n \geq 2)$, $P_n \cup P_n + 1 (n \geq 2)$, $K_{2,n}$, $C_n \odot K_1$, and that $C_3 \cup K_{1,n}$ an $a$-vertex consecutive edge bimagic labeling for $a = n + 3$ Vishnupriya, Manimekalai, and Babujee [1226] prove that bistars, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, and trees obtained subdividing each edge of a star have edge bimagic total labelings. Prathap and Babujee [941] obtain all possible edge magic total labelings and edge bimagic total labelings for the star $K_{1,n}$.

Magic labelings of directed graphs are discussed in [843] and [241].

# 6 Antimagic-type Labelings

## 6.1 Antimagic Labelings

Hartsfield and Ringel [511] introduced antimagic graphs in 1990. A graph with $q$ edges is called *antimagic* if its edges can be labeled with $1, 2, \ldots, q$ such that the sums of the labels of the edges incident to each vertex are distinct. Among the graphs they prove are antimagic are: $P_n$ $(n \geq 3)$, cycles, wheels, and $K_n$ $(n \geq 3)$. T. Wang [1238] has shown that the toroidal grids $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$ are antimagic and, more generally, graphs of the form $G \times C_n$ are antimagic if $G$ is an $r$-regular antimagic graph with $r > 1$. Cheng [319] proved that all Cartesian products or two or more regular graphs of positive degree are antimagic and that if $G$ is $j$-regular and $H$ has maximum degree at most $k$, minimum degree at least one ($G$ and $H$ need not be connected), then $G \times H$ is antimagic provided that $j$ is odd and $j^2 - j \geq 2k$, or $j$ is even and $j^2 > 2k$. Wang and Hsiao [1240] prove the following graphs are antimagic: $G \times P_n$ $(n > 1)$ where $G$ is regular; $G \times K_{1,n}$ where $G$ is regular; compositions $G[H]$ (see §2.3 for the definition) where $H$ is $d$-regular with $d > 1$; and the Cartesian product of any double star (two stars with an edge joining their centers) and a regular graph. In [318] Cheng proved that $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_t}$ $(t \geq 2)$ is antimagic. Cranston [342] used the Marriage Theorem to prove that every regular bipartite graph with degree at least 2 is anitmagic. Lee, Lin and Tsai [702] proved that $C_n^2$ is antimagic and the vertex sums form a set of successive integers when $n$ is odd.

Phanalasy, Miller, Rylands and Lieby [932] in 2011 showed that there is a relationship between completely separating systems and labeling of regular graphs. Based on this relationship they proved that some regular graphs are antimagic. Phanalasy, Miller, Iliopoulos, Pissis and Vaezpour [930] proved the Cartesian product of regular graphs obtained from [932] is antimagic. Ryan, Phanalasy, Miller and Rylands introduced the generalized web and flower graphs in [978] and proved that these families of graphs are antimagic. Rylands, Phanalasy, Ryan and Miller extended the concept of generalized web graphs to the single apex multi-generalized web graphs and they proved these graphs to be antimagic in [980]. Ryan, Phanalasy, Rylands and Miller extended the concept of generalized flower to the single apex multi-(complete) generalized flower graphs and constructed antimagic labeling for this family of graphs in [979]. For more about antimagicness of generalized web and flower graphs see [867]. Phanalasy, Ryan, Miller and Arumugam [931] introduced the concept of generalized pyramid graphs and they constructed antimagic labeling for these graphs. Bača, Miller, Phanalasy and Feňovčíková proved that some join graphs and incomplete join graphs are antimagic in [152].

A *split* graph is a graph that has a vertex set that can be partitioned into a clique and an independent set. Tyshkevich (see [202]) defines a canonically decomposable graph as follows. For a split graph $S$ with a given partition of its vertex set into an independent set $A$ and a clique $B$ (denoted by $S(A, B)$), and an arbitrary graph $H$ the composition $S(A, B) \circ H$ is the graph obtained by taking the disjoint union of $S(A, B)$ and $H$ and adding to it all edges having an endpoint in each of $B$ and $V(H)$. If $G$ contains nonempty induced subgraphs $H$ and $S$ and vertex subsets $A$ and $B$ such that $G = S(A, B) \circ H$, then $G$ is *canonically decomposable*; otherwise $G$ is *canonically indecomposable*. Barrus [202] proved that every connected graph on at least 3 vertices that is split or canonically decomposable is antimagic.

Hartsfield and Ringel [511] conjecture that every tree except $P_2$ is antimagic and, moreover, every connected graph except $P_2$ is antimagic. Alon, Kaplan, Lev, Roditty, and Yuster [58] use probabilistic methods and analytic number theory to show that this conjecture is true for all graphs with $n$ vertices and minimum degree $\Omega(\log n)$. They also prove that if $G$ is a graph with

$n \geq 4$ vertices and $\Delta(G) \geq n-2$, then $G$ is antimagic and all complete partite graphs except $K_2$ are antimagic. Chawathe and Krishna [307] proved that every complete $m$-ary tree is antimagic.

Kaplan, Lev, and Roditty [629] prove that every non-trivial rooted tree for which every vertex that is a not a leaf has at least two children is antimagic. For a graph $H$ with $m$ vertices and an Abelian group $G$ they define $H$ to be *$G$-antimagic* if there is a one-to-one mapping from the edges of $H$ to the nonzero elements of $G$ such that the sums of the labels of the edges incident to $v$, taken over all vertices $v$ of $H$, are distinct. For any $n \geq 2$ they show that a non-trivial rooted tree with $n$ vertices for which every vertex that is a not a leaf has at least two children is $Z_n$-antimagic if and only if $n$ is odd. They also show that these same trees are $G$-antimagic for elementary Abelian groups $G$ with prime exponent congruent to 1 (mod 3).

Sonntag [1125] has extended the notion of antimagic labelings to hypergraphs. He shows that certain classes of cacti, cycle, and wheel hypergraphs have antimagic labelings. In [145] Bača, MacDougall, Miller, Slamin, and Wallis survey results on antimagic, edge-magic total, and vertex-magic total labelings.

In [519] Hefetz, Mütze, and Schwartz investigate antimagic labelings of directed graphs. An *antimagic* labeling of a directed graph $D$ with $n$ vertices and $m$ arcs is a bijection from the set of arcs of $D$ to the integers $\{1, \ldots, m\}$ such that all $n$ oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of all edges entering that vertex minus the sum of labels of all edges leaving it. Hefetz et al. raise the questions "Is every orientation of any simple connected undirected graph antimagic? and "Given any undirected graph $G$, does there exist an orientation of $G$ which is antimagic?" They call such an orientation an *antimagic orientation* of $G$. Regarding the first question, they state that, except for $K_{1,2}$ and $K_3$, they know of no other counterexamples. They prove that there exists an absolute constant $C$ such that for every undirected graph on $n$ vertices with minimum degree at least $C\log\, n$ every orientation is antimagic. They also show that every orientation of $S_n,\ n \neq 2$, is antimagic; every orientation of $W_n$ is antimagic; and every orientation of $K_n,\ n \neq 3$, is antimagic. For the second question they prove: for odd $r$, every undirected $r$-regular graph has an antimagic orientation; for even $r$ every undirected $r$-regular graph that admits a matching that covers all but at most one vertex has an antimagic orientation; and if $G$ is a graph with $2n$ vertices that admits a perfect matching and has an independent set of size $n$ such that every vertex in the independent set has degree at least 3, then $G$ has an antimagic orientation. They conjecture that every connected undirected graph admits an antimagic orientation and ask if it true that every connected directed graph with at least 4 vertices is antimagic.

Hefetz [518] calls a graph with $q$ edges *$k$-antimagic* if its edges can be labeled with $1, 2, \ldots, q+k$ such that the sums of the labels of the edges incident to each vertex are distinct. In particular, antimagic is the same as 0-antimagic. More generally, given a weight function $\omega$ from the vertices to the natural numbers Hefetz calls a graph with $q$ edges *$(\omega, k)$-antimagic* if its edges can be labeled with $1, 2, \ldots, q+k$ such that the sums of the labels of the edges incident to each vertex and the weight assigned to each vertex by $\omega$ are distinct. In particular, antimagic is the same as $(\omega, 0)$-antimagic where $\omega$ is the zero function. Using Alon's combinatorial nullstellensatz [57] as his main tool, Hefetz has proved the following: a graph with $3^m$ vertices and a $K_3$ factor is antimagic; a graph with $q$ edges and at most one isolated vertex and no isolated edges is $(\omega, 2q-4)$-antimagic; a graph with $p > 2$ vertices that admits a 1-factor is $(p-2)$-antimagic; a graph with $p$ vertices and maximum degree $n-k$, where $k \geq 3$ is any function of $p$ is $(3k-7)$-antimagic and, in the case that $p \geq 6k^2$, is $(k-1)$-antimagic. Hefetz, Saluz, and Tran [520] improved the first of Hefetz's results by showing that a graph with $p^m$ vertices, where $p$ is an

odd prime and $m$ is positive, and a $C_p$ factor is antimagic.

The concept of an $(a,d)$-antimagic labelings was introduced by Bodendiek and Walther [244] in 1993. A connected graph $G=(V,E)$ is said to be $(a,d)$-*antimagic* if there exist positive integers $a$, $d$ and a bijection $f: E \to \{1,2,\ldots,|E|\}$ such that the induced mapping $g_f: V \to N$, defined by $g_f(v) = \sum\{f(uv)|\ uv \in E(G)\}$, is injective and $g_f(V) = \{a, a+d, \ldots, a+(|V|-1)d\}$. (In [795] Lin, Miller, Simanjuntak, and Slamim called these $(a,d)$-*vertex-antimagic edge labelings*). Bodendick and Walther ([246] and [247]) prove the Herschel graph is not $(a,d)$-antimagic and obtain both positive and negative results about $(a,d)$-antimagic labelings for various cases of graphs called *parachutes* $P_{g,p}$. ($P_{g,p}$ is the graph obtained from the wheel $W_{g+p}$ by deleting $p$ consecutive spokes.) In [130] Bača and Holländer prove that necessary conditions for $C_n \times P_2$ to be $(a,d)$-antimagic are $d=1,\ a=(7n+4)/2$ or $d=3,\ a=(3n+6)/2$ when $n$ is even, and $d=2,\ a=(5n+5)/2$ or $d=4,\ a=(n+7)/2$ when $n$ is odd. Bodendiek and Walther [245] conjectured that $C_n \times P_2$ $(n \geq 3)$ is $((7n+4)/2,1)$-antimagic when $n$ is even and is $((5n+5)/2,2)$-antimagic when $n$ is odd. These conjectures were verified by Bača and Holländer [130] who further proved that $C_n \times P_2$ $(n \geq 3)$ is $((3n+6)/2,3)$-antimagic when $n$ is even. Bača and Holländer [130] conjecture that $C_n \times P_2$ is $((n+7)/2,4)$-antimagic when $n$ is odd and at least 7. Bodendiek and Walther [245] also conjectured that $C_n \times P_2$ $(n \geq 7)$ is $((n+7)/2,4)$-antimagic. Miller and Bača [863] prove that the generalized Petersen graph $P(n,2)$ is $((3n+6)/2,3)$-antimagic for $n \equiv 0 \pmod 4$, $n \geq 8$ and conjectured that $P(n,k)$ is $((3n+6)/2,3)$-antimagic for even $n$ and $2 \leq k \leq n/2-1$ (see §2.7 for the definition of $P(n,k)$). This conjecture was proved for $k=3$ by Xu, Yang, Xi, and Li [1282]. Jirimutu and Wang proved that $P(n,2)$ is $((5n+5)/2,2)$-antimagic for $n \equiv 3 \pmod 4$ and $n \geq 7$. Xu, Xu, Lü, Baosheng, and Nan [1277] proved that $P(n,2)$ is $((3n+6)/2,2)$-antimagic for $n \equiv 2 \pmod 4$ and $n \geq 10$. Xu, Yang, Xi, and Li [1280] proved that $P(n,3)$ is $((3n+6)/2,3)$-antimagic for even $n \geq 10$.

Bodendiek and Walther [248] proved that the following graphs are not $(a,d)$-antimagic: even cycles; paths of even order; stars; $C_3^{(k)}$; $C_4^{(k)}$; trees of odd order at least 5 that have a vertex that is adjacent to three or more end vertices; $n$-ary trees with at least two layers when $d=1$; the Petersen graph; $K_4$ and $K_{3,3}$. They also prove: $P_{2k+1}$ is $(k,1)$-antimagic; $C_{2k+1}$ is $(k+2,1)$-antimagic; if a tree of odd order $2k+1$ $(k>1)$ is $(a,d)$-antimagic, then $d=1$ and $a=k$; if $K_{4k}$ $(k \geq 2)$ is $(a,d)$-antimagic, then $d$ is odd and $d \leq 2k(4k-3)+1$; if $K_{4k+2}$ is $(a,d)$-antimagic, then $d$ is even and $d \leq (2k+1)(4k-1)+1$; and if $K_{2k+1}$ $(k \geq 2)$ is $(a,d)$-antimagic, then $d \leq (2k+1)(k-1)$. Lin, Miller, Simanjuntak, and Slamin [795] show that no wheel $W_n$ $(n>3)$ has an $(a,d)$-antimagic labeling.

In [585] Ivančo, and Semaničová show that a 2-regular graph is super edge-magic if and only if it is $(a,1)$-antimagic. As a corollary we have that each of the following graphs are $(a,1)$-antimagic: $kC_n$ for $n$ odd and at least 3; $k(C_3 \cup C_n)$ for $n$ even and at least 6; $k(C_4 \cup C_n)$ for $n$ odd and at least 5; $k(C_5 \cup C_n)$ for $n$ even and at least 4; $k(C_m \cup C_n)$ for $m$ even and at least 6, $n$ odd, and $n \geq m/2+2$. Extending a idea of Kovář they prove if $G$ is $(a_1,1)$-antimagic and $H$ is obtained from $G$ by adding an arbitrary $2k$-factor then $H$ is $(a_2,1)$-antimagic for some $a_2$. As corollaries they observe that the following graphs are $(a,1)$-antimagic: circulant graphs of odd order; $2r$-regular Hamiltonian graphs of odd order; and $2r$-regular graphs of odd order $n<4r$. They further show that if $G$ is an $(a,1)$-antimagic $r$-regular graph of order $n$ and $n-r-1$ is a divisor of the non-negative integer $a+n(1+r-(n+1)/2)$, then $G \oplus K_1$ is supermagic. As a corollary of this result they have if $G$ is $(n-3)$-regular for $n$ odd and $n \geq 7$ or $(n-7)$-regular for $n$ odd and $n \geq 15$, then $G \oplus K_1$ is supermagic.

Bertault, Miller, Feria-Purón, and Vaezpour [222] approached labeling problems as combina-

torial optimization problems. They developed a general algorithm to determine whether a graph has a magic labeling, antimagic labeling, or an $(a,d)$-antimagic labeling. They verified that all trees with fewer than 10 vertices are super edge magic and all graphs of the form $P_2^r \times P_3^s$ with less than 50 vertices are antimagic.

Yegnanarayanan [1299] introduced several variations of antimagic labelings and provides some results about them.

The *antiprism* on $2n$ vertices has vertex set $\{x_{1,1},\ldots,x_{1,n},x_{2,1},\ldots,x_{2,n}\}$ and edge set $\{x_{j,i},x_{j,i+1}\}\cup\{x_{1,i},x_{2,i}\}\cup\{x_{1,i},x_{2,i-1}\}$ (subscripts are taken modulo $n$). For $n\geq 3$ and $n\not\equiv 2 \pmod 4$ Bača [107] gives $(6n+3,2)$-antimagic labelings and $(4n+4,4)$-antimagic labelings for the antiprism on $2n$ vertices. He conjectures that for $n\equiv 2 \pmod 4$, $n\geq 6$, the antiprism on $2n$ vertices has a $(6n+3,2)$-antimagic labeling and a $(4n+4,4)$-antimagic labeling.

Nicholas, Somasundaram, and Vilfred [908] prove the following: If $K_{m,n}$ where $m\leq n$ is $(a,d)$-antimagic, then $d$ divides $((m-n)(2a+d(m+n-1)))/4+dmn/2$; if $m+n$ is prime, then $K_{m,n}$, where $n>m>1$, is not $(a,d)$-antimagic; if $K_{n,n+2}$ is $(a,d)$-antimagic, then $d$ is even and $n+1\leq d<(n+1)^2/2$; if $K_{n,n+2}$ is $(a,d)$-antimagic and $n$ is odd, then $a$ is even and $d$ divides $a$; if $K_{n,n+2}$ is $(a,d)$-antimagic and $n$ is even, then $d$ divides $2a$; if $K_{n,n}$ is $(a,d)$-antimagic, then $n$ and $d$ are even and $0<d<n^2/2$; if $G$ has order $n$ and is unicylic and $(a,d)$-antimagic, then $(a,d)=(2,2)$ when $n$ is even and $(a,d)=(2,2)$ or $(a,d)=((n+3)/2,1)$ when $n$ is odd; a cycle with $m$ pendant edges attached at each vertex is $(a,d)$-antimagic if and only if $m=1$; the graph obtained by joining an endpoint of $P_m$ with one vertex of the cycle $C_n$ is $(2,2)$-antimagic if $m=n$ or $m=n-1$; if $m+n$ is even the graph obtained by joining an endpoint of $P_m$ with one vertex of the cycle $C_n$ is $(a,d)$-antimagic if and only if $m=n$ or $m=n-1$. They conjecture that for $n$ odd and at least 3, $K_{n,n+2}$ is $((n+1)(n^2-1)/2,n+1)$-antimagic and they have obtained several results about $(a,d)$-antimagic labelings of caterpillars.

In [1217] Vilfred and Florida proved the following: the one-sided infinite path is $(1,2)$-antimagic; $P_{2n}$ is not $(a,d)$-antimagic for any $a$ and $d$; $P_{2n+1}$ is $(a,d)$-antimagic if and only if $(a,d)=(n,1)$; $C_{2n+1}$ has an $(n+2,1)$-antimagic labeling; and that a 2-regular graph $G$ is $(a,d)$-antimagic if and only if $|V(G)|=2n+1$ and $(a,d)=(n+2,1)$. They also prove that for a graph with an $(a,d)$-antimagic labeling, $q$ edges, minimum degree $\delta$ and maximum degree $\Delta$, the vertex labels lie between $\delta(\delta+1)/2$ and $\Delta(2q-\Delta+1)/2$.

For $n>1$ and distinct odd integers $x,y$ and $z$ in [1,$n-1$] Javaid, Ismail, and Salman [588] define the *chordal ring* of order $n$, $CR_n(x,y,z)$, as the graph with vertex set $Z_n$, the additive group of integers modulo $n$, and edges $(i,i+x),(i,i+y),(i,i+z)$ for all even $i$. They prove that $CR_n(1,3,7)$ and $CR_n(1,5,n-1)$ have $(a,d)$-antimagic labelings when $n\equiv 0 \bmod 4$ and conjecture that for an odd integer $\Delta$, $3\leq\Delta\leq n-3, n\equiv 0 \bmod 4$, $CR_n((1,\Delta,n-1)$ has an $((7n+8)/4,1)$-antimagic labeling.

In [1218] Vilfred and Florida call a graph $G=(V,E)$ *odd antimagic* if there exist a bijection $f\colon E\to\{1,3,5,\ldots,2|E|-1\}$ such that the induced mapping $g_f\colon V\to N$, defined by $g_f(v)=\sum\{f(uv)|\ uv\in E(G)\}$, is injective and *odd* $(a,d)$*-antimagic* if there exist positive integers $a$, $d$ and a bijection $f\colon E\to\{1,3,5,\ldots,2|E|-1\}$ such that the induced mapping $g_f\colon V\to N$, defined by $g_f(v)=\sum\{f(uv)|\ uv\in E(G)\}$, is injective and $g_f(V)=\{a,a+d,a+2d,\ldots,a+(|V|-1)d\}$. Although every $(a,d)$-antimagic graph is antimagic, $C_4$ has an antimagic labeling but does not have an $(a,d)$-antimagic labeling. They prove: $P_{2n+1}$ is not odd $(a,d)$-antimagic for any $a$ and $d$; $C_{2n+1}$ has an odd $(2n+2,2)$-antimagic labeling; if a 2-regular graph $G$ has an odd $(a,d)$-antimagic labeling, then $|V(G)|=2n+1$ and $(a,d)=(2n+2,2)$; $C_{2n}$ is odd magic; and an odd magic graph with at least three vertices, minimum degree $\delta$, maximum degree $\Delta$, and $q\geq 2$

edges has all its vertex labels between $\delta^2$ and $\Delta(2q-\Delta)$.

In Tables 11, 12, and 13 use the abbreviations **A** to mean antimagic, $(a,d)$-**A** to mean that the graph has an $(a,d)$-antimagic labeling and $(a,d)$-**EAV** to mean that the graph has an $(a,d)$-antimagic vertex labeling A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 11: **Summary of Antimagic Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $P_n$ | A | for $n \geq 3$ [511] |
| $C_n$ | A | [511] |
| $W_n$ | A | [511] |
| $K_n$ | A | for $n \geq 3$ [511] |
| every tree except $K_2$ | A? | [511] |
| every connected graph except $K_2$ | A? | [511] |
| $n \geq 4$ vertices $\Delta(G) \geq n-2$ | A | [58] |
| all complete partite graphs except $K_2$ | A | [58] |

Table 12: **Summary of $(a, d)$-Edge-Antimagic Vertex Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $P_n$ | $(3, 2)$-EAV | [1087] |
| | not $(a, d)$-EAV | $d > 2$ [1087] |
| $P_{2n}$ | $(n + 2, 1)$-EAV | [1087] |
| $C_n$ | not $(a, d)$-EAV | $d > 1$ [140] |
| $C_{2n}$ | not $(a, d)$-EAV | [1087] |
| $C_{2n+1}$ | $(n + 2, 1)$-EAV | [1087] |
| | $(n + 3, 1)$-EAV | [1087] |
| $K_n$ | not $(a, d)$-EAV | for $n > 1$ [140] |
| $K_{n,n}$ | not $(a, d)$-EAV | for $n > 3$ [140] |
| $W_n$ | not $(a, d)$-EAV | [140] |
| $C_3^{(n)}$ (friendship graph) | $(a, 1)$-EAV | iff $n = 1, 3, 4, 5, 7$ [141] |
| generalized Petersen graph $P(n, k)$ | not $(a, d)$-EAV | $d > 1$ [140] |

Table 13: **Summary of $(a,d)$-Antimagic Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $C_{2n+1}$ | not $(n+2,1)$-A | $n$ even [248] |
| $P_{2n}$ | not $(a,d)$-A | [248] |
| $P_{2n+1}$ | $(n,1)$-A | [248] |
| stars | not $(a,d)$-A | [248] |
| $C_3^{(k)}, C_4^{(k)}$ | not $(a,d)$-A | [248] |
| $K_{n,n+2}$ | $\left(\frac{(n+1)(n^2-1)}{2}, n+1\right)$-A | $n \geq 3$, $n$ odd [248] |
| $K_{3,3}$ | not $(a,d)$-A | [248] |
| $K_4$ | not $(a,d)$-A | [248] |
| Petersen graph | not $(a,d)$-A | [248] |
| $W_n$ | not $(a,d)$-A | $n > 3$ [795] |
| antiprism on $2n$ vertices (see §6.1) | $(6n+3,2)$-A | $n \geq 3$, $n \not\equiv 2 \pmod 4$ [107] |
| | $(4n+4,4)$-A | $n \geq 3$, $n \not\equiv 2 \pmod 4$ [107] |
| | $(2n+5,6)$-A? | $n \geq 4$ [107] |
| | $(6n+3,2)$-A? | $n \geq 6$, $n \not\equiv 2 \pmod 4$ [107] |
| | $(4n+4,4)$-A? | $n \geq 6$, $n \not\equiv 2 \pmod 4$ [107] |
| Hershel graph (see [304]) | not $(a,d)$-A | [244], [246] |
| parachutes $P_{g,p}$ (see §6.1) | $(a,d)$-A | for certain classes [244], [246] |
| $C_n$ | not $(a,d)$-A | $n$ even [248] |
| prisms $C_n \times P_2$ | $((7n+4)/2,1)$-A | $n \geq 3$, $n$ even [245], [130] |
| | $((5n+5)/2,2)$-A | $n \geq 3$, $n$ odd [245], [130] |
| | $((3n+6)/2,3)$-A | $n \geq 3$, $n$ even [130] |
| | $((n+7)/2,4)$-A? | $n \geq 7$, [246], [130] |
| generalized Petersen graph $P(n,2)$ | $((3n+6)/2,3)$-A | $n \geq 8$, $n \equiv 0 \pmod 4$ [131] |

## 6.2 $(a,d)$-Antimagic Total Labelings

Bača, Bertault, MacDougall, Miller, Simanjuntak, and Slamin [123] introduced the notion of a $(a,d)$-vertex-antimagic total labeling in 2000. For a graph $G(V,E)$, an injective mapping $f$ from $V \cup E$ to the set $\{1,2,\ldots,|V|+|E|\}$ is a $(a,d)$-*vertex-antimagic total labeling* if the set $\{f(v)+\sum f(vu)\}$ where the sum is over all vertices $u$ adjacent to $v$ for all $v$ in $G$ is $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. In the case where the vertex labels are 1,2, ..., $|V|$, $(a,d)$-vertex-antimagic total labeling is called a *super* $(a,d)$-*vertex-antimagic total labeling.* Among their results are: every super-magic graph has an $(a,1)$-vertex-antimagic total labeling; every $(a,d)$-antimagic graph $G(V,E)$ is $(a+|E|+1, d+1)$-vertex-antimagic total; and, for $d>1$, every $(a,d)$-antimagic graph $G(V,E)$ is $(a+|V|+|E|, d-1)$-vertex-antimagic total. They also show that paths and cycles have $(a,d)$-vertex-antimagic total labelings for a wide variety of $a$ and $d$. In [124] Bača et al. use their results in [123] to obtain numerous $(a,d)$-vertex-antimagic total labelings for prisms, and generalized Petersen graphs (see §2.7 for the definition). (See also [133] and [1142] for more results on generalized Petersen graphs.)

Sugeng, Miller, Lin, and Bača [1142] prove: $C_n$ has a super $(a,d)$-vertex-antimagic total labeling if and only if $d=0$ or 2 and $n$ is odd, or $d=1$; $P_n$ has a super $(a,d)$-vertex-antimagic total labeling if and only if $d=2$ and $n \geq 3$ is odd, or $d=3$ and $n \geq 3$; no even order tree has a super $(a,1)$-vertex antimagic total labeling; no cycle with at least one tail and an even number of vertices has a super $(a,1)$-vertex-antimagic labeling; and the star $S_n$, $n \geq 3$, has no super $(a,d)$-super antimagic labeling. As open problems they ask whether $K_{n,n}$ has a super $(a,d)$-vertex-antimagic total labeling and the generalized Petersen graph has a super $(a,d)$-vertex-antimagic total labeling for specific values $a, d$, and $n$. Lin, Miller, Simanjuntak, and Slamin [795] have shown that for $n>20$, $W_n$ has no $(a,d)$-vertex-antimagic total labeling. Tezer and Cahit [1176] proved that neither $P_n$ nor $C_n$ has $(a,d)$-vertex-antimagic total labelings for $a \geq 3$ and $d \geq 6$. Kovář [677] has shown that every $2r$-regular graph with $n$ vertices has an $(s,1)$-vertex antimagic total labeling for $s \in \{(rn+1)(r+1)+tn \mid t=0,1,\ldots,r\}$.

Several papers have been written about vertex-antimagic total labeling of graphs that are the disjoint union of suns. The sun graph $S_n$ is $C_n \odot K_1$. Rahim and Sugeng [948] proved that $S_{n_1} \cup S_{n_2} \cup \ldots \cup S_{n_t}$ is $(a,0)$-vertex-antimagic total (or vertex magic total); Parestu, Silaban, and Sugeng [920] and [921] proved $S_{n_1} \cup S_{n_2} \cup \ldots \cup S_{n_t}$ is $(a,d)$-vertex-antimagic total for $d=1,2,3,4$, and 6 and particular values of $a$. In [950] Rahim, Ali, Kashif, and Javaid provide $(a,d)$-vertex antimagic total labelings of disjoint unions of cycles, sun graphs, and disjoint unions of sun graphs.

In [899] Ngurah, Baskova, and Simanjuntak provide $(a,d)$-vertex-antimagic total labelings for the generalized Petersen graphs $P(n,m)$ for the cases: $n \geq 3$, $1 \leq m \leq \lfloor (n-1)/2 \rfloor$, $(a,d)=(8n+3,2)$; odd $n \geq 5$, $m=2$, $(a,d)=((15n+5)/2,1)$; odd $n \geq 5$, $m=2$, $(a,d)=((21n+5)/2,1)$; odd $n \geq 7$, $m=3$, $(a,d)=((15n+5)/2,1)$; odd $n \geq 7$, $m=3$, $(a,d)=((21n+5)/2,1)$; odd $n \geq 9$, $m=4$, $(a,d)=((15n+5)/2,1)$; and $(a,d)=((21n+5)/2,1)$. They conjecture that for $n$ odd and $1 \leq m \leq \lfloor (m-1)/2 \rfloor$, $P(n,m)$ has an $((21n+5)/2,1)$-vertex-antimagic labeling. In [1147] Sugeng and Silaban show: the disjoint union of any number of odd cycles of orders $n_1, n_2, \ldots, n_t$, each at least 5, has a super $(3(n_1+n_2+\cdots+n_t)+2,1)$-vertex-antimagic total labeling; for any odd positive integer $t$, the disjoint union of $t$ copies of the generalized Petersen graph $P(n,1)$ has a super $(10t+2)n-\lfloor n/2 \rfloor+2,1)$-vertex-antimagic total labeling; and for any odd positive integers $t$ and $n$ $(n \geq 3)$, the disjoint union of $t$ copies of the generalized Petersen graph $P(n,2)$ has a super $(21tn+5)/2,1)$-vertex-antimagic total labeling.

Ail, Bača, Lin, and Semaničová-Feňovčiková [55] investigated super-$(a,d)$-vertex antimagic total labelings of disjoint unions of regular graphs. Among their results are: if $m$ and $(m-1)(r+1)/2$ are positive integers and $G$ is an $r$-regular graph that admits a super-vertex magic total labeling, then $mG$ has a super-$(a,2)$-vertex antimagic total labeling; if $G$ has a 2-regular super-$(a,1)$-vertex antimagic total labeling, then $mG$ has a super-$(m(a-2)+2,1),1)$-vertex antimagic total labeling; $mC_n$ has a super-$(a,d)$-vertex antimagic total labeling if and only if either $d$ is 0 or 2 and $m$ and $n$ are odd and at least 3 or $d=1$ and $n \geq 3$; and if $G$ is an even regular Hamilton graph, then $mG$ has a super-$(a,1)$-vertex antimagic total labeling for all positive integers $m$.

For $t \geq 2$ and $p \geq 4$, a *Harary* graph, $C_p^t$, is a graph obtained from a cycle $C_p$ by joining every two vertices at distance $t$. In [566] Hussain, Ali, Rahim, and Baskoro construct various $(a,d)$-vertex-antimagic labelings for Harary graphs and disjoint unions of identical Harary graphs. For $p$ odd and at least 5, Balbuena, Barker, Das, Lin, Miller, Ryan, Slamin, Sugeng, and Tkac [173] give a super $((17p+5)/2)$-vertex-antimagic total labeling of $C_p^t$.

Thirusangu, Nagar, and Rajeswari [1177] show that certain Cayley digraphs of dihedral groups have $(a,d)$-vertex-magic total labelings.

For a simple graph $H$ we say that $G(V,E)$ admits an *$H$-covering*, if every edge in $E(G)$ belongs to a subgraph of $G$ that is isomorphic to $H$. Inayah, Salman, and Simanjuntak [575] define an *$(a,d)$-$H$-antimagic total labeling* of $G$ as a bijective function $\xi$ from $V \cup E \to \{1,2,\ldots,|V|+|E|\}$ such that for all subgraphs $H'$ isomorphic to $H$, the $H$-weights $w(H') = \sum_{v \in V(H')} \xi(v) + \sum_{e \in E(H')} \xi(e)$ constitute an arithmetic progression $a, a+d, a+2d, \ldots, a+(t-1)d$ where $a$ and $d$ are positive integers and $t$ is the number of subgraphs of $G$ isomorphic to $H$. Such a labeling $\xi$ is called a *super $(a,d)$-$H$-antimagic total labeling*, if $\xi(V) = \{1,2,\ldots,|V|\}$. Inayah et al. study some basic properties of such labeling and give $(a,d)$-cycle-antimagic labelings of fans.

Simanjuntak, Bertault, and Miller [1087] define an *(a,d)-edge-antimagic vertex labeling* for a graph $G(V,E)$ as an injective mapping $f$ from $V$ onto the set $\{1,2,\ldots,|V|\}$ such that the set $\{f(u)+f(v)|uv \in E\}$ is $\{a, a+d, a+2d, \ldots, a+(|E|-1)d\}$. (The equivalent notion of *(a,d)-indexable* labeling was defined by Hegde in 1989 in his Ph. D. thesis–see [522].) Similarly, Simanjuntak et al. define an *(a,d)-edge-antimagic total labeling* for a graph $G(V,E)$ as an injective mapping $f$ from $V \cup E$ onto the set $\{1,2,\ldots,|V|+|E|\}$ such that the set $\{f(v)+f(vu)+f(v)|uv \in E\}$ where $v$ ranges over all of $V$ is $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. Among their results are: $C_{2n}$ has no $(a,d)$-edge-antimagic vertex labeling; $C_{2n+1}$ has a $(n+2,1)$-edge-antimagic vertex labeling and a $(n+3,1)$-edge-antimagic vertex labeling; $P_{2n}$ has a $(n+2,1)$-edge-antimagic vertex labeling; $P_n$ has a $(3,2)$-edge-antimagic vertex labeling; $C_n$ has $(2n+2,1)$- and $(3n+2,1)$-edge-antimagic total labelings; $C_{2n}$ has $(4n+2,2)$- and $(4n+3,2)$-edge-antimagic total labelings; $C_{2n+1}$ has $(3n+4,3)$- and $(3n+5,3)$-edge-antimagic total labelings; $P_{2n+1}$ has $(3n+4,2)$-, $(3n+4,3)$-, $(2n+4,4)$-, $(5n+4,2)$-, $(3n+5,2)$-, and $(2n+6,4)$-edge-antimagic total labelings; $P_{2n}$ has $(6n,1)$- and $(6n+2,2)$-edge-antimagic total labelings; and several parity conditions for $(a,d)$-edge-antimagic total labelings. They conjecture: $C_{2n}$ has a $(2n+3,4)$- or a $(2n+4,4)$-edge-antimagic total labeling; $C_{2n+1}$ has a $(n+4,5)$- or a $(n+5,5)$-edge-antimagic total labeling; paths have no $(a,d)$-edge-antimagic vertex labelings with $d > 2$; and cycles have no $(a,d)$-antimagic total labelings with $d > 5$. These last two conjectures were verified by Bača, Lin, Miller, and Simanjuntak [140] who proved that a graph with $v$ vertices and $e$ edges that has an $(a,d)$-edge-antimagic vertex labeling must satisfy $d(e-1) \leq 2v-1-a \leq 2v-4$. As a consequence, they obtain: for every path there is no $(a,d)$-edge-antimagic vertex labeling with

$d > 2$; for every cycle there is no $(a, d)$-edge-antimagic vertex labeling with $d > 1$; for $K_n$ $(n > 1)$ there is no $(a, d)$-edge-antimagic vertex labeling (the cases for $n = 2$ and $n = 3$ are handled individually); $K_{n,n}$ $(n > 3)$ has no $(a, d)$-edge-antimagic vertex labeling; for every wheel there is no $(a, d)$-edge-antimagic vertex labeling; for every generalized Petersen graph there is no $(a, d)$-edge-antimagic vertex labeling with $d > 1$. They also study the relationship between graphs with $(a, d)$-edge-antimagic labelings and magic and antimagic labelings. They conjecture that every tree has an $(a, 1)$-edge-antimagic total labeling.

In [140] Bača, Lin, Miller, and Simanjuntak prove: if $P_n$ has an $(a, d)$-edge-antimagic total labeling, then $d \leq 6$; $P_n$ has $(2n+2, 1)$-, $(3n, 1)$-, $(n+4, 3)$-, and $(2n+2, 3)$-edge-antimagic total labelings; $P_{2n+1}$ has $(3n + 4, 2)$-,$(5n + 4, 3)$-, $(2n + 4, 4)$-, and $(2n + 6, 4)$-edge-antimagic total labelings; and $P_{2n}$ has $(3n + 3, 2)$- and $(5n + 1, 2)$-edge-antimagic total labelings. Ngurah [897] proved $P_{2n+1}$ has $(4n + 4, 1)$-, $(6n + 5, 3)$-,$(4n + 4, 2)$-,$(4n + 5, 2)$-edge-antimagic total labelings and $C_{2n+1}$ has $(4n + 4, 2)$- and $(4n + 5, 2)$-edge-antimagic total labelings. Silaban and Sugeng [1086] prove: $P_n$ has $(n+4, 4)$- and $(6, 6)$-edge-antimagic total labelings; if $C_m \odot \overline{K_n}$ has an $(a, d)$-edge-antimagic total labeling, then $d \leq 5$; $C_m \odot \overline{K_n}$ has $(a, d)$-edge-antimagic total labelings for $m > 3, n > 1$ and $d = 2$ or 4; and $C_m \odot \overline{K_n}$ has no $(a, d)$-edge-antimagic total labelings for $m$ and $d$ and $n \equiv 1 \bmod 4$. They conjecture that $P_n$ $(n \geq 3)$ has $(a, 5)$-edge-antimagic total labelings. In [1148] Sugeng and Xie use adjacency methods to construct super edge magic graphs from $(a, d)$-edge-antimagic vertex graphs.

In [159] Bača and Youssef used parity arguments to find a large number of conditions on $p, q$ and $d$ for which a graph with $p$ vertices and $q$ edges cannot have an $(a, d)$-edge-antimagic total labeling or vertex-antimagic total labeling. Bača and Youssef [159] made the following connection between $(a, d)$-edge-antimagic vertex labelings and sequential labelings: if $G$ is a connected graph other than a tree that has an $(a, d)$-edge-antimagic vertex labeling, then $G+K_1$ has a sequential labeling.

In [1138] Sudarsana, Ismaimuza, Baskoro, and Assiyatun prove: for every $n \geq 2$, $P_n \cup P_{n+1}$ has a $(6n+1, 1)$- and a $(4n+3, 3)$-edge-antimagic total labeling, for every odd $n \geq 3$, $P_n \cup P_{n+1}$ has a $(6n, 1)$- and a $(5n + 1, 2)$-edge-antimagic total labeling, for every $n \geq 2$, $nP_2 \cup P_n$ has a $(7n, 1)$- and a $(6n + 1, 2)$-edge-antimagic total labeling. In [1136] the same authors show that $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ $(n \geq 2)$, and $nP_2 \cup P_{n+2}$ are super edge-magic total. They also show that under certain conditions one can construct new super edge-magic total graphs from existing ones by joining a particular vertex of the existing super edge-magic total graph to every vertex in a path or every vertex of a star and by joining one extra vertex to some vertices of the existing graph. Baskoro, Sudarsana, and Cholily [205] also provide algorithms for constructing new super edge-magic total graphs from existing ones by adding pendant vertices to the existing graph. A corollary to one of their results is that the graph obtained by attaching a fixed number of pendant edges to each vertex of a path of even length is super edge-magic. Baskoro and Cholily [203] show that the graphs obtained by attaching any numbers of pendant edges to a single vertex or a fix number of pendant edges to every vertex of the following graphs are super edge-magic total graphs: odd cycles, the generalized Petersen graphs $P(n, 2)$ ($n$ odd and at least 5), and $C_n \times P_m$ ($n$ odd, $m \geq 2$).

An $(a, d)$-edge-antimagic total labeling of $G(V, E)$ is called a *super (a,d)-edge-antimagic total* if the vertex labels are $\{1, 2, \ldots, |V(G)|\}$ and the edge labels are $\{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$. Bača, Baskoro, Simanjuntak, and Sugeng [122] prove the following: $C_n$ has a super $(a, d)$-edge-antimagic total labeling if and only if either $d$ is 0 or 2 and $n$ is odd, or $d = 1$; for odd $n \geq 3$ and $m = 1$ or 2, the generalized Petersen graph $P(n, m)$ has a

super $(11n+3)/2, 0)$-edge-antimagic total labeling and a super $((5n+5)/2, 2)$-edge-antimagic total labeling; for odd $n \geq 3$, $P(n,(n-1)/2)$ has a super $((11n+3)/2, 0)$-edge-antimagic total labeling and a super $((5n+5)/2, 2)$-edge-antimagic total labeling. They also prove: if $P(n,m), n \geq 3,\ 1 \leq m \leq \lfloor (n-1)/2 \rfloor$ is super $(a,d)$-edge-antimagic total, then $(a,d) = (4n+2, 1)$ if $n$ is even, and either $(a,d) = ((11n+3)/2, 0)$, or $(a,d) = (4n+2, 1)$, or $(a,d) = ((5n+5)/2, 2)$, if $n$ is odd; and for odd $n \geq 3$ and $m = 1, 2$, or $(n-1)/2$, $P(n,m)$ has an $(a,0)$-edge-antimagic total labeling and an $(a,2)$-edge-antimagic total labeling. (In a personal communication MacDougall argues that "edge-magic" is a better term than "$(a,0)$-edge-antimagic" for while the latter is technically correct, "antimagic" suggests different weights whereas "magic" emphasizes equal weights and that the edge-magic case is much more important, interesting, and fundamental rather than being just one subcase of equal value to all the others.) They conjecture that for odd $n \geq 9$ and $3 \leq m \leq (n-3)/2$, $P(n,m)$ has a $(a,0)$-edge-antimagic total labeling and an $(a,2)$-edge-antimagic total labeling. Ngurah and Baskoro [898] have shown that for odd $n \geq 3$, $P(n,1)$ and $P(n,2)$ have $((5n+5)/2, 2)$-edge-antimagic total labelings and when $n \geq 3$ and $1 \leq m < n/2, P(n,m)$ has a super $(4n+2, 1)$-edge-antimagic total labeling. In [899] Ngurah, Baskova, and Simanjuntak provide $(a,d)$-edge-antimagic total labelings for the generalized Petersen graphs $P(n,m)$ for the cases $m = 1$ or 2, odd $n \geq 3$, and $(a,d) = ((9n+5)/2, 2)$.

In [1137] Sudarsana, Baskoro, Uttunggadewa, and Ismaimuza show how to construct new larger super $(a,d)$-edge-antimagic-total graphs from existing smaller ones.

In [900] Ngurah, Baskoro, and Simanjuntak prove that $mC_n$ $(n \geq 3)$ has an $(a,d)$-edge-antimagic total in the following cases: $(a,d) = (5mn/2+2, 1)$ where $m$ is even; $(a,d) = (2mn+2, 2)$; $(a,d) = ((3mn+5)/2, 3)$ for $m$ and $n$ odd; and $(a,d) = ((mn+3), 4)$ for $m$ and $n$ odd; and $mC_n$ has a super $(2mn+2, 1)$-edge-antimagic total labeling.

Bača and Barrientos [112] have shown that $mK_n$ has a super $(a,d)$-edge-antimagic total labeling if and only if $(i)$ $d \in \{0, 2\}$, $n \in \{2, 3\}$ and $m \geq 3$ is odd, or $(ii)$ $d = 1$, $n \geq 2$ and $m \geq 2$, or $(iii)$ $d \in \{3, 5\}$, $n = 2$ and $m \geq 2$, or $(iv)$ $d = 4$, $n = 2$, and $m \geq 3$ is odd. In [111] Bača and Barrientos proved the following: if a graph with $q$ edges and $q+1$ vertices has an $\alpha$-labeling, than it has an $(a,1)$-edge-antimagic vertex labeling; a tree has a $(3,2)$-edge-antimagic vertex labeling if and only if it has an $\alpha$-labeling and the number of vertices in its two partite sets differ by at most 1; if a tree with at least two vertices has a super $(a,d)$-edge-antimagic total labeling, then $d$ is at most 3; if a graph has an $(a,1)$-edge-antimagic vertex labeling, then it also has a super $(a_1, 0)$-edge-antimagic total labeling and a super $(a_2, 2)$-edge-antimagic total labeling.

Bača and Youssef [158] proved the following: if $G$ is a connected $(a,d)$-edge-antimagic vertex graph that is not a tree, then $G + K_1$ is sequential; $mC_n$ has an $(a,d)$-edge-antimagic vertex labeling if and only if $m$ and $n$ are odd and $d = 1$; an odd degree $(p,q)$-graph $G$ cannot have a $(a,d)$-edge-antimagic total labeling if $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$, or $p \equiv 0 \pmod 4$, $q \equiv 2 \pmod 4$, and $d$ is even; a $(p,q)$-graph $G$ cannot have a super $(a,d)$-edge-antimagic total labeling if $G$ has odd degree, $p \equiv 2 \pmod 4$, $q$ is even, and $d$ is odd, or $G$ has even degree, $q \equiv 2 \pmod 4$, and $d$ is even; $C_n$ has a $(2n+2, 3)$- and an $(n+4, 3)$-edge-antimagic total labeling; a $(p,q)$-graph is not super $(a,d)$-vertex-antimagic total if: $p \equiv 2 \pmod 4$ and $d$ is even; $p \equiv 0 \pmod 4$, $q \equiv 2 \pmod 4$, and $d$ is odd; $p \equiv 0 \pmod 8$ and $q \equiv 2 \pmod 4$.

In [1138] Sudarsana, Ismaimuza, Baskoro, and Assiyatun prove: for every $n \geq 2$, $P_n \cup P_{n+1}$ has super $(n+4, 1)$- and $(2n+6, 3)$-edge antimagic total labelings; for every odd $n \geq 3$, $P_n \cup P_{n+1}$ has super $(4n+5, 1)$-,$(3n+6, 2)$-, $(4n+3, 1)$- and $(3n+4, 2)$-edge antimagic total labelings; for every $n \geq 2$, $nP_2 \cup P_n$ has super $(6n+2, 1)$- and $(5n+3, 2)$-edge antimagic total labelings; and

for every $n \geq 1$, $nP_2 \cup P_{n+2}$ has super $(6n+6,1)$- and $(5n+6,2)$-edge antimagic total labelings. They pose a number of open problems about constructing $(a,d)$-edge antimagic labelings and super $(a,d)$-edge antimagic labelings for the graphs $P_n \cup P_{n+1}$, $nP_2 \cup P_n$, and $nP_2 \cup P_{n+2}$ for specific values of $d$.

Dafik, Miller, Ryan, and Bača [344] investigated the super edge-antimagicness of the disconnected graph $mC_n$ and $mP_n$. For the first case they prove that $mC_n$, $m \geq 2$, has a super $(a,d)$-edge-antimagic total labeling if and only if either $d$ is 0 or 2 and $m$ and $n$ are odd and at least 3, or $d = 1$, $m \geq 2$, and $n \geq 3$. For the case of the disjoint union of paths they determine all feasible values for $m, n$ and $d$ for $mP_n$ to have a super $(a,d)$-edge-anti-magic total labeling except when $m$ is even and at least 2, $n \geq 2$, and $d$ is 0 or 2. In [346] Dafik, Miller, Ryan, and Bača obtain a number of results about super edge-antimagicness of the disjoint union of two stars and state three open problems.

In [136] Bača, Lascsáková, and Semaničová investigated the connection between graphs with $\alpha$-labelings and graphs with super $(a,d)$-edge-antimagic total labelings. Among their results are: If $G$ is a graph with $n$ vertices and $n-1$ edges ($n \geq 3$) and $G$ has an $\alpha$-labeling, then $mG$ is super $(a,d)$-edge-antimagic total if either $d$ is 0 or 2 and $m$ is odd, or $d = 1$ and $n$ is even; if $G$ has an $\alpha$-labeling and has $n$ vertices and $n-1$ edges with vertex bipartition sets $V_1$ and $V_2$ where $|V_1|$ and $|V_2|$ differ by at most 1, then $mG$ is super $(a,d)$–edge-antimagic total for $d = 1$ and $d = 3$. In the same paper Bača et al. prove: caterpillars with odd order at least 3 have super $(a,1)$-edge-antimagic total labelings; if $G$ is a caterpillar of odd order at least 3 and $G$ has a super $(a,1)$-edge-antimagic total labeling, then $mG$ has a super $(b,1)$-edge-antimagic total labeling for some $b$ that is a function of $a$ and $m$.

In [343] Dafik, Miller, Ryan, and Bača investigated the existence of antimagic labelings of disjoint unions of $s$-partite graphs. They proved: if $s \equiv 0$ or $1 \pmod 4$, $s \geq 4, m \geq 2, n \geq 1$ or $mn$ is even , $m \geq 2$, $n \geq 1$, $s \geq 4$, then the complete $s$-partite graph $mK_{n,n,\ldots,n}$ has no super $(a,0)$-edge-antimagic total labeling; if $m \geq 2$ and $n \geq 1$, then $mK_{n,n,n,n}$ has no super $(a,2)$-antimagic total labeling; and for $m \geq 2$ and $n \geq 1$, $mK_{n,n,n,n}$ has an $(8mn+2,1)$-edge-antimagic total labeling. They conjecture that for $m \geq 2$, $n \geq 1$ and $s \geq 5$, the complete $s$-partite graph $mK_{n,n,\ldots,n}$ has a super $(a,1)$-antimagic total labeling.

In [155] Bača, Muntaner-Batle, Semaničová-Feňovčiková, and Shafiq investigate super $(a,d)$-edge-antimagic total labelings of disconnected graphs. Among their results are: If $G$ is a (super) $(a,2)$-edge-antimagic total labeling and $m$ is odd, then $mG$ has a (super) $(a',2)$-edge-antimagic-total labeling where $a' = m(a-3) + (m+1)/2 + 2$; and if $d$ a positive even integer and $k$ a positive odd integer, $G$ is a graph with all of its vertices having odd degree, and the order and size of $G$ have opposite parity, then $2kG$ has no $(a,d)$-edge-antimagic total labeling. Bača and Brankovic [125] have obtained a number of results about the existence of super $(a,d)$-edge-antimagic totaling of disjoint unions of the form $mK_{n,n}$. In [126] Bača, Dafik, Miller, and Ryan provide $(a,d)$-edge-antimagic vertex labelings and super $(a,d)$-edge-antimagic total labelings for a variety of disjoint unions of caterpillars. Bača and Youssef [159] proved that $mC_n$ has an $(a,d)$-edge-antimagic vertex labeling if and only if $m$ and $n$ are odd and $d = 1$. Bača, Dafik, Miller, and Ryan [127] constructed super $(a,d)$-edge-antimagic total labeling for graphs of the form $m(C_n \odot \overline{K}_s)$ and $mP_n \cup kC_n$ while Dafik, Miller, Ryan, and Bača [345] do the same for graphs of the form $mK_{n,n,n}$ and $K_{1,m} \cup 2sK_{1,n}$. Both papers provide a number of open problems. In [144] Bača, Lin, and Muntaner-Batle provide super $(a,d)$-edge-antimagic total labeling of forests in which every component is a specific kind of tree. In [135] Bača, Kovvář, Semaničová-Feňovčiková, and Shafiq prove that every even regular graph and every odd regular

graph with a 1-factor are super $(a,1)$-edge-antimagic total and provide some constructions of non-regular super $(a,1)$-edge-antimagic total graphs. Bača, Lin and Semaničová-Feňovčiková [146] show: the disjoint union of $m$ graphs with super $(a,1)$-edge antimagic total labelings have super $(m(a-2)+2,1)$-edge antimagic total labelings; the disjoint union of $m$ graphs with super $(a,3)$-edge antimagic total labelings have super $(m(a-3)+3,3)$-edge antimagic total labelings; if $G$ has a $(a,1)$-edge antimagic total labelings then $mG$ has an $(b,1)$-edge antimagic total labeling for some $b$; and if $G$ has a $(a,3)$-edge antimagic total labelings then $mG$ has an $(b,3)$-edge antimagic total labeling for some $b$. For $t \geq 2$ and $n \geq 4$ the *Harary graph* $C_p^t$ is the graph obtained by joining every two vertices of $C_p$ that are at distance $t$ in $C_p$. In [950] Rahim, Ali, Kashif, and Javaid provide super $(a,d)$-edge antimagic total labelings for disjoint unions of Harary graphs and disjoint unions of cycles.

In [141] Bača, Lin, Miller, and Youssef prove: if the friendship $C_3^{(n)}$ is super $(a,d)$-antimagic total, then $d<3$; $C_3^{(n)}$ has an $(a,1)$-edge antimagic vertex labeling if and only if $n=1,3,4,5$, and 7; $C_3^{(n)}$ has a super $(a,d)$-edge-antimagic total labelings for $d=0$ and 2; $C_3^{(n)}$ has a super $(a,1)$-edge-antimagic total labeling; if a fan $F_n$ $(n \geq 2)$ has a super $(a,d)$-edge-antimagic total labeling, then $d<3$; $F_n$ has a super $(a,d)$-edge-antimagic total labeling if $2 \leq n \leq 6$ and $d=0,1$ or 2; the wheel $W_n$ has a super $(a,d)$-edge-antimagic total labeling if and only if $d=1$ and $n \not\equiv 1 \pmod 4$; $K_n$, $n \geq 3$, has a super $(a,d)$-edge-antimagic total labeling if and only if either $d=0$ and $n=3$, or $d=1$ and $n \geq 3$, or $d=2$ and $n=3$; and $K_{n,n}$ has a super $(a,d)$-edge antimagic total labeling if and only if $d=1$ and $n \geq 2$.

Bača, Lin, and Muntaner-Batle [142] have shown that if a tree with at least two vertices has a super $(a,d)$-edge-antimagic total labeling, then $d$ is at most three and $P_n$, $n \geq 2$, has a super $(a,d)$-edge-antimagic total labeling if and only if $d=0,1,2$, or 3. They also characterize certain path-like graphs in a grid that have super$(a,d)$-edge-antimagic total labelings.

Recall that $C_n^t$ denotes the graph obtained from the $n$-cycle by joining two vertices at a distance $t$. MacDougall and Wallis [837] have proved the following: $C_{4m+3}^t$, $m \geq 1$, has a super $(a,0)$-edge-antimagic total labeling for all possible values of $t$ with $a=10m+9$ or $10m+10$; $C_{4m+1}^t$, $m \geq 3$, has a super $(a,0)$-edge-antimagic total labeling for all possible values except $t=5,9,4m-4$, and $4m-8$ with $a=10m+4$ and $10m+5$; $C_{4m+1}^t$, $m \geq 1$, has a super $(10m+4,0)$-edge-antimagic total labeling for all $t \equiv 1 \pmod 4$ except $4m-3$; $C_{4m}^t$, $m>1$, has a super $(10m+2,0)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$; $C_{4m+2}^t$, $m>1$, has a super $(10m+7,0)$-edge-antimagic total labeling for all odd $t$ other than 5 and for $t=2$ or 6.

Bača and Murugan [156] have proved: if $C_n^t$, $n \geq 4, 2 \leq t \leq n-2$, is super $(a,d)$-edge-antimagic total, then $d=0,1$, or 2; for $n=2k+1 \geq 5$, $C_n^t$ has a super $(a,0)$-edge-antimagic total labeling for all possible values of $t$ with $a=5k+4$ or $5k+5$; for $n=2k+1 \geq 5$, $C_n^t$ has a super $(a,2)$-edge-antimagic total labeling for all possible values of $t$ with $a=3k+3$ or $3k+4$; for $n \equiv 0 \pmod 4$, $C_n^t$ has a super $(5n/2+2,0)$-edge-antimagic total labeling and a super $(3n/2+2,0)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$; for $n=10$ and $n \equiv 2 \pmod 4$, $n \geq 18$, $C_n^t$ has a super $(5n/2+2,0)$-edge-antimagic total labeling and a super $(3n/2+2,0)$-edge-antimagic total labeling for all $t \equiv 3 \pmod 4$ and for $t=2$ and 6; for odd $n \geq 5$, $C_n^t$ has a super $(2n+2,1)$-edge-antimagic total labeling for all possible values of $t$; for even $n \geq 6$, $C_n^t$ has a super $(2n+2,1)$-edge-antimagic total labeling for all odd $t \geq 3$; and for even $n \equiv 0 \pmod 4$, $n \geq 4$, $C_n^t$ has a super $(2n+2,1)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$. They conjecture that there is a super $(2n+2,1)$-edge-antimagic total labeling of $C_n^t$ for $n \equiv 0 \pmod 4$ and for $t \equiv 0 \pmod 4$ and for $n \equiv 2 \pmod 4$ and for $t$ even.

In [1141] Sugeng, Miller, and Bača prove that the ladder, $P_n \times P_2$, is super $(a,d)$-edge-antimagic total if $n$ is odd and $d = 0, 1,$ or 2 and $P_n \times P_2$ is super $(a,1)$-antimagic total if $n$ is even. They conjecture that $P_n \times P_2$ is super $(a,0)$- and $(a,2)$-edge-antimagic when $n$ is even. Sugeng, Miller, and Bača [1141] prove that $C_m \times P_2$ has a super $(a,d)$-edge-antimagic total labeling if and only if either $d = 0, 1$ or 2 and $m$ is odd and at least 3, or $d = 1$ and $m$ is even and at least 4. They conjecture that if $m$ is even, $m \geq 4,\ n \geq 3$, and $d = 0$ or 2, then $C_m \times P_n$ has a super $(a,d)$-edge-antimagic total labeling.

Sugeng, Miller, and Bača [1141] define a variation of a ladder, $\mathbb{L}_n$, as the graph obtained from $P_n \times P_2$ by joining each vertex $u_i$ of one path to the vertex $v_{i+1}$ of the other path for $i = 1, 2, \ldots, n-1$. They prove $\mathbb{L}_n,\ n \geq 2$, has a super $(a,d)$-edge-antimagic total labeling if and only if $d = 0, 1,$ or 2.

In [114] Bača, Bashir, and Semaničová showed that for $n \geq 4$ and $d = 0, 1, 2, 3, 4, 5,$ and 6 the antiprism $A_n$ has a super $d$-antimagic labeling of type $(1,1,1)$. The *generalized antiprism* $A_m^n$ is obtained from $C_m \times P_n$ by inserting the edges $\{v_{i,j+1}, v_{i+1,j}\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n-1$ where the subscripts are taken modulo $m$. Sugeng et al. prove that $A_m^n,\ m \geq 3,\ n \geq 2$, is super $(a,d)$-edge-antimagic total if and only if $d = 1$.

Baca, Miller, Phanalasy, and A. Semaničová-Feňovčíková [151] investigated the existence of (super) 1-antimagic labelings of type $(1,1,1)$ for disjoint union of plane graphs. They prove that if a plane graph $G(V,E,F)$ has a (super) 1-antimagic labeling $h$ of type $(1,1,1)$ such that $h(z_{ext}) = |V(G)| + |E(G)| + |F(G)|$ where $z_{ext}$ denotes the unique external face then, for every positive integer $m$, the graph $mG$ also admits a (super) 1-antimagic labeling of type $(1,1,1)$; and if a plane graph $G(V,E,F)$ has 4-sided inner faces and $h$ is a (super) $d$-antimagic labeling of type $(1,1,1)$ of $G$ such that $h(z_{ext}) = |V(G)| + |E(G)| + |F(G)|$ where $d = 1, 3, 5, 7, 9$ then, for every positive integer $m$, the graph $mG$ also admits a (super) $d$- antimagic labeling of type $(1,1,1)$. They also give a similar result about plane graphs with inner faces that are 3-sided.

Sugeng, Miller, Slamin, and Bača [1144] proved: the star $S_n$ has a super $(a,d)$-antimagic total labeling if and only if either $d = 0, 1$ or 2, or $d = 3$ and $n = 1$ or 2; if a nontrivial caterpillar has a super $(a,d)$-edge-antimagic total labeling, then $d \leq 3$; all caterpillars have super $(a,0)$-, $(a,1)$- and $(a,2)$-edge-antimagic total labelings; all caterpillars have a super $(a,1)$-edge-antimagic total labeling; if $m$ and $n$ differ by at least 2 the double star $S_{m,n}$ (that is, the graph obtained by joining the centers of $K_{1,m}$ and $K_{1,n}$ with an edge) has no $(a,3)$-edge-antimagic total labeling.

Sugeng and Miller [1139] show how to manipulate adjacency matrices of graphs with $(a,d)$-edge-antimagic vertex labelings and super $(a,d)$-edge-antimagic total labelings to obtain new $(a,d)$-edge-antimagic vertex labelings and super $(a,d)$-edge-antimagic total labelings. Among their results are: every graph can be embedded in a connected $(a,d)$-edge-antimagic vertex graph; every $(a,d)$-edge-antimagic vertex graph has a proper $(a,d)$-edge-antimagic vertex subgraph; if a graph has a $(a,1)$-edge-antimagic vertex labeling and an odd number of edges, then it has a super $(a,1)$-edge-antimagic total labeling; every super edge magic total graph has an $(a,1)$-edge-antimagic vertex labeling; and every graph can be embedded in a connected super $(a,d)$-edge-antimagic total graph.

Ajitha, Arumugan, and Germina [56] show that $(p, p-1)$ graphs with $\alpha$-labelings (see §3.1) and partite sets with sizes that differ by at most 1 have super $(a,d)$-edge antimagic total labelings for $d = 0, 1, 2$ and 3. They also show how to generate large classes of trees with super $(a,d)$-edge-antimagic total labelings from smaller graceful trees.

Bača, Lin, Miller, and Ryan [139] define a *Möbius grid*, $M_n^m$, as the graph with vertex set $\{x_{i,j} |\ i = 1, 2, \ldots, m+1, j = 1, 2, \ldots, n\}$ and edge set $\{x_{i,j}x_{i,j+1} |\ i = 1, 2, \ldots, m+1, j =$

$1, 2, \ldots, n-1\} \cup \{x_{i,j}x_{i+1,j} |\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n\} \cup \{x_{i,n}x_{m+2-i,1} |\ i = 1, 2, \ldots, m+1\}$. They prove that for $n \geq 2$ and $m \geq 4$, $M_n^m$ has no $d$-antimagic vertex labeling with $d \geq 5$ and no $d$-antimagic-edge labeling with $d \geq 9$.

Ali, Bača, and Bashir, [53] investigated super $(a, d)$-vertex-antimagic total labelings of the disjoint unions of paths. They prove: $mP_2$ has a super $(a, d)$-vertex-antimagic total labeling if and only if $m$ is odd and $d = 1$; $mP_3, m > 1$, has no super $(a, 3)$-vertex-antimagic total labeling; $mP_3$ has a super $(a, 2)$-vertex-antimagic total labeling for $m \equiv 1 \pmod 6$; and $mP_4$ has a super $(a, 2)$-vertex-antimagic total labeling for $m \equiv 3 \pmod 4$.

In [568] Ichishima, López, Muntaner-Batle and Rius-Font prove that if $G$ is tripartite and has a (super) $(a, d)$-edge antimagic total labeling, then $nG$ $(n \geq 3)$ has a (super) $(a, d)$-edge antimagic total labeling for $d = 1$ and for $d = 0, 2$ when $n$ is odd.

The book [150] by Bača and Miller has a wealth of material and open problems on super edge-antimagic labelings. In [121] Bača, Baskoro, Miller, Ryan, Simanjuntak, and Sugeng provide detailed survey of results on edge antimagic labelings and include many conjectures and open problems.

In Tables 14 and 15 we use the abbreviations

$(a, d)$-**VAT** $(a, d)$-vertex-antimagic total labeling

$(a, d)$-**SVAT** super $(a, d)$-vertex-antimagic total labeling

$(a, d)$-**EAT** $(a, d)$-edge-antimagic total labeling

$(a, d)$-**SEAT** super $(a, d)$-edge-antimagic total labeling

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 14: **Summary of $(a,d)$-Vertex-Antimagic Total and Super $(a,d)$-Vertex-Antimagic Total Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $P_n$ | $(a,d)$-VAT | wide variety of $a$ and $d$ [123] |
| $P_n$ | $(a,d)$-SVAT | iff $d=3,\ d=2, n\geq 3$ odd or $d=3,\ n\geq 3$ [1142] |
| $C_n$ | $(a,d)$-VAT | wide variety of $a$ and $d$ [122] |
| $C_n$ | $(a,d)$-SVAT | iff $d=0,2$ and $n$ odd or $d=1$ [1142] |
| generalized Petersen graph $P(n,k)$ | $(a,d)$-VAT<br>$(a,1)$-VAT | [124]<br>$n\geq 3,\ 1\leq k\leq n/2$ [1143] |
| prisms $C_n\times P_2$ | $(a,d)$-VAT | [124] |
| antiprisms | $(a,d)$-VAT | [124] |
| $S_{n_1}\cup\ldots\cup S_{n_t}$ | $(a,d)$-VAT | $d=1,2,3,4,6$ [921], citeRahSl |
| $W_n$ | not $(a,d)$-VAT | for $n>20$ [795] |
| $K_{1,n}$ | not $(a,d)$-SVAT | $n\geq 3$ [1142] |

Table 15: **Summary of $(a,d)$-Edge-Antimagic Total Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| trees | $(a,1)$-EAT? | [140] |
| $P_n$ | not $(a,d)$-EAT | $d>2$ [140] |
| $P_{2n}$ | $(6n,1)$-EAT | [1087] |
| | $(6n+2,2)$-EAT | [1087] |
| $P_{2n+1}$ | $(3n+4,2)$-EAT | [1087] |
| | $(3n+4,3)$-EAT | [1087] |
| | $(2n+4,4)$-EAT | [1087] |
| | $(5n+4,2)$-EAT | [1087] |
| | $(3n+5,2)$-EAT | [1087] |
| | $(2n+6,4)$-EAT | [1087] |
| $C_n$ | $(2n+2,1)$-EAT | [1087] |
| | $(3n+2,1)$-EAT | [1087] |
| | not $(a,d)$-EAT | $d>5$ [140] |
| $C_{2n}$ | $(4n+2,2)$-EAT | [1087] |
| | $(4n+3,2)$-EAT | [1087] |
| | $(2n+3,4)$-EAT? | [1087] |
| | $(2n+4,4)$-EAT? | [1087] |
| $C_{2n+1}$ | $(3n+4,3)$-EAT | [1087] |
| | $(3n+5,3)$-EAT | [1087] |
| | $(n+4,5)$-EAT? | [1087] |
| | $(n+5,5)$-EAT? | [1087] |

Table 15: **Summary of $(a,d)$-Edge-Antimagic Total Labelings continued**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $K_n$ | not $(a,d)$-EAT | $d > 5$ [140] |
| $K_{n,n}$ | $(a,d)$-EAT | iff $d = 1, n \geq 2$ [141] |
| caterpillars | $(a,d)$-EAT | $d \leq 3$ [1144] |
| $W_n$ | not $(a,d)$-EAT | $d > 4$ [140] |
| generalized Petersen graph $P(n,k)$ | not $(a,d)$-EAT<br>$((5n+5)/2, 2)$-EAT<br>super $(4n+2, 1)$-EAT | $d > 4$ [140]<br>for $n$ odd, $n \geq 3$ and $k = 1, 2$ [898]<br>for $n \geq 3$, and $1 \leq k \leq n/2$ [898] |

Table 16: **Summary of $(a,d)$-Super-Edge-Antimagic Total Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $C_n^+$ (see §2.2) | $(a,d)$-SEAT | variety of cases [104], [156] |
| $P_n \times P_2$ (ladders) | $(a,d)$-SEAT | $n$ odd, $d \leq 2$ [1141] |
| | | $n$ even, $d = 1$ [1141] |
| | $(a,d)$-SEAT? | $d = 0, 2,\ n$ even [1141] |
| $C_n \times P_2$ | $(a,d)$-SEAT | iff $d \leq 3$ $n$ odd [1141] |
| | | or $d = 1,\ n \geq 4$ even [1141] |
| $C_m \times P_n$ | $(a,d)$-SEAT? | $m \geq 4$ even, $n \geq 3,\ d = 0, 2$ [1141] |
| caterpillars | $(a,1)$-SEAT | [1144] |
| $C_3^{(n)}$ (friendship graphs) | $(a,d)$-SEAT | $d = 0, 1, 2$ [141] |
| $F_n$ $(n \geq 2)$ (fans) | $(a,d)$ SEAT | only if $d < 3$ [141] |
| | $(a,d)$-SEAT | $2 \leq n \leq 6, d = 0, 1, 2$ [141] |
| $W_n$ | $(a,d)$-SEAT | iff $d = 1, n \not\equiv 1 \pmod 4$ [141] |
| $K_n$ $(n \geq 3)$ | $(a,d)$ SEAT | iff $d = 0, n = 3$ [141] |
| | | $d = 1, n \geq 3$ [141] |
| | | $d = 2, n = 3$ [141] |
| trees | $(a,d)$-SEAT | only if $d \leq 3$ [142] |
| $P_n$ $(n > 1)$ | $(a,d)$-SEAT | iff $d \leq 3$ [142] |
| $mK_n$ | $(a,d)$-SEAT | iff $d \in \{0,2\}, n \in \{2,3\}, m \geq 3$ odd [112] |
| | | $d = 1, m, n \geq 2$ [112] |
| | | $d = 3$ or 5,$n = 2, m \geq 2$ [112] |
| | | $d = 4, n = 2,\ m \geq 3$ odd [112] |
| $C_n$ | $(a,d)$-SEAT | iff $d = 0$ or 2, $n$ odd [142] |
| | | $d = 1$ [122] |
| $P(m,n)$ | $(a,d)$-SEAT | many cases [122] |

## 6.3 Face Antimagic Labelings and $d$-antimagic Labeling of Type (1,1,1)

Bača [106] defines a connected plane graph $G$ with edge set $E$ and face set $F$ to be $(a,d)$-*face antimagic* if there exist positive integers $a$ and $d$ and a bijection $g\colon E \to \{1,2,\ldots,|E|\}$ such that the induced mapping $\psi_g\colon F \to \{a, a+d, \ldots, a+(|F(G)|-1)d\}$, where for a face $f$, $\psi_g(f)$ is the sum of all $g(e)$ for all edges $e$ surrounding $f$ is also a bijection. In [108] Bača proves that for $n$ even and at least 4, the prism $C_n \times P_2$ is $(6n+3,2)$-face antimagic and $(4n+4,4)$-face antimagic. He also conjectures that $C_n \times P_2$ is $(2n+5,6)$-face antimagic. In [137] Bača, Lin, and Miller investigate $(a,d)$-face antimagic labelings of the convex polytopes $P_{m+1} \times C_n$. They show that if these graphs are $(a,d)$-face antimagic then either $d=2$ and $a=3n(m+1)+3$, or $d=4$ and $a=2n(m+1)+4$, or $d=6$ and $a=n(m+1)+5$. They also prove that if $n$ is even, $n \geq 4$ and $m \equiv 1 \pmod 4$, $m \geq 3$, then $P_{m+1} \times C_n$ has a $(3n(m+1)+3,2)$-face antimagic labeling and if $n$ is at least 4 and even and $m$ is at least 3 and odd, or if $n \equiv 2 \pmod 4$, $n \geq 6$ and $m$ is even, $m \geq 4$, then $P_{m+1} \times C_n$ has a $(3n(m+1)+3,2)$-face antimagic labeling and a $(2n(m+1)+4,4)$-face antimagic labeling. They conjecture that $P_{m+1} \times C_n$ has $(3n(m+1)+3,2)$- and $(2n(m+1)+4,4)$-face antimagic labelings when $m \equiv 0 \pmod 4$, $n \geq 4$, and for $m$ even and $m \geq 4$, that $P_{m+1} \times C_n$ has a $(n(m+1)+5,6)$-face antimagic labeling when $n$ is even and at least 4.

In [148] Bača and Miller define the class $Q_n^m$ of convex polytopes with vertex set $\{y_{j,i} : i = 1,2,\ldots,n; j = 1,2,\ldots,m+1\}$ and edge set $\{y_{j,i}y_{j,i+1} : i = 1,2,\ldots,n; j = 1,2,\ldots,m+1\} \cup \{y_{j,i}y_{j+1,i} : i = 1,2,\ldots,n; j = 1,2,\ldots,m\} \cup \{y_{j,i+1}y_{j+1,i} : 1{+}1,2,\ldots,n; j = 1,2,\ldots,m, j \text{ odd}\} \cup \{y_{j,i}y_{j+1,i+1} : i = 1,2,\ldots,n; j = 1,2,\ldots,m, j \text{ even}\}$ where $y_{j,n+1} = y_{j,1}$. They prove that for $m$ odd, $m \geq 3, n \geq 3$, $Q_n^m$ is $(7n(m+1)/2+2,1)$-face antimagic and when $m$ and $n$ are even, $m \geq 4, n \geq 4$, $Q_n^m$ is $(7n(m+1)/2+2,1)$-face antimagic. They conjecture that when $n$ is odd, $n \geq 3$, and $m$ is even, then $Q_n^m$ is $((5n(m{+}1){+}5)/2,2)-$face antimagic and $((n(m{+}1){+}7)/2,4)$-face antimagic. They further conjecture that when $n$ is even, $n > 4, m > 1$ or $n$ is odd, $n > 3$ and $m$ is odd, $m > 1$, then $Q_n^m$ is $(3n(m{+}1)/2{+}3,3)$-face antimagic. In [110] Bača proves that for the case $m = 1$ and $n \geq 3$ the only possibilities for $(a,d)$-antimagic labelings for $Q_n^m$ are $(7n+2,1)$ and $(3n+3,3)$. He provides the labelings for the first case and conjectures that they exist for the second case. Bača [106] and Bača and Miller [147] describe $(a,d)$-face antimagic labelings for a certain classes of convex polytopes.

In [117] Bača et al. provide a detailed survey of results on face antimagic labelings and include many conjectures and open problems.

For a plane graph $G$, Bača and Miller [149] call a bijection $h$ from $V(G) \cup E(G) \cup F(G)$ to $\{1,2,\ldots,|V(G)|+|E(G)| \cup |F(G)|\}$ a *d-antimagic labeling of type* $(1,1,1)$ if for every number $s$ the set of $s$-sided face weights is $W_s = \{a_s, a_s+d, a_s+2d, \ldots, a_s+(f_s-1)d\}$ for some integers $a_s$ and $d$, where $f_s$ is the number of $s$-sided faces ($W_s$ varies with $s$). They show that the prisms $C_n \times P_2$ $(n \geq 3)$ have a 1-antimagic labeling of type $(1,1,1)$ and that for $n \equiv 3 \pmod 4$, $C_n \times P_2$ have a $d$-antimagic labeling of type $(1,1,1)$ for $d = 2,3,4$, and 6. They conjecture that for all $n \geq 3$, $C_n \times P_2$ has a $d$-antimagic labeling of type $(1,1,1)$ for $d = 2,3,4,5$, and 6. This conjecture has been proved for the case $d = 3$ and $n \neq 4$ by Bača, Miller, and Ryan [153] (the case $d = 3$ and $n = 4$ is open). The cases for $d = 2,4,5$, and 6 were done by Lin, Slamin, Bača, and Miller [796]. Bača, Lin, and Miller [138] prove: for $m, n > 8$, $P_m \times P_n$ has no $d$-antimagic edge labeling of type $(1,1,1)$ with $d \geq 9$; for $m \geq 2, n \geq 2$, and $(m,n) \neq (2,2)$, $P_m \times P_n$ has $d$-antimagic labelings of type $(1,1,1)$ for $d = 1,2,3,4$, and 6. They conjecture the same is true for $d = 5$.

Bača, Miller, and Ryan [153] also prove that for $n \geq 4$ the antiprism (see §6.1 for the definition) on $2n$ vertices has a $d$-antimagic labeling of type $(1,1,1)$ for $d = 1, 2$, and 4. They conjecture the result holds for $d = 3, 5$, and 6 as well. Lin, Ahmad, Miller, Sugeng, and Bača [793] did the cases that $d = 7$ for $n \geq 3$ and $d = 12$ for $n \geq 11$. Sugeng, Miller, Lin, and Bača [1143] did the cases: $d = 7, 8, 9, 10$ for $n \geq 5$; $d = 15$ for $n \geq 6$; $d = 18$ for $n \geq 7$; $d = 12, 14, 17, 20, 21, 24, 27, 30, 36$ for $n$ odd and $n \geq 7$; and $d = 16, 26$ for $n$ odd and $n \geq 9$.

Ali, Bača, Bashir, and Semaničová-Feňovčíková [54] investigated antimagic labelings for disjoint unions of prisms and cycles. They prove: for $m \geq 2$ and $n \geq 3$, $m(C_n \times P_2)$ has no super $d$-antimagic labeling of type $(1,1,1)$ with $d \geq 30$; for $m \geq 2$ and $n \geq 3$, $n \neq 4$, $m(C_n \times P_2)$ has super $d$-antimagic labeling of type $(1,1,1)$ for $d = 0, 1, 2, 3, 4$, and 5; and for $m \geq 2$ and $n \geq 3$, $mC_n$ has $(m(n+1)+3, 3)$- and $(2mn+2, 2)$-vertex-antimagic total labeling. Bača and Bashir [113] proved that for $m \geq 2$ and $n \geq 3$, $n \neq 4$, $m(C_n \times P_2)$ has super 7-antimagic labeling of type $(1,1,1)$ and for $n \geq 3$, $n \neq 4$ and $2 \leq m \leq 2n$ $m(C_n \times P_2)$ has super 6-antimagic labeling of type $(1,1,1)$.

Bača, Jendraľ, Miller, and Ryan [133] prove: for $n$ even, $n \geq 6$, the generalized Petersen graph $P(n,2)$ has a 1-antimagic labeling of type $(1,1,1)$; for $n$ even, $n \geq 6$, $n \neq 10$, and $d = 2$ or 3, $P(n,2)$ has a $d$-antimagic labeling of type $(1,1,1)$; and for $n \equiv 0 \pmod 4$, $n \geq 8$ and $d = 6$ or 9, $P(n,2)$ has a $d$-antimagic labeling of type $(1,1,1)$. They conjecture that there is an $d$-antimagic labeling of type (1,1,1) for $P(n,2)$ when $n \equiv 2 \pmod 4$, $n \geq 6$, and $d = 6$ or 9.

Bača, Baskoro, and Miller [119] have proved that hexagonal planar honeycomb graphs with an even number of columns have 2-antimagic and 4-antimagic labelings of type $(1,1,1)$. They conjecture that these honeycombs also have $d$-antimagic labelings of type $(1,1,1)$ for $d = 3$ and 5. They pose the odd number of columns case for $1 \leq d \leq 5$ as an open problem. Bača, Baskoro, and Miller [120] give $d$-antimagic labelings of a special class of plane graphs with 3-sided internal faces for $d = 0$, 2, and 4. Bača, Lin, Miller, and Ryan [139] prove for odd $n \geq 3$, $m \geq 1$ and $d = 0, 1, 2$ or 4, the Möbius grid $M_n^m$ has an $d$-antimagic labeling of type $(1,1,1)$.

Kathiresan and Ganesan [637] define a class of plane graphs denoted by $P_a^b$ $(a \geq 3, b \geq 2)$ as the graph obtained by starting with vertices $v_1, v_2, \ldots, v_a$ and for each $i = 1, 2 \ldots, a-1$ joining $v_i$ and $v_{i+1}$ with $b$ internally disjoint paths of length $i+1$. They prove that $P_a^b$ has $d$-antimagic labelings of type $(1,1,1)$ for $d = 0, 1, 2, 3, 4$, and 6. Lin and Sugen [797] prove that $P_a^b$ has a $d$-antimagic labeling of type $(1,1,1)$ for $d = 5, 7a-2, a+1, a-3, a-7, a+5, a-4, a+2, 2a-3, 2a-1, a-1, 3a-3, a+3, 2a+1, 2a+3, 3a+1, 4a-1, 4a-3, 5a-3, 3a-1, 6a-5, 6a-7, 7a-7$, and $5a-5$. Similarly, Bača, Baskoro, and Cholily [116] define a class of plane graphs denoted by $C_a^b$ as the graph obtained by starting with vertices $v_1, v_2, \ldots, v_a$ and for each $i = 1, 2 \ldots, a$ joining $v_i$ and $v_{i+1}$ with $b$ internally disjoint paths of length $i+1$ (subscripts are taken modulo $a$). In [116] and [115] they prove that for $a \geq 3$ and $b \geq 2$, $C_a^b$ has a $d$-antimagic labeling of type $(1,1,1)$ for $d = 0, 1, 2, 3, a+1, a-1, a+2$, and $a-2$.

In the table following we use the abbreviations

$(a,d)$-**FA** $(a,d)$-face antimagic labeling

$d$-**AT(1,1,1)** $d$-antimagic labeling of type $(1,1,1)$.

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 17: **Summary of Face Antimagic Labelings**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $Q_n^m$ (see §6.3) | $(7n(m+1)/2+2,1)$-FA | $m \geq 3$, $n \geq 3$, $m$ odd [148] |
| | $(7n(m+1)/2+2,1)$-FA | $m \geq 4$, $n \geq 4$, $m, n$ even [148] |
| | $((5n(m+1)+5)/2,2)$-FA? | $m \geq 2$, $n \geq 3$, $m$ even, $n$ odd [148] |
| | $((n(m+1)+7)/2,4)$-FA? | $m \geq 2$, $n \geq 3$, $m$ even, $n$ odd [148] |
| | $(3n(m+1)/2+3,3)$-FA? | $m > 1$, $n > 4$, $n$ even [148] |
| | $(3n(m+1)/2+3,3)$-FA? | $m > 1$, $n > 3$, $m$ odd, $n$ odd [148] |
| $C_n \times P_2$ | $(6n+3,2)$-FA | $n \geq 4$, $n$ even [108] |
| | $(4n+4,4)$-FA | $n \geq 4$, $n$ even [108] |
| | $(2n+5,6)$-FA? | [108] |
| $P_{m+1} \times C_n$ | $(3n(m+1)+3,2)$-FA | $n \geq 4$, $n$ even and [137] |
| | | $m \geq 3$, $m \equiv 1 \pmod 4$, |
| | $(3n(m+1)+3,2)$-FA and | $n \geq 4$, $n$ even and [137] |
| | $(2n(m+1)+4,4)$-FA | $m \geq 3$, $m$ odd [137], |
| | | or $n \geq 6$, $n \equiv 2 \pmod 4$ and |
| | | $m \geq 4$, $m$ even |
| | $(3n(m+1)+3,2)$-FA? | $m \geq 4$, $n \geq 4$, $m \equiv 0 \pmod 4$ [137] |
| | $(2n(m+1)+4,4)$-FA? | $m \geq 4$, $n \geq 4$, $m \equiv 0 \pmod 4$ [137] |
| | $(n(m+1)+5,6)$-FA? | $n \geq 4$, $n$ even [137] |

Table 18: **Summary of $d$-antimagic Labelings of Type (1,1,1)**

| *Graph* | *Labeling* | *Notes* |
|---|---|---|
| $P_m \times P_n$ | not $d$-AT(1,1,1) | $m, n, d \geq 9$, [138] |
| $P_m \times P_n$ | $d$-AT(1,1,1) | $d = 1, 2, 3, 4, 6$; $m, n \geq 2$, $(m, n) \neq (2, 2)$ [138] |
| $P_m \times P_n$ | 5-AT(1,1,1) | $m, n \geq 2$, $(m, n) \neq (2, 2)$ [138] |
| $C_n \times P_2$ | 1-AT(1,1,1) | [149] |
| | $d$-AT(1,1,1) | $d = 2, 3, 4$ and 6 [149] for $n \equiv 3 \pmod 4$ |
| | $d$-AT(1,1,1) | $d = 2, 4, 5, 6$ for $n \geq 3$ [796] |
| | $d$-AT(1,1,1) | $d = 3$ for $n \geq 5$ [153] |
| $P_m \times P_n$ | 5-AT(1,1,1)? | [796] |
| | not $d$-AT | $m, n > 8$, $d \geq 9$ [796] |
| antiprism on $2n$ vertices | $d$-AT(1,1,1) | $d = 1, 2$ and 4 for $n \geq 4$ [153] |
| | $d$-AT(1,1,1)? | $d = 3, 5$ and 6 for $n \geq 4$ [153] |
| $M_n^m$ (Möbius grids) | $d$-AT(1,1,1) | $n \geq 3$ odd, $d = 0, 1, 2, 4$ [139] |
| | | $d = 7$, $n \geq 3$ [793] |
| | | $d = 12$, $n \geq 11$ [793] |
| | | $d = 7, 8, 9, 10$, $n \geq 5$ [1143] |
| | | $d = 15$, $n \geq 6$ [1143] |
| | | $d = 18$ $n \geq 7$ [1143] |
| $P(n, 2)$ | $d$-AT(1,1,1) | $d = 1$; $d = 2, 3$, $n \geq 6$, $n \neq 10$ [133] |
| $P(4n, 2)$ | $d$-AT(1,1,1) | $d = 6, 9$, $n \geq 2$, $n \neq 10$ [133] |
| $P(4n + 2, 2)$ | $d$-AT(1,1,1)? | $d = 6, 9$, $n \geq 1$, $n \neq 10$ [133] |
| honeycomb graphs with even number of columns | $d$-AT(1,1,1) | $d = 2, 4$ [119] |
| | $d$-AT(1,1,1)? | $d = 3, 5$ [119] |
| $C_n \times P_2$ | $d$-AT(1,1,1) | $d = 1, 2, 4, 5, 6$ [796], [149] |
| $C_n \times P_2$ | 3-AT(1,1,1) | $n \neq 4$ [153] |

## 6.4 Product Antimagic Labelings

Figueroa-Centeno, Ichishima, and Muntaner-Batle [411] have introduced multiplicative analogs of magic and antimagic labelings. They define a graph $G$ of size $q$ to be *product magic* if there is a labeling from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that, at each vertex $v$, the product of the labels on the edges incident with $v$ is the same. They call a graph $G$ of size $q$ *product antimagic* if there is a labeling $f$ from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that the products of the labels on the edges incident at each vertex $v$ are distinct. They prove: a graph of size $q$ is product magic if and only if $q \leq 1$ (that is, if and only if it is $K_2, \overline{K_n}$ or $K_2 \cup \overline{K_n}$); $P_n$ $(n \geq 4)$ is product antimagic; every 2-regular graph is product antimagic; and, if $G$ is product antimagic, then so are $G + K_1$ and $G \odot \overline{K}_n$. They conjecture that a connected graph of size $q$ is product antimagic if and only if $q \geq 3$. Kaplan, Lev, and Roditty [628] proved the following graphs are product anti-magic: the disjoint union of cycles and paths where each path has least three edges; connected graphs with $n$ vertices and $m$ edges where $m \geq 4n \ln n$; graphs $G = (V, E)$ where each component has at least two edges and the minimum degree of $G$ is at least $8\sqrt{\ln |E| \ln (\ln |E|)}$; all complete $k$-partite graphs except $K_2$ and $K_{1,2}$; and $G \odot H$ where $G$ has no isolated vertices and $H$ is regular.

In [938] Pikhurko characterizes all large graphs that are product anti-magic graphs. More precisely, it is shown that there is an $n_0$ such that a graph with $n \geq n_0$ vertices is product anti-magic if and only if it does not belong to any of the following four classes: graphs that have at least one isolated edge; graphs that have at least two isolated vertices; unions of vertex-disjoint of copies of $K_{1,2}$; graphs consisting of one isolated vertex; and graphs obtained by subdividing some edges of the star $K_{1,k+l}$.

In [411] Figueroa-Centeno, Ichishima, and Muntaner-Batle also define a graph $G$ with $p$ vertices and $q$ edges to be *product edge-magic* if there is a labeling $f$ from $V(G) \cup E(G)$ onto $\{1, 2, \ldots, p + q\}$ such that $f(u) \cdot f(v) \cdot f(uv)$ is a constant for all edges $uv$ and *product edge-antimagic* if there is a labeling $f$ from $V(G) \cup E(G)$ onto $\{1, 2, \ldots, p+q\}$ such that for all edges $uv$ the products $f(u) \cdot f(v) \cdot f(uv)$ are distinct. They prove $K_2 \cup \overline{K}_n$ is product edge-magic, a graph of size $q$ without isolated vertices is product edge-magic if and only if $q \leq 1$ and every graph other than $K_2$ and $K_2 \cup \overline{K}_n$ is product edge-antimagic.

# 7 Miscellaneous Labelings

## 7.1 Sum Graphs

In 1990, Harary [505] introduced the notion of a sum graph. A graph $G(V, E)$ is called a *sum graph* if there is an bijection $f$ from $V$ to a set of positive integers $S$ such that $xy \in E$ if and only if $f(x) + f(y) \in S$. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. In 1991 Harary, Hentzel, and Jacobs [507] defined a *real sum graph* in an analogous way by allowing $S$ to be any finite set of positive real numbers. However, they proved that every real sum graph is a sum graph. Bergstrand, Hodges, Jennings, Kuklinski, Wiener, and Harary [215] defined a *product graph* analogous to a sum graph except that 1 is not permitted to belong to $S$. They proved that every product graph is a sum graph and vice versa.

For a connected graph $G$, let $\sigma(G)$, the *sum number of* $G$, denote the minimum number of isolated vertices that must be added to $G$ so that the resulting graph is a sum graph (some authors use $s(G)$ for the sum number of $G$). A labeling that makes $G$ together with $\sigma(G)$ isolated points a sum graph is called an *optimal sum graph* labeling. Ellingham [379] proved the conjecture of Harary [505] that $\sigma(T) = 1$ for every tree $T \neq K_1$. Smyth [1114] proved that there is no graph $G$ with $e$ edges and $\sigma(G) = 1$ when $n^2/4 < e \leq n(n-1)/2$. Smyth [1115] conjectures that the disjoint union of graphs with sum number 1 has sum number 1. More generally, Kratochvil, Miller, and Nguyen [680] conjecture that $\sigma(G \cup H) \leq \sigma(G) + \sigma(H) - 1$. Hao [504] has shown that if $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence of a graph $G$, then $\sigma(G) > \max(d_i - i)$ where the maximum is taken over all $i$. Bergstand et al. [214] proved that $\sigma(K_n) = 2n - 3$. Hartsfield and Smyth [512] claimed to have proved that $\sigma(K_{m,n}) = \lceil 3m + n - 3 \rceil / 2$ when $n \geq m$ but Yan and Liu [1284] found counterexamples to this assertion when $m \neq n$. Pyatkin [944], Liaw, Kuo, and Chang [790], Wang and Liu [1245], and He, Shen, Wang, Chang, Kang, and Yu [516] have shown that for $2 \leq m \leq n$, $\sigma(K_{m,n}) = \lceil \frac{n}{p} + \frac{(p+1)(m-1)}{2} \rceil$ where $p = \lceil \sqrt{\frac{2n}{m-1} + \frac{1}{4}} - \frac{1}{2} \rceil$ is the unique integer such that $\frac{(p-1)p(m-1)}{2} < n \leq \frac{(p+1)p(m-1)}{2}$.

Miller, Ryan, Slamin, and Smyth [871] proved that $\sigma(W_n) = \frac{n}{2} + 2$ for $n$ even and $\sigma(W_n) = n$ for $n \geq 5$ and $n$ odd (see also [1162]). Miller, Ryan, and Smyth [873] prove that the complete $n$-partite graph on $n$ sets of 2 nonadjacent vertices has sum number $4n-5$ and obtain upper and lower bounds on the complete $n$-partite graph on $n$ sets of $m$ nonadjacent vertices. Fernau, Ryan, and Sugeng [408] proved that the generalized friendship graphs $C_n^{(t)}$ (see §2.2) has sum number 2 except for $C_4$. Gould and Rödl [479] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [478] proved that $\sigma(K_n - \text{edge}) \leq 2n - 4$. The same group of six students also investigated the difference between the largest and smallest labels in a sum graph, which they called the *spum*. They proved spum of $K_n$ is $4n - 6$ and the spum of $C_n$ is at most $4n - 10$. Kratochvil, Miller, and Nguyen [680] have proved that every sum graph on $n$ vertices has a sum labeling such that every label is at most $4^n$.

At a conference in 2000 Miller [862] posed the following two problems: Given any graph $G$, does there exist an optimal sum graph labeling that uses the label 1; Find a class of graphs $G$ that have sum number of the order $|V(G)|^s$ for $s > 1$. (Such graphs were shown to exist for $s = 2$ by Gould and Rödl in [479]).

In [1102] Slamet, Sugeng, and Miller show how one can use sum graph labelings to distribute secret information to set of people so that only authorized subsets can reconstruct the secret.

Chang [294] generalized the notion of sum graph by permitting $x = y$ in the definition of

sum graph. He calls graphs that have this kind of labeling *strong sum graphs* and uses $i^*(G)$ to denote the minimum positive integer $m$ such that $G \cup mK_1$ is a strong sum graph. Chang proves that $i^*(K_n) = \sigma(K_n)$ for $n = 2, 3$, and 4 and $i^*(K_n) > \sigma(K_n)$ for $n \geq 5$. He further shows that for $n \geq 5$, $3n^{\log_2 3} > i^*(K_n) \geq 12\lfloor n/5 \rfloor - 3$.

In 1994 Harary [506] generalized sum graphs by permitting $S$ to be any set of integers. He calls these graphs *integral sum graphs*. Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [1053] has shown that $C_n$ and $W_n$ are integral sum graphs for all $n \neq 4$. Chen [313] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum graphs. In [313] and [315] Chen gives methods for constructing new connected integral sum graphs from given integral sum graphs by identifying vertices. Chen [315] has shown that every graph is an induced subgraph of a connected integral sum graph. Chen [315] calls a vertex of a graph *saturated* if it is adjacent to every other vertex of the graph. He proves that every integral sum graph except $K_3$ has at most two saturated vertices and gives the exact structure of all integral sum graphs that have exactly two saturated vertices. Chen [315] also proves that a connected integral sum graph with $p > 1$ vertices and $q$ edges and no saturated vertices satisfies $q \leq p(3p-2)/8 - 2$. Wu, Mao, and Le [1264] proved that $mP_n$ are integral sum graphs. They also show that the conjecture of Harary [506] that the sum number of $C_n$ equals the integral sum number of $C_n$ if and only if $n \neq 3$ or 5 is false and that for $n \neq 4$ or 6 the integral sum number of $C_n$ is at most 1. Vilfred and Nicholas [1210] prove that graphs $G$ of order $n$ with $\Delta(G) = n-1$ and $|V_\Delta(G)| > 2$ are not integral sum graphs, except $K_3$, and that integral sum graphs $G$ of order $n$ with $\Delta(G) = n-1$ and $|V_\Delta(G)| = 2$ exist and are unique up to isomorphism. Chen [317] proved that if $G(V, E)$ is an integral sum other than $K_3$ that has vertex of degree $|V| - 1$, then the edge-chromatic number of $G$ is $|V| - 1$.

He, Wang, Mi, Shen, and Yu [514] say that a graph has a *tail* if the graph contains a path for which each interior vertex has degree 2 and an end vertex of degree at least 3. They prove that every tree with a tail of length at least 3 is an integral sum graph.

B. Xu [1273] has shown that the following are integral sum graphs: the union of any three stars; $T \cup K_{1,n}$ for all trees $T$; $mK_3$ for all $m$; and the union of any number of integral sum trees. Xu also proved that if $2G$ and $3G$ are integral sum graphs, then so is $mG$ for all $m > 1$. Xu poses the question as to whether all disconnected forests are integral sum graphs. Nicholas and Somasundaram [906] prove that all banana trees (see Section 2.1 for the definition) and the union of any number of stars are integral sum graphs.

Liaw, Kuo, and Chang [790] proved that all caterpillars are integral sum graphs (see also [1264] and [1273] for some special cases of caterpillars). This shows that the assertion by Harary in [506] that $K(1,3)$ and $S(2,2)$ are not integral sum graphs is incorrect. They also prove that all cycles except $C_4$ are integral sum graphs and they conjecture that every tree is an integral sum graph. Singh and Santhosh show that the crowns $C_n \odot K_1$ are integral sum graphs for $n \geq 4$ [1096] and that the subdivision graphs of $C_n \odot K_1$ are integral sum graphs for $n \geq 3$ [995].

The *integral sum number*, $\zeta(G)$, of $G$ is the minimum number of isolated vertices that must be added to $G$ so that the resulting graph is an integral sum graph. Thus, by definition, $G$ is a integral sum graph if and only if $\zeta(G) = 0$. Harary [506] conjectured that $\zeta(K_n) = 2n - 3$ for $n \geq 4$. This conjecture was verified by Chen [312], by Sharary [1053], and by B. Xu [1273]. Yan and Liu proved: $\zeta(K_n - E(K_r)) = n - 1$ when $n \geq 6$, $n \equiv 0 \pmod 3$ and $r = 2n/3 - 1$ [1285]; $\zeta(K_{m.m}) = 2m - 1$ for $m \geq 2$ [1285]; $\zeta(K_n - \text{edge}) = 2n - 4$ for $n \geq 4$ [1285], [1273]; if $n \geq 5$ and $n - 3 \geq r$, then $\zeta(K_n - E(K_r)) \geq n - 1$ [1285]; if $\lceil 2n/3 \rceil - 1 > r \geq 2$, then

$\zeta(K_n - E(K_r)) \geq 2n - r - 2$ [1285]; and if $2 \leq m < n$, and $n = (i+1)(im - i + 2)/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = (m-1)(i+1)+1$ while if $(i+1)(im-i+2)/2 < n < (i+2)[(i+1)m-i+1]/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = \lceil((m-1)(i+1)(i+2)+2n)/(2i+2)\rceil$ [1285].

Nagamochi, Miller, and Slamin [891] have determined upper and lower bounds on the sum number a graph. For most graphs $G(V,E)$ they show that $\sigma(G) = \Omega(|E|)$. He, Yu, Mi, Sheng, and Wang [515] investigated $\zeta(K_n - E(K_r))$ where $n \geq 5$ and $r \geq 2$. They proved that $\zeta(K_n - E(K_r)) = 0$ when $r = n$ or $n-1$; $\zeta(K_n - E(K_r)) = n-2$ when $r = n-2$; $\zeta(K_n - E(K_r)) = n-1$ when $n - 3 \geq r \geq \lceil 2n/3\rceil - 1$; $\zeta(K_n - E(K_r)) = 3n - 2r - 4$ when $\lceil 2n/3\rceil - 1 > r \geq n/2$; $\zeta(K_n - E(K_r)) = 2n - 4$ when $\lceil 2n/3\rceil - 1 \geq n/2 > r \geq 2$. Moreover, they prove that if $n \geq 5, r \geq 2$, and $r \neq n-1$, then $\sigma(K_n - E(K_r)) = \zeta(K_n - E(K_r))$. Dou and Gao [370] prove that for $n \geq 3$, the fan $F_n = P_n + K_1$ is an integral sum graph, $\rho(F_4) = 1, \rho(F_n) = 2$ for $n \neq 4$, and $\sigma(F_4) = 2, \sigma(F_n) = 3$ for $n = 3$ or $n \geq 6$ and $n$ even, and $\sigma(F_n) = 4$ for $n \geq 6$ and $n$ odd.

Wang and Gao [1234] and [1235] determined the sum numbers and the integral sum numbers of the complements of paths, cycles, wheels, and fans as follows.
$0 = \zeta(\overline{P_4}) < \sigma(\overline{P_4}) = 1$; $1 = \zeta(\overline{P_5}) < \sigma(\overline{P_5}) = 2$;
$3 = \zeta(\overline{P_6}) < \sigma(\overline{P_6}) = 4$; $\zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 0,\ n = 1, 2, 3$;
$\zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 2n - 7,\ n \geq 7$.
$\zeta(\overline{C_n}) = \sigma(\overline{C_n}) = 2n - 7,\ n \geq 7$.
$\zeta(\overline{W_n}) = \sigma(\overline{W_n}) = 2n - 8,\ n \geq 7$.
$0 = \zeta(\overline{F_5}) < \sigma(\overline{F_5}) = 1$;
$2 = \zeta(\overline{F_6}) < \sigma(\overline{F_6}) = 3$; $\zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 0, n = 3, 4$;
$\zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 2n - 8,\ n \geq 7$.

Chen [312] has given some properties of integral sum labelings of graphs $G$ with $\Delta(G) < |V(G)| - 1$ whereas Nicholas, Somasundaram, and Vilfred [908] provided some general properties of connected integral sum graphs $G$ with $\Delta(G) = |V(G)| - 1$. They have shown that connected integral sum graphs $G$ other than $K_3$ with the property that $G$ has exactly two vertices of maximum degree are unique and that a connected integral sum graph $G$ other than $K_3$ can have at most two vertices with degree $|V(G)| - 1$ (see also [1223]).

Vilfred and Florida [1220] have examined one-point unions of pairs of small complete graphs. They show that the one-point union of $K_3$ and $K_2$ and the one-point union of $K_3$ and $K_3$ are integral sum graphs whereas the one-point union of $K_4$ and $K_2$ and the one-point union of $K_4$ and $K_3$ are not integral sum graphs. In [1221] Vilfred and Florida defined and investigated properties of maximal integral sum graphs.

Vilfred and Nicholas [1224] have shown that the following graphs are integral sum graphs: banana trees, the union of any number of stars, fans $P_n + K_1$ $(n \geq 2)$, Dutch windmills $K_3^{(m)}$, and the graph obtained by starting with any finite number of integral sum graphs $G_1, G_2, \ldots, G_n$ and any collections of $n$ vertices with $v_i \in G_i$ and creating a graph by identifying $v_1, v_2, \ldots, v_n$. The same authors [1225] also proved that $G + v$ where $G$ is a union of stars is an integral sum graph.

Melnikov and Pyatkin [859] have shown that every 2-regular graph except $C_4$ is an integral sum graph and that for every positive integer $r$ there exists an $r$-regular integral sum graph. They also show that the cube is not an integral sum graph. For any integral sum graph $G$, Melnikov and Pyatkin define the *integral radius of* $G$ as the smallest natural number $r(G)$ that has all its vertex labels in the interval $[-r(G), r(G)]$. For the family of all integral sum graphs of order $n$ they use $r(n)$ to denote maximum integral radius among all members of the family. Two questions they raise are: Is there a constant $C$ such that $r(n) \leq C_n$ and for $n > 2$, is $r(n)$

equal to the $(n-2)$th prime?

The concepts of sum number and integral sum number have been extended to hypergraphs. Sonntag and Teichert [1127] prove that every hypertree (i.e., every connected, non-trivial, cycle-free hypergraph) has sum number 1 provided that a certain cardinality condition for the number of edges is fulfilled. In [1128] the same authors prove that for $d \geq 3$ every $d$-uniform hypertree is an integral sum graph and that for $n \geq d+2$ the sum number of the complete $d$-uniform hypergraph on $n$ vertices is $d(n-d)+1$. They also prove that the integral sum number for the complete $d$-uniform hypergraph on $n$ vertices is 0 when $d = n$ or $n-1$ and is between $(d-1)(n-d-1)$ and $d(n-d)+1$ for $d \leq n-2$. They conjecture that for $d \leq n-2$ the sum number and the integral sum number of the complete $d$-uniform hypergraph are equal. Teichert [1173] proves that hypercycles have sum number 1 when each edge has cardinality at least 3 and that hyperwheels have sum number 1 under certain restrictions for the edge cardinalities. (A *hypercycle* $\mathcal{C}_n = (\mathcal{V}_n, \mathcal{E}_n)$ has $\mathcal{V}_n = \cup_{i=1}^{n}\{v_1^i, v_2^i, \ldots, v_{d_i-1}^i\}, \mathcal{E}_n = \{e_1, e_2, \ldots, e_n\}$ with $e_i = \{v_1^i, \ldots, v_{d_i}^i = v_1^{i+1}\}$ where $i+1$ is taken modulo $n$. A *hyperwheel* $\mathcal{W}_n = (\mathcal{V}_n', \mathcal{E}_n')$ has $\mathcal{V}_n' = \mathcal{V}_n \cup \{c\} \cup_{i=1}^{n} \{{v_2}^{n+i}, \ldots, {v_{d_{n+i}-1}}^{n+i}\}, \mathcal{E}_n' = \mathcal{E}_n \cup \{e_{n+1}, \ldots, e_{2n}\}$ with $e_{n+i} = \{{v_1}^{n+i} = c, {v_2}^{n+i}, \ldots, {v_{d_{n+i}-1}}^{n+i}, {v_{d_{n+i}}}^{n+i} = {v_1}^i\}$.)

Teichert [1172] determined an upper bound for the sum number of the $d$-partite complete hypergraph $K_{n_1,\ldots,n_d}^d$. In [1174] Teichert defines the *strong hypercycle* $\mathcal{C}_n^d$ to be the $d$-uniform hypergraph with the same vertices as $C_n$ where any $d$ consecutive vertices of $C_n$ form an edge of $\mathcal{C}_n^d$. He proves that for $n \geq 2d+1 \geq 5$, $\sigma(\mathcal{C}_n^d) = d$ and for $d \geq 2$, $\sigma(\mathcal{C}_{d+1}^d) = d$. He also shows that $\sigma(\mathcal{C}_5^3) = 3$; $\sigma(\mathcal{C}_6^3) = 2$, and he conjectures that $\sigma(\mathcal{C}_n^d) < d$ for $d \geq 4$ and $d+2 \leq n \leq 2d$.

In [909] Nicholas and Vilfred define the *edge reduced sum number* of a graph as the minimum number of edges whose removal from the graph results in a sum graph. They show that for $K_n$, $n \geq 3$, this number is $(n(n-1)/2 + \lfloor n/2 \rfloor)/2$. They ask for a characterization of graphs for which the edge reduced sum number is the same as its sum number. They conjecture that an integral sum graph of order $p$ and size $q$ exists if and only if $q \leq 3(p^2-1)/8 - \lfloor (p-1)/4 \rfloor$ when $p$ is odd and $q \leq 3(3p-2)/8$ when $p$ is even. They also define the *edge reduced integral sum number* in an analogous way and conjecture that for $K_n$ this number is $(n-1)(n-3)/8 + \lfloor (n-1)/4 \rfloor$ when $n$ is odd and $n(n-2)/8$ when $n$ is even.

For certain graphs $G$ Vilfred and Florida [1219] investigated the relationships among $\sigma(G), \zeta(G), \chi(G)$, and $\chi'(G)$ where $\chi(G)$ is the chromatic number of $G$ and $\chi'(G)$ is the edge chromatic number of $G$. They prove: $\sigma(C_4) = \zeta(C_4) > \chi(C_4) = \chi'(C_4)$; for $n \geq 3$, $\zeta(C_{2n}) < \sigma(C_{2n}) = \chi(C_{2n}) = \chi'(C_{2n})$; $\zeta(C_{2n+1}) < \sigma(C_{2n+1}) < \chi(C_{2n+1}) = \chi'(C_{2n+1})$; for $n \geq 4$, $\chi'(K_n) \leq \chi(K_n) < \zeta(K_n) = \sigma(K_n)$; and for $n \geq 2$, $\chi(P_n \times P_2) < \chi'(P_n \times P_2) = \zeta(P_n \times P_2) = \sigma(P_n \times P_2)$.

Alon and Scheinermann [59] generalized sum graphs by replacing the condition $f(x)+f(y) \in S$ with $g(f(x), f(y)) \in S$ where $g$ is an arbitrary symmetric polynomial. They called a graph with this property a *g-graph* and proved that for a given symmetric polynomial $g$ not all graphs are $g$-graphs. On the other hand, for every symmetric polynomial $g$ and every graph $G$ there is some vertex labeling such that $G$ together with at most $|E(G)|$ isolated vertices is a $g$-graph.

Boland, Laskar, Turner, and Domke [250] investigated a modular version of sum graphs. They call a graph $G(V,E)$ a *mod sum graph* (MSG) if there exists a positive integer $n$ and an injective labeling from $V$ to $\{1, 2, \ldots, n-1\}$ such that $xy \in E$ if and only if $(f(x)+f(y)) \pmod n = f(z)$ for some vertex $z$. Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [250] have shown the following graphs are MSG: all trees on 3 or more vertices; all cycles on 4 or more vertices; and $K_{2,n}$. They further proved that $K_p$ $(p \geq 2)$ is not MSG (see also [470]) and that $W_4$ is MSG. They conjecture that $W_p$ is

MSG for $p \geq 4$. This conjecture was refuted by Sutton, Miller, Ryan, and Slamin [1163] who proved that for $n \neq 4$, $W_n$ is not MSG (the case where $n$ is prime had been proved in 1994 by Ghoshal, Laskar, Pillone, and Fricke [470]. In the same paper Sutton et al. also showed that for $n \geq 3$, $K_{n,n}$ is not MSG. Ghoshal, Laskar, Pillone, and Fricke [470] proved that every connected graph is an induced subgraph of a connected MSG graph and any graph with $n$ vertices and at least two vertices of degree $n-1$ is not MSG.

Sutton, Miller, Ryan, and Slamin [1163] define the *mod sum number*, $\rho(G)$, of a connected graph $G$ to be the least integer $r$ such that $G \cup \overline{K_r}$ is MSG. Sutton and Miller [1161] define the cocktail party graph $H_{m,n}$, $m, n \geq 2$, as the graph with a vertex set $V = \{v_1, v_2, \ldots, v_{mn}\}$ partitioned into $n$ independent sets $V = \{I_1, I_2, \ldots, I_n\}$ each of size $m$ such that $v_i v_j \in E$ for all $i, j \in \{1, 2, \ldots, mn\}$ where $i \in I_p$, $j \in I_q$, $p \neq q$. The graphs $H_{m,n}$ can be used to model relational database management systems (see [1159]). Sutton and Miller prove that $H_{m,n}$ is not MSG for $n > m \geq 3$ and $\rho(K_n) = n$ for $n \geq 4$. In [1160] Sutton, Draganova, and Miller prove that for $n$ odd and $n \geq 5$, $\rho(W_n) = n$ and when $n$ is even, $\rho(W_n) = 2$. Wang, Zhang, Yu, and Shi [1244] proved that fan $F_n (n \geq 2)$ are not mod sum graphs and $\rho(F_n) = 2$ for even $n$ at least 6. They also prove that $\rho(K_{n,n}) = n$ for $n \geq 3$.

Dou and Gao [371] obtained exact values for $\rho(K_{m,n})$ and $\rho(K_m - E(K_n))$ for some cases of $m$ and $n$ and bounds in the remaining cases. They call a graph $G(V, E)$ a *mod integral sum graph* if there exists a positive integer $n$ and an injective labeling from $V$ to $\{0, 1, 2, \ldots, n-1\}$ (note that 0 is included) such that $xy \in E$ if and only if $(f(x) + f(y)) \pmod{n} = f(z)$ for some vertex $z$. They define the *mod integral sum number*, $\psi(G)$, of a connected graph $G$ to be the least integer $r$ such that $G \cup \overline{K_r}$ is a mod integral sum graph. They prove that for $m + n \geq 3$, $\psi(K_{m,n}) = \rho(K_{m,n})$ and obtained exact values for $\psi(K_m - E(K_n))$ for some cases of $m$ and $n$ and bounds in the remaining cases.

Wallace [1227] has proved that $K_{m,n}$ is MSG when $n$ is even and $n \geq 2m$ or when $n$ is odd and $n \geq 3m - 3$ and that $\rho(K_{m,n}) = m$ when $3 \leq m \leq n < 2m$. He also proves that the complete $m$-partite $K_{n_1,n_2,\ldots,n_m}$ is not MSG when there exist $n_i$ and $n_j$ such that $n_i < n_j < 2n_i$. He poses the following conjectures: $\rho(K_{m,n}) = n$ when $3m - 3 > n \geq m \geq 3$; if $K_{n_1,n_2,\ldots,n_m}$ where $n_1 > n_2 > \cdots > n_m$, is not MSG, then $(m-1)n_m \leq \rho(K_{n_1,n_2,\ldots,n_m}) \leq (m-1)n_1$; if $G$ has $n$ vertices, then $\rho(G) \leq n$; and determining the mod sum number of a graph is $NP$-complete (Sutton has observed that Wallace probably meant to say '$NP$-hard'). Miller [862] has asked if it is possible for the mod sum number of a graph $G$ be of the order $|V(G)|^2$.

In a sum graph $G$, a vertex $w$ is called a *working vertex* if there is an edge $uv$ in $G$ such that $w = u + v$. If $G = H \cup \overline{H_r}$ has a sum labeling such that $H$ has no working vertex the labeling is called an *exclusive sum labeling of $H$ with respect $G$*. The *exclusive sum number*, $\epsilon(H)$, of a graph $H$ is the smallest integer $r$ such that $G \cup \overline{K_r}$ has an exclusive sum labeling. The exclusive sum number is known in the following cases (see [866] and [872]): for $n \geq 3$, $\epsilon(P_n) = 2$; for $n \geq 3$, $\epsilon(C_n) = 3$; for $n \geq 3$, $\epsilon(K_n) = 2n - 3$; for $n \geq 4$, $\epsilon(F_n) = n$ (fan of order $n + 1$); for $n \geq 4$, $\epsilon(W_n) = n$; $\epsilon(C_3^{(n)}) = 2n$ (friendship graph–see §2.2); $m \geq 2$, $n \geq 2$, $\epsilon(K_{m,n}) = m+n-1$; for $n \geq 2$, $S_n = n$ (star of order $n + 1$); $\epsilon(S_{m,n}) = \max\{m, n\}$ (double star); $H_{2,n} = 4n - 5$ (cocktail party graph); and $\epsilon(\text{caterpillar } G) = \Delta(G)$. Vilfred and Florida [1222] proved that $\epsilon(P_3 \times P_3) = 4$ and $\epsilon(P_n \times P_2) = 3$. In [546] Hegde and Vasudeva provide an $O(n^2)$ algorithm that produces an exclusive sum labeling of a graph with $n$ vertices given its adjacency matrix.

In 2001 Kratochvil, Miller, and Nguyen proved that $\sigma(G \cup H) \leq \sigma(G) + \sigma(H) - 1$. In 2003 Miller, Ryan, Slamin, Sugeng, and Tuga [868] posed the problem of finding the exclusive sum number of the disjoint union of graphs. In 2010 Wang and Li [1236] proved the following. Let

$G_1$ and $G_2$ be graphs without isolated vertices, $L_i$ be an exclusive sum labeling of $G_i \cup \epsilon(G_i)K_1$, and $C_i$ be the isolated set of $L_i$ for $i = 1$ and 2. If $\max C_1$ and $\min C_2$ are relatively prime, then $\epsilon(G_1 \cup G_2) \leq \epsilon(G_1) + \epsilon(G_2) - 1$. Wang and Li also proved the following: $\epsilon(K_{r,s}) = s + r - 1$; $\epsilon(K_{r,s} - E(K_2)) = s - 1$; for $s \geq r \geq 2$, $\epsilon(K_{r,s} - E(rK_2)) = s + r - 3$. For $n \geq 5$ they prove: $\epsilon(K_n - E(K_n)) = 0$; $\epsilon(K_n - E(K_{n-1})) = n - 1$; for $2 \leq r < n/2$, $\epsilon(K_n - E(K_r)) = 2n - 4$; for $n/2 \leq r \leq n - 2$, $\epsilon(K_n - E(K_r)) = 3n - 2r - 4$, and $\epsilon(C_n \odot K_1)$ is 3 or 4. They show that $\epsilon(C_3 \odot K_1) = 3$ and guess that for $n \geq 4$, $\epsilon(C_n \odot K_1) = 4$. A survey of exclusive sum labelings of graphs is given by Ryan in [977].

If $\epsilon(G) = \Delta(G)$, then $G$ is said to be an $\Delta$-*optimum summable graph.* An exclusive sum labeling of a graph $G$ using $\Delta(G)$ isolates is called a $\Delta$-*optimum exclusive sum labeling* of $G$. Tuga, Miller, Ryan, and Ryjáček [1186] show that some families of trees that are $\Delta$-optimum summable and some that are not. They prove that if $G$ is a tree that has at least one vertex that has two or more neighbors that are not leaves then $\epsilon(G) = \Delta(G)$.

Grimaldi [497] has investigated labeling the vertices of a graph $G(V, E)$ with $n$ vertices with distinct elements of the ring $Z_n$ so that $xy \in E$ whenever $(x + y)^{-1}$ exists in $Z_n$.

In his 2001 Ph. D. thesis Sutton [1159] introduced two methods of graph labelings with applications to storage and manipulation of relational database links specifically in mind. He calls a graph $G = (V_p \cup V_i, E)$ a *sum* graph* of $G_p = (V_p, E_p)$ if there is an injective labeling $\lambda$ of the vertices of $G$ with non-negative integers with the property that $uv \in E_p$ if and only if $\lambda(u) + \lambda(v) = \lambda(z)$ for some vertex $z \in G$. The *sum* number*, $\sigma^*(G_p)$, is the minimum cardinality of a set of new vertices $V_i$ such that there exists a sum* graph of $G_p$ on the set of vertices $V_p \cup V_i$. A *mod sum* graph* of $G_p$ is defined in the identical fashion except the sum $\lambda(u) + \lambda(v)$ is taken modulo $n$ where the vertex labels of $G$ are restricted to $\{0, 1, 2, \ldots, n - 1\}$. The *mod sum* number*, $\rho^*(G_p)$, of a graph $G_p$ is defined in the analogous way. Sum* graphs are a generalization of sum graphs and mod sum* graphs are a generalization of mod sum graphs. Sutton shows that every graph is an induced subgraph of a connected sum* graph. Sutton [1159] poses the following conjectures: $\rho(H_{m,n}) \leq mn$ for $m, n \geq 2$; $\sigma^*(G_p) \leq |V_p|$; and $\rho^*(G_p) \leq |V_p|$.

The following table summarizing what is known about sum graphs, mod sum graphs, sum* graphs, and mod sum* graphs is reproduced from Sutton's Ph. D. thesis [1159]. It was updated by J. Gallian in 2006. A question mark indicates the value is unknown. The results on sum* and mod sum* graphs are found in [1159].

Table 19: **Summary of Sum Graph Labelings**

| **Graph** | $\sigma(G)$ | $\rho(G)$ | $\sigma^*(G)$ | $\rho^*(G)$ |
|---|---|---|---|---|
| $K_2 = S_1$ | 1 | 1 | 0 | 0 |
| stars, $S_n,\ n \geq 2$ | 1 | 0 | 0 | 0 |
| trees $T_n,\ n \geq 3$ when $T_n \neq S_n$ | 1 | 0 | 1 | 0 |
| $C_3$ | 2 | 1 | 1 | 0 |
| $C_4$ | 3 | 0 | 2 | 0 |
| $C_n,\ n > 4$ | 2 | 0 | 2 | 0 |
| $W_4$ | 4 | 0 | 2 | 0 |
| $W_n,\ n \geq 5,\ n$ odd | $n$ | $n$ | 2 | 0 |
| $W_n,\ n \geq 6,\ n$ even | $\frac{n}{2} + 2$ | 2 | 2 | 0 |
| fan, $F_4$, | 2 | 1 | 1 | 0 |
| fans, $F_n,\ n \geq 5,\ n$ odd | ? | 2 | 1 | 0 |
| fans, $F_n,\ n \geq 6,\ n$ even | 3 | 2 | 1 | 0 |
| $K_n,\ n \geq 4$ | $2n - 3$ | $n$ | $n - 2$ | 0 |
| cocktail party graphs, $H_{2,n}$ | $4n - 5$ | 0 | ? | 0 |
| $C_n^{(t)}\ (n,t) \neq (4,1)$ (see §2.2) | 2 | ? | ? | ? |
| $K_{n,n}$ | $\lceil \frac{4n-3}{2} \rceil$ | $n (n \geq 3)$ | ? | ? |
| $K_{m,n},\ 2nm \geq n \geq 3$ | ? | $n$ | ? | ? |
| $K_{m,n}\ m \geq 3n - 3,\ n \geq 3,\ m$ odd | ? | 0 | ? | 0 |
| $K_{m,n},\ m \geq 2n,\ n \geq 3,\ m$ even | ? | 0 | ? | 0 |
| $K_{m,n},\ m < n$<br>$k = \lceil \sqrt{1 + (8m + n - 1)(n - 1)/2}\ \rceil$ | $\lceil (kn - k)/2 + m/(k - 1) \rceil$ | ? | ? | ? |
| $K_{n,n} - E(nK_2),\ n \geq 6$ | $2n - 3$ | $n - 2$ | ? | ? |

## 7.2 Prime and Vertex Prime Labelings

The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla [1179]. A graph with vertex set $V$ is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge $xy$ the labels assigned to $x$ and $y$ are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. So far, there has been little progress towards proving this conjecture. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders (i.e., trees with one vertex of degree at least 3 and with all other vertices with degree at most 2), olive trees (i.e., a rooted tree consisting of $k$ branches such that the $i$th branch is a path of length $i$), all trees of order up to 50, palm trees (i.e., trees obtained by appending identical stars to each vertex of a path), banana trees, and binomial trees (the binomial tree $B_0$ of order 0 consists of a single vertex; the binomial tree $B_n$ of order $n$ has a root vertex whose children are the roots of the binomial trees of order $0, 1, 2, \ldots, n-1$ (see [935], [937], [1179], [433] and [969]).

Other graphs with prime labelings include all cycles and the disjoint union of $C_{2k}$ and $C_n$ [356]. The complete graph $K_n$ does not have a prime labeling for $n \geq 4$ and $W_n$ is prime if and only if $n$ is even (see [781]).

Seoud, Diab, and Elsakhawi [1010] have shown the following graphs are prime: fans; helms; flowers (see §2.2); stars; $K_{2,n}$; and $K_{3,n}$ unless $n = 3$ or 7. They also shown that $P_n + \overline{K_m}$ $(m \geq 3)$ is not prime. Tout, Dabboucy, and Howalla [1179] proved that $C_m \odot \overline{K_n}$ is prime for all $m$ and $n$.

For $m$ and $n$ at least 3, Seoud and Youssef [1017] define $S_n^{(m)}$, the $(m, n)$-*gon star*, as the graph obtained from the cycle $C_n$ by joining the two end vertices of the path $P_{m-2}$ to every pair of consecutive vertices of the cycle such that each of the end vertices of the path is connected to exactly one vertex of the cycle. Seoud and Youssef [1017] have proved the following graphs have prime labelings: books; $S_n^{(m)}$; $C_n \odot P_m$; $P_n + \overline{K}_2$ if and only if $n = 2$ or $n$ is odd; and $C_n \odot K_1$ with a complete binary tree of order $2^k - 1$ $(k \geq 2)$ attached at each pendant vertex. They also prove that every spanning subgraph of a prime graph is prime and every graph is a subgraph of a prime graph. They conjecture that all unicycle graphs have prime labelings. Seoud and Youssef [1017] proved the following graphs are not prime: $C_m + C_n$; $C_n^2$ for $n \geq 4$; $P_n^2$ for $n = 6$ and for $n \geq 8$; and Möbius ladders $M_n$ for $n$ even (see §2.3 for the definition). They also give an exact formula for the maximum number of edges in a prime graph of order $n$ and an upper bound for the chromatic number of a prime graph.

Youssef and Elsakhawi [1311] have shown: the union of stars $S_m \cup S_n$, are prime; the union of cycles and stars $C_m \cup S_n$ are prime; $K_m \cup P_n$ is prime if and only if $m$ is at most 3 or if $m = 4$ and $n$ is odd; $K_n \odot K_1$ is prime if and only if $n \leq 7$; $K_n \odot \overline{K_2}$ is prime if and only if $n \leq 16$; $6K_m \cup S_n$ is prime if and only if the number of primes less than or equal to $m + n + 1$ is at least $m$; and that the complement of every prime graph with order at least 20 is not prime.

Salmasian [989] has shown that every tree with $n$ vertices $(n \geq 50)$ can be labeled with $n$ integers between 1 and $4n$ such that every two adjacent vertices have relatively prime labels. Pikhurko [937] has improved this by showing that for any $c > 0$ there is an $N$ such that any tree of order $n > N$ can be labeled with $n$ integers between 1 and $(1 + c)n$ such that labels of adjacent vertices are relatively prime.

Varkey and Singh (see [1200]) have shown the following graphs have prime labelings: ladders, crowns, cycles with a chord, books, one point unions of $C_n$, and $L_n + K_1$. Varkey [1200] has

shown that graph obtained by connecting two points with internally disjoint paths of equal length are prime. Varkey defines a *twig* as a graph obtained from a path by attaching exactly two pendent edges to each internal vertex of the path. He proves that twigs obtained from a path of odd length (at least 3) and lotus inside a circle (see §5.1 for the definition) graphs are prime.

Babujee and Vishnupriya [168] proved the following graphs have prime labelings: $nP_2, P_n \cup P_n \cup \cdots P_n$, bistars (that is, the graphs obtained by joining the centers of two identical stars with an edge), and the graph obtained by subdividing the edge joining edge of a bistar. Babujee [161] obtained prime labelings for the graphs: $(P_m \cup nK_1) + \overline{K_2}$, $(C_m \cup nK_1) + \overline{K_2}$, $(P_m \cup C_n \cup \overline{K}_r) + \overline{K_2}$, $C_n \cup C_{n+1}$, $(2n-2)C_{2n}$ $(n > 1)$, $C_n \cup mP_k$ and the graph obtained by subdividing each edge of a star once. In [90] Babujee and Jagadesh prove the following graphs have prime labelings: bistars $B_m, n$; $P_3 \odot K_{1,n}$; the union of $K_{1,n}$ and the graph obtained from $K_{1,n}$ by appending a pendent edge to every pendent edge of $K_{1,n}$; and the graph obtained by identifying the center of $K_{1,n}$ with the two end points and the middle vertex of $P_5$.

Given a collection of graphs $G_1, \ldots, G_n$ and some fixed vertex $v_i$ from each $G_i$, Lee, Wui, and Yeh [781] define $Amal\{(G_i, v_i)\}$, the amagamation of $\{(G_i, v_i)|\ i = 1, \ldots, n\}$, as the graph obtained by taking the union of the $G_i$ and identifying $v_1, v_2, \ldots, v_n$. Lee, Wui, and Yeh [781] have shown $Amal\{(G_i, v_i)\}$ has a prime labeling when $G_i$ are paths and when $G_i$ are cycles. They also showed that the amagamation of any number of copies of $W_n$, $n$ odd, with a common vertex is not prime. They conjecture that for any tree $T$ and any vertex $v$ from $T$, the amagamation of two or more copies of $T$ with $v$ in common is prime. They further conjecture that the amagamation of two or more copies of $W_n$ that share a common point is prime when $n$ is even $(n \neq 4)$. Vilfred, Somasundaram, and Nicholas [1216] have proved this conjecture for the case that $n \equiv 2 \pmod 4$ where the central vertices are identified.

Vilfred, Somasundaram, and Nicholas [1216] have also proved the following: helms are prime; the grid $P_m \times P_n$ is prime when $m \leq 3$ and $n$ is a prime greater than $m$; the double cone $C_n + \overline{K_2}$ is prime only for $n = 3$; the double fan $P_n \times \overline{K_2}$ $(n \neq 2)$ is prime if and only if $n$ is odd or $n = 2$; and every cycle with a $P_k$-chord is prime. They conjecture that the grid $P_m \times P_n$ is prime when $n$ is prime and $n > m$. This conjecture was proved by Sundaram, Ponraj, and Somasundaram [1156]. In the same article they also showed that $P_n \times P_n$ is prime when $n$ is prime. Kanetkar [622] proved: $P_6 \times P_6$ is prime; that $P_{n+1} \times P_{n+1}$ is prime when $n$ is a prime with $n \equiv 3$ or $9 \pmod{10}$ and $(n+1)^2 + 1$ is also prime; and $P_n \times P_{n+2}$ is prime when $n$ is an odd prime with $n \not\equiv 2 \pmod 7$.

For any finite collection $\{G_i, u_i v_i\}$ of graphs $G_i$, each with a fixed edge $u_i v_i$, Carlson [288] defines the edge amalgamation $Edgeamal\{(G_i, u_i v_i)\}$ as the graph obtained by taking the union of all the $G_i$ and identifying their fixed edges. The case where all the graphs are cycles she calls *generalized books*. She proves that all generalized books are prime graphs. Moreover, she shows that graphs obtained by taking the union of cycles and identifying in each cycle the path $P_n$ are also prime. Carlson also proves that $C_m$-snakes are prime (see §2.2) for the definition).

In [82] Babujee proves that the maximum number of edges in a simple graph with $n$ vertices that has a prime labeling is $\sum_{k=2}^{n} \phi(k)$. He also shows that the planar graphs having $n$ vertices and $3(n-2)$ edges (i.e., the maximum number of edges for a planar graph with $n$ vertices) obtained from $K_n$ $(n \geq 5)$ with vertices $v_1, v_2, \ldots, v_n$ by deleting the edges joining $v_s$ and $v_t$ for all $s$ and $t$ satisfying $3 \leq s \leq n-2$ and $s+2 \leq t \leq n$ has a prime labeling if and only if $n$ is odd.

Yao, Cheng, Zhongfu, and Yao [1297] have shown: a tree of order $p$ with maximum degree at

least $p/2$ is prime; a tree of order $p$ with maximum degree at least $p/2$ has a vertex subdivision that is prime; if a tree $T$ has an edge $u_1u_2$ such that the two components $T_1$ and $T_2$ of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and $d_{T_2}(u_2) \geq |T_2|/2$, then $T$ is prime when $|T_1| + |T_2|$ is prime; if a tree $T$ has two edges $u_1u_2$ and $u_2u_3$ such that the three components $T_1, T_2$, and $T_3$ of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$, $d_{T_2}(u_2) \geq |T_2|/2$, and $d_{T_3}(u_3) \geq |T_3|/2$, then $T$ is prime when $|T_1| + |T_2| + |T_3|$ is prime.

A dual of prime labelings has been introduced by Deretsky, Lee, and Mitchem [356]. They say a graph with edge set $E$ has a *vertex prime labeling* if its edges can be labeled with distinct integers $1, \ldots, |E|$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky, Lee, and Mitchem show the following graphs have vertex prime labelings: forests; all connected graphs; $C_{2k} \cup C_n$; $C_{2m} \cup C_{2n} \cup C_{2k+1}$; $C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$; and $5C_{2m}$. They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles. Let $G = \bigcup_{i=1}^{t} C_{2n_i}$ and $N = \sum_{i=1}^{t} n_i$. In [252] Borosh, Hensley and Hobbs proved that there is a positive constant $n_0$ such that the conjecture of Deretsky et al. is true for the following cases: $G$ is the disjoint union of at most seven cycles; $G$ is a union of cycles all of the same even length $2n$ where $n \leq 150\,000$ or where $n \geq n_0$; $n_i \geq (\log N)^{4 \log\log\log n}$ for all $i = 1, \ldots, t$; and when each $C_{2n_i}$ is repeated at most $n_i$ times. They end their paper with a discussion of graphs whose components are all even cycles, and of graphs with some components that are not cycles and some components that are odd cycles.

Jothi [619] calls a graph $G$ *highly vertex prime* if its edges can be labeled with distinct integers $\{1, 2, \ldots, |E|\}$ such that the labels assigned to any two adjacent edges are relatively prime. Such labeling is called a *highly vertex prime* labeling. He proves: if $G$ is highly vertex prime then the line graph of $G$ is prime; cycles are highly vertex prime; paths are highly vertex prime; $K_n$ is highly vertex prime if and only if $n \leq 3$; $K_{1,n}$ is highly vertex prime if and only if $n \leq 2$; even cycles with a chord are highly vertex prime; $C_p \cup C_q$ is not highly vertex prime when both $p$ and $q$ are odd; and crowns $C_n \odot K_1$ are highly vertex prime.

The tables following summarize the state of knowledge about prime labelings and vertex prime labelings. In the table
**P** means prime labeling exists
**VP** means vertex prime labeling exists.

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property.

Table 20: **Summary of Prime Labelings**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $P_n$ | P | [433] |
| stars | P | [433] |
| caterpillars | P | [433] |
| complete binary trees | P | [433] |
| spiders | P | [433] |
| trees | P? | [781] |
| $C_n$ | P | [356] |
| $C_n \cup C_{2m}$ | P | [356] |
| $K_n$ | P | iff $n \leq 3$ [781] |
| $W_n$ | P | iff $n$ is even [781] |
| helms | P | [1010], [1216] |
| fans | P | [1010] |
| flowers | P | [1010] |
| $K_{2,n}$ | P | [1010] |
| $K_{3,n}$ | P | $n \neq 3, 7$ [1010] |
| $P_n + \overline{K_m}$ | not P | $n \geq 3$ [1010] |
| $P_n + \overline{K_2}$ | P | iff $n = 2$ or $n$ is odd [1010] |

Table 20: **Summary of Prime Labelings continued**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| books | P | [1017] |
| $C_n \odot P_m$ | P | [1017] |
| unicyclic graphs | P? | [1017] |
| $C_m + C_n$ | not P | [1017] |
| $C_n^2$ | not P | $n \geq 4$ [1017] |
| $P_n^2$ | not P | $n \geq 6,\ n \neq 7$ [1017] |
| $M_n$ (Möbius ladders) | not P | $n$ even [1017] |
| $S_m \cup S_n$ | P | [1311] |
| $C_m \cup S_n$ | P | [1311] |
| $K_m \cup S_n$ | P | iff number of primes $\leq m+n+1$ is at least $m$ [1311] |
| $K_n \cdot K_1$ | P | iff $n \leq 7$ [1311] |
| $P_n \times P_2$ (ladders) | P | [1200] |
| $P_m \times P_n$ (grids) | P | $m \leq 3,\ m > n,\ n$ prime [1216] |
| $C_n \odot K_1$ (crowns) | P | [1200] |
| cycles with a chord | P | [1200] |

Table 20: **Summary of Prime Labelings continued**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| wheels | P | [1200] |
| $C_n \odot \overline{K_2}$ | P | iff $n = 3$ [1216] |
| $P_n \odot \overline{K_2}$ | P | iff $n \neq 2$ [1216] |
| $C_m$-snakes (see §2.2) | P | [288] |
| unicyclic | P? | [1010] |
| $C_m \odot P_n$ | P | [1017] |
| $K_{1,n} + \overline{K_2}$ | P | [1098] |
| $K_{1,n} + K_2$ | P | $n$ prime, $n \geq 4$ [1098] |
| $P_n \odot K_1$ (combs) | P | $n \geq 2$ [1098] |
| $P_1 \cup P_2 \cup \cdots \cup P_n$ | P | [1098] |
| $P_n \times P_2$ (ladders) | P<br>P? | $n \geq 3,\ 2n+1$ prime [1098]<br>$n \geq 3$ [1098] |
| $C_m^{(n)}$ (see §2.2 | P | $n(m-1)+1$ prime [1098] |
| triangular snakes | P | [1098] |
| quadrilateral snakes | P | [1098] |

Table 21: **Summary of Vertex Prime Labelings**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $C_m + C_n$ | not P | [1017] |
| $C_n^2$ | not P | $n \geq 4$ [1017] |
| $P_n$ | not P | $n = 6,\ n \geq 8$ [1017] |
| $M_{2n}$ (Möbius ladders) | not P | [1017] |
| connected graphs | VP | [356] |
| forests | VP | [356] |
| $C_{2m} \cup C_n$ | VP | [356] |
| $C_{2m} \cup C_{2n} \cup C_{2k+1}$ | VP | [356] |
| $C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$ | VP | [356] |
| $5C_{2m}$ | VP | [356] |
| $G \cup H$ | VP | if $G,\ H$ are connected and one is not an odd cycle [356] |
| 2-regular graph $G$ | not VP<br>VP? | $G$ has at least 2 odd cycles [356]<br>iff $G$ has at most 1 odd cycle [356] |

## 7.3 Edge-graceful Labelings

In 1985, Lo [815] introduced the notion of edge-graceful graphs. A graph $G(V, E)$ is said to be *edge-graceful* if there exists a bijection $f$ from $E$ to $\{1, 2, \ldots, |E|\}$ such that the induced mapping $f^+$ from $V$ to $\{0, 1, \ldots, |V| - 1\}$ given by $f^+(x) = (\sum f(xy)) \pmod{|V|}$ taken over all edges $xy$ is a bijection. Note that an edge-graceful graph is antimagic (see §6.1). A necessary condition for a graph with $p$ vertices and $q$ edges to be edge-graceful is that $q(q+1) \equiv p(p+1)/2 \pmod{p}$. Lee [706] notes that this necessary condition extends to any multigraph with $p$ vertices and $q$ edges. It was conjectured by Lee [706] that any connected simple $(p, q)$-graph with $q(q+1) \equiv p(p-1)/2 \pmod{p}$ vertices is edge-graceful. Lee, Kitagaki, Young, and Kocay [709] prove that the conjecture is true for maximal outerplanar graphs. Lee and Murthy [701] proved that $K_n$ is edge-graceful if and only if $n \not\equiv 2 \pmod 4$. (An edge-graceful labeling given in [815] for $K_n$ for $n \not\equiv 2 \pmod 4$ is incorrect.) Lee [706] notes that a multigraph with $p \equiv 2 \pmod 4$ vertices is not edge-graceful and conjectures that this condition is sufficient for the edge-gracefulness of connected graphs. Lee [705] has conjectured that all trees of odd order are edge-graceful. Small [1111] has proved that spiders for which every vertex has odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [642] proved that all spiders of odd order with exactly three end vertices are edge-graceful. Cabaniss, Low, and Mitchem [271] have shown that regular spiders of odd order are edge-graceful.

Lee and Seah [745] have shown that $K_{n,n,\ldots,n}$ is edge-graceful if and only if $n$ is odd and the number of partite sets is either odd or a multiple of 4. Lee and Seah [744] have also proved that $C_n^k$ (the $k$th power of $C_n$) is edge-graceful for $k < \lfloor n/2 \rfloor$ if and only if $n$ is odd and $C_n^k$ is edge-graceful for $k \geq \lfloor n/2 \rfloor$ if and only if $n \not\equiv 2 \pmod 4$ (see also [271]). Lee, Seah, and Wang [750] gave a complete characterization of edge-graceful $P_n^k$ graphs. Shiu, Lam, and Cheng [1067] proved that the composition of the path $P_3$ and any null graph of odd order is edge-graceful.

Lo [815] proved that all odd cycles are edge-graceful and Wilson and Riskin [1259] proved the Cartesian product of any number of odd cycles is edge-graceful. Lee, Ma, Valdes, and Tong [722] investigated the edge-gracefulness of grids $P_m \times P_n$. The necessity condition of Lo [815] that a $(p, q)$ graph must satisfy $q(q+1) \equiv 0$ or $p/2 \pmod p$ severely limits the possibilities. Lee et al. prove the following: $P_2 \times P_n$ is not edge-graceful for all $n > 1$; $P_3 \times P_n$ is edge-graceful if and only if $n = 1$ or $n = 4$; $P_4 \times P_n$ is edge-graceful if and only if $n = 3$ or $n = 4$; $P_5 \times P_n$ is edge-graceful if and only if $n = 1$; $P_{2m} \times P_{2n}$ is edge-graceful if and only if $m = n = 2$. They conjecture that for all $m, n \geq 10$ of the form $m = (2k+1)(4k+1)$, $n = (2k+1)(4k+3)$, the grids $P_m \times P_n$ are edge-graceful. Riskin and Weidman [968] proved: if $G$ is an edge-graceful $2r$-regular graph with $p$ vertices and $q$ edges and $(r, kp) = 1$, then $kG$ is edge-graceful when $k$ is odd; when $n$ and $k$ are odd, $kC_n^r$ is edge-graceful; and if $G$ is the cartesian product of an odd number of odd cycles and $k$ is odd, then $kG$ is edge-graceful. They conjecture that the disjoint union of an odd number of copies of a $2r$-regular edge-graceful graph is edge-graceful.

Shiu, Lee, and Schaffer [1073] investigated the edge-gracefulness of multigraphs derived from paths, combs, and spiders obtained by replacing each edge by $k$ parallel edges. Lee, Ng, Ho, and Saba [731] construct edge-graceful multigraphs starting with paths and spiders by adding certain edges to the original graphs. Lee and Seah [746] have also investigated edge-gracefulness of various multigraphs.

Lee and Seah (see [706]) define a *sunflower* graph $SF(n)$ as the graph obtained by starting with an $n$-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ and creating new vertices $w_1, w_2, \ldots, w_n$

with $w_i$ connected to $v_i$ and $v_{i+1}$ ($v_{n+1}$ is $v_1$). In [747] they prove that $SF(n)$ is edge-graceful if and only if $n$ is even. In the same paper they prove that $C_3$ is the only triangular snake that is edge-graceful. Lee and Seah [744] prove that for $k \leq n/2$, $C_n^k$ is edge-graceful if and only if $n$ is odd, and for $k \geq n/2$, $C_n^k$ is edge-graceful if and only if $n \not\equiv 2 \pmod 4$. Lee, Seah, and Lo (see [706]) have proved that for $n$ odd, $C_{2n} \cup C_{2n+1}, C_n \cup C_{2n+2}$, and $C_n \cup C_{4n}$ are edge-graceful. They also show that for odd $k$ and odd $n$, $kC_n$ is edge-graceful. Lee and Seah (see [706]) prove that the generalized Petersen graph $P(n,k)$ (see Section 2.7 for the definition) is edge-graceful if and only if $n$ is even and $k < n/2$. In particular, $P(n,1) = C_n \times P_2$ is edge-graceful if and only if $n$ is even.

Schaffer and Lee [997] proved that $C_m \times C_n$ $(m > 2, n > 2)$ is edge-graceful if and only if $m$ and $n$ are odd. They also showed that if $G$ and $H$ are edge-graceful regular graphs of odd order then $G \times H$ is edge-graceful and that if $G$ and $H$ are edge-graceful graphs where $G$ is $c$-regular of odd order $m$ and $H$ is $d$-regular of odd order $n$, then $G \times H$ is edge-magic if $\gcd(c,n) = \gcd(d,m) = 1$. They further show that if $H$ has odd order, is $2d$-regular and edge-graceful with $\gcd(d,m) = 1$, then $C_{2m} \times H$ is edge-magic, and if $G$ is odd-regular, edge-graceful of even order $m$ that is not divisible by 3, and $G$ can be partitioned into 1-factors, then $G \times C_m$ is edge-graceful.

In 1987 Lee (see [748]) conjectured that $C_{2m} \cup C_{2n+1}$ is edge-graceful for all $m$ and $n$ except for $C_4 \cup C_3$. Lee, Seah, and Lo [748] have proved this for the case that $m = n$ and $m$ is odd. They also prove: the disjoint union of an odd number copies of $C_m$ is edge-graceful when $m$ is odd; $C_n \cup C_{2n+2}$ is edge-graceful; and $C_n \cup C_{4n}$ is edge-graceful for $n$ odd. Bu [257] gave necessary and sufficient conditions for graphs of the form $mC_n \cup P_{n-1}$ to be edge-graceful.

Kendrick and Lee (see [706]) proved that there are only finitely many $n$ for which $K_{m,n}$ is edge-graceful and they completely solve the problem for $m = 2$ and $m = 3$. Ho, Lee, and Seah [551] use $S(n; a_1, a_2, \ldots, a_k)$ where $n$ is odd and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k < n/2$ to denote the $(n, nk)$-multigraph with vertices $v_0, v_1, \ldots, v_{n-1}$ and edge set $\{v_i v_j |\ i \neq j, i - j \equiv a_t \pmod n$ for $t = 1, 2, \ldots, k\}$. They prove that all such multigraphs are edge-graceful. Lee and Pritikin (see [706]) prove that the Möbius ladders (see §2.2 for definition) of order $4n$ are edge-graceful. Lee, Tong, and Seah [763] have conjectured that the total graph of a $(p,p)$-graph is edge-graceful if and only if $p$ is even. They have proved this conjecture for cycles.

Kuang, Lee, Mitchem, and Wang [685] have conjectured that unicyclic graphs of odd order are edge-graceful. They have verified this conjecture in the following cases: graphs obtained by identifying an end point of a path $P_m$ with a vertex of $C_n$ when $m + n$ is even; crowns with one pendant edge deleted; graphs obtained from crowns by identifying an endpoint of $P_m$, $m$ odd, with a vertex of degree 1; amalgamations of a cycle and a star obtained by identifying the center of the star with a cycle vertex where the resulting graph has odd order; graphs obtained from $C_n$ by joining a pendant edge to $n - 1$ of the cycle vertices and two pendant edges to the remaining cycle vertex.

Gayathri and Subbiah [459] say a graph $G(V,E)$ has a *strong edge graceful* labeling if there is an injection $f$ from the $E$ to $\{1, 2, 3, \ldots, \langle 3|E|/2 \rangle\}$ such that the induced mapping $f^+$ from $V$ defined by $f^+(u) = (\Sigma f(uv)) \pmod{2|V|}$ taken all edges $uv$ is an injection. They prove the following graphs have strong edge graceful labelings: $P_n (n \geq 3), C_n, K_{1,n} (n \geq 2)$, crowns $C_n \oplus K_1$, and fans $P_n + K_1 (n \geq 2)$.

Hefetz [518] has shown that a graph $G = (V,E)$ of the form $G = H \cup f_1 \cup f_2 \cup \cdots \cup f_r$ where $H = (V, E')$ is edge-graceful and the $f_i$'s are 2-factors is also edge-graceful and that a regular graph of even degree that has a 2-factor consisting of $k$ cycles each of length $t$ where $k$ and $t$ are

odd is edge-graceful.

Bača and Holländer [131] investigated a generalization of edge-graceful labeling called $(a,b)$-consecutive labelings. A connected graph $G(V,E)$ is said to have an $(a,b)$-*consecutive labeling* where $a$ is a nonnegative integer and $b$ is a positive proper divisor of $|V|$, if there is a bijection from $E$ to $\{1,2,\ldots,|E|\}$ such that if each vertex $v$ is assigned the sum of all edges incident to $v$ the vertex labels are distinct and they can be partitioned into $|V|/b$ intervals

$$W_j = [w_{\min} = (j-1)b + (j-1)a, w_{\min} + jb + (j-1)a - 1],$$

where $1 \le j \le p/b$ and $w_{\min}$ is the minimum value of the vertices. They present necessary conditions for $(a,b)$-consecutive labelings and describe $(a,b)$-consecutive labelings of the generalized Petersen graphs for some values of $a$ and $b$.

A graph with $p$ vertices and $q$ edges is said to be *$k$-edge-graceful* if its edges can be labeled with $k, k+1, \ldots, k+q-1$ such that the sums of the edges incident to each vertex are distinct modulo $p$. In [766] Lee and Wang show that for each $k \neq 1$ there are only finitely many trees that are $k$-edge graceful (there are infinitely many 1-edge graceful trees). They describe completely the $k$-edge-graceful trees for $k = 0, 2, 3, 4,$ and 5.

In 1991 Lee [706] defined the *edge-graceful spectrum* of a graph $G$ as the set of all nonnegative integers $k$ such that $G$ has a $k$-edge graceful labeling. In [770] Lee, Wang, Ng, and Wang determine the edge-graceful spectrum of the following graphs: $G \odot K_1$ where $G$ is an even cycle with one chord; two even cycles of the same order joined by an edge; and two even cycles of the same order sharing a common vertex with an arbitrary number of pendant edges attached at the common vertex (*butterfly graph*). Lee, Chen, and Wang [708] have determined the edge-graceful spectra for various cases of cycles with a chord and for certain cases of graphs obtained by joining two disjoint cycles with an edge (i.e., *dumbbell graphs*). More generally, Shiu, Ling, and Low [1075] call a connected with $p$ vertices and $p+1$ edges *bicyclic*. In particular, the family of bicyclic graphs includes the one-point union of two cycles, two cycles joined by a path and cycles with one cord. In [1076] they determine the edge-graceful spectra of bicyclic graphs that do not have pendant edges. Kang, Lee, and Wang [625] determined the edge-graceful spectra of wheels and Wang, Hsiao, and Lee [1241] determined the edge-graceful spectra of the square of $P_n$ for odd $n$ (see also Lee, Wang, and Hsiao [768]). Results about the edge-graceful spectra of three types of $(p, p+1)$-graphs are given by Chen, Lee, and Wang [309].

Lee, Levesque, Lo, and Schaffer [717] investigate the edge-graceful spectra of cylinders. They prove: for odd $n \ge 3$ and $m \equiv 2 \pmod 4$, the spectra of $C_n \times P_m$ is $\emptyset$; for $m = 3$ and $m \equiv 0, 1$ or 3 (mod 4), the spectra of $C_4 \times P_m$ is $\emptyset$; for even $n \ge 4$, the spectra of $C_n \times P_2$ is all natural numbers; the spectra of $C_n \times P_4$ is all odd positive integers if and only if $n \equiv 3 \pmod 4$; and $C_n \times P_4$ is all even positive integers if and only if $n \equiv 1 \pmod 4$. They conjecture that $C_4 \times P_m$ is $k$-edge-graceful for some $k$ if and only if $m \equiv 2 \pmod 4$. Shiu, Ling, and Low [1076] determine the edge-graceful spectra of all connected bicyclic graphs without pendant edges.

A graph $G(V,E)$ is called *super edge-graceful* if there is a bijection $f$ from $E$ to $\{0, \pm1, \pm2, \ldots, \pm(|E|-1)/2\}$ when $|E|$ is odd and from $E$ to $\{\pm1, \pm2, \ldots, \pm|E|/2\}$ when $|E|$ is even such that the induced vertex labeling $f^*$ defined by $f^*(u) = \Sigma f(uv)$ over all edges $uv$ is a bijection from $V$ to $\{0, \pm1, \pm2, \ldots, \pm(p-1)/2\}$ when $p$ is odd and from $V$ to $\{\pm1, \pm2, \ldots, \pm p/2\}$ when $p$ is even. Lee, Wang, and Nowak [771] proved the following: $K_{1,n}$ is super-edge-magic if and only if $n$ is even; the double star $DS(m,n)$ (that is, the graph obtained by joining the centers of $K_{1,m}$ and $K_{1,n}$ by an edge) is super edge-graceful if and only if $m$ and $n$ are both odd. They conjecture that all trees of odd order are super edge-graceful.

Lee, Sun, Wei, Wen, and Yiu [758] proved that trees obtained by starting with the paths the $P_{2n+2}$ or $P_{2n+3}$ and identifying each internal vertex with an end point of a path of length 2 are super edge-graceful.

Shiu [1061] has shown that $C_n \times P_2$ is super-edge-graceful for all $n \geq 2$. More generally, he defines a family of graphs that includes $C_n \times P_2$ and generalized Petersen graphs are follows. For any permutation $\theta$ on $n$ symbols without a fixed point the *$\theta$-Petersen graph* $P(n;\theta)$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i u_{i+1}, u_i w_i, w_i w_{\theta(i)} \mid 1 \leq i \leq n\}$ where addition of subscripts is done modulo $n$. (The graph $P(n;\theta)$ need not be simple.) Shiu proves that $P(n;\theta)$ is super-edge-graceful for all $n \geq 2$. He also shows that certain other families of connected cubic multigraphs are super-edge-graceful and conjectures that every connected cubic of multigraph except $K_4$ and the graph with 2 vertices and 3 edges is super-edge-graceful.

In [1065] Shiu and Lam investigated the super-edge-gracefulness of fans and wheel-like graphs. They showed that fans $F_{2n}$ and wheels $W_{2n}$ are super-edge-graceful. Although $F_3$ and $W_3$ are not super-edge-graceful the general cases $F_{2n+1}$ and $W_{2n+1}$ are open. For a positive integer $n_1$ and even positive integers $n_2, n_3, \ldots, n_m$ they define an *$m$-level wheel* as follows. A wheel is a 1-level wheel and the cycle of the wheel is the 1-level cycle. An $i$-level wheel is obtained from an $(i-1)$-level wheel by appending $n_i/2$ pairs of edges from any number of vertices of the $i-1$-level cycle to $n_i$ new vertices that form the vertices in the $i$-level cycle. They prove that all $m$-level wheels are super-edge-graceful. They also prove that for $n$ odd $C_m \odot \overline{K_n}$ is super-edge-graceful, for odd $m \geq 3$ and even $n \geq 2$ $C_m \odot \overline{K_n}$ is edge-graceful, and for $m \geq 2$ and $n \geq 1$ $C_m \odot \overline{K_n}$ is super-edge-graceful. For a cycle $C_m$ with consecutive vertices $v_1, v_2, \ldots, v_m$ and nonnegative integers $n_1, n_2, \ldots, n_m$ they define the graph $A(m; n_1, n_2, \ldots, n_m)$ as the graph obtained from $C_m$ by attaching $n_i$ edges to the vertex $v_i$ for $1 \leq i \leq m$. They prove $A(m; n_1, n_2, \ldots, n_m)$ is super-edge-graceful if $m$ is odd and $A(m; n_1, n_2, \ldots, n_m)$ is super-edge-graceful if $m$ is even and all the $n_i$ are positive and have the same parity. Chung, Lee, Gao, and Schaffer [336] provide super edge-graceful labelings for various even order paths, spiders and disjoint unions of two stars. In [334] Chung and Lee characterize spiders of even orders that are not super-edge-graceful and exhibit some spiders of even order of diameter at most four that are super-edge-graceful. They raised the question of which paths are super edge-graceful. This was answered by Cichacz, Fronček and Xu [338] who showed that the only paths that are not super edge-graceful are $P_2$ and $P_4$. Gao and Zhang [452] proved that some cases of caterpillars are super edge-graceful.

Although it is not the case that a super edge-graceful graph is edge-graceful, Lee, Chen, Yera, and Wang [707] proved that if $G$ is a super edge-graceful with $p$ vertices and $q$ edges and $q \equiv -1 \pmod{p}$ when $q$ is even, or $q \equiv 0 \pmod{p}$ when $q$ is odd, then $G$ is also edge-graceful. They also prove: the graph obtained from a connected super edge-graceful unicyclic graph of even order by joining any two nonadjacent vertices by an edge is super edge-graceful; the graph obtained from a super edge-graceful graph with $p$ vertices and $p+1$ edges by appending two edges to any vertex is super edge-graceful; and the one-point union of two identical cycles is super edge-graceful.

Gayathri, Duraisamy, and Tamilselvi [455] calls a $(p,q)$-graph with $q \geq p$ *even edge-graceful* if there is an injection $f$ from the set of edges to $\{1, 2, 3, \ldots, 2q\}$ such that the values of the induced mapping $f^+$ from the vertex set to $\{0, 1, 2, \ldots, 2q-1\}$ given by $f^+(x) = (\Sigma f(xy)) \pmod{2q}$ over all edges $xy$ are distinct and even. In [455] and [454] Gayathri et al. prove the following: cycles are even edge-graceful if and only if the cycles are odd; even cycles with one pendant edge are even edge-graceful; wheels are even edge-graceful; gears (see §2.2 for the definition) are

not even edge-graceful; fans $P_n + K_1$ are even edge-graceful; $C_4 \cup P_m$ for all $m$ are even edge-graceful; $C_{2n+1} \cup P_{2n+1}$ are even edge-graceful; crowns $C_n \odot K_1$ are even edge-graceful; $C_n^{(m)}$ (see §2.2 for the definition) are even edge-graceful; sunflowers (see §3.7 for the definition) are even edge-graceful; triangular snakes (see §2.2 for the definition) are even edge-graceful; closed helms (see §2.2 for the definition) with the center vertex removed are even edge-graceful; graphs decomposable into two odd Hamiltonian cycles are even edge-graceful; and odd order graphs that are decomposable into three Hamiltonian cycles are even edge-graceful.

In [454] Gayathri and Duraisamy generalized the definition of even edge-graceful to include $(p, q)$-graphs with $q < p$ by changing the modulus from $2q$ the maximum of $2q$ and $2p$. With this version of the definition, they have shown that trees of even order are not even edge-graceful whereas, for odd order graphs, the following are even edge-graceful: banana trees (see §2.1 for the definition); graphs obtained joining the centers of two stars by a path; $P_n \odot K_{1,m}$; graphs obtained by identifying an end point from each of any number of copies of $P_3$ and $P_2$; bistars (that is, graphs obtained by joining the centers of two stars with an edge); and graphs obtained by appending the end point of a path to the center of a star. They define odd edge-graceful graphs in the analogous way and provide a few results about such graphs.

Lee, Pan, and Tsai [736] call a graph $G$ with $p$ vertices and $q$ edges *vertex-graceful* if there exists a labeling $f\ V(G) \to \{1, 2, \ldots, p\}$ such that the induced labeling $f^+$ from $E(G)$ to $Z_q$ defined by $f^+(uv) = f(u) + f(v) \pmod q$ is a bijection. Vertex-graceful graphs can be viewed the dual of edge-graceful graphs. They call a vertex-graceful graph *strong vertex-graceful* if the values of $f^+(E(G)$ are consecutive. They observe that the class of vertex-graceful graphs properly contains the super edge-magic graphs and strong vertex-graceful graphs are super edge-magic. They provide vertex-graceful and strong vertex-graceful labelings for various $(p, p + 1)$-graphs of small order and their amalgamations.

As a dual to super edge-graceful graphs Lee and Wei [774] define a graph $G(V, E)$ to be *super vertex-graceful* if there is a bijection $f$ from $V$ to $\{\pm 1, \pm 2, \ldots, \pm(|V| - 1)/2\}$ when $|V|$ is odd and from $V$ to $\{\pm 1, \pm 2, \ldots, \pm|V|/2\}$ when $|V|$ is even such that the induced edge labeling $f^*$ defined by $f^+(uv) = f(u) + f(v)$ over all edges $uv$ is a bijection from $E$ to $\{0, \pm 1, \pm 2, \ldots, \pm(|E| - 1)/2\}$ when $|E|$ is odd and from $E$ to $\{\pm 1, \pm 2, \ldots, \pm|E|/2\}$ when $|E|$ is even. They show: for $m$ and $n_1, n_2, \ldots, n_m$ each at least 3, $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_m}$ is not super vertex-graceful; for $n$ odd, books $K_{1,n} \times P_2$ are not super vertex-graceful; for $n \geq 3$, $P_n^2 \times P_2$ is super vertex-graceful if and only if $n = 3, 4$, or 5; and $C_m \times C_n$ is not super vertex-graceful. They conjecture that $P_n \times P_n$ is super vertex-graceful for $n \geq 3$.

In [778] Lee and Wong generalize super edge-vertex graphs by defining a graph $G(V, E)$ to be $P(a)Q(1)$-*super vertex-graceful* if there is a bijection $f$ from $V$ to $\{0, \pm a, \pm(a + 1), \ldots, \pm(a - 1 + (|V| - 1)/2)\}$ when $|V|$ is odd and from $V$ to $\{\pm a, \pm(a + 1), \ldots, \pm(a - 1 + |V|/2)\}$ when $|V|$ is even such that the induced edge labeling $f^*$ defined by $f^+(uv) = f(u) + f(v)$ over all edges $uv$ is a bijection from $E$ to $\{0, \pm 1, \pm 2, \ldots, \pm(|E| - 1)/2\}$ when $|E|$ is odd and from $E$ to $\{\pm 1, \pm 2, \ldots, \pm|E|/2\}$ when $|E|$ is even. They show various classes of unicyclic graphs are $P(a)Q(1)$-super vertex-graceful. In [716] Lee, Leung, and Ng more simply refer to $P(1)Q(1)$-super vertex-graceful graphs as *super vertex-graceful* and show how to construct a variety of unicyclic graphs that are super vertex-graceful. They conjecture that every unicyclic graph is an induced subgraph of a super vertex-graceful unicyclic graph. Lee and Leung [715] determine which trees of diameter at most 6 are super vertex-graceful graphs and propose two conjectures. Lee, Ng, and Sun [732] found many classes of caterpillars that are super vertex-graceful.

In [324] Chopra and Lee define a graph $G(V, E)$ to be $Q(a)P(b)$-*super edge-graceful* if there

is a bijection $f$ from $E$ to $\{\pm a, \pm(a+1), \ldots, \pm(a+(|E|-2)/2)\}$ when $|E|$ is even and from $E$ to $\{0, \pm a, \pm(a+1), \ldots, \pm(a+(|E|-3)/2)\}$ when $|E|$ is odd and $f^+(u)$ is equal to the sum of $f(uv)$ over all edges $uv$ is a bijection from $V$ to $\{\pm b, \pm(b+1), \ldots, (|V|-2)/2\}$ when $|V|$ is even and from $V$ to $\{0, \pm b, \pm(b+1), \ldots, \pm(|V|-3)/2\}$ when $|V|$ is odd. They say a graph is *strongly super edge-graceful* if it is $Q(a)P(b)$-super edge-graceful for all $a \geq 1$. Among their results are: a star with $n$ pendent edges is strongly super edge-graceful if and only if $n$ is even; wheels with $n$ spokes are strongly super edge-graceful if and only if $n$ is even; coronas $C_n \odot K_1$ are strongly super edge-graceful for all $n \geq 3$; and double stars $DS(m,n)$ are strongly super edge-graceful in the case that $m$ is odd and at least 3 and $n$ is even and at least 2 and in the case that both $m$ and $n$ are odd and one of them is at least 3. Lee, Song, and Valdés [753] investigate the $Q(a)P(b)$-super edge-gracefulness of wheels $W_n$ for $n = 3, 4, 5$, and 6.

In [775] Lee, Wang, and Yera proved that some Eulerian graphs are super edge-graceful, but not edge-graceful, and that some are edge-graceful, but not super edge-graceful. They also showed that a Rosa-type condition for Eulerian super edge-graceful graphs does not exist and pose some conjectures.

In 1997 Yilmaz and Cahit [1301] introduced a weaker version of edge-graceful called $E$-cordial. Let $G$ be a graph with vertex set $V$ and edge set $E$ and let $f$ a function from $E$ to $\{0,1\}$. Define $f$ on $V$ by $f(v) = \sum\{f(uv)|uv \in E\} \pmod 2$. The function $f$ is called an *$E$-cordial* labeling of $G$ if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits an $E$-cordial labeling is called *$E$-cordial*. Yilmaz and Cahit prove the following graphs are $E$-cordial: trees with $n$ vertices if and only if $n \not\equiv 2 \pmod 4$; $K_n$ if and only if $n \not\equiv 2 \pmod 4$; $K_{m,n}$ if and only if $m+n \not\equiv 2 \pmod 4$; $C_n$ if and only if $n \not\equiv 2 \pmod 4$; regular graphs of degree 1 on $2n$ vertices if and only if $n$ is even; friendship graphs $C_3^{(n)}$ for all $n$ (see §2.2 for the definition); fans $F_n$ if and only if $n \not\equiv 1 \pmod 4$; and wheels $W_n$ if and only if $n \not\equiv 1 \pmod 4$. They observe that graphs with $n \equiv 2 \pmod 4$ vertices can not be $E$-cordial. They generalize $E$-cordial labelings to $E_k$-cordial ($k > 1$) labelings by replacing $\{0,1\}$ by $\{0,1,2,\ldots,k-1\}$. Of course, $E_2$-cordial is the same as $E$-cordial.

Devaraj [359] has shown that $M(m,n)$, the mirror graph of $K(m,n)$ (see §2.3 for the definition), is $E$-cordial when $m+n$ is even and the generalized Petersen graph $P(n,k)$ is $E$-cordial when $n$ is even. (Recall that $P(n,1)$ is $C_n \times P_2$.)

The table following summarizes the state of knowledge about edge-graceful labelings. In the table **EG** means edge-graceful labeling exists. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property.

Table 22: **Summary of Edge-graceful Labelings**

| *Graph* | *Types* | *Notes* |
|---|---|---|
| $K_n$ | EG | iff $n \not\equiv 2 \pmod 4$ [701] |
| odd order trees | EG? | [705] |
| $K_{n,n,\ldots,n}$ ($k$ terms) | EG | iff $n$ is odd or $k \not\equiv 2 \pmod 4$ [745] |
| $C_n^k,\ k < \lfloor n/2 \rfloor$ | EG | iff $n$ is odd [744] |
| $C_n^k,\ k \geq \lfloor n/2 \rfloor$ | EG | iff $n \not\equiv 2 \pmod 4$ [744] |
| $P_3[K_n]$ | EG | $n$ is odd [744] |
| $M_{4n}$ (Möbius ladders) | EG | [706] |
| odd order dragons | EG | [685] |
| odd order unicycilc graphs | EG? | [685] |
| $P_{2m} \times P_{2n}$ | EG | iff $m = n = 2$ [722] |
| $C_n \cup P_2$ | EG | $n$ even [748] |
| $C_{2n} \cup C_{2n+1}$ | EG | $n$ odd [748] |
| $C_n \cup C_{2n+2}$ | EG | [748] |
| $C_n \cup C_{4n}$ | EG | $n$ odd [748] |
| $C_{2m} \cup C_{2n+1}$ | EG? | $(m,n) \neq (4,3)$ odd [749] |
| $P(n,k)$ generalized Petersen graph | EG | $n$ even, $k < n/2$ [706] |
| $C_m \times C_n$ | EG? | $(m,n) \neq (4,3)$ [749] |

## 7.4 Radio Labelings

In 2001 Chartrand, Erwin, Zhang, and Harary [300] were motivated by regulations for channel assignments of FM radio stations to introduce radio labelings of graphs. A *radio labeling* of a connected graph $G$ is an injection $c$ from the vertices of $G$ to the natural numbers such that $d(u,v)+|c(u)-c(v)| \geq 1+diam(G)$ for every two distinct vertices $u$ and $v$ of $G$. The *radio number* of $c$, $rn(c)$, is the maximum number assigned to any vertex of $G$. The *radio number* of $G$, $rn(G)$, is the minimum value of $rn(c)$ taken over all radio labelings $c$ of $G$. Chartrand et al. and Zhang [1322] gave bounds for the radio numbers of cycles. The exact values for the radio numbers for paths and cycles were reported by Liu and Zhu [804] as follows: for odd $n \geq 3$, $rn(P_n) = (n-1)^2/2+2$; for even $n \geq 4$, $rn(P_n) = n^2/2-n+1; rn(C_{4k}) = (k+2)(k-2)/2+1$; $rn(C_{4k+1}) = (k+1)(k-1)/2$; $rn(C_{4k+2}) = (k+2)(k-2)/2+1$; and $rn(C_{4k+3}) = (k+2)(k-1)/2$. However, Chartrand, Erwin, and Zhang [299] obtained different values than Liu and Zhu for $P_4$ and $P_5$. Chartrand, Erwin, and Zhang [299] proved: $rn(P_n) \leq (n-1)(n-2)/2+n/2+1$ when $n$ is even; $rn(P_n) \leq n(n-1)/2+1$ when $n$ is odd; $rn(P_n) < rn(P_{n+1})$ $(n > 1)$; for a connected graph $G$ of diameter $d$, $rn(G) \geq (d+1)^2/4+1$ when $d$ is odd; and $rn(G) \geq d(d+2)/4+1$ when $d$ is even. In [800] Liu obtained lower bounds for the radio number of trees and the radio number of spiders (trees with at most one vertex of degree greater than 2) and characterized the graphs that achieve these bounds.

Chartrand, Erwin, Zhang, and Harary [300] proved: $rn(K_{n_1,n_2,\ldots,n_k}) = n_1+n_2+\cdots+n_k+k-1$; if $G$ is a connected graph of order $n$ and diameter 2, then $n \leq rn(G) \leq 2n-2$; and for every pair of integers $k$ and $n$ with $n \leq k \leq 2n-2$, there exists a connected graph of order $n$ and diameter 2 with $rn(G) = k$. They further provide a characterization of connected graphs of order $n$ and diameter 2 with prescribed radio number.

Fernandez, Flores, Tomova, and Wyels [407] proved $rn(K_n) = n; rn(W_n) = n+2$; and the radio number of the gear graph obtained from $W_n$ by inserting a vertex between each vertex of the rim is $4n+2$.

Liu and Xie [802] investigated the radio numbers of squares of cycles. Letting $n = 4k+r$ where $r = 0, 1, 2$ or 3, they proved:
$rn(C_n^2) = (2k^2+5k-1)/2$, if $r = 0$ and $k$ is odd;
$rn(C_n^2) = (2k^2+3k)/2$, if $r = 0$ and $k$ is even;
$rn(C_n^2) \geq (k^2+k)$, if $r = 1$ and $k \equiv 1 \pmod 4$;
$rn(C_n^2) = (k^2+k)$, if $r = 1$ and $k \equiv 3 \pmod 4$;
$rn(C_n^2) = (k^2+2k)$, if $r = 1$ and $k$ is even;
$rn(C_n^2) = (k^2+5k+1)$, if $r = 2$ and $k$ is odd;
$rn(C_n^2) = (k^2+4k+1)$, if $r = 2$ and $k \equiv 2 \pmod 4$;
$rn(C_n^2) \geq (2k^2+7k+3)/2$, if $r = 3$ and $k$ is odd;
$rn(C_n^2) \geq (2k^2+9k+4)/2$, if $r = 3$ and $k$ is even;
$rn(C_n^2) = (2k^2+5k-1)/2$, if $r = 2$ and $k$ is odd;
$rn(C_n^2) = (2k^2+9k+4)/2$, if $r = 3$ and $k \equiv 0 \pmod 4$;
$rn(C_n^2) = (2k^2+9k+4)/2$, if $r = 3$ and $k = 4m+2$ for some $m \not\equiv 5 \pmod 7$;
$rn(C_n^2) = (2k^2+7k+3)/2$ if $r = 3$ and $k = 4m+1$ where $m \equiv 0$ or $1 \pmod 3$;
$rn(C_n^2) = (2k^2+7k+5)/2$ if $r = 3$ and $k = 4m+1$ where $m \equiv 2 \pmod 3$;
$rn(C_n^2) \geq (2k^2+7k+5)/2$ if $r = 3$ and $k = 4m+3$ where $m \equiv 0 \pmod 3$;
$rn(C_n^2) \geq (2k^2+7k+3)/2$ if $r = 3$ and $k = 4m+3$ where $m \equiv 1$ or $2 \pmod 3$.
They also prove that if $r = 3$ for some $k = 4m+2$ and $m \equiv 5 \pmod 7$, then $(2k^2+9k+4)/2 \leq$

$rn(C_n^2) \leq 2k^2 + 9k + 10)/2$ and conjecture that if $k \equiv 3 \pmod 4$, then $rn(C_{4k+1}^2) = k^2 + k + 2$.

In [803] Liu and Xie proved that $rn(P_n^2)$ is $\lfloor n/2 \rfloor + 2$ if $n \equiv 1 \pmod 4$ and $n \geq 9$ and $rn(P_n^2)$ is $\lfloor n/2 + 1 \rfloor$ otherwise. In [801] Liu found a lower bound for the radio number of trees and characterizes the trees that achieve the bound. She also provides a lower bound for the radio number of spiders in terms of the lengths of their legs and characterizes the spiders that achieve this bound. Sweetly and Joseph [1169] prove that the radio number of the graph obtained from the wheel $W_n$ by subdividing each edge of the rim exactly twice is $5n - 3$. Marinescu-Ghemeci [842] determined the radio number of the caterpillar obtained from a path by attaching a new terminal vertex to each non-terminal vertex of the path and the graph obtained from a star by attaching $k$ new terminal vertices to each terminal vertex of the star.

For any connected graph $G$ and positive integer $k$ Chartrand, Erwin, and Zhang, [298] define a *radio k-coloring* as an injection $f$ from the vertices of $G$ to the natural numbers such that $d(u, v) + |f(u) - f(v)| \geq 1 + k$ for every two distinct vertices $u$ and $v$ of $G$. Using $rc_k(f)$ to denote the maximum number assigned to any vertex of $G$ by $f$, the *radio k-chromatic number* of $G$, $rc_k(G)$, is the minimum value of $rc_k(f)$ taken over all radio $k$-colorings of $G$. Note that $rc_1(G)$ is $\chi(G)$, the chromatic number of $G$, and when $k = diam(G)$, $rc_k(G)$ is $rn(G)$, the radio number of $G$. Chartrand, Nebesky, and Zang [305] gave upper and lower bounds for $rc_k(P_n)$ for $1 \leq k \leq n - 1$. Kchikech, Khennoufa, and Togni [640] improved Chartrand et al.'s lower bound for $rc_k(P_n)$ and Kola and Panigrahi [662] improved the upper bound for certain special cases of $n$. The exact value of $rc_{n-2}(P_n)$ for $n \geq 5$ was given by Khennoufa and Togni in [644] and the exact value of $rc_{n-3}(P_n)$ for $n \geq 8$ was given by Kola and Panigrahi in [662]. Kola and Panigrahi [662] gave the exact value of $rc_{n-4}(P_n)$ when $n$ is odd and $n \geq 11$ and an upper bound for $rc_{n-4}(P_n)$ when $n$ is even and $n \geq 12$. Kchikech, Khennoufa, and Togni [641] gave upper and lower bounds for $rc_k(G \times H)$ and $rc_k(Q_n)$. In [640] the same authors proved that $rc_k(K_{1,n} = n(k-1) + 2$ and for any tree T and $k \geq 2$, $rc_k(T) \leq (n-1)(k-1)$.

Sooryanarayana and Raghunath [1129] say a graph with $n$ vertices is *radio graceful* if $rn(G) = n$. They determine the radio number of the cube of $C_n$ for all $n \leq 20$ and for $n \equiv 0$ or 2 or 4 (mod 6). They also determine the values of $n$ for which $C_n^3$ is radio graceful.

The survey article by Panigrahi [917] includes background information and further results about radio $k$-colorings.

## 7.5 Line-graceful Labelings

Gnanajothi [471] has defined a concept similar to edge-graceful. She calls a graph with $n$ vertices *line-graceful* if it is possible to label its edges with $0, 1, 2, \ldots, n$ such that when each vertex is assigned the sum modulo $n$ of all the edge labels incident with that vertex the resulting vertex labels are $0, 1, \ldots, n - 1$. A necessary condition for the line-gracefulness of a graph is that its order is not congruent to 2 (mod 4). Among line-graceful graphs are (see [471, pp. 132–181]) $P_n$ if and only if $n \not\equiv 2 \pmod 4$; $C_n$ if and only if $n \not\equiv 2 \pmod 4$; $K_{1,n}$ if and only if $n \not\equiv 1 \pmod 4$; $P_n \odot K_1$ (combs) if and only if $n$ is even; $(P_n \odot K_1) \odot K_1$ if and only if $n \not\equiv 2 \pmod 4$; (in general, if $G$ has order $n$, $G \odot H$ is the graph obtained by taking one copy of $G$ and $n$ copies of $H$ and joining the $i$th vertex of $G$ with an edge to every vertex in the $i$th copy of $H$); $mC_n$ when $mn$ is odd; $C_n \odot K_1$ (crowns) if and only if $n$ is even; $mC_4$ for all $m$; complete $n$-ary trees when $n$ is even; $K_{1,n} \cup K_{1,n}$ if and only if $n$ is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete

$n$-ary tree when $n$ is even; trees for which exactly one vertex has even degree. She conjectures that all trees with $p \not\equiv 2 \pmod 4$ vertices are line-graceful and proved this conjecture for $p \leq 9$.

Gnanajothi [471] has investigated the line-gracefulness of several graphs obtained from stars. In particular, the graph obtained from $K_{1,4}$ by subdividing one spoke to form a path of even order (counting the center of the star) is line-graceful; the graph obtained from a star by inserting one vertex in a single spoke is line-graceful if and only if the star has $p \not\equiv 2 \pmod 4$ vertices; the graph obtained from $K_{1,n}$ by replacing each spoke with a path of length $m$ (counting the center vertex) is line-graceful in the following cases: $n = 2$; $n = 3$ and $m \not\equiv 3 \pmod 4$; and $m$ is even and $mn + 1 \equiv 0 \pmod 4$.

Gnanajothi studied graphs obtained by joining disjoint graphs $G$ and $H$ with an edge. She proved such graphs are line-graceful in the following circumstances: $G = H$; $G = P_n, H = P_m$ and $m + n \not\equiv 0 \pmod 4$; and $G = P_n \odot K_1$, $H = P_m \odot K_1$ and $m + n \not\equiv 0 \pmod 4$.

## 7.6 Representations of Graphs modulo $n$

In 1989 Erdős and Evans [392] defined a *representation* modulo $n$ of a graph $G$ with vertices $v_1, v_2, \ldots, v_r$ as a set $\{a_1, \ldots, a_r\}$ of distinct, nonnegative integers each less than $n$ satisfying $\gcd(a_i - a_j, n) = 1$ if and only if $v_i$ is adjacent to $v_j$. They proved that every finite graph can be represented modulo some positive integer. The *representation number*, $\mathrm{Rep}(G)$, is smallest such integer. Obviously the representation number of a graph is prime if and only if a graph is complete. Evans, Fricke, Maneri, McKee, and Perkel [401] have shown that a graph is representable modulo a product of a pair of distinct primes if and only if the graph does not contain an induced subgraph isomorphic to $K_2 \cup 2K_1$, $K_3 \cup K_1$, or the complement of a chordless cycle of length at least five. Nešetřil and Pultr [894] showed that every graph can be represented modulo a product of some set of distinct primes. Evans et al. [401] proved that if $G$ is representable modulo $n$ and $p$ is a prime divisor of $n$, then $p \geq \chi(G)$. Evans, Isaak, and Narayan [402] determined representation numbers for specific families as follows (here we use $q_i$ to denote the $i$th prime and for any prime $p_i$ we use $p_{i+1}, p_{i+2}, \ldots, p_{i+k}$ to denote the next $k$ primes larger than $p_i$): $\mathrm{Rep}(P_n) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2(n-1) \rceil}$; $\mathrm{Rep}(C_4) = 4$ and for $n \geq 3$, $\mathrm{Rep}(C_{2n}) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2(n-1) \rceil + 1}$; $\mathrm{Rep}(C_5) = 3 \cdot 5 \cdot 7 = 105$ and for $n \geq 4$ and not a power of 2, $\mathrm{Rep}(C_{2n+1}) = 3 \cdot 5 \cdot \cdots \cdot q_{\lceil \log_2 n \rceil + 1}$; if $m \geq n \geq 3$, then $\mathrm{Rep}(K_m - P_n) = p_i p_{i+1}$ where $p_i$ is the smallest prime greater than or equal to $m - n + \lceil n/2 \rceil$; if $m \geq n \geq 4$, and $p_i$ is the smallest prime greater than or equal to $m - n + \lceil n/2 \rceil$, then $\mathrm{Rep}(K_m - C_n) = q_i q_{i+1}$ if $n$ is even and $\mathrm{Rep}(K_m - C_n) = q_i q_{i+1} q_{i+2}$ if $n$ is odd; if $n \leq m - 1$, then $\mathrm{Rep}(K_m - K_{1,n}) = p_s p_{s+1} \cdots p_{s+n-1}$ where $p_s$ is the smallest prime greater than or equal to $m - 1$; $\mathrm{Rep}(K_m)$ is the smallest prime greater than or equal to $m$; $\mathrm{Rep}(nK_2) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2 n \rceil + 1}$; if $n, m \geq 2$, then $\mathrm{Rep}(nK_m) = p_i p_{i+1} \cdots p_{i+m-1}$, where $p_i$ is the smallest prime satisfying $p_i \geq m$, if and only if there exists a set of $n - 1$ mutually orthogonal Latin squares of order $m$; $\mathrm{Rep}(mK_1) = 2m$; and if $t \leq (m-1)!$, then $\mathrm{Rep}(K_m + tK_1) = p_s p_{s+1} \cdots p_{s+m-1}$ where $p_s$ is the smallest prime greater than or equal to $m$. Narayan [893] proved that for $r \geq 3$ the maximum value for $\mathrm{Rep}(G)$ over all graphs of order $r$ is $p_s p_{s+1} \cdots p_{s+r-2}$, where $p_s$ is the smallest prime that is greater than or equal to $r - 1$.

Evans [400] used matrices over the additive group of a finite field to obtain various bounds for the representation number of graphs of the form $nK_m$. Among them are $\mathrm{Rep}(4K_3) = 3 \cdot 5 \cdot 7 \cdot 11$; $\mathrm{Rep}(7K_5) = 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$; and $\mathrm{Rep}((3q-1)/2)K_q) \leq p_q p_{q+1} \cdots p_{(3q-1)/2}$ where $q$ is a prime power with $q \equiv 3 \pmod 4$, $p_q$ is the smallest prime greater than or equal to $q$, and the remaining terms are the next consecutive $(3q-3)/2$ primes; $\mathrm{Rep}(2q-2)K_q) \leq p_q p_{q+1} \cdots p_{(3q-3)/2}$

where $q$ is a prime power with $q \equiv 3 \bmod 4$, and $p_q$ is the smallest prime greater than or equal to $q$; $\mathrm{Rep}((2q-2)K_q) \leq p_q p_{q+1} \cdots p_{2q-3}$.

In [892] Narayan asked for the values of $\mathrm{Rep}(C_{2^k+1})$ when $k \geq 3$ and $\mathrm{Rep}(G)$ when $G$ is a complete multipartite graph or a disjoint union of complete graphs. He also asked about the behavior of the representation number for random graphs.

## 7.7 $k$-sequential Labelings

In 1981 Bange, Barkauskas, and Slater [183] defined a *$k$-sequential* labeling $f$ of a graph $G(V,E)$ as one for which $f$ is a bijection from $V \cup E$ to $\{k, k+1, \ldots, |V \cup E| + k - 1\}$ such that for each edge $xy$ in $E$, $\quad f(xy) = |f(x) - f(y)|$. This generalized the notion of *simply sequential* where $k = 1$ introduced by Slater. Bange, Barkauskas, and Slater showed that cycles are 1-sequential and if $G$ is 1-sequential, then $G + K_1$ is graceful. Hegde and Shetty [533] have shown that every $T_p$-tree (see §4.4 for the definition) is 1-sequential. In [1106], Slater proved: $K_n$ is 1-sequential if and only if $n \leq 3$; for $n \geq 2$, $K_n$ is not $k$-sequential for any $k \geq 2$; and $K_{1,n}$ is $k$-sequential if and only if $k$ divides $n$. Acharya and Hegde [25] proved: if $G$ is $k$-sequential, then $k$ is at most the independence number of $G$; $P_{2n}$ is $n$-sequential for all $n$ and $P_{2n+1}$ is both $n$-sequential and $(n+1)$-sequential for all $n$; $K_{m,n}$ is $k$-sequential for $k = 1, m$, and $n$; $K_{m,n,1}$ is 1-sequential; and the join of any caterpillar and $\overline{K_t}$ is 1-sequential. Acharya [13] showed that if $G(E,V)$ is an odd graph with $|E| + |V| \equiv 1$ or 2 (mod 4) when $k$ is odd or $|E| + |V| \equiv 2$ or 3 (mod 4) when $k$ is even, then $G$ is not $k$-sequential. Acharya also observed that as a consequence of results of Bermond, Kotzig, and Turgeon [220] we have: $mK_4$ is not $k$-sequential for any $k$ when $m$ is odd and $mK_2$ is not $k$-sequential for any odd $k$ when $m \equiv 2$ or 3 (mod 4) or for any even $k$ when $m \equiv 1$ or 2 (mod 4). He further noted that $K_{m,n}$ is not $k$-sequential when $k$ is even and $m$ and $n$ are odd, whereas $K_{m,k}$ is $k$-sequential for all $k$. Acharya points out that the following result of Slater's [1107] for $k = 1$ linking $k$-graceful graphs and $k$-sequential graphs holds in general: A graph is $k$-sequential if and only if $G + v$ has a $k$-graceful labeling $f$ with $f(v) = 0$. Slater [1106] also proved that a $k$-sequential graph with $p$ vertices and $q > 0$ edges must satisfy $k \leq p - 1$. Hegde [523] proved that every graph can be embedded as an induced subgraph of a simply sequential graph. In [13] Acharya conjectured that if $G$ is a connected $k$-sequential graph of order $p$ with $k > \lfloor p/2 \rfloor$, then $k = p - 1$ and $G = K_{1,p-1}$ and that, except for $K_{1,p-1}$, every tree in which all vertices are odd is $k$-sequential for all odd positive integers $k \leq p/2$. In [523] Hegde gave counterexamples for both of these conjectures.

In [532] Hegde and Miller prove the following: for $n > 1$, $K_n$ is $k$-sequentially additive if and only if $(n,k) = (2,1), (3,1)$ or (3,2); $K_{1,n}$ is $k$-sequentially additive if and only if $k$ divides $n$; caterpillars with bipartition sets of sizes $m$ and $n$ are $k$-sequentially additive for $k = m$ and $k = n$; and if an odd-degree $(p,q)$-graph is $k$-sequentially additive, then $(p+q)(2k+p+q-1) \equiv 0 \bmod 4$. As corollaries of the last result they observe that when $m$ and $n$ are odd and $k$ is even $K_{m,n}$ is not $k$-sequentially additive and if an odd-degree tree is $k$-sequentially additive then $k$ is odd.

## 7.8 IC-colorings

For a subgraph $H$ of a graph $G$ with vertex set $V$ and a coloring $f$ from $V$ to the natural numbers define $f_s(H) = \Sigma f(v)$ over all $v \in H$. The coloring $f$ is called an IC-*coloring* if for any integer $k$ between 1 and $f_s(G)$ there is a connected subgraph $H$ of $G$ such that $f_s(H) = k$. The IC-*index* of a graph $G$, $M(G)$, is $\max\{f_s |\ f_s$ is an IC-coloring of $G\}$. Salehi, Lee, and Khatirinejad [987]

obtained the following: $M(K_n) = 2^n - 1$; for $n \geq 2$, $M(K_{1,n}) = 2^n + 2$; if $\Delta$ is the maximum degree of a connected graph $G$, then $M(G) \geq 2^\Delta + 2$; if $ST(n; 3^n)$ is the graph obtained by identifying the end points of $n$ paths of length 3, then $ST(n; 3^n)$ is at least $3^n + 3$ (they conjecture that equality holds for $n \geq 4$); for $n \geq 2$, $M(K_{2,n}) = 3 \cdot 2^n + 1$; $M(P_n) \geq (2 + \lfloor n/2 \rfloor)(n - \lfloor n/2 \rfloor) + \lfloor n/2 \rfloor - 1$; for $m, n \geq 2$, the IC-index of the double star $DS(m, n)$ is at least $(2^{m-1} + 1)(2^{n-1} + 1)$ (they conjecture that equality holds); for $n \geq 3$, $n(n+1)/2 \leq M(C_n) \leq n(n-1) + 1$; and for $n \geq 3$, $2^n + 2 \leq M(W_n) \leq 2^n + n(n-1) + 1$. They pose the following open problems: find the IC-index of the graph obtained by identifying the end points of $n$ paths of length $b$; find the IC-index of the graph obtained by identifying the end points of $n$ paths; and find the IC-index of $K_{m,n}$. Shiue and Fu [1083] completed the partial results by Penrice [926] Salehi, Lee, and Khatirinejad [987] by proving $M(K_{m,n}) = 3 \cdot 2^{m+n-2} - 2^{m-2} + 2$ for any $2 \leq m \leq n$.

## 7.9 Product Cordial Labelings

Sundaram and Somasundaram [1158] introduced the notion of product cordial labelings. A *product cordial labeling* of a graph $G$ with vertex set $V$ is a function $f$ from $V$ to $\{0, 1\}$ such that if each edge $uv$ is assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

A graph with a product cordial labeling is called a *product cordial* graph. In [1153] and [1157] Sundaram, Ponraj, and Somasundaram prove the following graphs are product cordial: trees; unicyclic graphs of odd order; triangular snakes; dragons; helms; $P_m \cup P_n$; $C_m \cup P_n$; $P_m \cup K_{1,n}$; $W_m \cup F_n$ ($F_n$ is the fan $P_n + K_1$); $K_{1,m} \cup K_{1,n}$; $W_m \cup K_{1,n}$; $W_m \cup P_n$; $W_m \cup C_n$; the total graph of $P_n$ (the total graph of $P_n$ has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in $P_n$); $C_n$ if and only if $n < 4$; $C_n^{(t)}$, the one-point union of $t$ copies of $C_n$, provided $t$ is even or both $t$ and $n$ are even; $K_2 + mK_1$ if and only if $m$ is odd; $C_m \cup P_n$ if and only if $m + n$ is odd; $K_{m,n} \cup P_s$ if $s > mn$; $C_{n+2} \cup K_{1,n}$; $K_n \cup K_{n,(n-1)/2}$ when $n$ is odd; $K_n \cup K_{n-1,n/2}$ when $n$ is even; and $P_n^2$ if and only if $n$ is odd. They also prove that $K_{m,n}$ $(m, n > 2)$, $P_m \times P_n$ $(m, n > 2)$ and wheels are not product cordial and if a $(p, q)$-graph is product cordial graph, then $q < (p-1)(p+1)/4$.

Sundaram and Somasundaram [1158] also have introduced the notion of total product cordial labelings. A *total product cordial labeling* of a graph $G$ with vertex set $V$ is a function $f$ from $V$ to $\{0, 1\}$ such that if each edge $uv$ is assigned the label $f(u)f(v)$ the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a *total product cordial* graph. In [1158] and [1155] Sundaram, Ponraj, and Somasundaram prove the following graphs are total product cordial: every product cordial graph of even order or odd order and even size; trees; all cycles except $C_4$; $K_{n,2n-1}$; $C_n$ with $m$ edges appended at each vertex; fans; double fans; wheels; helms; $C_2 \times P_2$; $K_{2,n}$ if and only if $n \equiv 2 \pmod 4$; $P_m \times P_n$ if and only if $(m, n) \neq (2, 2)$; $C_n + 2K_1$ if and only if $n$ is even or $n \equiv 1 \pmod 3$; $\overline{K_n} \times 2K_2$ if $n$ is odd, or $n \equiv 0$ or $2 \pmod 6$, or $n \equiv 2 \pmod 8$.

Ramanjaneyulu, Venkaiah, and Kothapalli [954] give total product cordial labeling for a family of planar graphs for which each face is a 4-cycle.

## 7.10 Prime Cordial Labelings

Sundaram, Ponraj, and Somasundaram [1154] have introduced the notion of prime cordial labelings. A *prime cordial labeling* of a graph $G$ with vertex set $V$ is a bijection $f$ from $V$ to $\{1, 2, \ldots, |V|\}$ such that if each edge $uv$ is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. In [1154] Sundaram, Ponraj, and Somasundram prove the following graphs are prime cordial: $C_n$ if and only if $n \geq 6$; $P_n$ if and only if $n \neq 3$ or 5; $K_{1,n}$ ($n$ odd); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \geq 3$; bistars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders; $K_{1,n}$ if $n$ is even and there exists a prime $p$ such that $2p < n + 1 < 3p$; $K_{2,n}$ if $n$ is even and if there exists a prime $p$ such that $3p < n + 2 < 4p$; and $K_{3,n}$ if $n$ is odd and if there exists a prime $p$ such that $5p < n + 3 < 6p$. They also prove that if $G$ is a prime cordial graph of even size, then the graph obtained by identifying the central vertex of $K_{1,n}$ with the vertex of $G$ labeled with 2 is prime cordial, and if $G$ is a prime cordial graph of odd size, then the graph obtained by identifying the central vertex of $K_{1,2n}$ with the vertex of $G$ labeled with 2 is prime cordial. They further prove that $K_{m,n}$ is not prime cordial for a number of special cases of $m$ and $n$. Sundaram and Somasundaram [1158] and Youssef [1310] observed that for $n \geq 3$, $K_n$ is not prime cordial provided that the inequality $\phi(2) + \phi(3) + \cdots + \phi(n) \geq n(n-1)/4 + 1$ is valid for $n \geq 3$. This inequality was proved by Yufei Zhao [1324].

Seoud and Salim [1014] give an upper bound for the number of edges of a graph with a prime cordial labeling as a function of the number of vertices. For bipartite graphs they give a stronger bound. They prove that $K_n$ does not have a prime cordial labeling for $2 < n < 500$ and conjecture that $K_n$ is not prime cordial for all $n > 2$. They determine all prime cordial graphs of order at most 6. In [166] Babujee and Shobana proved sun graphs $C_n \odot K_1$; $C_n$ with a path of length $n - 3$ attached to a vertex; and $P_n$ ($n \geq 6$) with $n - 3$ pendent edges attached to a pendent vertex of $P_n$ have prime cordial labelings. Additional results on prime cordial labelings are given in [167].

## 7.11 Geometric Labelings

If $a$ and $r$ are positive integers at least 2, we say a $(p, q)$-graph $G$ is $(a, r)$-*geometric* if its vertices can be assigned distinct positive integers such that the value of the edges obtained as the product of the end points of each edge is $\{a, ar, ar^2, \ldots, ar^{q-1}\}$. Hegde [526] has shown the following: no connected bipartite graph, except the star, is $(a, a)$-geometric where $a$ is a prime number or square of a prime number; any connected $(a, a)$-geometric graph where $a$ is a prime number or square of a prime number, is either a star or has a triangle; $K_{a,b}$, $2 \leq a \leq b$ is $(k, k)$-geometric if and only if $k$ is neither a prime number nor the square of a prime number; a caterpillar is $(k, k)$-geometric if and only if $k$ is neither a prime number nor the square of a prime number; $K_{a,b,1}$ is $(k, k)$-geometric for all integers $k \geq 2$; $C_{4t}$ is $(a, a)$-geometric if and only if $a$ is neither a prime number nor the square of a prime number; for any positive integers $t$ and $r \geq 2$, $C_{4t+1}$ is $(r^{2t}, r)$-geometric; for any positive integer $t$, $C_{4t+2}$ is not geometric for any values of $a$ and $r$; and for any positive integers $t$ and $r \geq 2$, $C_{4t+3}$ is $(r^{2t+1}, r)$-geometric. Hegde [528] has also shown that every $T_p$-tree and the subdivision graph of every $T_p$-tree are $(a, r)$-geometric for some values of $a$ and $r$ (see Section 3.2 for the definition of a $T_p$-tree). He conjectures that all trees are $(a, r)$-geometric for some values of $a$ and $r$.

Hegde and Shankaran [534] prove: a graph with an $\alpha$-labeling (see §3.1 for the definition)

where $m$ is the fixed integer that is between the end points of each edge has an $(a^{m+1}, a)$-geometric for any $a > 1$; for any integers $m$ and $n$ both greater than 1 and $m$ odd, $mP_n$ is $(a^r, a)$-geometric where $r = (mn+3)/2$ if $n$ is odd and $(a^r, a)$-geometric where $r = (m(n+1)+3)/2$ if $n$ is even; for positive integers $k > 1, d \geq 1$, and odd $n$, the generalized closed helm (see §5.3 for the definition) $CH(t, n)$ is $(k^r, k^d)$-geometric where $r = (n-1)d/2$; for positive integers $k > 1, d \geq 1$, and odd $n$, the generalized web graph (see §5.3 for the definition) $W(t, n)$ is $(k^r, a)$-geometric where $a = k^d$ and $r = (n-1)d/2$; for positive integers $k > 1, d \geq 1$, the generalized $n$-crown $(P_m \times K_3) \odot K_{1,n}$ is $(a, a)$-geometric where $a = k^d$; and $n = 2r+1$, $C_n \odot P_3$ is $(k^r, k)$-geometric.

If $a$ and $r$ are positive integers and $r$ is at least 2 Arumugan, Germina, and Anadavally [76] say a $(p, q)$-graph $G$ is *additively* $(a, r)$*-geometric* if its vertices can be assigned distinct integers such that the value of the edges obtained as the sum of the end points of each edge is $\{a, ar, ar^2, \ldots, ar^{q-1}\}$. In the case that the vertex labels are nonnegative integers the labeling is called *additively* $(a, r)*$*-geometric*. They prove: for all $a$ and $r$ every tree is additively $(a, r)*$-geometric; a connected additively $(a, r)$-geometric graph is either a tree or unicyclic graph with the cycle having odd size; if $G$ is a connected unicyclic graph and not a cycle, then $G$ is additively $(a, r)$-geometric if and only if either $a$ is even or $a$ is odd and $r$ is even; connected unicyclic graphs are not additively $(a, r)*$-geometric; if a disconnected graph is additively $(a, r)$-geometric, then each component is a tree or a unicyclic graph with an odd cycle; and for all even $a$ at least 4, every disconnected graph for which every component is a tree or unicyclic with an odd cycle has an additively $(a, r)$-geometric labeling.

Vijayakumar [1208] calls a graph $G$ (not necessarily finite) *arithmetic* if its vertices can be assigned distinct natural numbers such that the value of the edges obtained as the sum of the end points of each edge is an arithmetic progression. He proves [1207] and [1208] that a graph is arithmetic if and only if it is $(a, r)$-geometric for some $a$ and $r$.

## 7.12 Sequentially Additive Graphs

Bange, Barkauskas, and Slater [184] defined a *k-sequentially additive labeling* $f$ of a graph $G(V, E)$ to be a bijection from $V \cup E$ to $\{k, \ldots, k + |V \cup E| - 1\}$ such that for each edge $xy$, $f(xy) = f(x) + f(y)$. They proved: $K_n$ is 1-sequentially additive if and only if $n \leq 3$; $C_{3n+1}$ is not $k$-sequentially additive for $k \equiv 0$ or 2 (mod 3); $C_{3n+2}$ is not $k$-sequentially additive for $k \equiv 1$ or 2 (mod 3); $C_n$ is 1-sequentially additive if and only if $n \equiv 0$ or 1 (mod 3); and $P_n$ is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive. Hegde [525] proved that $K_{1,n}$ is $k$-sequentially additive if and only if $k$ divides $n$.

Hajnal and Nagy [502] investigated 1-sequentially additive labelings of 2-regular graphs. They prove: $kC_3$ is 1-sequentially additive for all $k$; $kC_4$ is 1-sequentially additive if and only if $k \equiv 0$ or 1 (mod 3); $C_{6n} \cup C_{6n}$ and $C_{6n} \cup C_{6n} \cup C_3$ are 1-sequentially additive for all $n$; $C_{12n}$ and $C_{12n} \cup C_3$ are 1-sequentially additive for all $n$. They conjecture that every 2-regular simple graph on $n$ vertices is 1-sequentially additive where $n \equiv 0$ or 1 (mod 3).

Acharya and Hegde [27] have generalized $k$-sequentially additive labelings by allowing the image of the bijection to be $\{k, k+d, \ldots, (k+|V \cup E|-1)d\}$. They call such a labeling *additively* $(k, d)$*-sequential*.

## 7.13 Strongly Multiplicative Graphs

Beineke and Hegde [211] call a graph with $p$ vertices *strongly multiplicative* if the vertices of $G$ can be labeled with distinct integers $1, 2, \ldots, p$ such that the labels induced on the edges

by the product of the end vertices are distinct. They prove the following graphs are strongly multiplicative: trees; cycles; wheels; $K_n$ if and only if $n \leq 5$; $K_{r,r}$ if and only if $r \leq 4$; and $P_m \times P_n$. They then consider the maximum number of edges a strongly multiplicative graph on $n$ vertices can have. Denoting this number by $\lambda(n)$, they show: $\lambda(4r) \leq 6r^2; \lambda(4r+1) \leq 6r^2+4r; \lambda(4r+2) \leq 6r^2+6r+1$; and $\lambda(4r+3) \leq 6r^2+10r+3$. Adiga, Ramaswamy, and Somashekara [37] give the bound $\lambda(n) \leq n(n+1)/2 + n - 2 - \lfloor (n+2)/4 \rfloor - \sum_{i=2}^{n} i/p(i)$ where $p(i)$ is the smallest prime dividing $i$. For large values of $n$ this is a better upper bound for $\lambda(n)$ than the one given by Beineke and Hegde. It remains an open problem to find a nontrivial lower bound for $\lambda(n)$.

Seoud and Zid [1024] prove the following graphs are strongly multiplicative: wheels; $rK_n$ for all $r$ and $n$ at most 5; $rK_n$ for $r \geq 2$ and $n = 6$ or 7; $rK_n$ for $r \geq 3$ and $n = 8$ or 9; $K_{4,r}$ for all $r$; and the corona of $P_n$ and $K_m$ for all $n$ and $2 \leq m \leq 8$.

Germina and Ajitha [464] (see also [20]) prove that $K_2 + \overline{K_t}$, quadrilateral snakes, Petersen graphs, ladders, and unicyclic graphs are strongly multiplicative. Acharya, Germina, and Ajitha [20] have shown that $C_k^{(n)}$ (see §2.2 for the definition) is strongly multiplicative and that every graph can be embedded as an induced subgraph of a strongly multiplicative graph. Germina and Ajitha [464] define a graph with $q$ edges and a strongly multiplicative labeling to be *hyper strongly multiplicative* if the induced edge labels are $\{2, 3, \ldots, q+1\}$. They show that every hyper strongly multiplicative graph has exactly one nontrivial component that is either a star or has a triangle and every graph can be embedded as an induced subgraph of a hyper strongly multiplicative graph.

Krawec [664] calls a graph $G$ on $n$ edges *modular multiplicative* if the vertices of $G$ can be labeled with distinct integers $0, 1, \ldots, n-1$ (with one exception if $G$ is a tree) such that the labels induced on the edges by the product of the end vertices modulo $n$ are distinct. He proves that every graph can be embedded as an induced subgraph of a modular multiplicative graph on prime number of edges. He also shows that if $G$ is a modular multiplicative graph on prime number of edges $p$ then for every integer $k \geq 2$ there exist modular multiplicative graphs on $p^k$ and $kp$ edges that contain $G$ as a subgraph. In the same paper, Krawec also calls a graph $G$ on $n$ edges *$k$-modular multiplicative* if the vertices of $G$ can be labeled with distinct integers $0, 1, \ldots, n+k-1$ such that the labels induced on the edges by the product of the end vertices modulo $n+k$ are distinct. He proves that every graph is $k$-modular multiplicative for some $k$ and also shows that if $p = 2n+1$ is prime then the path on $n$ edges is $(n+1)$-modular multiplicative. He also shows that if $p = 2n+1$ is prime then the cycle on $n$ edges is $(n+1)$-modular multiplicative if there does not exist $\alpha \in \{2, 3, \ldots, n\}$ such that $\alpha^2 + \alpha - 1 \equiv 0 \bmod p$. He concludes with four open problems.

## 7.14 Mean Labelings

Somasundaram and Ponraj [1121] have introduced the notion of mean labelings of graphs. A graph $G$ with $p$ vertices and $q$ edges is called a *mean graph* if there is an injective function $f$ from the vertices of $G$ to $\{0, 1, 2, \ldots, q\}$ such that when each edge $uv$ is labeled with $(f(u)+f(v))/2$ if $f(u)+f(v)$ is even, and $(f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd, then the resulting edge labels are distinct. In [1121], [1122], [1123], [1124], and [940] they prove the following graphs are mean graphs: $P_n$, $C_n$, $K_{2,n}$, $K_2 + mK_1$, $\overline{K_n} + 2K_2$, $C_m \cup P_n$, $P_m \times P_n$, $P_m \times C_n$, $C_m \odot K_1$, $P_m \odot K_1$, triangular snakes, quadrilateral snakes, $K_n$ if and only if $n < 3$, $K_{1,n}$ if and only if $n < 3$, bistars $B_{m,n}$ $(m > n)$ if and only if $m < n+2$, the subdivision graph of the star $K_{1,n}$ if and only if

$n < 4$, and the friendship graph $C_3^{(t)}$ if and only if $t < 2$. They also prove that $W_n$ is not a mean graph for $n > 3$ and enumerate all mean graphs of order less than 5.

Ramya, Ponraj, and Jeyanthi [957] called a mean graph *super mean* if vertex labels and the edge labels are $\{1, 2, \ldots, p+q\}$. They prove following graphs are super mean: paths, combs, odd cycles, $P_n^2$, $L_n \odot K_1$, $C_m \cup P_n$ $(n \geq 2)$, the bistars $B_{n,n}$ and $B_{n+1,n}$. They also prove that unions of super mean graphs are super mean and $K_n$ and $K_{1,n}$ are not super mean when $n > 3$. In [599] Jeyanthi, Ramya, and Thangavelu prove the following are super mean: $nK_{1,4}$; the graphs obtained by identifying an endpoint of $P_m$ $(m \geq 2)$ with each vertex of $C_n$; the graphs obtained by identifying an endpoint of two copies of $P_m$ $(m \geq 2)$ with each vertex of $C_n$; the graphs obtained by identifying an endpoint of three copies of $P_m$ $(m \geq 2)$; and the graphs obtained by identifying an endpoint of four copies of $P_m$ $(m \geq 2)$. In [597] Jeyanthi and Ramya prove the following graphs have super mean labelings: the graph obtained by identifying the endpoints of two or more copies of $P_5$; the graph obtained from $C_n$ $(n \geq 4)$ by joining two vertices of $C_n$ distance 2 apart with a path of length 2 or 3; Jeyanthi, Ramya, Thangavelu [600] give super mean labelings for $C_m \cup C_n$ and $k$-super mean labelings for a variety of graphs.

In JeRaTh10 Jayanthi, Ramya, and Thangavelu proved the following graphs have super mean labelings: the one point union of any two cycles, graphs obtained by joining any two cycles by an edge (dumbbell graphs), $C_{2n+1} \odot C_{2m+1}$, graphs obtained by identifying a copy of an odd cycle $C_m$ with each vertex of $C_n$, the quadrilateral snake $Q_n$, where $n$ is odd, and the graphs obtained from an odd cycle $u_1, u_2, \ldots, u_n$ by joining the vertices $u_i$ and $u_{i+1}$ by the path $P_m$ ($m$ is odd) for $1 \leq i \leq n-1$ and joining vertices $u_n$ and $u_1$ by the path $P_m$. They also prove the following graphs have $k$-super mean labelings: $C_{2n}$, $C_{2n+1} \times P_m$, grids $P_m \times P_n$ with one arbitrary crossing edge in every square, and antiprisms on $2n$ vertices $(n > 4)$. (Recall an antiprism on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ where subscripts are taken modulo $n$).

Jeyanthi, Ramya, Maheswari [598] prove that $T_p$-trees (see §3.2 for the definition), graphs of the form $T \odot \overline{K_n}$ where $T$ is a $T_p$-tree, and the graph obtained from $P_m$ and $m$ copies of $K_{1,n}$ by identifying a noncentral vertex of $i$th copy of $K_{1,n}$ with $i$th vertex of $P_m$ are mean graphs. Jeyanthi, Ramya, Thangavelu, and Aditanar [599] give super mean labelings of $C_m \cup C_n$ and $T_p$-trees and show how to construct $k$-super mean graphs from existing ones.

In [596] Jayanthi and Ramya define $S_{m,n}$ as the graph obtained by identifying one endpoint of each of $n$ copies of $P_m$ and $< S_{m,n} : P_m >$ as a graph obtained by identifying one endpoint of a path $P_m$ with the vertex of degree $n$ of a copy of $S_{m,n}$ and the other endpoint of the same path to the vertex of degree $n$ of another copy of $S_{m,n}$. They prove the following graphs have super mean labelings: caterpillars, $< S_{m,n} : P_{m+1} >$, and the graphs obtained from $P_{2m}$ and $2m$ copies of $K_{1,n}$ by identifying a leaf of $i$th copy of $K_{1,n}$ with $i$th vertex of $P_{2m}$. They further establish that if $T$ is a $T_p$-tree, then $T \odot K_1$, $T \odot \overline{K_2}$, and, when $T$ has an even number of vertices, $T \odot \overline{K_n}$ $(n \geq 3)$ are super mean graphs.

Gayathri and Amuthavalli [456] say a $(p,q)$-graph $G$ has a $(k,d)$-*odd mean* labeling if there exists an injection $f$ from the vertices of $G$ to $\{0, 1, 2, \ldots, 2k-1+2(q-1)d\}$ such that the induced map $f^*$ defined on the edges of $G$ by $f^*(uv) = \lceil (f(u)+f(v))/2 \rceil$ is a bijection from edges of $G$ to $\{2k-1, 2k-1+2d, 2k-1+4d, \ldots, 2k-1+2(q-1)d\}$. When $d = 1$ a $(k,d)$-odd mean labeling is called $k$-*odd mean*. For $n \geq 2$ they prove the following graphs are $k$-odd mean for all $k$: $P_n$; combs $P_n \odot K_1$; crowns $C_n \odot K_1$ $(n \geq 4)$; bistars $B_{n,n}$; $P_m \odot \overline{K_n}$ $(m \geq 2)$; $C_m \odot \overline{K_n}$; $K_{2,n}$; $C_n$ except for $n = 3$ or 6; the one-point union of $C_n$ $(n \geq 4)$ and an end point of any path; grids $P_m \times P_n$ $(m \geq 2)$; $(P_n \times P_2) \odot K_1$; arbitrary unions of paths; arbitrary unions of stars; arbitrary

unions of cycles; the graphs obtained by joining two copies of $C_n$ ($n \geq 4$) by any path; and the graph obtained from $P_m \times P_n$ by replacing each edge by a path of length 2. They prove the following graphs are not $k$-odd mean for any $k$: $K_n; K_n$ with an edge deleted; $K_{3,n}$ ($n \geq 3$); wheels; fans; friendship graphs; triangular snakes; Möbius ladders; books $K_{1,m} \times P_2$ ($m \geq 4$); and webs. For $n \geq 3$ they prove $K_{1,n}$ is $k$-odd mean if and only if $k \geq n-1$. Gayathri and Amuthavalli [457] prove that the graph obtained by joining the centers of stars $K_{1,m}$ and $K_{1,n}$ are $k$-odd mean for $m = n, n+1, n+2$ and not $k$-odd mean for $m > n+2$. For $n \geq 2$ the following graphs have a $(k,d)$-mean labeling [460]: $C_m \cup P_n$ ($m \geq 4$) for all $k$; arbitrary unions of cycles for all $k$; $P_{2m}$; $P_{2m+1}$ for $k \geq d$ ($P_{2m+1}$ is not $(k,d)$-mean when $k < d$); combs $P_n \odot K_1$ for all $k$; $K_{1,n}$ for $k \geq d$; $K_{2,n}$ for $k \geq d$; bistars for all $k$; $nC_4$ for all $k$; and quadrilateral snakes for $k \geq d$.

Gayathri and Tamilselvi [460] say a $(p,q)$-graph $G$ has a $(k,d)$-*super mean* labeling if there exists an injection $f$ from the vertices of $G$ to $\{k, k+1, \ldots, p+q+k-1\}$ such that the induced map $f^*$ defined on the edges of $G$ by $f^*(uv) = \lceil (f(u)+f(v))/2 \rceil$ has the property that the vertex labels and the edge labels together are the integers from $k$ to $p+q+k-1$. When $d=1$ a $(k,d)$-super mean labeling is called $k$-*super mean*. For $n \geq 2$ they prove the following graphs are $k$-super mean for all $k$: odd cycles; $P_n$; $C_m \cup P_n$; the one-point union of a cycle and the end point of $P_n$; the union of any two cycles excluding $C_4$; and triangular snakes. For $n \geq 2$ they prove the following graphs are $(k,d)$-super mean for all $k$ and $d$: $P_n$; odd cycles; combs $P_n \odot K_1$; and bistars.

Gayathri and Tamilselvi [460] say a $(p,q)$-graph $G$ has a $k$-*super edge mean* labeling if there exists an injection $f$ from the edges of $G$ to $\{k, k+1, \ldots, k+2(p+q)\}$ such that the induced map $f^*$ from the vertices of $G$ to $\{k, k+1, \ldots, k+2(p+q)\}$ defined by $f^*(v) = \lceil (\Sigma f(vu))/2 \rceil$ taken all edges $vu$ incident to $v$ is an injection. For $n \geq 3$ they prove the following graphs are $k$-super edge mean for all $k$: paths; cycles; combs $P_n \odot K_1$; triangular snakes; crowns $C_n \odot K_1$; the one-point union of $C_3$ and an end point of $P_n$; and $P_n \odot K_2$.

## 7.15 Irregular Total Labelings

Motivated by the notion of the irregularity strength of a graph introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba [302] in 1988 and various kinds of other total labelings, Bača, Jendroľ, Miller, and Ryan [134] introduced the *total edge irregularity strength* of a graph as follows. For a graph $G(V,E)$ a labeling $\partial : V \cup E \to \{1, 2, \ldots, k\}$ is called an *edge irregular total $k$-labeling* if for every pair of distinct edges $uv$ and $xy$, $\partial(u)+\partial(uv)+\partial(v) \neq \partial(x)+\partial(xy)+\partial(y)$. Similarly, $\partial$ is called an *vertex irregular total $k$-labeling* if for every pair of distinct vertices $u$ and $v$, $\partial(u) + \sum \partial(e)$ over all edges $e$ incident to $u \neq \partial(v) + \sum \partial(e)$ over all edges $e$ incident to $v$. The minimum $k$ for which $G$ has an edge (vertex) irregular total $k$-labeling is called the *total edge (vertex) irregularity strength* of $G$. The total edge (vertex) irregular strength of $G$ is denoted by $\mathrm{tes}(G)$ ($\mathrm{tvs}(G)$). They prove: for $G(V,E)$, $E$ not empty, $\lceil (|E|+2)/3 \rceil \leq \mathrm{tes}(G) \leq |E|$; $\mathrm{tes}(G) \geq \lceil (\Delta(G)+1)/2 \rceil$ and $\mathrm{tes}(G) \leq |E| - \Delta(G)$, if $\Delta(G) \leq (|E|-1)/2$; $\mathrm{tes}(P_n) = \mathrm{tes}(C_n) = \lceil (n+2)/3 \rceil$; $\mathrm{tes}(W_n) = \lceil (2n+2)/3 \rceil$; $\mathrm{tes}(C_3^n)$ (friendship graph) $= \lceil (3n+2)/3 \rceil$; $\mathrm{tvs}(C_n) = \lceil (n+2)/3 \rceil$; for $n \geq 2$, $\mathrm{tvs}(K_n) = 2$; $\mathrm{tvs}(K_{1,n}) = \lceil (n+1)/2 \rceil$; and $\mathrm{tvs}(C_n \times P_2) = \lceil (2n+3)/4 \rceil$. Jendroľ, Miškul, and Soták [589] (see also [590]) proved: $\mathrm{tes}(K_5) = 5$; for $n \geq 6$, $\mathrm{tes}(K_n) = \lceil (n^2 - n + 4)/6 \rceil$; and that $\mathrm{tes}(K_{m,n}) = \lceil (mn+2)/3 \rceil$. They conjecture that for any graph $G$ other than $K_5$, $\mathrm{tes}(G) = \max\{\lceil (\Delta(G)+1)/2 \rceil, \lceil (|E|+2)/3 \rceil\}$. Ivančo and Jendroľ [580] proved that this conjecture is true for all trees. Jendroľ, Miškuf, and Soták [589] prove the conjecture for complete graphs

and complete bipartite graphs. Ahmad and Bača [46] proved the conjecture holds for the categorical product of two paths. (The categorical product $P_m \times P_n$ has vertex set the Cartesian product of $P_m$ and $P_n$ and edge set $((u,x),(v,y))$ for all $(u,v)$ in $P_m$ and $(x,y)$ in $P_n$.) Brandt, Mišškuf, and Rautenbach [253] proved the conjecture for large graphs whose maximum degree is not too large relative to its order and size. In particular, using the probabilistic method they prove that if $G(V,E)$ is a multigraph without loops and with nonzero maximum degree less than $|E|/10^3\sqrt{8|V|}$, then $\mathrm{tes}(G) = (\lceil |E|+2)/3\rceil$. As corollaries they have: if $G(V,E)$ satisfies $|E| \geq 3\cdot 10^3|V|^{3/2}$, then $\mathrm{tes}(G) = \lceil (|E|+2)/3\rceil$; if $G(V,E)$ has minimum degree $\delta > 0$ and maximum degree $\Delta$ such that $\Delta < \delta\sqrt{|V|}/10^3\cdot 4\sqrt{2}$ then $\mathrm{tes}(G) = \lceil (|E|+2)/3\rceil$; and for every positive integer $\Delta$ there is some $n(\Delta)$ such that every graph $G(V,E)$ without isolated vertices with $|V| \geq n(\Delta)$ and maximum degree at most $\Delta$ satisfies $\mathrm{tes}(G) = \lceil (|E|+2)/3\rceil$. Notice that this last result includes $d$-regular graphs of large order. They also prove that if $G(V,E)$ has maximum degree $\Delta \geq 2|E|/3$, then $G$ has an edge irregular total $k$-labeling with $k = \lceil(\Delta+1)/2\rceil$. Pfender [929] proved the conjecture for graphs with at least $7\times 10^{10}$ edges and proved for graphs $G(V,E)$ with $\Delta(G) \leq E(G)/4350$ we have $\mathrm{tes}(G) = (\lceil |E|+2)/3\rceil$.

Nurdin, Baskoro, Salman, and Gaos [912] determine the total vertex irregularity strength of trees with no vertices of degree 2 or 3; improve some of the bounds given in [134]; and show that $\mathrm{tvs}(P_n) = \lceil (n+1)/3)\rceil$. In [915] Nurdin, Salman, Gaos, and Baskoro prove that for $t \geq 2$, $\mathrm{tvs}(tP_1) = t$; $\mathrm{tvs}(tP_2) = t+1$; $\mathrm{tvs}(tP_3) = t+1$; and for $n \geq 4$, $\mathrm{tvs}(tP_n) = \lceil (nt+1)/3)\rceil$. Anholcer, Kalkowski, and Przybylo [72] prove that for every graph with $\delta(G) > 0$, $\mathrm{tvs}(G) \leq \lceil 3n/\delta\rceil + 1$. Anholcer, Karoński, and Pfender [71] prove that for every forest $F$ with no vertices of degree 2 and no isolated vertices $\mathrm{tvs}(F) = \lceil (n_1+1)/2\rceil$, where $n_1$ is the number of vertices in $F$ of degree 1. They also prove that for every forest with no isolated vertices and at most one vertex of degree 2, $\mathrm{tvs}(F) = \lceil (n_1+1)/2\rceil$.

In [1178] Tong, Lin, Yang, and Wang give the exact values of the total edge irregularity strength and total vertex irregularity strength of the toroidal grid $C_m \times C_n$. Chunling, Xiaohui, Yuansheng, and Liping, [337] showed $\mathrm{tvs}(K_p) = 2$ $(p \geq 2)$ and for the generalized Petersen graph $P(n,k)$ they proved $\mathrm{tvs}(P(n,k)) = \lceil n/2\rceil + 1$ if $k \leq n/2$ and $\mathrm{tvs}(P(n,n/2)) = n/2+1$. They also obtained the exact values for the total vertex strengths for ladders, Möbius ladders, and Knödel graphs. For graphs with no isolated vertices, Przybylo [945] gave bounds for $\mathrm{tvs}(G)$ in terms of the order and minimum and maximum degrees of $G$. For $d$-regular $(d > 0)$ graphs, Przybylo [946] gave bounds for $\mathrm{tvs}(G)$ in terms $d$ and the order of $G$. Ahmad, Ahtsham, H. B. Imran, and A.Q. Gaig [42] determined the exact values of the total vertex irregularity strength for five families of cubic plane graphs. In [44] Ahmad and Bača determine that the total edge-irregular strength of the categorical product of $C_n$ and $P_m$ where $m \geq 2$, $n \geq 4$ and $n$ and $m$ are even is $\lceil (2n(m-1)+2)/3\rceil$. They leave the case where at least one of $n$ and $m$ is odd as an open problem.

In [914] Nurdin, Salman, and Baskoro determine the total edge-irregular strength of the following graphs: for any integers $m \geq 2$, $n \geq 2$, $\mathrm{tes}(P_m \odot P_n) = \lceil (2mn+1)/3\rceil$; for any integers $m \geq 2$, $n \geq 3$, $\mathrm{tes}(P_m \odot C_n) = \lceil ((2n+1)m+1)/3\rceil$; for any integers $m \geq 2$, $n \geq 2$, $\mathrm{tes}(P_m \odot K_{1,n}) = \lceil (2m(n+1)+1)/3\rceil$; for any integers $m \geq 2$ and $n \geq 3$, $\mathrm{tes}(P_m \odot G_n) = \lceil (m(5n+2)+1)/3\rceil$ where $G_n$ is the gear graph obtained from the wheel $W_n$ by subdividing every edge on the $n$-cycle of the wheel; for any integers $m \geq 2$, $n \geq 2$, $\mathrm{tes}(P_m \odot F_n) = \lceil m(5n+2)+1\rceil$, where $F_n$ is the friendship graph obtained from $W_{2m}$ by subdividing every other rim edge; for any integers $m \geq 2$ and $n \geq 3$: and $\mathrm{tes}(P_m \odot W_n) = \lceil ((3n+2)m+1)/3\rceil$.

In [912] Nurdin, Baskoro, Salman, and Gaos determine the total edge-irregular strength of

the following types of trees: for $G$ isomorphic to a complete $n$-ary tree with depth $d$, $\text{tvs}(G)= \lceil (n^d+1)/2 \rceil$; $\text{tvs}(K_{1,n})= \lceil (n+1)/3 \rceil$ $(n \geq 3)$; and if $G$ is isomorphic to the caterpillar obtained by starting with $P_m$ and $m$ copies of $P_n$ denoted by $P_{n,1}, P_{n,2}, \ldots, P_{n,m}$, where $m \geq 2,\ n \geq 2$, then joining the $i$-th vertex of $P_m$ to an end vertex of the path $P_{n,i}$, $\text{tvs}(G)= \lceil (mn+3)/3 \rceil$.

Ahmad and Bača [45] proved $\text{tvs}(J_{n,2})= \lceil (n+1)/2) \rceil$ $(n \geq 4)$ and conjectured that for $n \geq 3$ and $m \geq 3$, $\text{tvs}(J_{n,m})= \max\{\lceil (n(m-1)+2)/3 \rceil, \lceil (nm+2)/4 \rceil\}$. They also proved that for the circulant graph (see § 5.1 for the definition) $C_n(1,2),\ n \geq 5$, $\text{tvs}(C_n(1,2))= \lceil (n+4)/5 \rceil$. They conjecture that for the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ with degree at least 5 and $n \geq 5,\ 1 \leq a_i \leq \lfloor n/2 \rfloor$, $\text{tvs}(C_n(a_1, a_2, \ldots, a_m)= \lceil (n+r)/(1+r) \rceil$.

In [41] Ahmad shows that the total vertex irregularity strength of the antiprism graph $A_n$ $(n \geq 3)$ is $\lceil (2n+4)/5 \rceil$ (see §5.5) for the definition and gives the vertex irregularity strength of three other families convex polytope graphs.

For a simple graph $G$ with no isolated edges and at most one isolated vertex Anholcer [70] calls a labeling $w : E(G) \to \{1, 2, \ldots, m\}$ *product-irregular*, if all product degrees $pd_G(v) = \prod_{e \ni v} w(e)$ are distinct. Analogous to the notion of irregularity strength the goal is to find a product-irregular labeling that minimizes the maximum label. This minimum value is called the *product irregularity strength* of $G$. He provides bounds for the product irregularity strength of paths, cycles, cartesian products of paths, and cartesian products of cycles.

## 7.16 Minimal $k$-rankings

A *$k$-ranking* of a graph is a labeling of the vertices with the integers 1 to $k$ inclusively such that any path between vertices of the same label contains a vertex of greater label. The *rank number of a graph $G$*, $\chi_r(G)$, is the smallest possible number of labels in a ranking. A $k$-ranking is *minimal* if no label can be replaced by a smaller label and still be a $k$-ranking. The concept of the rank number arose in the study of the design of very large scale integration (VLSI) layouts and parallel processing (see [347], [782] and [1004]). Ghoshal, Laskar, and Pillone [469] were the first to investigate minimal $k$-rankings from a mathematical perspective. Laskar and Pillone [696] proved that the decision problem corresponding to minimal $k$-rankings is NP-complete. It is HP-hard even for bipartite graphs [355]. Bodlaender, Deogun, Jansen, Kloks, Kratsch, Müller, Tuza [242] proved that the rank number of $P_n$ is $\chi_r(P_n) = \lfloor \log_2(n) \rfloor + 1$ and satisfies the recursion $\chi_r(P_n) = 1 + \chi_r(P_{\lceil (n-1)/2 \rceil})$ for $n > 1$. The following results are given in [355]: $\chi_r(S_n) = 2$; $\chi_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2$; $\chi_r(W_n) = \lfloor \log_2(n-3) \rfloor + 3 (n > 3)$; $\chi_r(K_n) = n$; the complete $t$-partite graph with $n$ vertices has ranking number $n+1$ - the cardinality of the largest partite set; and a split graph with $n$ vertices has ranking number $n+1$ - the cardinality of the largest independent set (a *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set.) Wang proved that for any graphs $G$ and $H$ $\chi_r(G+H) = \min\{|V(G)| + \chi_r(H)\}, |V(H) + \chi_r(G)\}$.

In 2009 Novotny, Ortiz, and Narayan [910] determined the rank number of $P_n^2$ from the recursion $\chi_r(P_n^2) = 2 + \chi_r(P(_{\lceil (n-2)/2 \rceil})$ for $n > 2$. They posed the problem of determining $\chi_r(P_m \times P_n)$ and $\chi(P_n^k)$. In 2009 [61] and [60] Alpert determined the rank numbers of $P_n^k,\ C_n^k,\ P_2 \times C_n,\ K_m \times P_n,\ P_3 \times P_n$, Möbius ladders and found bounds for rank numbers of general grid graphs $P_m \times P_n$. About the same time as Alpert and independently, Chang, Kuo, and Lin [293] determined the rank numbers of $P_n^k,\ C_n^k,\ P_2 \times P_n, P_2 \times C_n$. Chang et al. also determined the rank numbers of caterpillars and proved that for any graphs $G$ and $H$ $\chi_r(G[H]) = \chi_r(H) + |V(H)|(\chi_r(G) - 1)$.

In 2010 J. Jacob, D. Narayan, E. Sergel, P. Richter, and A. Tran [586] investigated $k$-rankings of paths and cycles with pendant paths of length 1 or 2. Among their results are: for any caterpillar $G$ $\chi_r(P_n) \leq \chi_r(G) \leq \chi_r(P_n)+1$ and both cases occur; if $2^m \leq n \leq 2^{m+1}$ then for any graph $G$ obtained by appending edges to an $n$-cycle we have $m+2 \leq \chi_r(G) \leq m+3$ and both cases occur; if $G$ is a lobster with spine $P_n$ then $\chi_r(P_n) \leq \chi_r(G) \leq \chi_r(P_n)+2$ and all three cases occur; if $G$ a graph obtained from the cycle $C_n$ by appending paths of length 1 or 2 to any number of the vertices of the cycle then $\chi_r(P_n) \leq \chi(G) \leq \chi(P_n)+2$ and all three cases occur; and if $G$ the graph obtained from the comb obtained from $P_n$ by appending one path of length $m$ to each vertex of $P_n$ then $\chi_r(G) = \chi_r(P_n) + \chi_r(P_{m+1}) - 1$.

## 7.17 Set Graceful and Set Sequential Graphs

The notions of set graceful and set sequential graphs were introduced by Acharaya in 1983 [14]. A graph is called *set graceful* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph such that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge, the assignment of sets to the vertices is injective, and the assignment to the edges is bijective. A graph is called *set sequential* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph such that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge and the the assignment of sets to the vertices and the edges is bijective. The following has been shown: $P_n$ $(n > 3)$ is not set graceful [527]; $C_n$ is not set sequential [26]; $C_n$ is set graceful if and only if $n = 2^m - 1$ [529] and [14]; $K_n$ is set graceful if and only if $n = 2, 3$ or 6 [879]; $K_n$ $(n \geq 2)$ is set sequential if and only if $n = 2$ or 5 [529]; $K_{a,b}$ is set sequential if and only if $(a+1)(b+1)$ is a positive power of 2 [529]; a necessary condition for $K_{a,b,c}$ to be set sequential is that $a, b$, and $c$ cannot have the same parity [527]; $K_{1,b,c}$ is not set sequential when $b$ and $c$ even [529]; $K_{2,b,c}$ is not set sequential when $b$ and $c$ are odd [527]; no theta graph is set graceful [527]; the complete nontrivial $n$-ary tree is set sequential if and only if $n+1$ is a power of 2 and the number of levels is 1 [527]; a tree is set sequential if and only if it is set graceful [527]; the nontrivial plane triangular grid graph $G_n$ is set graceful if and only if $n = 2$ [529]; every graph can be embedded as an induced subgraph of a connected set sequential graph [527]; every graph can be embedded as an induced subgraph of a connected set graceful graph [527], every planar graph can be embedded as an induced subgraph of a set sequential planar graph [529]; every tree can be embedded as an induced subgraph of a set sequential tree [529]; and every tree can be embedded as an induced subgraph of a set graceful tree [529]. Hegde conjectures [529] that no path is set sequential. Hegde's conjecture [530] every complete bipartite graph that has a set graceful labeling is a star was proved by Vijayakumar [1209]

Germina, Kumar, and Princy [463] prove: if a $(p, q)$-graph is set-sequential with respect to a set with $n$ elements, then the maximum degree of any vertex is $2^{n-1} - 1$; if $G$ is set-sequential with respect to a set with $n$ elements other than $K_5$, then for every edge $uv$ with $d(u) = d(v)$ one has $d(u) + d(v) < 2^{n-1} - 1$; $K_{1,p}$ is set-sequential if and only if $p$ has the form $2^{n-1} - 1$ for some $n \geq 2$; binary trees are not set-sequential; hypercubes $Q_n$ are not set-sequential for $n > 1$; wheels are not set-sequential; and uniform binary trees with an extra edge appended at the root are set-graceful and set graceful.

Acharya [14] has shown: a connected set graceful graph with $q$ edges and $q+1$ vertices is a tree of order $p = 2^m$ and for every positive integer $m$ such a tree exists; if $G$ is a connected set sequential graph, then $G + K_1$ is set graceful; and if a graph with $p$ vertices and $q$ edges is

set sequential, then $p+q = 2^m - 1$. Acharya, Germina, Princy, and Rao [22] proved: if $G$ is set graceful, then $G \cup \overline{K_t}$ is set sequential for some $t$; if $G$ is a set graceful graph with $n$ edges and $n+1$ vertices, then $G + \overline{K_t}$ is set graceful if and only if $m$ has the form $2^t - 1$; $P_n + \overline{K_m}$ is set graceful if $n = 1$ or 2 and $m$ has the form $2^t - 1$; $K_{1,m,n}$ is set graceful if and only if $m$ has the form $2^t - 1$ and $n$ has the form $2^s - 1$; $P_4 + \overline{K_m}$ is not set graceful when $m$ has the form $2^t - 1$ ($t \geq 1$); $K_{3,5}$ is not set graceful; if $G$ is set graceful, then graph obtained from $G$ by adding for each vertex $v$ in $G$ a new vertex $v'$ that is adjacent to every vertex adjacent to $v$ is not set graceful; and $K_{3,5}$ is not set graceful.

## 7.18 Difference Graphs

Analogous to a sum graph, Harary [505] calls a graph a *difference graph* if there is an bijection $f$ from $V$ to a set of positive integers $S$ such that $xy \in E$ if and only if $|f(x) - f(y)| \in S$. Bloom, Hell, and Taylor [237] have shown that the following graphs are difference graphs: trees, $C_n, K_n, K_{n,n}, K_{n,n-1}$, pyramids, and $n$-prisms. Gervacio [466] proved that wheels $W_n$ are difference graphs if and only if $n = 3, 4$, or 6. Sonntag [1126] proved that cacti (that is, graphs in which every edge is contained in at most one cycle) with girth at least 6 are difference graphs and he conjectures that all cacti are difference graphs. Sugeng and Ryan [1145] provided difference labelings for cycles, fans, cycles with chords, graphs obtained by the one-point union of $K_n$ and $P_m$; and graphs made from any number of copies of a given graph $G$ that has a difference labeling by identifying one vertex the first with a vertex of the second, a different vertex of the second with the third and so on.

Hegde and Vasudeva [545] call a simple digraph a *mod difference digraph* if there is a positive integer $m$ and a labeling $L$ from the vertices to $\{1, 2, \ldots, m\}$ such that for any vertices $u$ and $v$, $(u, v)$ is an edge if and only if there is a vertex $w$ such that $L(v) - L(u) \equiv L(w) \pmod{m}$. They prove that the complete symmetric digraph and unidirectional cycles and paths are mod difference digraphs.

## 7.19 Square Sum Labelings

Ajitha, Arumugam, and Germina [56] call a labeling $f$ from a graph $G(p, q)$ to $\{1, 2, \ldots, q\}$ a *square sum labeling* if the induced edge labeling $f^*(uv) = (f(u))^2 + (f(v))^2$ is injective. They say a square sum labeling is a *strongly square sum labeling* if the $q$ edge labels are the first $q$ consecutive integers of the form $a^2 + b^2$ where $a$ and $b$ are less than $p$ and distinct. They prove the following graphs have square sum labelings: trees; cycles; $K_2 + mK_1$; $K_n$ if and only if $n \leq 5$; $C_n^{(t)}$ (the one-point union of $t$ copies of $C_n$); grids $P_m \times P_n$; and $K_{m,n}$ if $m \leq 4$. They also prove that every strongly square sum graph except $K_1, K_2$, and $K_3$ contains a triangle.

## 7.20 Permutation and Combination Graphs

Hegde and Shetty [540] define a graph $G$ with $p$ vertices to be a *permutation graph* if there exists a injection $f$ from the vertices of $G$ to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function $g_f$ defined by $g_f(uv) = f(u)!/|f(u) - f(v)|!$ is injective. They say a graph $G$ with $p$ vertices is a *combination graph* if there exists a injection $f$ from the vertices of $G$ to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function $g_f$ defined as $g_f(uv) = f(u)!/|f(u) - f(v)|!f(v)!$ is injective. They prove: $K_n$ is a permutation graph if and only if $n \leq 5$; $K_n$ is a combination graph if and only if $n \leq 5$; $C_n$ is a combination graph for $n > 3$; $K_{n,n}$ is a combination graph if and only if $n \leq 2$;

$W_n$ is a not a combination graph for $n \leq 6$; and a necessary condition for a $(p,q)$-graph to be a combination graph is that $4q \leq p^2$ if $p$ is even and $4q \leq p^2 - 1$ if $p$ is odd. They strongly believe that $W_n$ is a combination graph for $n \geq 7$ and all trees are combinations graphs. Babujee and Vishnupriya [169] prove the following graphs are permutation graphs: $P_n$; $C_n$; stars; graphs obtained adding a pendent edge to each edge of a star; graphs obtained by joining the centers of two identical stars with an edge or a path of length 2); and complete binary trees with at least three vertices.

## 7.21 Strongly *-graphs

A variation of strong multiplicity of graphs is a strongly *-graph. A graph of order $n$ is said to be a *strongly *-graph* if its vertices can be assigned the values $1, 2, \ldots, n$ in such a way that, when an edge whose vertices are labeled $i$ and $j$ is labeled with the value $i + j + ij$, all edges have different labels. Adiga and Somashekara [38] have shown that all trees, cycles, and grids are strongly *-graphs. They further consider the problem of determining the maximum number of edges in any strongly *-graph of given order and relate it to the corresponding problem for strongly multiplicative graphs.

Babujee and Vishnupriya [169] have proved the following are strongly *-graphs: $C_n \times P_2$, $(P_2 \cup \overline{K}_m) + \overline{K}_2$, windmills $K_3^{(n)}$, and *jelly fish* graphs $J(m,n)$ obtained from a 4-cycle $v_1, v_2, v_3, v_4$ by joining $v_1$ and $v_3$ with an edge and appending $m$ pendent edges to $v_2$ and $n$ pendent edges to $v_4$.

Babujee and Beaula [84] prove that cycles and complete bipartite graphs are vertex strongly *-graphs. Babujee, Kannan, and Vishnupriya [91] prove that wheels, paths, fans, crowns, $(P_2 \cup mK_1) + \overline{K_2}$, and umbrellas (graphs obtained by appending a path to the central vertex of a fan) are vertex strongly *-graphs.

## 7.22 Triangular Sum Graphs

S. Hegde and P. Shankaran [536] call a labeling of graph with $q$ edges a *triangular sum labeling* if the vertices can be assigned distinct non-negative integers in such a way that, when an edge whose vertices are labeled $i$ and $j$ is labeled with the value $i + j$, the edges labels are $\{k(k+1)/2 |\ k = 1, 2, \ldots, q\}$. They prove the following graphs have triangular sum labelings: paths, stars, complete $n$-ary trees, and trees obtained from a star by replacing each edge of the star by a path. They also prove that $K_n$ has a triangular sum labeling if and only if $n$ is 1 or 2 and the friendship graphs $C_3^{(t)}$ do not have a triangular sum labeling. They conjecture that $K_n$ $(n \geq 5)$ are forbidden subgraphs of graph with triangular sum labelings.

Vaidya, Prajapati, and Vihol [1196] proved that several classes of graphs do not have triangular sum labelings. Among them are: helms, graphs obtained by joining the centers of two wheels to a new vertex, and graphs in which every edge is an edge of a triangle. As corollaries of the latter result they have that fans, wheels, friendship graphs, flowers, $P_m + \overline{K_n}$, and $W_m + \overline{K_n}$ do not have triangular sum labelings. They also show that every cycle, every cycle with exactly one chord, and every cycle with exactly two chords that form a triangle with an edge of the cycle can be embedded as an induced subgraph of a graph with a triangular sum labeling.

## 7.23 Divisor Graphs

G. Santhosh and G. Singh [996] call a graph $G(V,E)$ a *divisor graph* if $V$ is a set of integers and

$uv \in E$ if and only if $u$ divides $v$ or vice versa. They prove the following are divisor graphs: trees; $mK_n$; induced subgraphs of divisor graphs; cocktail party graphs $H_{m,n}$ (see §7.1 for the definition); the one-point union of complete graphs of different orders; complete bipartite graphs; $W_n$ for $n$ even and $n > 2$; and $P_n + \overline{K}_t$. They also prove that $C_n$ ($n \geq 4$) is a divisor graph if and only if $n$ is even and if $G$ is a divisor graph then for all $n$ so is $G + K_n$.

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