Rogers-Ramanujan-Slater Type Identities

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Abstract

In this survey article, we present an expanded version of Lucy Slater's famous list of identities of the Rogers-Ramanujan type, including identities of similar type, which were discovered after the publication of Slater's papers, and older identities (such as those in Ramanujan's lost notebook) which were not included in Slater's papers. We attempt to supply the earliest known reference for each identity. Also included are identities of false theta functions, along with their relationship to Rogers-Ramanujan type identities. We also describe several ways in which pairs/larger sets of identities may be related, as well as dependence relationships between identities.

1 Introduction

1.1 Theta Functions

Ramanujan's theta function [9, p. 11, Eq. (1.1.5)] is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{1.1.1}$$

for $|ab| < 1$. It is called a theta function, despite the lack of a theta in the notation, because it is equivalent, via change of variable, to the theta function of Jacobi [46, p. 463]:

$$\vartheta(z, w) := \sum_{n=-\infty}^{\infty} (-1)^n w^{n^2} e^{2niz}, \tag{1.1.2}$$

where $|w| < 1$.

The following special cases of $f(a, b)$ arise so often that they were given there own special notation by Ramanujan [9, p. 11]:

$$\varphi(q) := f(q, q) \tag{1.1.3}$$
$$\psi(q) := f(q, q^3) \tag{1.1.4}$$
$$f(-q) := f(-q, -q^2). \tag{1.1.5}$$

Ramanujan further defines

$$\chi(q) := \frac{f(-q^2, -q^2)}{f(-q)}. \tag{1.1.6}$$

One of the most important results in the theory of theta functions is that they can be expressed as infinite products:

Jacobi's triple product identity. *For* $|ab| < 1$*,*

$$f(a, b) = (-a, -b, ab; ab)_\infty, \tag{1.1.7}$$

where

$$(A; w)_\infty := \prod_{n=0}^{\infty} (1 - Aw^n),$$

and

$$(A_1, A_2, \ldots, A_r; w)_\infty := (A_1; w)_\infty (A_2; w)_\infty \cdots (A_r; w)_\infty.$$

An immediate corollary of (1.1.7) is thus

Corollary 1.

$$f(-q) = (q; q)_\infty \qquad \textit{(Euler [20]))} \tag{1.1.8}$$
$$\varphi(-q) = \frac{(q; q)_\infty}{(-q; q)_\infty} \qquad \textit{(Gauß [23])} \tag{1.1.9}$$
$$\psi(-q) = \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \qquad \textit{(Gauß [23])} \tag{1.1.10}$$
$$\chi(-q) = (q; q^2)_\infty \tag{1.1.11}$$

Sometimes a linear combination of two theta series can be expressed as a single infinite product. The quintuple product identity has probably been (re)discovered more than any other identity in the theory of q-series. The earliest published occurrence appears to be due to Fricke [21, p. 207, Eq. (6)]. See Cooper's survey paper for a history and many proofs of the quintuple product identity [17].

Quintuple product identity.

$$\begin{aligned}
Q(w,x) &:= f(-wx^3,-w^2x^{-3}) + xf(-wx^{-3},-w^2x^3)\\
&= \frac{f(wx^{-1},x)f(-wx^{-2},-wx^2)}{f(-w^2,-w^4)}\\
&= (-wx^{-1},-x,w;w)_\infty(wx^{-2},wx^2;w^2)_\infty. \qquad (1.1.12)
\end{aligned}$$

Bailey [14, p. 220, Eq. (4.1)] showed how certain linear combinations of theta series can be simplified to a single theta series:

$$f(qz^2,q^3z^{-2}) + zf(q^3z^3,qz^{-2}) = f(z,qz^{-1}). \qquad (1.1.13)$$

1.2 Bailey Pairs and Bailey's lemma

Of central importance are the two Rogers-Ramanujan identities, which each assert the equality of a certain q-series with a ratio of theta functions:

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^2,-q^3)}{f(-q)} \qquad (1.2.1)$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q,-q^4)}{f(-q)}, \qquad (1.2.2)$$

where

$$(a;q)_n := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n\in\mathbb{Z}_+\\ 1 & \text{if } n=0\\ \left[(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{n})\right]^{-1} & \text{if } n\in\mathbb{Z}_-. \end{cases}$$

Note that it is customary to write $(A)_n$ for $(A;q)_n$ and $(A)_\infty$ for $(A;q)_\infty$.

Using (1.1.7), and after some simplification, the right hand sides of (1.2.1) and (1.2.2) can be expressed as, respectively, the infinite products

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})}$$

and

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}.$$

Normally, an "identity of Rogers-Ramanujan type" asserts the equality of a certain q-series with a ratio of a theta series (or perhaps a linear combination of several theta series) to one of $f(-q^m)$, $\varphi(-q^m)$, or $\psi(\pm q^m)$ for some positive integer m. Sometimes, however, the term is applied to other types of q-series—product identities, or polynomial generalizations thereof. In particular, we limit ourselves to considering q-series in which the power of q is quadratic in the summation variable. L. J. Rogers [36, 37] and S.

Ramanujan [9, 10] discovered a number of identities of this type. W. N. Bailey studied Rogers's work and as a result was able to simplify and extend it in [12, 13]. L. J. Slater used and extended Bailey's methods to obtain a large collection of identities of Rogers-Ramanujan type in [40, 41].

Bailey's transform. *If*

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.2.3}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \tag{1.2.4}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

Proof.

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \sum_{r=n}^{\infty} \alpha_n \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \sum_{n=0}^{r} \delta_r \alpha_n u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \delta_r \beta_r$$

□

Bailey does not use his transform in the most general setting, but rather specializes $u_n = 1/(q)_n$ and $v_n = 1/(xq)_n$. Such a pair (α_n, β_n) which satisfies (1.2.3) is called a *Bailey pair* in the subsequent literature.

Bailey's lemma. *If* $(\alpha_n(x,q), \beta_n(x,q))$ *satisfies* (1.2.3), *again with* $u_n = 1/(q)_n$ *and* $v_n = 1/(xq)_n$, *then so does* $(\alpha'_n(x,q), \beta'_n(x,q))$, *where*

$$\alpha'_r(x,q) = \frac{(\rho_1)_r (\rho_2)_r}{(xq/\rho_1)_r (xq/\rho_2)_r} \left(\frac{xq}{\rho_1 \rho_2} \right)^r \alpha_r \tag{1.2.5}$$

and

$$\beta'_n(x,q) = \frac{1}{(xq/\rho_1)_n (xq/\rho_2)_n} \sum_{j=0}^{n} \frac{(\rho_1)_j (\rho_2)_j (xq/\rho_1\rho_2)_{n-j}}{(q;q)_{n-j}} \left(\frac{xq}{\rho_1 \rho_2} \right)^j \beta_j(x,q). \tag{1.2.6}$$

Equivalently, if $(\alpha_n(x,q), \beta_n(x,q))$ *is a Bailey pair, then*

$$\begin{aligned} &\frac{1}{(xq/\rho_1)_n (xq/\rho_2)_n} \sum_{j=0}^{n} \frac{(\rho_1)_j (\rho_2)_j (xq/\rho_1\rho_2)_{n-j}}{(q)_{n-j}} \left(\frac{xq}{\rho_1 \rho_2} \right)^j \beta_j(x,q) \\ &\qquad = \sum_{r=0}^{n} \frac{(\rho_1)_r (\rho_2)_r}{(q)_{n-r} (xq)_{n+r} (xq/\rho_1)_r (xq/\rho_2)_r} \left(\frac{xq}{\rho_1 \rho_2} \right)^r \alpha_r(x,q). \end{aligned} \tag{1.2.7}$$

Proof. See Andrews [8, pp. 25–27, Theorem 3.3]. □

Corollary 2.

$$\sum_{n=0}^{\infty} x^n q^{n^2} \beta_n(x,q) = \frac{1}{(xq)_\infty} \sum_{r=0}^{\infty} x^r q^{r^2} \alpha_r(x,q) \tag{1.2.8}$$

$$\sum_{n=0}^{\infty} x^n q^{n^2} (-q;q^2)_n \beta_n(x,q^2) = \frac{(-xq;q^2)_\infty}{(xq^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{x^r q^{r^2} (-q;q^2)_r}{(-xq;q^2)_r} \alpha_r(x,q^2) \tag{1.2.9}$$

$$\sum_{n=0}^{\infty} x^n q^{n(n+1)/2} (-1)_n \beta_n(x,q) = \frac{(-xq)_\infty}{(xq)_\infty} \sum_{r=0}^{\infty} \frac{x^r q^{r(r+1)/2} (-1)_r}{(-xq)_r} \alpha_r(x,q) \tag{1.2.10}$$

Proof. Let $n, \rho_2 \to \infty$ in (1.2.7) to obtain

$$\sum_{j=0}^{\infty} \frac{(-1)^j x^j q^{j(j+1)/2} (\rho_1)_j}{\rho_1^j} \beta_j(x,q) = \frac{(xq/\rho_1)_\infty}{(xq)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r x^r q^{r(r+1)/2} (\rho_1)_r}{\rho_1^r (xq/\rho_1)_r} \alpha_r(x,q). \tag{1.2.11}$$

To obtain (1.2.9), let $\rho_1 \to \infty$ in (1.2.11). To obtain (1.2.9), in (1.2.11), set $\rho_1 = -\sqrt{q}$ and replace q by q^2 throughout. To obtain (1.2.10), in (1.2.11) set $\rho_1 = -q$. □

Corollary 3.

$$\sum_{n=0}^{\infty} q^{n^2} \beta_n(1,q) = \frac{1}{f(-q)} \sum_{r=0}^{\infty} q^{r^2} \alpha_r(1,q) \tag{PBL}$$

$$\sum_{n=0}^{\infty} q^{n^2} (-q;q^2)_n \beta_n(1,q^2) = \frac{1}{\psi(-q)} \sum_{r=0}^{\infty} q^{r^2} \alpha_r(1,q^2) \tag{TBL}$$

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} (-1)_n \beta_n(1,q) = \frac{2}{\varphi(-q)} \sum_{r=0}^{\infty} \frac{q^{r(r+1)/2}}{1+q^r} \alpha_r(1,q) \tag{S1BL}$$

Proof. To obtain (PBL), (TBL), and (S1BL), set $x = 1$ in (1.2.8), (1.2.9), and (1.2.10) respectively. □

Corollary 4.

$$\frac{1}{1-q} \sum_{n=0}^{\infty} q^{n(n+1)/2} (-q)_n \beta_n(q,q) = \frac{1}{\varphi(-q)} \sum_{r=0}^{\infty} q^{r(r+1)/2} \alpha_r(q,q) \tag{S2BL}$$

$$\frac{1}{1-q} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} (q)_n \beta_n(q,q) = \sum_{r=0}^{\infty} (-1)^r q^{r(r+1)/2} \alpha_r(q,q). \tag{FBL}$$

Proof. To obtain (S2BL), set $x = q$ and $\rho_2 = -q$ in (1.2.11). To obtain (FBL), set $x = \rho_2 = q$ in (1.2.11). $\square$

Example 5. The pair $(\alpha_n(x,q), \beta_n(x,q))$, where

$$\alpha_n(x,q) = \frac{(-1)^n x^n q^{n(3n-1)/2}(1-xq^{2n})(x)_n}{(1-x)(q)_n}$$

and

$$\beta_n(x,q) = \frac{1}{(q)_n}$$

form a Bailey pair.

Proof.

$$\begin{aligned}
\beta_n(x,q) &= \sum_{r=0}^{n} \frac{\alpha_r(x,q)}{(q)_{n-r}(xq)_{n+r}} \\
&= \frac{1}{(q)_n(xq)_n} \sum_{r=0}^{n} \frac{(q^{-n})_r}{(xq^{n+1})_r}(-1)^r q^{nr-r(r-1)/2}\alpha_r(x,q) \\
&= \frac{1}{(q)_n(xq)_n} \sum_{r=0}^{n} \frac{(x)_r(1-xq^{2r})(q^{-n})_r}{(q)_r(1-x)(xq^{n+1})_r} x^r q^{2r^2} \\
&= \frac{1}{(q)_n(xq)_n} \lim_{b\to\infty} \sum_{r=0}^{n} \frac{(x)_r(q\sqrt{x})_r(-q\sqrt{x})_r(q^{-n})_r(b)_r^2}{(\sqrt{x})_r(-\sqrt{x})_r(xq^{n+1})_r(xq/b)_r^2}\left(\frac{xq^{n+1}}{b^2}\right)^r \\
&= \frac{1}{(q)_n(xq)_n} \lim_{b\to\infty} \frac{(xq)_n(xq/b^2)_n}{(xq/b)_n^2} \text{ (by Jackson's } {}_6\phi_5 \text{ sum; see [22, Eq. (II.21)])} \\
&= \frac{1}{(q)_n}
\end{aligned}$$

$\square$

Now that we have a Bailey pair in hand, it can be inserted into the various limiting cases of Bailey's lemma to yield series-product identities. Notice that

$$\beta_n(1,q) = 1/(q)_n \tag{1.2.12}$$

and

$$\alpha_n(1,q) = \begin{cases} (-1)^n q^{n(3n-1)/2}(1+q^n) & \text{when } n > 0 \\ 1 & \text{when } n = 0. \end{cases} \tag{1.2.13}$$

Upon inserting (1.2.12) and (1.2.13) into (PBL), we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{f(-q)}\left(1 + \sum_{n=1}^{\infty}(-1)^n q^{n(5n-1)/2}(1+q^n)\right)$$

$$= \frac{1}{f(-q)} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n-1)/2}$$

$$= \frac{f(-q^2,-q^3)}{f(-q)} \qquad \text{(by (1.1.7))}$$

the extremes of which are the first Rogers-Ramanujan identity.

Upon inserting (1.2.12) and (1.2.13) into (TBL), we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{\psi(-q)} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{n(4n-1)}(1+q^{2n}) \right)$$

$$= \frac{1}{\psi(-q)} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(4n-1)}$$

$$= \frac{f(-q^3,-q^5)}{\psi(-q)} \qquad \text{(by (1.1.7))}$$

the extremes of which are the first Ramanujan-Slater/Göllnitz-Gordon identity.

Finally, inserting (1.2.12) and (1.2.13) into (S1BL), we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1)_n}{(q)_n} = \frac{2}{\varphi(-q)} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{2r^2}(1+q^{2n}) \right)$$

$$= \frac{1}{\varphi(-q)} \sum_{n=-\infty}^{\infty} (-1)^n q^{2r^2}$$

$$= \frac{f(-q^2,-q^2)}{\varphi(-q)} \qquad \text{(by (1.1.7))}$$

the extremes of which are a special case of Lebesgue's identity (L), which appears in Slater's list [41] as Eq. (12).

1.3 Some More General q-Series Identities

A number of Rogers-Ramanujan type identities are special cases of the identities in this subsection.

Identity 6 (Euler).

$$(-z)_\infty = \sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)/2}}{(q)_n} \tag{E}$$

([20]; cf. Andrews [6, p. 19, Eq. (2.2.6)])

Identity 7 (Cauchy).

$$\frac{1}{(z)_\infty} = \sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)}}{(q)_n (z)_n} \tag{C}$$

([6, p. 20, Eq. (2.2.8)])

Identity 8 (Lebesgue).

$$(aq;q^2)_\infty(-q)_\infty = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(a)_n}{(q)_n} \tag{L}$$

(Lebesgue [30]; cf. Andrews [6, p. 21, Cor. 2.7])

Identity 9 (Heine's q-analog of Gauß's sum).

$$\frac{(c/a)_\infty}{(c)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n c^n q^{n(n-1)/2}(a)_n}{a^n (q)_n (c)_n} \tag{H}$$

([26]; cf. Gasper and Rahman [22, Eq. (II.8)])

Identity 10 (Andrews' q-analog of Gauß's ${}_2F_1(\frac{1}{2})$ sum).

$$\frac{(aq;q^2)_\infty(bq;q^2)_\infty}{(q;q^2)_\infty(abq;q^2)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n q^{n(n+1)/2}}{(q)_n(abq;q^2)_n} \tag{AG}$$

(Andrews [4, p. 526, Eq. (1.8)]; cf. Gasper and Rahman [22, Eq. (II.11)])

Identity 11 (Andrews' q-analog of Bailey's ${}_2F_1(\frac{1}{2})$ sum).

$$\frac{(cq/b;q^2)_\infty(bc;q^2)_\infty}{(c)_\infty} = \sum_{n=0}^{\infty} \frac{(b)_n(q/b)_n c^n q^{\binom{n}{2}}}{(c)_n(q^2;q^2)_n} \tag{AB}$$

(Andrews [4, p. 526, Eq. (1.9)]; cf. Gasper and Rahman [22, Eq. (II.10)])

Identity 12 (Ramanujan).

$$\frac{f(aq^3,a^{-1}q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}(-a^{-1}q;q^2)_n(-aq;q^2)_n}{(q^2;q^2)_{2n}} \tag{R1}$$

$$\frac{f(aq^2,a^{-1}q^2)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-a^{-1}q;q^2)_n(-aq;q^2)_n}{(q;q^2)_n(q^4;q^4)_n} \tag{R2}$$

The preceding result of Ramanujan [35, p. 33] yields infinitely many identities of Rogers-Ramanujan-Slater type when a is set to $\pm q^r$ for $r \in \mathbb{Q}$.

2 Rogers-Ramanujan Type Identities

The subsections are numbered to correspond to the modulus associated with the product side of the identities, and identities are numbered sequentially within each subsection. Just as Mozart's compositions are identified according to their listing in the Köchel catalog [29] with a designation of the form "K.n," Rogers-Ramanujan type identities will likely always be associated with their appearance in Slater's list [41]. Accordingly, each identity below that appears in [41] is designated with a "Slater number" **S.**n. The designation "S.$n-$" means that q has been replaced by $-q$ in the nth identity in Slater's list. The designation "**S.**nc" means that a corrected form of the nth identity in Slater's list is being presented. The earliest known occurrence of each identity is indicated.

2.1 q-series Expansions of Constants and Theta Functions

$$0 = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q)_n} \qquad \text{((E) with } z=-1) \qquad (2.1.1)$$

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_{n+1}} \qquad \text{(Rogers [37, p. 333 (4)])} \qquad (2.1.2)$$

$$1 = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^2;q^2)_{n+1}}{(-q^3;q^3)_{n+1}(q)_n} \qquad \text{(M.-S.-Z. [34, Eq. (2.3)])} \qquad (2.1.3)$$

$$2 = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(-q)_n} \qquad (2.1.4)$$

$$f(-q) = \sum_{n=0}^{\infty} (-1)^n q^{n(3n-1)/2}(1+q^n) \qquad \text{(Euler; } \mathbf{S.\ 1}) \qquad (2.1.5)$$

$$\frac{1}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2} \qquad \text{(Euler [20]; (C) with } z=q) \qquad (2.1.6)$$

$$\varphi(-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(q)_n}{(-q)_n}$$
$$\text{(Starcher [44, p. 805, (3.6)]; (H) with } a=-c=q) \qquad (2.1.7)$$

$$\frac{1}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1)_n}{(q)_n^2}$$
$$\text{(Starcher [44, p. 805, (3.7)]; (H) with } a=-1, c=q) \qquad (2.1.8)$$

$$\psi(q) = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n-1}(1-q)}{(q^2;q^2)_n(1-q^{2n-1})} \qquad \text{(Starcher [44, p. 807, (3.14)])} \qquad (2.1.9)$$

$$\frac{1}{\psi(q)} = \sum_{n=1}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2} \qquad \text{(Ramanujan [10, Entry 4.2.6])} \qquad (2.1.10)$$

$$(-q)_\infty = \sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}(1+q^{2n})(-q)_{n-1}}{(q)_n} \qquad \text{(Starcher [44, p. 809, (3.29)])} \qquad (2.1.11)$$

2.2 Mod 2 Identities

$$\frac{f(-q,-q)}{f(-q)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} \qquad \text{((E) with } z=-\sqrt{q}; \mathbf{S.\ 3}) \qquad (2.2.1)$$

$$\frac{f(-q,-q)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} \qquad \text{(Ramanujan [10, Entry 5.3.6]; } \textbf{S. 4}\text{)} \tag{2.2.2}$$

$$\frac{f(1,q^2)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-q;q^2)_n}{(q)_{2n}} \qquad \text{(H)} \tag{2.2.3}$$

$$= 2\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q)_{2n+1}} \qquad \text{(H)} \tag{2.2.4}$$

2.3 Mod 3 Identities

$$\frac{f(-q)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} \qquad \text{((E) with } z=q\text{; } \textbf{S. 2}\text{)} \tag{2.3.1}$$

$$\frac{f(-q)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)}}{(-q;q^2)_{n+1}(q^2;q^2)_n} \qquad \text{(}\textbf{S. 5}\text{)} \tag{2.3.2}$$

$$\frac{f(q)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1)_n}{(q)_n(q;q^2)_n} \qquad \text{(Ramanujan [10, Ent. 4.2.8]; } \textbf{S. 6}\text{c)} \tag{2.3.3}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q)_n}{(q)_n(q;q^2)_{n+1}} \qquad \text{(Ramanujan [10, Entry 4.2.9])} \tag{2.3.4}$$

$$\frac{f(q)}{\psi(q)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n^2}{(q^2;q^2)_{2n}} \qquad \text{(Ramanujan [10, Entry 5.3.3])} \tag{2.3.5}$$

$$\frac{f(-q,q^2)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^2)_n}{(q)_{2n}} \qquad \text{(Ramanujan [10, 4.2.10]; } \textbf{S. 48}\text{)} \tag{2.3.6}$$

2.4 Mod 4 Identities

$$\frac{f(-q,-q^3)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \qquad \text{((E) with } z=q\text{; } \textbf{S. 7}\text{)} \tag{2.4.1}$$

$$\frac{f(-q^2,-q^2)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q^2)_n}{(q^4;q^4)_n} \qquad \text{((H) with } a^2=-c=q\text{; } \textbf{S. 4-}\text{)} \tag{2.4.2}$$

$$\frac{f(-q,-q^3)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n}{(q)_n} \qquad \text{((L) with } a=-q\text{; } \textbf{S. 8}\text{)} \tag{2.4.3}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q)_{2n+1}} \qquad \text{((H) with } -aq=c=q^{3/2}\text{; } \textbf{S. 51}\text{)} \tag{2.4.4}$$

$$\frac{f(-q^2,-q^2)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1)_n}{(q)_n} \qquad \text{((L); SR[10, Ent. 1.7.14]; } \textbf{S. 12}\text{)} \tag{2.4.5}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_{n+1}}{(q)_n} \qquad \text{((L) with } a=-q^2) \tag{2.4.6}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q^2;q^2)_n}{(q)_{2n+1}} \qquad \text{((H); Ramanujan [10, Ent. 1.7.13])} \tag{2.4.7}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^2)_n}{(q)_{2n}} \qquad \text{((H) with } a=-1, c=\sqrt{q}) \tag{2.4.8}$$

$$\frac{f(q,q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n+1}} \qquad \text{(Jackson [28, p. 179, line -3]; } \textbf{S. 9}) \tag{2.4.9}$$

$$\frac{f(q,q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1)_{2n}}{(q^2;q^2)_n(q^2;q^4)_n} \qquad (\textbf{S. 10c}) \tag{2.4.10}$$

$$\frac{f(q,-q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^4)_n(-q;q^2)_n}{(q^2;q^2)_{2n}} \qquad (\textbf{S. 66}) \tag{2.4.11}$$

$$\frac{f(-q,q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-1;q^4)_n(-q;q^2)_n}{(q^2;q^2)_{2n}} \qquad (\textbf{S. 67}) \tag{2.4.12}$$

$$\frac{f(q,q^3)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q)_{2n}}{(q;q^2)_{n+1}(q^4;q^4)_n} \qquad (\textbf{S. 11}) \tag{2.4.13}$$

$$\frac{f(q,-q^3)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^4)_n}{(q)_{2n+1}(-q;q^2)_n} \qquad (\textbf{S. 65}) \tag{2.4.14}$$

$$\frac{f(-q^2,-q^2)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^2;q^2)_{n+1}}{(-q^3;q^3)_{n+1}(q)_n} \qquad \text{(M.-S.-Z. [34, Eq. (2.3)])} \tag{2.4.15}$$

2.5 Mod 5 Identities

$$H(q) = \frac{f(-q,-q^4)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} \qquad \text{(Rogers [36, p. 330 (2)]; } \textbf{S. 14}) \tag{2.5.1}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q^2)_{n+1}}{(q)_n(q;q^2)_n} \qquad \text{(M.-S.-Z. [34, Eq. (2.5)])} \tag{2.5.2}$$

$$G(q) = \frac{f(-q^2,-q^3)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} \qquad \text{(Rogers [36, p. 328 (2)]; } \textbf{S. 18}) \tag{2.5.3}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q)_{n+1}}{(q^2;q^2)_n} \qquad \text{(M.-S.-Z. [34, Eq. (2.6)])} \tag{2.5.4}$$

$$\chi(-q)H(q) = \frac{f(-q,-q^4)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n-2)}}{(-q;q^2)_n (q^4;q^4)_n}$$

(Rogers [37, p. 330 (5)]; **S. 15**) (2.5.5)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+2)}}{(-q;q^2)_{n+1} (q^4;q^4)_n}$$

(Ramanujan [9, p. 252 (11.2.7)]) (2.5.6)

$$\chi(-q)G(q) = \frac{f(-q^2,-q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(-q;q^2)_n (q^4;q^4)_n}$$

(Rogers [36, p. 339, Ex. 2]; **S. 19**) (2.5.7)

$$\chi(-q^2)H(q) = \frac{f(-q,-q^4)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4;q^4)_n}$$

(Rogers [37, p. 331, abv (7)]; **S. 16**) (2.5.8)

$$\chi(-q^2)G(q) = \frac{f(-q^2,-q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}$$

(Rogers [36, p. 330]; **S. 20**) (2.5.9)

$$\frac{f(1,q^5)}{\psi(q)} = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+2)} (q;q^2)_n}{(-q;q^2)_{n+1} (q^4;q^4)_n}$$

(M.-S.-Z. [34, Eq. (2.7)]) (2.5.10)

$$\frac{f(q,q^4)}{\psi(q)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+2)} (q;q^2)_n}{(-q;q^2)_n (q^4;q^4)_n}$$

(B.-M.-S. [16, Eq. (2.17)]) (2.5.11)

$$\frac{f(q^2,q^3)}{\psi(q)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q;q^2)_n (q^4;q^4)_n} \qquad (\textbf{S. 21}) \qquad (2.5.12)$$

$$\frac{H(q)\chi(-q)}{\chi(-q^2)} = \frac{f(-q,-q^4)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n (-q;q^2)_{n+1}} \qquad (\textbf{S. 17}) \qquad (2.5.13)$$

$$\frac{G(q)\chi(-q)}{\chi(-q^2)} = \frac{f(-q^2,-q^3)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n (-q;q^2)_n} \qquad (\textbf{S. 99-}) \qquad (2.5.14)$$

2.6 Mod 6 Identities

$$\frac{f(-q^3,-q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2} (q;q^2)_n}{(-q)_{2n} (q^2;q^2)_n}$$

(B.-M.-S. [16, Eq. (2.13)]) (2.6.1)

$$\frac{f(-q,-q^5)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q^4;q^4)_n}$$

(Ramanujan [10, Entry 4.2.11], Stanton [43]) (2.6.2)

$$\frac{f(-q^3,-q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} \qquad \text{(Ramanujan [10, Ent. 4.2.7]; } \textbf{S. 25}\text{)} \tag{2.6.3}$$

$$\frac{f(-q,-q^5)}{\psi(q)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} \qquad \text{((E) with } z=-\sqrt{q}\text{; } \textbf{S. 23}\text{)} \tag{2.6.4}$$

$$\frac{f(-q,-q^5)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q)_n}{(q;q^2)_{n+1}(q)_n}$$

(Ramanujan [10, Entry 4.2.12], Bailey [11, p. 72, Eq. (10)]; **S. 22**) (2.6.5)

$$\frac{f(-q^3,-q^3)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q)_n}{(q;q^2)_{n+1}(q)_n} \qquad (\textbf{S. 26}) \tag{2.6.6}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}(-1)_n}{(q;q^2)_n(q)_n} \qquad \text{(M.-S.-Z. [34])} \tag{2.6.7}$$

$$\frac{f(q,q^5)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(-q;q^2)_n}{(q)_{2n+1}(-q^2;q^2)_n} \qquad (\textbf{S. 27}) \tag{2.6.8}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^{-1};q^2)_n(-q^3;q^2)_n}{(q^2;q^2)_{2n}}$$

((R1) with $a=q$; Stanton [43]) (2.6.9)

$$= \sum_{n=0}^{\infty} \frac{q^{2n(n-1)}(-q;q^2)_n}{(q)_{2n}(-1;q^2)_{n+1}} \tag{2.6.10}$$

$$\frac{f(q^3,q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q;q^2)_n}{(q;q^2)_n(q^4;q^4)_n}$$

(B.-M.-S. [16, Eq. (2.13)]) (2.6.11)

$$\frac{f(q,q^5)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q)_{2n+1}}$$

(Ramanujan [9, p. 254, (11.3.5)], [10, Ent. 4.2.13]; **S. 28**) (2.6.12)

$$\frac{f(q^3,q^3)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^2)_n}{(q)_{2n}} \qquad \textbf{(S. 48-)}$$

(2.6.13)

$$\frac{f(q^2,q^4)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q)_{2n}}$$

(Ramanujan [7, p. 178 (3.1)], [10, Ent. 5.2.3]; **S. 29**) (2.6.14)

$$\frac{1}{(q,q^4,q^5;q^6)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}(q^2;q^6)_n}{(q)_{3n}}$$

(special case of (H); cf. Corteel and Savage [19]) (2.6.15)

$$\frac{1}{(q,q^2,q^5;q^6)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}(q^4;q^6)_n}{(q)_{3n+1}}$$

(special case of (H); cf. Corteel and Savage [19]) (2.6.16)

2.7 The Rogers-Selberg Mod 7 Identities

$$\frac{f(-q,-q^6)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q)_{2n+1}}$$ (Rogers [37, p. 331 (6)]; **S. 31**) (2.7.1)

$$\frac{f(-q^2,-q^5)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q)_{2n}}$$ (Rogers [36, p. 342]; **S. 32**) (2.7.2)

$$\frac{f(-q^3,-q^4)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q)_{2n}}$$ (Rogers [36, p. 339]; **S. 33**) (2.7.3)

2.8 Mod 8 Identities

2.8.1 Triple Products

$$\frac{f(-q,-q^7)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q^2;q^2)_n}$$

(Ramanujan [10, Entry 1.7.12]; **S. 34**) (2.8.1)

$$\frac{1}{(q^2,q^3,q^7;q^8)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q^2;q^2)_n}$$

(Göllnitz [25, (2.24)]; (L) with $a=-q^{1/2}$) (2.8.2)

$$\frac{1}{(q,q^5,q^6;q^8)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^{-1};q^2)_n}{(q^2;q^2)_n}$$

(Göllnitz [25, (2.22)]; (L) with $a=-q^{-1/2}$) (2.8.3)

$$\frac{f(-q^3,-q^5)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}$$

(Ramanujan [10, Ent. 1.7.11, 4.2.15]; **S. 36**) (2.8.4)

$$\frac{f(-q,-q^7)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}(-q;q^2)_n(-q)_n}{(q)_{2n+1}}$$

(Ramanujan [10, Entry 1.7.6]; **S. 35**) (2.8.5)

$$\frac{f(-q^2,-q^6)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q^2;q^2)_n}{(q;q^2)_{n+1}(q)_n}$$

(Ramanujan [10, Entry 1.7.5]) (2.8.6)

$$\frac{f(-q^3,-q^5)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q^2)_n(-q)_n}{(q)_{2n+1}}$$

(Ramanujan [10, Entry 1.7.8]; **S. 37**) (2.8.7)

$$\frac{f(-q^4,-q^4)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q^2)_n}{(q)_n(q;q^2)_n}$$

(Ramanujan [10, Entry 1.7.4]) (2.8.8)

$$\frac{f(q,q^7)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q)_{2n+1}}$$

(**S. 38**) (2.8.9)

$$\frac{f(q^3,q^5)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}}$$

(Jackson [28, p. 170, 5th Eq.]; **S. 39**) (2.8.10)

$$\frac{f(q,q^7)}{\psi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^{-1};q^4)_n(-q^5;q^4)_n}{(q^8;q^8)_n(q^2;q^4)_n}$$

((R2) with $a = q^{3/2}$) (2.8.11)

$$\frac{f(q^3,q^5)}{\psi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q;q^2)_{2n}}{(q^8;q^8)_n(q^2;q^4)_n}$$

((R2) $a = q^{\frac{1}{2}}$; Gessel-Stanton [24, (7.24)]) (2.8.12)

$$\frac{f(-q,-q^7)}{\varphi(-q^4)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(-q^4;q^4)_n(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}}$$

((AG) with $a = q^{3/4}, b = q^{5/4}$) (2.8.13)

$$\frac{f(-q^3,-q^5)}{\varphi(-q^4)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(-q^4;q^4)_n(q^{-1};q^2)_{2n}}{(q^4;q^4)_{2n}}$$

((AG) with $a = q^{-1/4}, b = q^{1/4}$) (2.8.14)

2.8.2 Quintuple products

$$\frac{Q(q^4,q)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^2)_n}{(q)_{2n}} \qquad \text{((H) with } a=-1,\ c=\sqrt{q}\text{; } \mathbf{S.\ 47}\text{)} \tag{2.8.15}$$

2.9 Mod 9 Identities

$$\frac{f(-q,-q^8)}{f(-q^3)} = \sum_{n=0}^{\infty} \frac{q^{3n(n+1)}(q)_{3n+1}}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} \tag{2.9.1}$$

(Bailey [12, p. 422, (1.7)]; **S. 40c**)

$$\frac{f(-q^2,-q^7)}{f(-q^3)} = \sum_{n=0}^{\infty} \frac{q^{3n(n+1)}(q)_{3n}(1-q^{3n+2})}{(q^3;q^3)_n(q^3;q^3)_{2n+1}} \tag{2.9.2}$$

(Bailey [12, p. 422, (1.8)]; **S. 41c**)

$$\frac{f(-q^4,-q^5)}{f(-q^3)} = \sum_{n=0}^{\infty} \frac{q^{3n^2}(q)_{3n}}{(q^3;q^3)_n(q^3;q^3)_{2n}} \tag{2.9.3}$$

((R1) $a=-q^{\frac{1}{2}}$; Bailey [12, p. 422, (1.6)]; **S. 42c**)

$$\frac{f(-q^3,-q^6)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^6)_n(-q;q^2)_n}{(-1;q^2)_n(q^2;q^2)_{2n}} \tag{2.9.4}$$

(M.-S. [33, Eq. (1.13)])

$$\frac{f(-q^3,q^6)}{\psi(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^6;q^6)_{n-1}(-q;q^2)_n}{(q^2;q^2)_{2n}(q^2;q^2)_{n-1}} \qquad (\mathbf{S.\ 113}) \tag{2.9.5}$$

2.10 Mod 10 Identities

2.10.1 Triple Products

$$\frac{f(-q,-q^9)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}(-q)_n}{(q;q^2)_{n+1}(q)_n}$$

(Rogers [37, p. 330 (4), line 2]; **S. 43**) (2.10.1)

$$\frac{f(-q^3,-q^7)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n}{(q;q^2)_{n+1}(q)_n}$$

(Rogers [37, p. 330 (4), line 1]; **S. 45**) (2.10.2)

$$\frac{f(-q^5,-q^5)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1)_n}{(q;q^2)_n(q)_n}$$

(Rogers [37, p. 330 (4), line 3, corrected]) (2.10.3)

$$\frac{H(q^2)}{\chi(-q)} = \frac{f(-q^2,-q^8)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{3n(n+1)/2}}{(q;q^2)_{n+1}(q)_n}$$

(Rogers [37, p. 330 (2), line 2]; **S. 44**) (2.10.4)

$$\frac{G(q^2)}{\chi(-q)} = \frac{f(-q^4,-q^6)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q;q^2)_n(q)_n}$$

(Rogers [36, p. 341, Ex. 1]; **S. 46**) (2.10.5)

$$= \sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}}{(q;q^2)_{n+1}(q)_n}$$

(Rogers [37, p. 330 (2), line 1]) (2.10.6)

2.10.2 Quintuple Products

$$\frac{G(q^2)}{\chi(-q)} = \frac{Q(q^5,-q)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}(-q)_n}{(q)_{2n+1}} \quad (\textbf{S. 62}) \tag{2.10.7}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}(-q)_n}{(q)_{2n}}$$

(Rogers [37, p. 332 (12), 1st Eq.]) (2.10.8)

$$\frac{H(q^2)}{\chi(-q)} = \frac{Q(q^5,-q^2)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{3n(n+1)/2}(-q)_n}{(q)_{2n+1}}$$

(Rogers [37, p. 332 (12), 2nd Eq.]; **S. 63**) (2.10.9)

2.12 Mod 12 identities

2.12.1 Triple Products

$$\frac{f(-q,-q^{11})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^2;q^2)_n(1-q^{n+1})}{(q)_{2n+2}} \quad (\textbf{S. 49c}) \tag{2.12.1}$$

$$\frac{f(-q^2,-q^{10})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q)_{2n+1}}$$

(Ramanujan [10, Ent. 3.4.4]; **S. 50**) (2.12.2)

$$\frac{f(-q^3,-q^9)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q)_{2n+1}} \quad (\textbf{S. 28}) \tag{2.12.3}$$

$$\frac{f(-q^4,-q^8)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q)_{2n+1}}$$

((H) with $aq = -c = q^{3/2}$; **S. 51**) (2.12.4)

$$\frac{f(-q^5,-q^7)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q^2;q^2)_{n-1}(1+q^n)}{(q)_{2n}} \qquad (\mathbf{S.\ 54c}) \qquad (2.12.5)$$

$$\frac{f(-q^6,-q^6)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q)_{2n}}$$

(Ramanujan [9, Ent. 11.3.1]; **S. 29**) (2.12.6)

$$\frac{f(-q,-q^{11})}{f(-q^4)} = \sum_{n=0}^{\infty} \frac{q^{4n(n+1)}(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}} \qquad (\mathbf{S.\ 55}) \qquad (2.12.7)$$

$$\frac{f(-q^5,-q^7)}{f(-q^4)} = \sum_{n=0}^{\infty} \frac{q^{4n^2}(q;q^2)_{2n}}{(q^4;q^4)_{2n}}$$

((R1) with $a=-\sqrt{q}$; **S. 53**) (2.12.8)

$$\frac{f(-q^3,-q^9)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(-q;q^2)_n}{(q;q^2)_{n+1}(q^4;q^4)_n} \qquad (\mathbf{S.\ 27}) \qquad (2.12.9)$$

$$\frac{f(-q^4,-q^8)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}(-q;q^2)_n}{(q^2;q^2)_n(q^2;q^4)_n} \qquad (\mathbf{S.\ 52}) \qquad (2.12.10)$$

$$\frac{f(-q^5,-q^7)}{\psi(-q^3)} = \sum_{n=0}^{\infty} \frac{q^{3n^2}(q^2;q^2)_{3n}}{(q^{12};q^{12})_n(q^3;q^3)_{2n}}$$

((R2) $a=-q^{1/3}$; Dyson [13, p. 9, Eq. (7.5)]) (2.12.11)

$$\frac{f(-q,-q^{11})}{\psi(-q^3)} = \sum_{n=0}^{\infty} \frac{q^{3n^2}(q^2;q^2)_{3n+1}}{(q^{12};q^{12})_n(q^3;q^3)_{2n}(q^{6n}-q^2)}$$

(Dyson [13, p. 9, Eq. (7.6)]) (2.12.12)

$$\frac{f(q,q^{11})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q)_n}{(q;q^2)_{n+1}(q)_{n+1}} \qquad (\mathbf{S.\ 56})$$

(2.12.13)

$$\frac{f(q^3,q^9)}{f(-q)} = \frac{1+q^3}{(1-q)(1-q^2)} + \sum_{n=1}^{\infty} \frac{q^{n(n+2)}(-q)_{n-1}(-q)_{n+2}}{(q)_{2n+2}} \qquad [34,\ \text{Eq. } (2.10)] \qquad (2.12.14)$$

$$\frac{f(q^5,q^7)}{f(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(-q)_{n-1}}{(q;q^2)_n(q)_n} \qquad (\mathbf{S.\ 58c}) \qquad (2.12.15)$$

$$\frac{f(q,q^{11})}{f(-q^4)} = \sum_{n=0}^{\infty} \frac{q^{4n(n+1)}(-q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}} \qquad (\mathbf{S.\ 57}) \qquad (2.12.16)$$

$$\frac{f(q^3,q^9)}{f(-q^4)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q^4;q^4)_n} \qquad (\text{Ramanujan [10, Entry 5.3.7]}) \qquad (2.12.17)$$

$$\frac{f(q^5,q^7)}{f(-q^4)} = \sum_{n=0}^{\infty} \frac{q^{4n^2}(-q;q^2)_{2n}}{(q^4;q^4)_{2n}} \qquad \text{((R1) with } a=\sqrt{q}\text{; } \textbf{S. 53-}) \tag{2.12.18}$$

2.12.2 Quintuple Products

$$\frac{Q(q^6,-q)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q^3)_n(-q)_n}{(q)_{2n}(-1)_n} \qquad \text{(M.-S. [32, Eq. (1.22)])} \tag{2.12.19}$$

$$\frac{Q(q^6,-q^2)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q^3;q^3)_n}{(q)_{2n+1}} \qquad \text{(M.-S. [32, Eq. (1.24)])} \tag{2.12.20}$$

$$\frac{Q(q^6,q)}{\varphi(-q)} = 1+\sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}(q^3;q^3)_{n-1}(-q)_n}{(q)_{2n}(q)_{n-1}} \qquad \text{(M.-S. [32, Eq. (1.27)])} \tag{2.12.21}$$

$$\frac{Q(q^6,q^2)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(q^3;q^3)_n(-q)_n}{(q)_{2n+1}(q)_n} \qquad \text{(Dyson [32, Eq. (D2)])} \tag{2.12.22}$$

2.14 Mod 14 Identities

2.14.1 Triple Products

$$\frac{f(-q^2,-q^{12})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q;q^2)_{n+1}(q)_n} \qquad \text{(Rogers [37, p. 329 (1)]; } \textbf{S. 59}) \tag{2.14.1}$$

$$\frac{f(-q^4,-q^{10})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q^2)_{n+1}(q)_n} \qquad \text{(Rogers [37, p. 329 (1)]; } \textbf{S. 60}) \tag{2.14.2}$$

$$\frac{f(-q^6,-q^8)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_n(q)_n} \qquad \text{(Rogers [36, p. 341, Ex. 2]; } \textbf{S. 61}) \tag{2.14.3}$$

2.14.2 Quintuple Products

$$\frac{Q(q^7,-q)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n}{(q)_{2n}} \qquad (\textbf{S. 81}) \tag{2.14.4}$$

$$\frac{Q(q^7,-q^2)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n}{(q)_{2n+1}} \qquad (\textbf{S. 80}) \tag{2.14.5}$$

$$\frac{Q(q^7,-q^3)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}(-q)_n}{(q)_{2n+1}} \qquad (\textbf{S. 82}) \tag{2.14.6}$$

2.15 Mod 15 Identities

$$\frac{f(-q,-q^{14})}{f(-q^5)} = 1-\sum_{n=1}^{\infty}\frac{q^{5n^2-4}(q;q^5)_{n-1}(q^4;q^5)_{n+1}}{(q^5;q^5)_{2n}} \quad \text{((R1) with } a=-q^{13/5}) \quad (2.15.1)$$

$$\frac{f(-q^2,-q^{13})}{f(-q^5)} = 1-\sum_{n=1}^{\infty}\frac{q^{5n^2-3}(q^2;q^5)_{n-1}(q^3;q^5)_{n+1}}{(q^5;q^5)_{2n}} \quad \text{((R1) with } a=-q^{11/5}) \quad (2.15.2)$$

$$\frac{f(-q^3,-q^{12})}{f(-q^5)} = 1-\sum_{n=1}^{\infty}\frac{q^{5n^2-2}(q^3;q^5)_{n-1}(q^2;q^5)_{n+1}}{(q^5;q^5)_{2n}} \quad \text{((R1) with } a=-q^{9/5}) \quad (2.15.3)$$

$$\frac{f(-q^4,-q^{11})}{f(-q^5)} = 1-\sum_{n=1}^{\infty}\frac{q^{5n^2-1}(q;q^5)_{n-1}(q;q^5)_{n+1}}{(q^5;q^5)_{2n}} \quad \text{((R1) with } a=-q^{7/5}) \quad (2.15.4)$$

$$\frac{f(-q^6,-q^{9})}{f(-q^5)} = \sum_{n=0}^{\infty}\frac{q^{5n^2}(q;q^5)_{n}(q^4;q^5)_{n}}{(q^5;q^5)_{2n}} \quad \text{((R1) with } a=-q^{3/5}) \quad (2.15.5)$$

$$\frac{f(-q^7,-q^{8})}{f(-q^5)} = \sum_{n=0}^{\infty}\frac{q^{5n^2}(q^2;q^5)_{n}(q^3;q^5)_{n}}{(q^5;q^5)_{2n}} \quad \text{((R1) with } a=-q^{1/5}) \quad (2.15.6)$$

2.16 Mod 16 Identities

2.16.1 Triple Products

$$\frac{f(-q^2,-q^{14})}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n(-q^4;q^4)_n}{(-q^2;q^2)_{n+1}(q^2;q^4)_{n+1}(q^2;q^2)_n} \quad (\textbf{S. 68}) \quad (2.16.1)$$

$$\frac{f(-q^4,-q^{12})}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q^2;q^4)_n}{(q;q^2)_{n+1}(q^4;q^4)_n} \quad (\textbf{S. 70}) \quad (2.16.2)$$

$$\frac{f(-q^6,-q^{10})}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n^2}(-q^4;q^4)_{n-1}}{(q)_{2n}(-q^2;q^2)_{n-1}} \quad (\textbf{S. 71}) \quad (2.16.3)$$

$$\frac{f(-q^8,-q^{8})}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n^2}(-q^2;q^4)_n}{(q;q^2)_n(q^4;q^4)_n}$$

(S. [39, Eq. (5.5)]; (AB) $b=ic=iq^{1/2}$) (2.16.4)

$$\frac{f(q^2,q^{14})}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n(n+2)}}{(q)_{2n+2}} \quad (\textbf{S. 69}) \quad (2.16.5)$$

$$= \sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}(-q)_n}{(q)_{n+1}} \quad \text{(Gessel-Stanton [24, Eq. (7.15)])} \quad (2.16.6)$$

$$= 1 + \frac{q}{(1-q)(1-q^2)} + \sum_{n=2}^{\infty} \frac{q^{n^2-2}(-q^2;q^2)_{n-2}(1+q^{2n+2})}{(q)_{2n}} \tag{2.16.7}$$

(M.-S.-Z. [34])

$$\frac{f(q^6,q^{10})}{\psi(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(-q)_{2n-1}}{(q^2;q^4)_n(q^2;q^2)_n} \qquad (\textbf{S. 72}) \tag{2.16.8}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}(-q)_{n-1}}{(q)_n} \qquad \text{(Gessel-Stanton [24, Eq. (7.13)])} \tag{2.16.9}$$

2.16.2 Quintuple Products

$$\frac{Q(q^8,-q)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}} \qquad (\textbf{S. 83}) \tag{2.16.10}$$

$$\frac{Q(q^8,-q^2)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n+1}} \qquad (\textbf{S. 84}) \tag{2.16.11}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q)_{2n}} \qquad \text{(Starcher [44, p. 809, Eq. (3.29)];} \textbf{ S. 85}) \tag{2.16.12}$$

$$\frac{Q(q^8,-q^3)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q)_{2n+1}} \qquad (\textbf{S. 86}) \tag{2.16.13}$$

2.18 Mod 18 Identities

2.18.1 Triple Products

$$\frac{f(-q^3,-q^{15})}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}(q^3;q^3)_n(-q)_{n+1}}{(q)_{2n+2}(q)_n}$$

$$\text{(Dyson [12, p. 434 (D1)];} \textbf{ S. 76}) \tag{2.18.1}$$

$$\frac{f(-q^6,-q^{12})}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n(q^3;q^3)_n}{(q)_n(q)_{2n+1}}$$

$$\text{(Dyson [12, p. 434 (D2)];} \textbf{ S. 77c}) \tag{2.18.2}$$

$$\frac{f(-q^9,-q^9)}{\varphi(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}(q^3;q^3)_{n-1}(-1)_{n+1}}{(q)_{2n}(q)_{n-1}}$$

$$\text{(Dyson [12, p. 434 (D3)];} \textbf{ S. 78}) \tag{2.18.3}$$

2.18.2 Quintuple Products

$$\frac{Q(q^9,-q)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^3)_n}{(-1)_n(q)_{2n}} \qquad \text{(M.-S. [32], Eq. (1.3))} \tag{2.18.4}$$

$$\frac{Q(q^9,-q^2)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^3)_n}{(-1)_n(q)_{2n}} \qquad \text{(M.-S. [32], Eq. (1.4))} \qquad (2.18.5)$$

$$\frac{Q(q^9,-q^3)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q)_n(q)_{2n+1}} \qquad \text{(M.-S. [32], Eq. (1.5))} \qquad (2.18.6)$$

$$\frac{Q(q^9,-q^4)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3;q^3)_n(1-q^{n+1})}{(-q)_n(q)_{2n+2}} \qquad \text{(M.-S. [32], Eq. (1.6))} \qquad (2.18.7)$$

$$\frac{Q(q^9,q)}{f(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}(2+q^n)}{(q)_{n-1}(q)_{2n}} \qquad \text{(M.-S. [32], Eq. (1.7))} \qquad (2.18.8)$$

$$\frac{Q(q^9,q^2)}{f(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}(1+2q^n)}{(q)_{n-1}(q)_{2n}} \qquad \text{(M.-S. [32], Eq. (1.8))} \qquad (2.18.9)$$

$$\frac{Q(q^9,q^3)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^3;q^3)_n}{(q)_n(q)_{2n+1}} \qquad \text{(Dyson [12], Eq. (B3))} \qquad (2.18.10)$$

$$\frac{Q(q^9,q^4)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^3)_n}{(q)_n^2(q^{n+2})_{n+1}} \qquad \text{(M.-S. [32], Eq. (1.10))} \qquad (2.18.11)$$

2.20 Mod 20 Identities

2.20.1 Triple Products

$$\frac{H(q^4)}{\chi(-q)} = \frac{f(-q^4,-q^{16})}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q)_{2n+1}} \qquad \text{(Rogers [37, p. 330 (3), 2nd Eq.])} \qquad (2.20.1)$$

$$\frac{G(q^4)}{\chi(-q)} = \frac{f(-q^8,-q^{12})}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_{2n}} \qquad \text{(Rogers [36, p. 330 (3), 1st Eq.]; } \textbf{S. 79}\text{)} \qquad (2.20.2)$$

2.20.2 Quintuple Products

$$\frac{G(-q)}{\chi(-q)} = \frac{Q(q^{10},-q)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_{2n}}$$

(Rogers [36, p. 332, Eq. (13)]; **S. 99**) (2.20.3)

$$\frac{G(q^4)}{\chi(-q)} = \frac{Q(q^{10},-q^2)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_{2n}}$$

(Rogers [36, p. 331 above (5)]; **S. 98**) (2.20.4)

$$\frac{H(-q)}{\chi(-q)} = \frac{Q(q^{10},-q^3)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_{2n+1}}$$

(Rogers [36, p. 331, Eq. (6)]; **S. 94**) (2.20.5)

$$\frac{H(q^4)}{\chi(-q)} = \frac{Q(q^{10},-q^4)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q)_{2n+1}}$$

(Rogers [36, p. 331, Eq. (7)]; **S. 96**) (2.20.6)

$$\frac{Q(q^{10},-q)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{3n^2}(-q;q^2)_n}{(q^2;q^2)_{2n}} \quad (\textbf{S. 100c}) \quad (2.20.7)$$

$$\frac{Q(q^{10},-q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(3n-2)}(-q;q^2)_n}{(q^2;q^2)_{2n}} \quad (\textbf{S. 95}) \quad (2.20.8)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(3n+2)}(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}} \quad (\textbf{S. 97c}) \quad (2.20.9)$$

2.24 Mod 24 Identities

$$\frac{Q(q^{12},-q)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n}$$

(M.-S. [32, Eq. (1.12)]) (2.24.1)

$$\frac{Q(q^{12},-q^2)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}}$$

(Ramanujan [10, Entry 5.3.8]) (2.24.2)

$$\frac{Q(q^{12},-q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n(-1;q^6)_n}{(-1;q^2)_n(q^2;q^2)_{2n}}$$

(M.-S. [32, Eq. (1.13)]) (2.24.3)

$$\frac{Q(q^{12},-q^4)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3;q^6)_n}{(q^2;q^2)_{2n}(1-q^{2n+1})}$$

(M.-S. [32, Eq. (1.14)]) (2.24.4)

$$\frac{Q(q^{12},-q^5)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_{n+1}(-q^6;q^6)_n(1-q^{2n+2})}{(-q^2;q^2)_n(q^2;q^2)_{2n+2}}$$

(M.-S. [32, Eq. (1.15)]) (2.24.5)

$$\frac{Q(q^{12},q)}{\psi(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}(2+q^{2n})}{(q^2;q^2)_{2n}(q^2;q^2)_{n-1}}$$

(M.-S. [32, Eq. (1.17)]) (2.24.6)

$$\frac{Q(q^{12},q^2)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(q^3;q^6)_n}{(q;q^2)_n^2(q^4;q^4)_n}$$

(Ramanujan [10, Entry 5.3.9]) (2.24.7)

$$\frac{Q(q^{12},q^3)}{\psi(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}(1+2q^{2n})}{(q^2;q^2)_{2n}(q^2;q^2)_{n-1}} \qquad \text{(M.-S. [32, Eq. (1.18)])}$$

(2.24.8)

$$\frac{Q(q^{12},q^4)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}(q;q^2)_n} \qquad (\textbf{S. 110c})$$

(2.24.9)

$$\frac{Q(q^{12},q^5)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^6;q^6)_n(-q;q^2)_{n+1}(1-q^{2n+2})}{(q^2;q^2)_{2n+2}(q^2;q^2)_n} \qquad (\textbf{S. 108c})$$

(2.24.10)

$$\frac{Q(q^{12},-q)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q)_{2n}(-q)_{2n+1}(-q;q^2)_n} \qquad \text{(M.-S. [32, Eq. (1.21)])}$$

(2.24.11)

$$\frac{Q(q^{12},-q^3)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n} \qquad \text{(M.-S. [32, Eq. (1.23)])}$$

(2.24.12)

$$\frac{Q(q^{12},-q^5)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+3)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n} \qquad \text{(M.-S. [32, Eq. (1.25)])}$$

(2.24.13)

$$\frac{Q(q^{12},q)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(q^3;q^6)_n}{(q)_{2n+1}(-q)_{2n}(q;q^2)_n} \qquad \text{(M.-S. [32, Eq. (1.26)])}$$

(2.24.14)

$$\frac{Q(q^{12},q^3)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^3;q^6)_n(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n} \qquad (\textbf{S. 107})$$

(2.24.15)

$$\frac{Q(q^{12},q^5)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+3)}(q^3;q^6)_n(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n} \qquad \text{(M.-S. [32, Eq. (1.30)])}$$

(2.24.16)

2.27 Dyson's Mod 27 Identities

$$\frac{f(-q^3,-q^{24})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+3)}(q^3;q^3)_n}{(q)_{2n+2}(q)_n} \qquad \text{(Dyson [12, p. 434 (B1)]; } \textbf{S. 90}) \qquad (2.27.1)$$

$$\frac{f(-q^6,-q^{21})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^3)_n}{(q)_{2n+2}(q)_n} \qquad \text{(Dyson [12, p. 434 (B2)]; } \textbf{S. 91}) \qquad (2.27.2)$$

$$\frac{f(-q^9,-q^{18})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^3;q^3)_n}{(q)_{2n+1}(q)_n} \qquad \text{(Dyson [12, p. 434 (B3)]; } \mathbf{S.\ 92}) \qquad (2.27.3)$$

$$\frac{f(-q^{12},-q^{15})}{f(-q)} = 1+\sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q)_{2n-1}(q)_n} \qquad \text{(Dyson [12, p. 434 (B4)]; } \mathbf{S.\ 93}) \qquad (2.27.4)$$

2.28 Mod 28 Identities

$$\frac{Q(q^{14},-q)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q^2;q^2)_{2n}} \qquad (\mathbf{S.\ 118}) \qquad (2.28.1)$$

$$\frac{Q(q^{14},-q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_{2n}} \qquad (\mathbf{S.\ 117}) \qquad (2.28.2)$$

$$\frac{Q(q^{14},-q^5)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}} \qquad (\mathbf{S.\ 119}) \qquad (2.28.3)$$

2.32 Mod 32 Identities

$$\frac{Q(q^{16},-q^2)}{f(-q)} = 1+\sum_{n=1}^{\infty} \frac{q^{n^2}(-q^2;q^2)_{n-1}}{(q)_{2n}} \qquad (\mathbf{S.\ 121}) \qquad (2.32.1)$$

$$\frac{Q(q^{16},-q^4)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q;q^2)_n}{(q)_{2n+1}} \qquad \text{((H) with } aq=-c=q^{3/2}) \qquad (2.32.2)$$

$$\frac{Q(q^{16},-q^6)}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^2;q^2)_n}{(q)_{2n+2}} \qquad (\mathbf{S.\ 123}) \qquad (2.32.3)$$

2.36 Mod 36 Identities

2.36.1 Triple Products

$$\frac{f(-q^3,-q^{33})}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+4)}(q^6;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+2}(q^2;q^2)_n}$$
$$\text{(Dyson [12, p. 434, Eq. (C1)]; } \mathbf{S.\ 116}) \qquad (2.36.1)$$

$$\frac{f(-q^9,-q^{27})}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(q^6;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+2}(q^2;q^2)_n}$$
$$\text{(Dyson [12, p. 434, Eq. (C2)]; } \mathbf{S.\ 115c}) \qquad (2.36.2)$$

$$\frac{f(-q^{12},-q^{24})}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}(q;q^2)_n}$$
$$(\mathbf{S.\ 110c}) \qquad (2.36.3)$$

$$\frac{f(-q^{15},-q^{21})}{\psi(-q)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^6;q^6)_{n-1}(-q;q^2)_n}{(q^2;q^2)_{2n-1}(q^2;q^2)_n}$$

(Dyson [12, p. 434, Eq. (C3)]; **S. 114**) (2.36.4)

2.36.2 Quintuple Products

$$\frac{Q(q^{18},q)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q^3;q^6)_n}{(q^2;q^2)_{2n}(q;q^2)_{n+1}}$$

(M.-S. [33]) (2.36.5)

$$\frac{Q(q^{18},q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}(q^3;q^6)_n}{(q^2;q^2)_{2n}(q;q^2)_n}$$

(Ramanujan [10, Ent. 5.3.4]) (2.36.6)

$$\frac{Q(q^{18},q^5)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q^3;q^6)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n}$$

(**S. 124**) (2.36.7)

$$\frac{Q(q^{18},q^7)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+2)}(q^3;q^6)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n}$$

(**S. 125**) (2.36.8)

3 q-series expansions of sums of ratios of theta functions

Slater's list contains quite a few identities between q-series and *sums* of two or more ratios of theta functions.

3.8 Mod 8 Identity

$$\frac{f(-q,-q^7)+f(-q^3,-q^5)}{\varphi(-q^4)} = \sum_{n=0}^{\infty} \frac{q^{2n(n-1)}(-q^4;q^4)_n(q;q^2)_{2n}}{(q^4;q^4)_{2n}}$$

(M.-S.-Z. [34, Eq. (2.4)]) (3.8.1)

3.10 Mod 10 Identity

$$\frac{f(-q^5,-q^5)-f(-q,-q)}{2q\varphi(-q)} = \frac{f(-q^3,-q^7)f(-q^4,-q^{16})_\infty}{\varphi(-q)f(-q^8,-q^{12})} = \sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}(-q)_n}{(q;q^2)_{n+1}(q)_{n+1}}$$

(B.-M.-S. [16, Eq. (2.22)]) (3.10.1)

3.12 Mod 12 Identities

$$\frac{f(-q^5,-q^7)+qf(-q,-q^{11})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^2)_n}{(q)_{2n}} \qquad (\textbf{S. 47}) \qquad (3.12.1)$$

$$\frac{f(-q^5,-q^7)-qf(-q,-q^{11})}{f(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^2)_n}{(q)_{2n}} \qquad (\textbf{S. 48}) \qquad (3.12.2)$$

3.15 Mod 15 Identities

$$\frac{f(q^7,q^8)-qf(q^2,q^{13})}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}(-q)_n}{(q)_{2n+1}} \qquad (\textbf{S. 62}) \qquad (3.15.1)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}(-q)_n}{(q)_{2n}} \qquad \text{(Rogers [37, p. 332])} \qquad (3.15.2)$$

$$\frac{f(q^4,q^{11})-qf(q,q^{14})}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{q^{3n(n+1)/2}(-q)_n}{(q)_{2n+1}} \text{(Rogers [37, p. 332]; } \textbf{S. 63}\text{)} \qquad (3.15.3)$$

3.16 Mod 16 Identities

$$\frac{f(q^6,q^{10})+qf(q^2,q^{14})}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q)_{2n}}{(q)_{2n+1}(q^4;q^4)_n} \qquad (\textbf{S. 64}) \qquad (3.16.1)$$

$$\frac{f(-q^6,-q^{10})+qf(-q^2,-q^{14})}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^4)_n}{(q)_{2n+1}(-q;q^2)_n} \qquad (\textbf{S. 65}) \qquad (3.16.2)$$

$$\frac{f(-q^6,-q^{10})+qf(-q^2,-q^{14})}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^4)_n(-q;q^2)_n}{(q^2;q^2)_{2n}} \qquad (\textbf{S. 66}) \qquad (3.16.3)$$

$$\frac{f(-q^6,-q^{10})-qf(-q^2,-q^{14})}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-1;q^4)_n(-q;q^2)_n}{(q^2;q^2)_{2n}} \qquad (\textbf{S. 67}) \qquad (3.16.4)$$

3.18 Mod 18 Identities

$$\frac{f(-q^6,-q^{12})+f(-q^9,-q^9)}{\varphi(-q)} = 2+\sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}(-q)_n(q^3;q^3)_{n-1}}{(q)_n(q)_{2n-1}} \qquad (\textbf{S. 73}) \qquad (3.18.1)$$

$$\frac{f(-q^6,-q^{12})+f(-q^9,-q^9)-qf(-q^3,-q^{15})}{\varphi(-q)}$$

$$= 1+\sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}(-q)_n(q^3;q^3)_{n-1}}{(q)_{n-1}(q)_{2n}} \qquad (\textbf{S. 74}) \qquad (3.18.2)$$

$$\frac{f(-q^9,-q^9)-qf(-q^3,-q^{15})}{\varphi(-q)} = 1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}(-q)_n(q^3;q^3)_{n-1}}{(q)_{n-1}(q)_{2n}} \quad (\textbf{S. 75}) \qquad (3.18.3)$$

3.21 Mod 21 Identities

$$\frac{f(q^{10},q^{11})-qf(q^4,q^{17})}{\varphi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q)_n}{(q)_{2n}} \quad (\text{Rogers [37, p. 331 (1)]; } \textbf{S. 81}) \quad (3.21.1)$$

$$\frac{f(q^8,q^{13})-q^2f(q,q^{20})}{\varphi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q)_n}{(q)_{2n+1}} \quad (\text{Rogers [37, p. 331 (1)]; } \textbf{S. 80}) \quad (3.21.2)$$

$$\frac{f(q^5,q^{16})-qf(q^2,q^{19})}{\varphi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+3)/2}(-q)_n}{(q)_{2n+1}} \quad (\text{Rogers [37, p. 331 (1)]; } \textbf{S. 82}) \quad (3.21.3)$$

3.24 Mod 24 Identities

$$\frac{f(q^{11},q^{13})-qf(q^5,q^{10})}{f(-q)} = \sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q)_{2n}} \quad (\textbf{S. 83}) \qquad (3.24.1)$$

$$\frac{f(q^{10},q^{14})-q^2f(q^2,q^{22})}{f(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(2n+1)}}{(q)_{2n+1}} \quad (\textbf{S. 84}) \qquad (3.24.2)$$

$$= \sum_{n=0}^{\infty}\frac{q^{n(2n-1)}}{(q)_{2n}} \quad (\textbf{S. 85}) \qquad (3.24.3)$$

$$\frac{f(q^7,q^{17})-q^2f(q,q^{23})}{f(-q)} = \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q)_{2n+1}} \quad (\textbf{S. 86}) \qquad (3.24.4)$$

$$\frac{f(q^8,q^{16})+qf(q^4,q^{20})}{f(-q^2)} = \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q;q^2)_n}{(q)_{2n+1}(-q^2;q^2)_n} \quad (\textbf{S. 87}) \qquad (3.24.5)$$

3.27 Mod 27 Identities

$$\frac{f(q^{15},q^{12})-qf(q^6,q^{21})}{f{-q}} = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-1;q^3)_n}{(-1)_n(q)_{2n}} \quad (\text{M.-S. [33, Eq. (1.3)]})$$

(3.27.1)

$$\frac{f(q^{12},q^{15})-q^2f(q^3,q^{24})}{f(-q)} = \sum_{n=0}^{\infty}\frac{q^{n^2}(-1;q^3)_n}{(-1)_n(q)_{2n}} \quad (\text{M.-S. [33, Eq. (1.4)]})$$

(3.27.2)

$$\frac{f(q^9,q^{18})-q^3f(1,q^{27})}{f(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q)_n(q)_{2n+1}} \quad (\text{M.-S. [33, Eq. (1.5)]})$$

(3.27.3)

$$\frac{f(q^6,q^{21})-q^4f(q^{-3},q^{30})}{f(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q^3;q^3)_n(1-q^{n+1})}{(-q)_n(q)_{2n+2}}\quad \text{(M.-S. [33, Eq. (1.6)])} \tag{3.27.4}$$

$$\frac{f(-q^6,-q^{21})-q^2f(-q^3,-q^{24})}{f(-q)}=\sum_{n=1}^{\infty}\frac{q^{n^2-1}(q^3;q^3)_{n-1}(1-q^{n+1})}{(q)_{2n}(q)_{n-1}}\quad (\textbf{S. 88c}) \tag{3.27.5}$$

$$\frac{f(-q^{12},-q^{15})-qf(-q^6,-q^{21})}{f(-q)}=1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}(q^3;q^3)_{n-1}}{(q)_{2n}(q)_{n-1}}\quad (\textbf{S. 89c}) \tag{3.27.6}$$

3.30 Mod 30 Identities

$$\frac{f(q^{17},q^{13})-qf(q^7,q^{23})}{f(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_{2n}}\quad \text{(Rogers [36, p. 333]; } \textbf{S. 99}) \tag{3.30.1}$$

$$\frac{f(q^{14},q^{16})-q^2f(q^4,q^{26})}{f(-q)}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_{2n}}\quad (\textbf{S. 98}) \tag{3.30.2}$$

$$\frac{f(q^{11},q^{19})-q^3f(q,q^{29})}{f(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q)_{2n+1}}\quad \text{(Rogers [36, p. 334]); } \textbf{S. 94}) \tag{3.30.3}$$

$$\frac{f(q^8,q^{22})-q^4f(q^{-2},q^{32})}{f(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q)_{2n+1}}\quad (\textbf{S. 96}) \tag{3.30.4}$$

$$\frac{f(q^{17},q^{13})-qf(q^7,q^{23})}{\psi(-q)}=\sum_{n=0}^{\infty}\frac{q^{3n^2}(-q;q^2)_n}{(q^2;q^2)_{2n}}\quad (\textbf{S. 100c}) \tag{3.30.5}$$

$$\frac{f(q^{11},q^{19})-q^3f(q,q^{29})}{\psi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(3n-2)}(-q;q^2)_n}{(q^2;q^2)_{2n}}\quad (\textbf{S. 95}) \tag{3.30.6}$$

$$=\sum_{n=0}^{\infty}\frac{q^{n(3n+2)}(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}}\quad (\textbf{S. 97c}) \tag{3.30.7}$$

3.32 Mod 32 Identities

$$\frac{f(q^8,q^{24})-q^3f(1,q^{32})}{\varphi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+3)/2}(-q^2;q^2)_n(-q)_{n+1}}{(q)_{2n+2}}\quad (\textbf{S. 103}) \tag{3.32.1}$$

$$\frac{f(q^{10},q^{22})-qf(q^6,q^{26})}{\varphi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+3)/2}(-q;q^2)_n(-q)_n}{(q)_{2n+1}}\quad (\textbf{S. 106}) \tag{3.32.2}$$

$$\frac{f(q^{12},q^{20})-q^2f(q^4,q^{28})}{\varphi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q^2;q^2)_n(-q)_{n+1}(1-q^{n+1})}{(q)_{2n+2}} \tag{3.32.3}$$

$$\frac{f(q^{14},q^{18})-q^3f(q^2,q^{30})}{\varphi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q;q^2)_n(-q)_n}{(q)_{2n+1}} \quad (\textbf{S. 105}) \tag{3.32.4}$$

$$\frac{f(q^{16},q^{16})-qf(q^8,q^{24})}{\varphi(-q)}=1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}(-q^2;q^2)_{n-1}(-q)_n}{(q)_{2n}} \quad (\textbf{S. 104}) \tag{3.32.5}$$

$$\frac{f(q^{16},q^{16})+f(q^{12},q^{20})-qf(q^8,q^{24})-q^2f(q^4,q^{28})}{\varphi(-q)}=2+\sum_{n=1}^{\infty}\frac{q^{n(n-1)/2}(-q^2;q^2)_{n-1}(-q)_n}{(q)_{2n}}$$
$$(\textbf{S. 101c}) \tag{3.32.6}$$

$$\frac{f(q^{12},q^{20})+qf(q^8,q^{24})-q^2f(q^4,q^{28})-q^4f(1,q^{32})}{\varphi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q^2;q^2)_n(-q)_{n+1}}{(q)_{2n+2}}$$
$$(\textbf{S. 102}) \tag{3.32.7}$$

3.36 Mod 36 Identities

$$\frac{f(-q^{15},-q^{21})+q^3f(-q^3,-q^{33})}{\varphi(-q^2)}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(q^3;q^6)_n(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n} \quad (\textbf{S. 107}) \tag{3.36.1}$$

$$\frac{f(-q^{9},-q^{27})+q^5f(-q^{-3},-q^{39})}{\varphi(-q^2)}=\sum_{n=0}^{\infty}\frac{q^{n(n+3)}(q^3;q^6)_n(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n} \quad ([33,\text{ Eq. }(1.30)]) \tag{3.36.2}$$

$$\frac{f(-q^{9},-q^{27})+q^3f(-q^{3},-q^{33})}{\psi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(q^6;q^6)_n(-q;q^2)_{n+2}}{(q^2;q^2)_{2n+2}(q^2;q^2)_n} \quad (\textbf{S. 112}) \tag{3.36.3}$$

$$\frac{f(-q^{12},-q^{24})+q^4f(-1,-q^{36})}{\psi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(q^3;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}(q;q^2)_n} \quad (\textbf{S. 110}) \tag{3.36.4}$$

$$\frac{f(-q^{15},-q^{21})-qf(-q^{9},-q^{27})}{\psi(-q)}=1+\sum_{n=1}^{\infty}\frac{q^{n(n+2)}(q^6;q^6)_{n-1}(-q;q^2)_n}{(q^2;q^2)_{2n}(q^2;q^2)_{n-1}} \quad (\textbf{S. 111}) \tag{3.36.5}$$

$$\frac{f(-q^{15},-q^{21})-q^3f(-q^{3},-q^{33})}{\psi(-q)}=1+\sum_{n=1}^{\infty}\frac{q^{n^2}(q^6;q^6)_{n-1}(-q;q^2)_n}{(q^2;q^2)_{2n}(q^2;q^2)_{n-1}} \quad (\textbf{S. 113}) \tag{3.36.6}$$

3.42 Mod 42 Identities

$$\frac{f(q^{17},q^{25})-qf(q^{11},q^{31})}{\psi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(q^2;q^2)_{2n}} \qquad (\textbf{S. 118}) \qquad (3.42.1)$$

$$\frac{f(q^{19},q^{23})-q^3f(q^{5},q^{37})}{\psi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_{2n}} \qquad (\textbf{S. 117}) \qquad (3.42.2)$$

$$\frac{f(q^{13},q^{29})-q^5f(q^{-1},q^{43})}{\psi(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}} \qquad (\textbf{S. 119}) \qquad (3.42.3)$$

3.48 Mod 48 Identities

$$\frac{f(q^{22},q^{26})-qf(q^{14},q^{34})}{f(-q)}=1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}(-q^2;q^2)_{n-1}}{(q)_{2n}} \qquad (\textbf{S. 120}) \qquad (3.48.1)$$

$$\frac{f(q^{22},q^{26})-q^2f(q^{10},q^{38})}{f(-q)}=1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q^2;q^2)_{n-1}}{(q)_{2n}} \qquad (\textbf{S. 121}) \qquad (3.48.2)$$

$$\frac{f(q^{10},q^{38})-q^3f(q^{2},q^{46})}{f(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+3)}(-q^2;q^2)_{n}}{(q)_{2n+2}} \qquad (\textbf{S. 122}) \qquad (3.48.3)$$

$$\frac{f(q^{20},q^{28})-q^4f(q^{4},q^{44})}{f(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q;q^2)_{n}}{(q)_{2n+1}}$$

$$((\mathrm{H})\text{ with } aq=-c=q^{3/2}) \qquad (3.48.4)$$

$$\frac{f(q^{34},q^{14})-q^6f(q^{-2},q^{50})}{f(-q)}=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q^2;q^2)_{n}}{(q)_{2n+2}} \qquad (\textbf{S. 123}) \qquad (3.48.5)$$

3.54 Mod 54 Identities

$$\frac{f(-q^{27},-q^{27})+q^3f(-q^{9},-q^{54})}{f(-q^2)}=\sum_{n=0}^{\infty}\frac{q^{2n^2}(q^3;q^6)_{n}}{(q^2;q^2)_{2n}(q;q^2)_n} \qquad (3.54.1)$$

$$\frac{f(-q^{33},-q^{21})+q^5f(-q^{3},-q^{51})}{f(-q^2)}=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(q^3;q^6)_{n}}{(q^2;q^2)_{2n+1}(q;q^2)_n} \qquad (\textbf{S. 124}) \qquad (3.54.2)$$

$$\frac{f(-q^{39},-q^{15})+q^7f(-q^{-3},-q^{57})}{f(-q^2)}=\sum_{n=0}^{\infty}\frac{q^{2n(n+2)}(q^3;q^6)_{n}}{(q^2;q^2)_{2n+1}(q;q^2)_n} \qquad (\textbf{S. 125c}) \qquad (3.54.3)$$

3.64 Mod 64 Identities

$$\frac{f(q^{28},q^{36})-q^3f(q^{12},q^{52})}{\psi(-q)}=1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(-q^4;q^4)_{n-1}}{(q^2;q^2)_{2n}} \qquad (\textbf{S. 126}) \qquad (3.64.1)$$

$$\frac{f(q^{28},q^{36})-qf(q^{20},q^{44})}{\psi(-q)} = 1+\sum_{n=1}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n(-q^4;q^4)_{n-1}}{(q^2;q^2)_{2n}} \quad (\textbf{S. 127}) \quad (3.64.2)$$

$$\frac{f(q^{20},q^{44})-q^5f(q^{4},q^{60})}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_{n+1}(-q^4;q^4)_{n}}{(q^2;q^2)_{2n+2}} \quad (\textbf{S. 128c}) \quad (3.64.3)$$

$$\frac{f(q^{12},q^{52})-q^3f(q^{4},q^{60})}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+4)}(-q;q^2)_{n+1}(-q^4;q^4)_{n}}{(q^2;q^2)_{2n+2}} \quad (\textbf{S. 129}) \quad (3.64.4)$$

$$\frac{f(q^{32},q^{32})+qf(q^{24},q^{40})-q^5f(q^{8},q^{56})-q^8f(1,q^{64})}{\psi(-q)}$$
$$= \sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_{n+1}(-q^2;q^4)_{n}}{(q^2;q^2)_{2n+1}} \quad (\textbf{S. 130}) \quad (3.64.5)$$

3.108 Mod 108 Identities

$$\frac{f(-q^{3},-q^{24})-2q^5f(q^{15},q^{93})+2q^{13}f(q^{-3},q^{111})}{f(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+3)}(-q^3;q^3)_n}{(-q)_n(q)_{2n+2}}$$
(B.-M.-S. [16, Eq. (3.18)]) (3.108.1)

$$\frac{f(-q^{6},-q^{21})-2q^4f(q^{21},q^{87})+2q^{11}f(q^{3},q^{105})}{f(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q^3;q^3)_n}{(-q)_n(q)_{2n+2}}$$
(B.-M.-S. [16, Eq. (3.17)]) (3.108.2)

$$\frac{f(-q^{9},-q^{18})-2q^3f(q^{27},q^{81})+2q^{9}f(q^{9},q^{99})}{f(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q)_n(q)_{2n+1}}$$
(B.-M.-S. [16, Eq. (3.16)]) (3.108.3)

$$\frac{f(-q^{12},-q^{15})-2q^2f(q^{33},q^{75})+2q^{7}f(q^{15},q^{93})}{f(-q)} = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q^3;q^3)_{n-1}}{(-q)_n(q)_{2n-1}}$$
(B.-M.-S. [16, Eq. (3.15)]) (3.108.4)

3.144 Mod 144 Identities

$$\frac{f(-q^{3},-q^{33})-2q^7f(q^{18},q^{126})+2q^{12}f(q^{6},q^{138})}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n(n+4)}(-q;q^2)_{n+1}(-q^6;q^6)_n}{(-q^2;q^2)_n(q^2;q^2)_{2n+2}}$$
(B.-M.-S. [16, Eq. (3.39)]) (3.144.1)

$$\frac{f(-q^{15},-q^{21})-2q^3f(q^{42},q^{102})+2q^{10}f(q^{18},q^{126})}{\psi(-q)} = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_{n}(-q^6;q^6)_{n-1}}{(-q^2;q^2)_n(q^2;q^2)_{2n-1}}$$
(B.-M.-S. [16, Eq. (3.38)]) (3.144.2)

4 Equivalent Products in Slater's List

There is no canonical way to represent ratios of theta functions, and it often happens that two seemingly different products are in fact equivalent. The table below summarizes such occurrences in Slater's list. If a particularly nice form of the infinite product was observed, then that form of the product is recorded in the table.

$\prod_{n\geq 1}(1-q^n)$	(S. 1)
$\prod_{n\geq 1}(1+q^n)$	(S. 2) = (S. 5-) = (S. 9) = (S. 52) = (S. 84) = (S. 85)
$\prod_{n\geq 1}(1-q^{2n-1})$	(S. 3) = (S. 23)
$\prod_{n\geq 1}(1-q^{2n-1})(1-q^{4n-2})$	(S. 4)
$\prod_{n\geq 1}(1+q^n)$	(S. 5-) = (S. 9) = (S. 52) = (S. 84) = (S. 85)
$\prod_{n\geq 1}(1+q^{3n-1})(1+q^{3n-2})(1-q^{3n})/(1-q^n)$	(S. 6)
$\prod_{n\geq 1}(1+q^{2n})$	(S. 7)
$\prod_{n\geq 1} 1/(1-q^n); n \not\equiv 0 \pmod 4$	(S. 8) = (S. 11) = =(S. 51) = (S. 64)
$\prod_{n\geq 1}(1+q^n)$	(S. 9) = (S. 5-) = (S. 52) = (S. 84) = (S. 85)
$\prod_{n\geq 1}(1-q^n)(1+q^{2n-1})$	(S. 10) = (S. 47)
$\prod_{n\geq 1} 1/(1-q^n); n \not\equiv 0 \pmod 4$	(S. 11) = (S. 8) = (S. 51) = (S. 64)
$\prod_{n\geq 1}(1+q^{2n-1})/(1-q^{2n-1})$	(S. 12)
	(S. 13) = (S. 8) + (S. 12)
$H(q) = \prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 2 \pmod 5$	(S. 14)
$H(q)/(-q)_\infty$	(S. 15)
$H(q)/(-q^2;q^2)_\infty$	(S. 16)
	(S. 17) = (S. 94-)
$G(q) = \prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 1 \pmod 5$	(S. 18)
$G(q)/(-q)_\infty$	(S. 19)
$G(q)/(-q^2;q^2)_\infty$	(S. 20)
$\prod_{n\geq 1}(1-q^{2n-1})$	(S. 23) = (S. 3)
	(S. 24) = (S. 30-)
$\prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 1, \pm 4 \pm 5 \pmod{12}$	(S. 27)= (S. 87)
$\prod_{n\geq 1} 1/(1-q^n); n \not\equiv 0, \pm 3 \pmod{12}$	(S. 28)
	(S. 30) = (S. 24-)
$\prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 3, 4 \pmod 8$	(S. 34)
	(S. 35) = (S. 106)
$\prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 1, 4 \pmod 8$	(S. 36)
	(S. 37) =(S. 105)
$\prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 1, \pm 4, \pm 6, \pm 7 \pmod{16}$	(S. 38) = (S. 86)
$\prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}$	(S. 39) = (S. 83)
$\prod_{n\geq 1} 1/(1-q^n); n \not\equiv 0, \pm 2 \pmod{10}$	(S. 44) = (S. 63)

$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 4 \pmod{10}$	(S. 46) = (S. 62)
$\prod_{n\geq 1}(1-q^n)(1+q^{2n-1})$	(S. 47) = (S.10) = (S. 54)+$q\times$(S. 49)
	(S. 48) = (S.54)$-q\times$(S. 49)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 1 \pmod{12}$	(S. 49)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 2 \pmod{12}$	(S. 50)
$\prod_{n\geq 1} 1/(1-q^n); n \not\equiv 0 \pmod{4}$	(S. 51) = (S. 8) = (S. 11) = (S. 64)
$\prod_{n\geq 1}(1+q^n)$	(S. 52) = (S. 5-) = (S. 9) = (S. 84) = (S. 85)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 5 \pmod{12}$	(S. 54)
	(S. 55) = (S. 57-)
	(S. 57) = (S. 55-)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 2 \pmod{14}$	(S. 59)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 4 \pmod{14}$	(S. 60)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 6 \pmod{14}$	(S. 61)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 4 \pmod{10}$	(S. 62) = (S. 46)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 2 \pmod{10}$	(S. 63) = (S. 44)
$\prod_{n\geq 1} 1/(1-q^n); n \not\equiv 0 \pmod{4}$	(S. 64) = (S. 8) = (S. 11) = (S. 51)
	(S. 65) = (S. 37)+$\sqrt{q}\times$(S. 35)
	(S. 66) = (S. 71)+$q\times$(S. 68)
	(S. 67) = (S. 71)$-q\times$(S. 68)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 4, \pm 6 \pm 10, 16 \pmod{32}$	(S. 69)= (S. 123)
$\prod_{n>1} 1/(1-q^n); n$ odd or $n \equiv 8 \pmod{16}$	(S. 70)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 2, \pm 12, \pm 14, 16 \pmod{32}$	(S. 72)=(S. 121)
	(S. 73)= (S. 77)+(S. 78)
	(S. 74)= (S. 77)+(S. 78) $-q\times$(S. 76)
	(S. 75) =(S. 78)$-q\times$(S. 76)
	(S. 78) = (S. 75)+$q\times$(S. 76)
$\prod_{n>1} 1/(1-q^n); n$ odd or $n \equiv \pm 4 \pmod{20}$	(S. 79)= (S. 98)
$\prod_{n>1} 1/(1-q^n); n \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}$	(S. 83) = (S. 39)
$\prod_{n\geq 1}(1+q^n)$	(S. 84) = (S. 5-) = (S. 9) = (S. 52) = (S. 85)
$\prod_{n\geq 1}(1+q^n)$	(S. 85) = (S. 5-) = (S. 9) = (S. 52) = (S. 84)

$\prod_{n>1} 1/(1-q^n); n \equiv \pm 1, \pm 4, \pm 6, \pm 7 \pmod{16}$	(S. 86) = (S. 38)
$\prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 1, \pm 4 \pm 5 \pmod{12}$	(S. 87) = (S. 27)
	(S. 88)= (S. 91)$-q^2\times$(S. 90)
	(S. 89)= (S. 93)$-q\times$(S. 91)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 3 \pmod{27}$	(S. 90)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 6 \pmod{27}$	(S. 91)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0 \pmod{9}$	(S. 92)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 12 \pmod{27}$	(S. 93)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, 10, \pm 3, \pm 4, \pm 7 \pmod{20}$	(S. 94) = (S. 17-)
	(S. 95)= (S. 97)
$\prod_{n>1} 1/(1-q^n); n$ odd or $n \equiv \pm 8 \pmod{20}$	(S. 96)
	(S. 97) = (S. 95)
$\prod_{n>1} 1/(1-q^n); n$ odd or $n \equiv \pm 4 \pmod{20}$	(S. 98) = (S. 79)
$\prod_{n>1} 1/(1-q^n); n \equiv \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7 \pmod{20}$	(S. 99)
	(S. 105) = (S. 37)
	(S. 106) = (S. 35)
	(S. 108) = (S. 115) $-q^2\times$(S. 116)
$\prod_{n\geq 1} 1/(1-q^n); n \not\equiv 0, 2, 6, 10 \pmod{12}$	(S. 110c)
	(S. 112) = (S. 115)$+q^3\times$(S. 116)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 2, \pm 6, \pm 10, \pm 14, \pm 15, 18 \pmod{36}$	(S. 114)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 2, \pm 6, \pm 9, \pm 10, \pm 14, 18 \pmod{36}$	(S. 115)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 2, \pm 3, \pm 6, \pm 10, \pm 14, 18 \pmod{36}$	(S. 116)
$\prod_{n>1} 1/(1-q^n); n \not\equiv 0, \pm 2, \pm 12, \pm 14, 16 \pmod{32}$	(S. 121) = (S. 72)
$\prod_{n\geq 1} 1/(1-q^n); n \not\equiv 0, \pm 4, \pm 6 \pm 10, 16 \pmod{32}$	(S. 123) = (S. 69)
$\prod_{n>1} 1/(1-q^n); n \equiv \pm 2, \pm 4, \pm 5, \pm 6 \pmod{18}$	(S. 124)
$\prod_{n\geq 1} 1/(1-q^n); n \equiv \pm 2, \pm 6, \pm 7, \pm 8 \pmod{18}$	(S. 125)
	(S. 127) = (S. 71) $-q\times$(S. 128)
	(S. 129) = q^{-2} $\times\Big($(S. 128)$-$(S. 68)$\Big)$

Table 1: Products and cross-references in Slater's list

5 False Theta Series Identities

Noting that Ramanujan's theta series

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} + \sum_{n=1}^{\infty} a^{n(n-1)/2} b^{n(n+1)/2},$$

let us define the corresponding *false theta series* as

$$\Psi(a,b) := \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} - \sum_{n=1}^{\infty} a^{n(n-1)/2} b^{n(n+1)/2}.$$

Notice that although $f(a,b) = f(b,a)$, in general, $\Psi(a,b) \neq \Psi(b,a)$.

L.J. Rogers [37] studied q-series expansions for many instances of $f(\pm q^{\alpha}, \pm q^{\beta})$ and $\Psi(\pm q^{\alpha}, \pm q^{\beta})$, which he called *theta (resp. false theta) series of order* $(\alpha+\beta)/2$.

A false theta series identity for the series $\Psi(\pm q^{\alpha}, \pm q^{\beta})$ arises from the same Bailey pair as the Rogers-Ramanujan type identity with product $f(\pm q^{\alpha}, \pm q^{\beta})/\varphi(-q)$. A designation of the form (**F.** n) means that the identity is the false theta analog of (**S.** n), the nth identity in Slater's list [41]. (Slater did not record any false theta function identities.)

5.0 False Theta Series Identities of Order $\frac{3}{2}$

$$\Psi(q^2,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} \qquad \text{(Rogers [37, p. 333, (5)]; \textbf{F. 2})} \qquad (5.0.1)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q)_{2n+1}} \qquad \text{(Ramanujan [9, p. 233, Entry 9.4.3]; \textbf{F. 5})} \qquad (5.0.2)$$

$$= 2 - \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(-q)_{2n}} \qquad \text{(Ramanujan [9, p. 233, Entry 9.4.4])} \qquad (5.0.3)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(-q^2;q^2)_n} \qquad \text{(Ramanujan [9, p. 235, Entry 9.4.7])} \qquad (5.0.4)$$

5.1 False Theta Series Identity of Order 2

$$\Psi(-q^3,-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (-q;q^2)_n}{(q;q^2)_{n+1}(-q^2;q^2)_n} \qquad \text{(M.-S.-Z. [34, Eq. (2.12)]; \textbf{F. 11})} \qquad (5.1.1)$$

5.2 False Theta Series Identities of Order $\frac{5}{2}$

$$\Psi(q^4,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(-q)_{2n+1}} \qquad \text{(Rogers [37, p. 334 (7)]; \textbf{F. 17})} \qquad (5.2.1)$$

$$\Psi(q^3,q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(-q)_{2n}} \qquad \text{(Rogers [37, p. 334 (7)])} \qquad (5.2.2)$$

5.3 False Theta Series Identities of Order 3

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_{n+1}} \qquad \text{(Rogers [37, p. 333 (4)]; }\mathbf{F.\ 26}\text{)} \tag{5.3.1}$$

$$\Psi(-q^5,-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q;q^2)_{n+1}} \qquad \text{(Rogers [37, p. 333 (4)]; }\mathbf{F.\ 22}\text{; }\mathbf{F.\ 28}\text{)} \tag{5.3.2}$$

$$\Psi(q^5,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}(q)_{3n+1}}{(q^3;q^3)_{2n+1}} \qquad \text{(Dyson [13, p. 9, Eq. (7.8)])} \tag{5.3.3}$$

$$\Psi(q^4,q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}(q)_{3n}(1-q^{3n+2})}{(q^3;q^3)_{2n+1}} \qquad \text{(Dyson [13, p. 9, Eq. (7.9)])} \tag{5.3.4}$$

5.4 False Theta Series Identities of Order 4

$$\Psi(-q^2,-q^6) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(-q)_n}{(q;q^2)_{n+1}}$$
$$\text{(Rogers [37, p. 333 (5)])} \tag{5.4.1}$$

$$\Psi(-q^6,-q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}}$$
$$\text{(Rogers [37, p. 333 (5)])} \tag{5.4.2}$$

$$\Psi(-q^5,-q^3) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(q)_n(-q;q^2)_n}{(q;q)_{2n+1}}$$
$$\text{(Ramanujan [9, p. 257, Eq. (11.5.3)]; }\mathbf{F.\ 37}\text{)} \tag{5.4.3}$$

$$\Psi(-q^7,-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}(q)_n(-q;q^2)_n}{(q;q)_{2n+1}}$$
$$\text{(Ramanujan [9, p. 257, Eq. (11.5.4)]; }\mathbf{F.\ 35}\text{)} \tag{5.4.4}$$

$$\Psi(q^7,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n+1)}(q^4;q^4)_n(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}}$$
$$\text{(Ramanujan [9, p. 257, Eq. (11.5.5)])} \tag{5.4.5}$$

5.5 False Theta Series Identities of Order 5

$$\Psi(-q^7,-q^3) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q^2)_{n+1}} \qquad \text{(Rogers [37, p. 333 (3)]; }\mathbf{F.\ 45}\text{)} \tag{5.5.1}$$

$$\Psi(-q^9,-q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}}{(q;q^2)_{n+1}} \qquad \text{(Rogers [37, p. 333 (3)]; }\mathbf{F.\ 43}\text{)} \tag{5.5.2}$$

5.8 False Theta Series Identities of Order $\frac{15}{2}$

$$\Psi(q^8,q^7) - q\Psi(q^2,q^{13}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n-1)/2}(q)_n}{(q)_{2n}} \quad \text{(Rogers [37, p. 333 (2)])} \tag{5.8.1}$$

$$\Psi(q^7,q^8) + q\Psi(q^2,q^{13}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q)_n}{(q)_{2n+1}} \quad \text{(Rogers [37, p. 333 (2)]; \textbf{F. 62})} \tag{5.8.2}$$

$$\Psi(q^4,q^{11}) + q\Psi(q,q^{14}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}(q)_n}{(q)_{2n+1}} \quad \text{(Rogers [37, p. 333 (2)])} \tag{5.8.3}$$

5.9 False Theta Series Identities of Order 9

$$\Psi(q^{15},-q^3) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}(-q^3;q^3)_n}{(q;q^2)_{n+1}(-q)_n(-q)_{n+1}} \quad \text{(M.-S. [32, Eq. (1.32)])} \tag{5.9.1}$$

$$\Psi(q^{12},-q^6) + q^2\Psi(q^{18},-1) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(-q^3;q^3)_n}{(q;q^2)_{n+1}(-q)_n^2} \quad \text{(M.-S. [32, Eq. (1.34)])} \tag{5.9.2}$$

$$\Psi(q^{15},q^3) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}(q^3;q^3)_n}{(1+q^{n+1})(q)_{2n+1}} \quad \text{(Dyson [12, J6; p. 434, Eq. (E1)]; \textbf{F. 76})} \tag{5.9.3}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}(q^3;q^3)_n(1-q^{n+1})}{(q)_{2n+2}} \quad \text{(M.-S.-Z. [34, Eq. (2.13)]; \textbf{F. 75})} \tag{5.9.4}$$

$$\Psi(q^{12},q^6) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(q^3;q^3)_n}{(q)_{2n+1}} \quad \text{(Dyson [12, p. 434, Eq. (E2)]; \textbf{F. 77})} \tag{5.9.5}$$

5.11 False Theta Series Identities of Order $\frac{21}{2}$

$$\Psi(q^8,q^{13}) + q^2\Psi(q,q^{20}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(q)_n}{(q)_{2n+1}} \quad \text{(Rogers [37, p. 332 (1)]; \textbf{F. 80})} \tag{5.11.1}$$

$$\Psi(q^{10},q^{11}) - q\Psi(q^4,q^{17}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (q)_n}{(q)_{2n}} \quad \text{(Rogers [37, p. 332 (1)]; \textbf{F. 81})} \tag{5.11.2}$$

$$\Psi(q^{5},q^{16}) + q\Psi(q^2,q^{19}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2} (q)_n}{(q)_{2n+1}} \quad \text{(Rogers [37, p. 332 (1)]; \textbf{F. 82})} \tag{5.11.3}$$

5.16 False Theta Series Identities of Order 16

$$\Psi(-q^8,-q^{24}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2} (q)_{n+1} (-q^2;q^2)_n}{(q)_{2n+2}}$$
$$\text{(M.-S.-Z. [34, Eq. (2.14)]; \textbf{F. 103})} \tag{5.16.1}$$

$$\Psi(q^{22},q^{10}) + q\Psi(q^{26},q^6) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2} (q)_n (-q;q^2)_n}{(q)_{2n+1}}$$
$$\text{(M.-S.-Z. [34, Eq. (2.15)]; \textbf{F. 106})} \tag{5.16.2}$$

5.18 False Theta Series Identities of Order 18

$$\Psi(q^{21},-q^{15}) - q\Psi(q^{27},-q^9) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (-q^3;q^6)_n}{(q^2;q^4)_n (-q;q)_{2n+1}}$$
$$\text{(M.-S. [32, Eq. (1.31)])} \tag{5.18.1}$$

$$\Psi(q^{21},-q^{15}) + q^3\Psi(q^{33},-q^3) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (-q^3;q^6)_n}{(q^2;q^4)_{n+1} (-q;q)_{2n}}$$
$$\text{(M.-S. [32, Eq. (1.33)])} \tag{5.18.2}$$

$$\Psi(q^{27},-q^{9}) + q^2\Psi(q^{33},-q^3) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)} (-q^3;q^6)_n}{(q^2;q^4)_{n+1} (-q;q)_{2n}}$$
$$\text{(M.-S. [32, Eq. (1.35)])} \tag{5.18.3}$$

$$\Psi(q^{21},q^{15}) + q\Psi(q^{27},q^9) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (q^3;q^6)_n}{(q^2;q^4)_n (-q^2;q^2)_n (q;q^2)_{n+1}}$$
$$\text{(M.-S. [32, Eq. (1.36)])} \tag{5.18.4}$$

$$\Psi(q^{21},q^{15}) - q^3\Psi(q^{33},q^3) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (q^3;q^6)_n}{(q^{2n+2};q^2)_{n+1} (q;q^2)_n}$$
$$\text{(M.-S. [32, Eq. (1.38)]; \textbf{F. 107})} \tag{5.18.5}$$

$$\Psi(q^{27},q^{9}) - q^5\Psi(q^{39},q^{-3}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)} (q^3;q^6)_n}{(q^{2n+2};q^2)_{n+1} (q;q^2)_n}$$

(M.-S. [32, Eq. (1.40)]) (5.18.6)

6 Inter-Dependence Between Identities

A number of pairs of identities on Slater's list are easily seen to be equivalent, either by simply replacing q by $-q$ (for example, **S24** and **S30**), replacing q by q^2 (for example, **S2** and **S7**), specializing a free parameter in a general series = product identity (for example, **S8**) or rearranging the finite q-products on the series side (for example, **S10** and **S47**).

After eliminating one of each such pair of identities from the list of identities, a natural question is: how independent from each other are the identities in the remaining set? In this section we describe a number of non-trivial ways in which pairs, or larger sets, of identities are dependent.

6.1 Series-Equivalent Identities

Suppose we have an identity of the form

$$\text{Infinite Series}_1 = \text{Infinite Product} \times \text{Infinite Series}_2, \tag{6.1.1}$$

where each side contains one or more free parameters. It is immediately clear that if, for particular values of the parameters, either series has a representation as an infinite product, then so does the other. We say two identities of Rogers-Ramanujan type are *series-equivalent*, if one can be derived from the other by specializing the free parameters in an identity of the type at (6.1.1).

We next list some general series transformations. Most can be derived as limiting cases of transformations between basic hypergeometric series. Let a, b, c, d, γ and $q \in \mathbb{C}$, $|q| < 1$. Then

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n q^{n(n-1)/2}\,(-c\gamma/ab)^n}{(c,\gamma,q;q)_n} = \frac{(c\gamma/ab;q)_\infty}{(\gamma;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/a,c/b;q)_n q^{n(n-1)/2}(-\gamma)^n}{(c,c\gamma/ab,q;q)_n}. \tag{6.1.2}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(b;q)_n(q;q)_n} = \frac{(-\gamma;q)_\infty}{(b;q)_\infty} \sum_{n=0}^{\infty} \frac{(-a\gamma/b;q)_n q^{n(n-1)/2}(-b)^n}{(-\gamma;q)_n(q;q)_n}. \tag{6.1.3}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(q;q)_n} = (-\gamma;q)_\infty \sum_{n=0}^{\infty} \frac{(-a\gamma)^n q^{n(n-1)}}{(-\gamma;q)_n(q;q)_n}. \tag{6.1.4}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(-a\gamma;q)_n(q;q)_n} = \frac{(-\gamma;q)_\infty}{(-a\gamma;q)_\infty}. \tag{6.1.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n(n-1)/2}\gamma^n}{(\gamma;q^2)_n(q;q)_n} = \frac{1}{(\gamma;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2-n}\gamma^n}{(q^2;q^2)_n}. \tag{6.1.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-\gamma)^n}{(\gamma q;q^2)_n(q^2;q^2)_n} = \frac{1}{(\gamma q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-\gamma)^n}{(q;q)_n}. \tag{6.1.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-\gamma)^n}{(\gamma/q;q^2)_n(q^2;q^2)_n} = \frac{1}{(\gamma/q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-2n}(-\gamma)^n}{(q;q)_n}. \tag{6.1.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\gamma^n}{(\gamma;q)_n(q;q)_n} = \frac{1}{(\gamma;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2-n}\gamma^{2n}}{(q^2;q^2)_n}. \tag{6.1.9}$$

$$\sum_{n=0}^{\infty} \frac{(b;q^2)_n q^{n(n-1)/2}(-\gamma)^n}{(b;q)_n(q;q)_n} = (\gamma;q)_\infty \sum_{n=0}^{\infty} \frac{q^{4n^2-2n}(b\gamma^2)^n}{(q^2;q^2)_n(bq;q^2)_n(\gamma;q)_{2n}}. \tag{6.1.10}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(-\gamma)^n}{(b;q)_n(q;q)_n} = (\gamma;q)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2-3n)/2}(-b\gamma)^n}{(q;q)_n(b;q)_n(\gamma;q)_n}. \tag{6.1.11}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}\gamma^n}{(q/b;q)_n(q;q)_n} = (-\gamma q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}\gamma^n(-q/b;q)_{2n}}{(q^2;q^2)_n(q^2/b^2;q^2)_n(-\gamma q^2;q^2)_n}. \tag{6.1.12}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n^2+n}(b\gamma/a)^n}{(-bq;q)_n(-\gamma q;q)_n(q;q)_n} = \frac{1}{(-\gamma q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-bq/a;q)_n q^{n(n+1)/2}\gamma^n}{(-bq;q)_n(q;q)_n}. \tag{6.1.13}$$

If n is a positive integer, then

$$(-bq^n;q^n)_\infty \sum_{m=0}^{\infty} \frac{q^{(m^2+m)/2}a^m}{(-bq^n;q^n)_m(q;q)_m} = (-aq;q)_\infty \sum_{m=0}^{\infty} \frac{q^{n(m^2+m)/2}b^m}{(-aq;q)_{nm}(q^n;q^n)_m}. \tag{6.1.14}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}a^n}{(q^2;q^2)_n(1+aq^{2n+1})} = (-aq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}(-a)^n}{(-aq;q)_n}. \tag{6.1.15}$$

$$\sum_{n=0}^{\infty} \frac{(b/a;q)_n q^{(n^2+n)/2}a^n}{(q;q)_n(aq;q)_n} = \frac{(bq;q^2)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(a^2q/b;q^2)_n q^{n^2+n}(-b)^n}{(q^2;q^2)_n(bq;q^2)_n}. \tag{6.1.16}$$

$$\sum_{n=0}^{\infty} \frac{(d;q)_{2n} q^{n^2-n}(-c^2/d^2)^n}{(q^2;q^2)_n(c;q)_{2n}} = \frac{(c^2/d^2;q^2)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-c)^n}{(q;q)_n(-c/d;q)_n}. \tag{6.1.17}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n^2-2n}(-a^2)^n}{(q^2;q^2)_n(a;q)_{2n}} = \frac{1}{(a;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-a)^n}{(q;q)_n}. \tag{6.1.18}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n^2-n}(-b)^n}{(q;q)_n(ab;q^2)_n} = \frac{(b;q^2)_\infty}{(ab;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n q^{n^2-n}(-bq)^n}{(q^2;q^2)_n(b;q^2)_n}. \tag{6.1.19}$$

$$\sum_{n=0}^{\infty} \frac{(a^2,b;q)_n q^{n^2+n}(-a^2/b)^n}{(q;q)_n(a^2q/b;q)_n} = \frac{(a^2q;q)_\infty}{(-aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq/b,-a;q)_n q^{(n^2-n)/2}(aq)^n}{(a^2q/b,aq,q;q)_n}. \tag{6.1.20}$$

$$\sum_{n=0}^{\infty}\frac{(a^2;q)_n q^{(3n^2+n)/2}a^{2n}}{(q;q)_n}=\frac{(a^2q;q)_\infty}{(-aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-a;q)_n q^{(n^2-n)/2}(aq)^n}{(aq,q;q)_n}. \tag{6.1.21}$$

Some of the identities above are derived from other identities above, as limiting cases. However, we list them explicitly to have the available for what follows below. Several follow from identities in Andrews' two papers [1], [2] and some also follow from identities in Ramanujan's notebooks.

The transformation at (6.1.2) is a limiting case of a q-analogue of the Kummer-Thomae-Whipple formula (see [22], page 72, equation 3.2.7), which in turn is a limiting case of Sear's ${}_4\phi_3$ transformation formula, [40].

The identity at (6.1.3) is found in Ramanujan's lost notebook [35] and a proof can be found in the recent book by Andrews and Berndt [9]. Equation 6.1.4 follows from 6.1.3 upon letting $b \to 0$. The identity at (6.1.5), which follows upon setting $b = -a\gamma$ in (6.1.3), is also found in Ramanujan's notebooks (see [15], Chapter 27, Entry 1, page 262). This identity is also equivalent to a result found in Andrews [3], where Andrews attributes it to Cauchy.

Proofs of (6.1.6), (6.1.7), (6.1.8) and (6.1.9) can be found in [24], and alternative proofs can be found in [16]. Identities (6.1.7), (6.1.8) and (6.1.9) were also stated by Ramanujan in the lost notebook (see Entries **1.5.1** and **1.5.2** in [10]). Proofs of (6.1.10), (6.1.11) and (6.1.12) also are to be found in [24]. Transformation 6.1.11 was also stated by Ramanujan (see Entry **2.24** of [10]).

The transformation at (6.1.13) is a limiting case of Jackson's transformation, [27] (see also [22], page 14).

A limiting case of a transformation due to Andrews [1] leads to the identity at (6.1.14), which was also given by Ramanujan in the lost notebook (see [10], Entry **1.4.12**).

Identity 6.1.15 can be found in Ramanujan's lost notebook, and a proof is given in [10], Entry **1.6.5**. Likewise, (6.1.16) is also from Ramanujan's lost notebook (see [10], Entry **1.7.3**).

The transformation at (6.1.17) follows from a series transformation relating two ${}_8\phi_7$'s in [22] ((3.5.4) on pages 77–78, after replacing c with aq/c, then letting $a \to 0$ and finally letting $b \to \infty$). The identity at (6.1.18) follows from that at (6.1.17), upon letting $d \to \infty$ and then replacing c with a.

The identity at (6.1.19) follows from a result of Andrews in [2] (see also Corollary 1.2.3 of [10], where it follows after replacing t by t/b, then letting $b \to \infty$ and finally replacing t by b).

A special case of Watson's transformation [45] of a terminating very-well-poised ${}_8\phi_7$ yields (6.1.20) (see also [22], page 43, where the transformation follows upon letting $n \to \infty$, replacing a by a^2, setting $c = a$, $d = -a$ and finally letting $e \to \infty$). The transformation at (6.1.21) follows from that at (6.1.20), after letting $b \to \infty$.

Several identities on Slater's list follow directly from some of the transformations above, in that particular values of the parameters make one of the series identically equal to 1, so that what remains is a "series = product" identity. We list such identities, along with the transformations from which they derive in Table 2. Series-equivalent identities are listed in Table 3.

Note that the case $a = q$ in (6.1.5) implies that, for $|q| < 1$ and $\gamma \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\gamma^n}{(-\gamma, q)_{n+1}} = 1.$$

The tables are to be understood as follows: let the parameters have the specified values in the specified transformations (listed in the first column), and then, if applicable, make the indicated base changes in either both sides of the transformation or else recognize that one of the series equals the series in the corresponding identity, after the indicated base change.

For example, from row one of Table 3, if the indicated substitutions are made in Transformation 6.1.13, we get that

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2+n}(1/q)^n}{(-q^{1/2};q)_n (q^{1/2};q)_n (q;q)_n} = \frac{1}{(q^{1/2};q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{1/2};q)_n q^{n(n+1)/2}(-q^{-1/2})^n}{(-q^{1/2};q)_n (q;q)_n}.$$

From **S.6** on Slater's list, the left side equals $(-q,-q^2,q^3;q)_\infty/(q;q)_\infty$. Replace q by q^2 and we have that

$$\frac{(-q^2,-q^4;q^6;q^6)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{n^2}(-1)^n}{(-q;q^2)_n (q^2;q^2)_n}.$$

Finally, replace q by $-q$ and we have Identity **S.29** from Slater's list.

As a second example, from row 5 of the table, if the indicated value $\gamma = q^{3/2}$ is substituted in Transformation 6.1.9, we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)/2}}{(q^{3/2};q)_n (q;q)_n} = \frac{1}{(q^{3/2};q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n}.$$

The series on the right is the series in Identity **S.14** (with q replaced by q^2). Thus it follows that

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)/2}}{(q^{3/2};q)_n (q;q)_n} = \frac{1}{(q^{3/2};q)_\infty (q^4,q^6;q^{10})_\infty}.$$

Replacing q by q^2 and dividing both sides by $1-q$ leads to Identity **S.96**.

The transformations also imply some identities which we believe to be new. For example, if we set $c = -q^2$, $d = q^{1/2}$ and then replace q by q^2 in (6.1.17), the series on the right side becomes that in **S38** (up to a multiple of $1-q$) and leads to the following identity:

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n} q^{2n^2+4n}(-1)^n}{(q^8;q^8)_n (-q^2;q^4)_{n+1}} = (-q^9,-q^7,q^8;q^8)_\infty \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{6.1.22}$$

A similar pairing of **S39** and (6.1.17) leads to the following:

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n} q^{2n^2}(-1)^n}{(q^8;q^8)_n (-q^2;q^4)_n} = (-q^3,-q^5,q^8;q^8)_\infty \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{6.1.23}$$

These identities are somewhat reminiscent of the following identities of Gessel and Stanton [24]:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_{2n} q^{2n^2}}{(q^8;q^8)_n (q^2;q^4)_n} = (-q^3,-q^5,q^8;q^8)_\infty \frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{6.1.24}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n+1} q^{2n^2+2n}}{(q^2;q^2)_{2n+1}(-q^2;q^4)_{n+1}} = \frac{(-q,-q^7,q^8;q^8)_\infty(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{6.1.25}$$

Transform.	Identity	a	b	γ	base change
6.1.4	**S.2**	0		q	
6.1.4	**S.3**	1		q	
6.1.3	**S.4**	$-q^{1/2}$	$-q$	$-q^{1/2}$	$q \to q^2$
6.1.4	**S.9**	1		$-q^3$	$q \to q^2$
6.1.3	**S.10**	-1	q	q	$q \to q^2$
6.1.3	**S.11**	$-q$	q^3	q^2	$q \to q^2$
6.1.4	**S.52**	1		$-q^{1/2}$	$q \to q^2$
6.1.4	**S.47**	-1		$q^{1/2}$	$q \to q^2$

Table 2: Identities deriving directly from general transformations.

6.2 Inter-dependence of Identities via the Jacobi Triple Product Identity

Let $m \geq 2$ be a positive integer. By considering sums in the m arithmetic progressions $mk+r$, $0 \leq r < m$ and $k \in \mathbb{Z}$, we easily get that

$$\sum_{k=-\infty}^{\infty} x^k q^{k(k-1)/2} = \sum_{r=0}^{m-1} q^{r(r-1)/2} x^r \sum_{k=-\infty}^{\infty} \left(x^m q^{(m^2-m+2mr)/2}\right)^k q^{m^2(k^2-k)/2},$$

and thus, after applying the Jacobi triple product, that

$$(-x,-q/x,q;q)_\infty = \sum_{r=0}^{m-1} q^{r(r-1)/2} x^r \left(-x^m q^{(m^2-m+2mr)/2}, -x^{-m} q^{(m^2+m-2mr)/2}, q^{m^2}; q^{m^2}\right)_\infty. \tag{6.2.1}$$

The case $m=2$ is of course Bailey's expression

$$(-x,-q/x,q;q)_\infty = \left(-x^2 q, -x^{-2} q^3, q^4; q^4\right)_\infty + x\left(-x^2 q^3, -x^{-2} q, q^4; q^4\right)_\infty.$$

If we replace q by q^3 and set $x=-q$ in (6.2.1), then

$$(q:q)_\infty = (-q^5,-q^7,q^{12};q^{12})_\infty - q(-q^{11},-q,q^{12};q^{12})_\infty, \qquad (m=2) \tag{6.2.2}$$

Transf.	Left	Right	a	b	γ	base change
6.1.13	**S6**	**S29**	-1	$q^{-1/2}$	$-q^{-1/2}$	$q \to q^2$,$q \to -q$
6.1.6	**S44**	**S14**			q^3	
6.1.7	**S17**	**S14**			$-q^2$	
6.1.8	**S16**	**S14**			$-q^3$	
6.1.11	**S16**	**S97**		$-q$	$q^{3/2}$	$q \to q^2$,$q \to -q$
6.1.9	**S96**	**S14**			$q^{3/2}$	$(q \to q^2)$ $q \to q^2$
6.1.6	**S46**	**S18**			q	
6.1.7	**S20**	**S18**			$-q$	
6.1.8	**S99**	**S18**			$-q^2$	
6.1.9	**S79**	**S18**			$q^{1/2}$	$(q \to q^2)$ $q \to q^2$
6.1.11	**S20**	**S19**		$-q$	$q^{-1/2}$	$q \to q^2$
6.1.13	**S22**	**S50**	$-q$	$q^{-1/2}$	$q^{1/2}$	$q \to q^2$
6.1.3	**S25**	**S48**	$-q^{1/2}$	$-q$	$q^{1/2}$	$q \to q^2$
6.1.13	**S27**	**S28**	$-q^{1/2}$	$-q^{1/2}$	1	$q \to q^2$
6.1.3	**S28**	(2.6.2)	$-q$	$q^{3/2}$	q	$q \to q^2$
6.1.16	**S29**	**S48**	$q^{-1/2}$	-1		$q \to q^{1/2}$
6.1.16	**S50**	(2.6.2)	$q^{-1/2}$	$-q$		$q \to q^{1/2}$
6.1.4	**S34**	**S38**	$-q^{1/2}$		$q^{3/2}$	$q \to q^2$
6.1.4	**S36**	**S39**	$-q^{1/2}$		$q^{1/2}$	$q \to q^2$
6.1.12	**S38**	**6.1.25**		$q^{-1/2}$	q	$q \to q^2$
6.1.12	**S39**	**6.1.24**		$q^{1/2}$	1	$q \to q^2$
6.1.11	**S44**	(2.20.1)		$q^{3/2}$	$-q^{3/2}$	$q \to q^2$
6.1.11	**S46**	**S79**		$q^{1/2}$	$-q^{1/2}$	$q \to q^2$
6.1.3	**S94**	**S16**	0	$q^{3/2}$	q	$q \to q^2$ $(q \to -q)$
6.1.11	**S94**	**S97**		$q^{3/2}$	$-q$	$q \to q^2$
6.1.11	**S96**	**S44**		$q^{3/2}$	$-q^{3/2}$	$q \to q^2$
6.1.3	**S99**	**S20**	0	$q^{1/2}$	q	$q \to q^2$ $(q \to -q)$
6.1.11	**S99**	**S100**		$q^{1/2}$	$-q$	$q \to q^2$
6.1.14	**S80**	**S118**	1	$-q$		$(n = 2)$
6.1.14	**S81**	**S117**	1	$-1/q$		$(n = 2)$
6.1.14	**S82**	**S119**	q	$-q$		$(n = 2)$
6.1.18	**2.5.3**	**S14**	$-q^2$			
6.1.18	**S19**	**S18**	$-q^2$			
6.1.19	**S6**	**S48**	-1	q		$q \to -q$
6.1.19	**S22**	(2.6.2)	$-q$	$-q^2$		
6.1.21	**1.3.3** $(a = q)$	**S50**	$q^{1/2}$			$q \to q^2$

Table 3: Identities equivalent to each other via some general transformation

$$
\begin{aligned}
&= (q^{12}, q^{15}, q^{27}; q^{27})_\infty - q(q^{21}, q^6, q^{27}; q^{27})_\infty \\
&\qquad\qquad - q^2(q^3, q^{24}, q^{27}; q^{27})_\infty, && (m=3) \\
&= (-q^{22}, -q^{26}, q^{48}; q^{48})_\infty - q(-q^{34}, -q^{14}, q^{48}; q^{48})_\infty \\
&\quad - q^2(-q^{10}, -q^{38}, q^{48}; q^{48})_\infty + q^5(-q^{46}, -q^2, q^{48}; q^{48})_\infty, && (m=4) \\
&\qquad \vdots && \vdots
\end{aligned}
$$

Note that the right sides required some elementary manipulations in the cases $m = 3$ and $m = 4$.

Now suppose we have a finite set of "Series S_i = Product P_i" identities ($1 \le i \le n$). Clearly

$$\sum_{i=1}^{n} \alpha_i S_i = \sum_{i=1}^{n} \alpha_i P_i \tag{6.2.3}$$

holds for any set of complex constants α_i. If, for a certain set of α_i, (6.2.3) follows as a consequence of the Jacobi triple product identity, then we say the identities $S_i = P_i$, $1 \le i \le n$ are *JTP-dependent*, in the sense that any one identity can be derived from the other $n-1$ identities taken together with the Jacobi triple product identity. As an example, consider the following three identities from Slater's list:

$$1 + \sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1} q^{n^2}}{(q;q)_{2n-1}(q;q)_n} = \frac{(q^{12}, q^{15}, q^{27}; q^{27})_\infty}{(q;q)_\infty} \tag{\textbf{S93}}$$

$$\sum_{n=0}^{\infty} \frac{(q^3;q^3)_{n} q^{n(n+2)}}{(q;q)_{2n+2}(q;q)_n} = \frac{(q^{6}, q^{15}, q^{21}; q^{27})_\infty}{(q;q)_\infty} \tag{\textbf{S91}}$$

$$\sum_{n=0}^{\infty} \frac{(q^3;q^3)_{n} q^{n(n+3)}}{(q;q)_{2n+2}(q;q)_n} = \frac{(q^{3}, q^{24}, q^{27}; q^{27})_\infty}{(q;q)_\infty}. \tag{\textbf{S90}}$$

One easily checks that the sum side of **S93** - $q\times$**S91** - $q^2\times$**S90** is identically 1, while the product side being identically 1 follows from the $m = 2$ case of (6.2.2). Thus the identities **S90**, **S91** and **S93** are JTP-dependent.

Before compiling a table of sets of identities which are JTP-dependent, we first exhibit two other q-products which, like $(q;q)_\infty$ at (6.2.2), have infinitely many expressions as sums of triple products.

Firstly, if we replace q by q^4 and set $x = -q$, we get that

$$
\begin{aligned}
\frac{(q^2:q^2)_\infty}{(-q:q^2)_\infty} &= (-q^6, -q^{10}, q^{16}; q^{16})_\infty - q(-q^{14}, -q^2, q^{16}; q^{16})_\infty, && (m=2) \qquad (6.2.4) \\
&= (q^{15}, q^{21}, q^{36}; q^{36})_\infty - q(q^{27}, q^9, q^{36}; q^{36})_\infty \\
&\qquad\qquad - q^3(q^3, q^{33}, q^{36}; q^{36})_\infty, && (m=3) \\
&= (-q^{28}, -q^{36}, q^{64}; q^{64})_\infty - q(-q^{44}, -q^{20}, q^{64}; q^{64})_\infty \\
&\quad - q^3(-q^{12}, -q^{58}, q^{64}; q^{64})_\infty + q^6(-q^{60}, -q^4, q^{64}; q^{64})_\infty, && (m=4) \\
&\qquad \vdots && \vdots
\end{aligned}
$$

Here also the right sides required some elementary manipulations in the cases $m = 3$ and $m = 4$.

Secondly, replace q by q^2 and set $x = -q$ to get

$$\begin{aligned}\frac{(q:q)_\infty}{(-q:q)_\infty} &= (-q^4, -q^4, q^8; q^8)_\infty - 2q(-q^8, -q^8, q^8; q^8)_\infty, && (m = 2) \quad (6.2.5)\\ &= (q^9, q^9, q^{18}; q^{18})_\infty - 2q(q^{15}, q^3, q^{18}; q^{18})_\infty && (m = 3)\\ &= (-q^{16}, -q^{16}, q^{32}; q^{32})_\infty - 2q(-q^{24}, -q^8, q^{32}; q^{32})_\infty \\ &\qquad + 2q^4(-q^{32}, -q^{32}, q^{32}; q^{32})_\infty, && (m = 4)\\ &\qquad\vdots && \vdots\end{aligned}$$

The table is to be understood as follows: in each case it is easily checked that the combinations of q-series from the indicated identities sums to the indicated value; that the same combination of the corresponding products in the indicated identities sums to the same value follows from the stated identity multiplied by the given q-product, for the stated value of m. See the example at (**S93**) above for more details.

Applying the Jacobi triple product in the ways described above can also lead to new identities. For example, considering **S53**(with q replaced by $-q$) $-q\times$**S57** in conjunction with the $m = 2$ case of (6.2.2) leads to the following identity:

$$1 + \sum_{n=1}^{\infty} \frac{q^{(2n-1)^2}(-q; q^2)_{2n-1}(-1 + q^{4n-1} + q^{8n} + q^{8n-2})}{(q^4; q^4)_{2n}} = \frac{(q; q)_\infty}{(q^4; q^4)_\infty}.$$

Series Identity	Product Identity
S58-$q\times$**S56**=1	$(6.2.2)\vert_{m=2} \times \dfrac{1}{(q;q)_\infty}$
S54+$q\times$**S49**=**S47**	$(6.2.2)\vert_{m=2,q\to -q} \times \dfrac{1}{(q;q)_\infty}$
S93-$q\times$**S91**-$q^2\times$**S90**=1	$(6.2.2)\vert_{m=3} \times \dfrac{1}{(q;q)_\infty}$
S120-$q^2\times$**S122**=1	$(6.2.2)\vert_{m=4} \times \dfrac{1}{(q;q)_\infty}$
S72-$q\times$**S69**=1	$(6.2.4)\vert_{m=2} \times \dfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$
S121-$q\times$**S123**=1	$(6.2.4)\vert_{m=2} \times \dfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$
S114-$q\times$**S115**-$q^3\times$**S116**=1	$(6.2.4)\vert_{m=3} \times \dfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$
S111+**S113**-**S114**=1	$(6.2.4)\vert_{m=3} \times \dfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$
S113-$q\times$**S115**=1	$(6.2.4)\vert_{m=3} \times \dfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$

S111-q^3×**S116**=1	$(6.2.4)\|_{m=3} \times \dfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$
S126-q×**S128**=1	$(6.2.4)\|_{m=4} \times \dfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$
S127-q^3×**S129**=1	$(6.2.4)\|_{m=4} \times \dfrac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$
S78-$2q$×**S76**=1	$(6.2.5)\|_{m=3} \times \dfrac{(-q;q)_\infty}{(q;q)_\infty}$
S104-q×**S103**=1	$(6.2.5)\|_{m=4} \times \dfrac{(-q;q)_\infty}{(q;q)_\infty}$

Table 4: Sets of JTP-dependent identities.

A similar consideration of **S53** $+q$×**S55** yields

$$\sum_{n=0}^{\infty} \frac{q^{4n^2}(-q;q^2)_{2n}(1-q^{4n+1}-q^{8n+2}-q^{8n+4})}{(q^4;q^4)_{2n+1}} = \frac{(q;q)_\infty}{(q^4;q^4)_\infty}.$$

Likewise, considering **S40**, **S41** and **S42** (each with q replaced by q^3), together with the $m=3$ case of (6.2.2), yields

$$\sum_{n=0}^{\infty} \frac{q^{9n^2}(q^3;q^3)_{3n}(1-q^{1+9n}-q^{2+9n}+q^{5+18n}+q^{7+18n}-q^{9+18n})}{(q^9;q^9)_n(q^9;q^9)_{2n+1}} = \frac{(q;q)_\infty}{(q^9;q^9)_\infty}.$$

6.3 Inter-dependence Via an Identity of Weierstrass

The identity in question is given in the lemma below. We first define

$$[x;q]_\infty := (x, q/x;q)_\infty, \qquad [x_1,\dots x_n;q]_\infty = [x_1;q]_\infty \dots [x_n;q]_\infty,$$

and note that $[x^{-1};q]_\infty = -x^{-1}[x;q]_\infty$.

Lemma 13. *Let $a_1, a_2, \dots, a_n$; $b_1, b_2, \dots, b_n$ be non-zero complex numbers such that*
i) $a_i \neq q^n a_j$, for all $i \neq j$ and all $n \in \mathbb{Z}$,
ii) $a_1a_2\dots a_n = b_1b_2\dots b_n$. Then

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n}[a_ib_j^{-1};q]_\infty}{\prod_{j=1,j\neq i}^{n}[a_ia_j^{-1};q]_\infty} = 0. \tag{6.3.1}$$

This lemma appears in [46] (page 45) and a proof can be found in [31]. If $n=3$, then (6.3.1) can be written as

$$\frac{[a_1/b_1,a_1/b_2,a_1/b_3;q]_\infty}{[a_1/a_2,a_1/a_3;q]_\infty} + \frac{[a_2/b_1,a_2/b_2,a_2/b_3;q]_\infty}{[a_2/a_1,a_2/a_3;q]_\infty} + \frac{[a_3/b_1,a_3/b_2,a_3/b_3;q]_\infty}{[a_3/a_1,a_3/a_2;q]_\infty} = 0. \tag{6.3.2}$$

The dependence of identities via (6.3.1) is more tenuous than dependence via series-equivalence or the Jacobi triple product. We give the following example to illustrate the concept. If we replace q by q^{10} in (6.3.2) and set

$$\{a_1, a_2, a_3; b_1, b_2, b_3\} = \{1, -q^5, q; q^{-1}, -q^{-2}, q^9\},$$

then after a little simplification we get that

$$q[q, q, -q^2, -q^4; q^{10}]_\infty + [q, q^3, -q^4, -q^4; q^{10}]_\infty - [q^2, q^2, -q^3, -q^5; q^{10}]_\infty = 0. \tag{6.3.3}$$

This identity can be rearranged to give

$$(-q, -q, q^2; q^2)_\infty = \frac{(q^{10}; q^{10})_\infty}{(q, q^9, -q^2, -q^8; q^{10})_\infty} + \frac{q(q^{10}; q^{10})_\infty}{(q^3, q^7, -q^4, -q^6; q^{10})_\infty} \tag{6.3.4}$$

Similarly, if we replace q by q^{10} in (6.3.2) and set

$$\{a_1, a_2, a_3; b_1, b_2, b_3\} = \{1, -q^5, q^6; q^{-1}, -q^{-2}, q^{14}\},$$

we get that

$$-q[q, -q, -q^2, q^4; q^{10}]_\infty + [-q, q^3, q^4, -q^4; q^{10}]_\infty - [q^2, -q^2, q^3, -q^5; q^{10}]_\infty = 0. \tag{6.3.5}$$

This identity can likewise be rearranged to give

$$(-q^5, -q^5, q^{10}; q^{10})_\infty = \frac{(q^{10}; q^{10})_\infty}{(q, q^9, -q^2, -q^8; q^{10})_\infty} - \frac{q(q^{10}; q^{10})_\infty}{(q^3, q^7, -q^4, -q^6; q^{10})_\infty} \tag{6.3.6}$$

Finally, if we replace q by $-q$ and subtract, we get that

$$(q^5, q^5, q^{10}; q^{10})_\infty - (q, q, q^2; q^2)_\infty = \frac{2q(q^{10}; q^{10})_\infty}{(-q^3, -q^7, -q^4, -q^6; q^{10})_\infty} \tag{6.3.7}$$

See the paper by Cooper and Hirschhorn [18] for these and similar identities.

We now exhibit two identities of Rogers-Ramanujan type which are related via (6.3.1). The first appears in [16]:

$$\sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-q; q)_n}{(q; q^2)_{n+1}(q; q)_{n+1}} = \frac{(q^{10}; q^{10})_\infty}{(q; q)_\infty (q; q^2)_\infty (-q^3, -q^4, -q^6, -q^7; q^{10})_\infty}. \tag{6.3.8}$$

The second is an identity of Rogers (see [38]):

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1; q)_n}{(q; q)_n (q; q^2)_n} = \frac{(q^5, q^5, q^{10}; q^{10})_\infty}{(q; q)_\infty (q; q^2)_\infty}. \tag{6.3.9}$$

One easily checks (by re-indexing the second sum) that

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1; q)_n}{(q; q)_n (q; q^2)_n} - 2q \sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-q; q)_n}{(q; q^2)_{n+1}(q; q)_{n+1}} = 1.$$

That
$$\frac{(q^5,q^5,q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty} - 2q\frac{(q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty(-q^3,-q^4,-q^6,-q^7;q^{10})_\infty} = 1$$
follows from (6.3.7). We say that pairs of identities which are dependent via (6.3.1) are *W-dependent*.

We give one other example of a pair of identities are W-dependent. The first is **S.56**:
$$\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+2)}}{(q;q^2)_{n+1}(q;q)_{n+1}} = \frac{(-q,-q^{11},q^{12};q^{12})_\infty}{(q;q)_\infty}. \tag{6.3.10}$$
The second is
$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2}}{(q;q^2)_n(q;q)_n} = \frac{(q^3,q^3,q^6;q^6)_\infty(-q;q)_\infty}{(q;q)_\infty}. \tag{6.3.11}$$
It is easy to check (by re-indexing, as above) that
$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2}}{(q;q^2)_n(q;q)_n} - 2q\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+2)}}{(q;q^2)_{n+1}(q;q)_{n+1}} = 1.$$
That
$$\frac{(q^3,q^3,q^6;q^6)_\infty(-q;q)_\infty}{(q;q)_\infty} - 2q\frac{(-q,-q^{11},q^{12};q^{12})_\infty}{(q;q)_\infty} = 1$$
follows from the following identity (proved by Cooper and Hirschhorn in [18], again making use of (6.3.2)):
$$(q^3,q^3,q^6;q^6)_\infty - (q,q,q^2;q^2)_\infty = 2q(-q,-q^{11},q^{12};q^{12})_\infty(q;q^2)_\infty.$$
It is possible that new, as yet unknown identities of the Rogers-Ramanujan type may be derivable from existing identities through W-dependence.

7 Bailey pairs

7.1 The Bailey Pairs of Slater

Rogers [37] categorized the series transformations he discovered into seven groups labeled A through G. With hindsight, each of these series transformations can be seen to correspond to a Bailey pair. Accordingly, when Slater [40, 41] tabulated the Bailey pairs she used to produce her list of Rogers-Ramanujan type identities, she retained Rogers' designations, and added Groups H through M which contained the new Bailey pairs she had discovered; those that did not correspond to Rogers' series transformations. The following table summarizes the use of Slater's Bailey pairs to produce identities via insertion into various limiting cases of Bailey's lemma (PBL)–(FBL).

BP	(PBL)	(TBL)	(S1BL)	(S2BL)	(FBL)
A1	S98	S117			
A2	S94			S80	(5.11.1)
A3	S99	S118		S81	(5.11.2)
A4	S96	S119		S82	(5.11.3)
A5	S83	S100		(3.15.2)	(5.8.1)
A6	S84			S62	(5.8.2)
A7	S85	S95			(5.8.3)
A8		S97		S63	
B1	S18	S36	S12	S13	
B2	S14	S34			
B3				S8	
C1	S61	S79	(2.10.3)		
C3	S60			S45	(5.5.1)
C4	S59	(2.20.1)		S43	(5.5.2)
C5	(2.10.6)	S52			
C6	S46			S26	(5.3.1)
C7	S44			S22	(5.3.2)
D5					(5.4.1)
D6				(2.8.6)	(5.4.2)
E1	S3-	S25			
E2	S3	S4			
E3	S7	(2.6.2)		S2	(5.0.1)
F1	S39	S29, S23-			
F2	S38			S28	(5.3.2)
F3	S9	(2.2.4)			
G1	S33	S20, S21-			
G2	S31			S17	(5.2.1)
G3	S32	S16			
G4	S19				
G5				S5	(5.0.2)
G6	S15				
H1	S6	S10			
H2	S39-	S23-			
H17	S1				
H19	S27			S11	(5.1.1)
I7	S47	S66			
I8	S48	S67			
I9	S58	S72			
I14	S54	S71			
I15	S87			S64	
I16				S65	

I17	S56 S69			S37	(5.4.3)
I18	S50	S68		S35	(5.4.4)
J1	S93	S114		S73	
J2	S88	S108, S113		S74	
J3	S89	S111		S75	(5.9.4)
J4	S124	S109		S107	(5.18.5)
J5	S125	S110			
J6	S90	S112		S76	(5.9.3)
K1	S121	S126		S101	
K2	S120	S127		S104	
K3	S51	S130		S105	
K4		S70		S106	(5.16.2)
K5	S123	S128		S102	
K6	S122	S129		S103	(5.16.1)
L2	S53	(2.8.12)			
M2	S57				
M3	S55				

Table 5: Slater's Bailey pairs and associated identities

7.2 Duality of Bailey pairs

We recall that Andrews showed in [5] that if $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair relative to a, then $(\alpha_n^*(a,q), \beta_n^*(a,q))$ is also a Bailey pair relative to a, where

$$\alpha_n^*(a,q) = a^n q^{n^2} \alpha_n(1/a, 1/q),$$
$$\beta_n^*(a,q) = a^{-n} q^{-n^2-n} \beta_n(1/a, 1/q).$$

The pair $(\alpha_n^*(a,q), \beta_n^*(a,q))$ is called *the dual* of $(\alpha_n(a,q), \beta_n(a,q))$. Note that the dual of $(\alpha_n^*(a,q), \beta_n^*(a,q))$ is $(\alpha_n(a,q),\ \beta_n(a,q))$. Identities that arise from Bailey pairs that are duals of each other may be regarded in some sense as being related, so for completeness sake we give a table that matches Bailey pairs with their duals.

Before giving this table, we make some comments. Firstly, Slater's list of Bailey pairs contains several pairs whose duals are not listed, so we list the new Bailey pairs that are dual to these. Secondly, some pairs, such as **H7** and **E2** are in fact the same pair, and we indicate the occurrence of such identical pairs. Thirdly, Slater's list of Bailey pairs contains a good many typographic errors. Sometimes retracing Slater's steps (making the same choices that she did for the parameters in the identities she used to derive the Bailey pairs) will recover the correct Bailey pair, but this does not always work as in several cases Slater employs an additional unstated manipulation to derive the Bailey pair(for example, the same choice of parameters give rise to pairs **B1** and **B2**, but only **B1** follows directly from substituting these values into Equation (4.1) of [40]).

In this latter case we can insert the stated sequence $\alpha_n(a,q)$ (under the assumption that the stated value for $\alpha_n(a,q)$ is correct, and that the error is in the stated value for

$\beta_n(a,q))$ into the equation

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{1}{(q;q)_{n-r}(aq;q)_{n+r}} \alpha_r(a,q), \tag{7.2.1}$$

use *Mathematica* to compute $\beta_n(a,q)$ for low values of n and hopefully recognize the form of $\beta_n(a,q)$ for arbitrary n. If this does not work, then we try substituting the stated value for $\beta_n(a,q)$ into (7.2.1), and use *Mathematica* to compute successive values for $\alpha_n(a,q)$, again try to find the formula for $\alpha_n(a,q)$. Of course there is the possibility that the pair derived in this way is not the pair Slater intended. This could happen if, for example, Slater incorrectly stated $\alpha_n(a,q)$, but this $\alpha_n(a,q)$ is part of an entirely different Bailey pair, so that the attempt to find Slater's correct pair would instead find the other Bailey pair (for example, Slater lists the same $\alpha_n(a,q)$ for both **F1** and **H1**, and assuming that the $\alpha_n(a,q)$ in **H1** is correct would lead to a Bailey pair different from what Slater had in mind for **H1**).

Slater's Bailey Pairs with their respective duals

Slater's BP	Dual Pair	Slater's BP	Dual Pair
A.1	A.5	H.1*	H.1*
A.2	A.8	H.2	7.2.6
A.3	A.7	H.5	H.9
A.4	A.6	H.6	H.6
B.1	H.4	H.10	H.11
B.2	H.3	H.12	H.14
B.3	7.2.2	H.13	H.13
B.4	7.2.3	H.15	H.16
C.1	C.5	H.17	H.17
C.2	7.2.4	H.18	H.18
C.3	C.7	H.19=I.15*	H.19=I.15*
C.4	C.6	I.1	I.4
E.1	E.2 = H.7	I.2*	I.3
E.3	E.7*	I.5	I.6
E.4	E.5= H.8	I.7	I.8*
E.6*	7.2.5	I.9	I.9
F.1	F.3	I.10	I.11
F.2	F.4	I.12	I.13
G.1	G.4	I.14	I.14
G.2*	G.5*	I.16*	I.16*
G.3	G.6	I.17	I.17
J.1*	J.1*	I.18	I.18
J.2	J.3	M.1	M.1
J.4	J.5	M.2*	M.2*
J.6	J.6	M.3*	M.3*

K.1	K.2	L.1	L.3
K.3	K.4	L.2	L.2
K.5	K.6	L.4*	L.6
		L.5*	L.5*

Table 6: A * indicates a typographic error in Slater's Paper, corrected versions of these Bailey pairs are given below. The five new Bailey pairs (7.2.2)—(7.2.6) are also stated below.

The new Bailey pairs found (by deriving the duals of Bailey pairs given by Slater whose duals were not given by her) are the following.

$$\alpha_n = \frac{(-1)^n q^{\frac{-n(n+3)}{2}}(1-q^{2n+1})}{(1-q)}, \qquad (a=q) \tag{7.2.2}$$
$$\beta_n = \frac{(-1)^n q^{-n(n+3)/2}}{(q;q)_n}.$$

$$\alpha_n = \frac{(-1)^n (q^{(-n^2+n+4)/2}(1-q^n) + q^{-n(n+5)/2}(1-q^{n+1}))}{(1-q)}, \qquad (a=q) \tag{7.2.3}$$
$$\beta_n = \frac{(-1)^n q^{\frac{-n(n+5)}{2}}}{(q;q)_n}.$$

$$\begin{aligned} \alpha_{2n} &= (-1)^n q^{n^2}\left(q^n + q^{-n}\right), \qquad (a=1) \\ \alpha_{2n+1} &= (-1)^n q^{n^2-n-1}(1-q^{4n+2}) \\ \beta_n &= \frac{q^{\frac{n(n-3)}{2}}}{(q;q)_n (q;q^2)_n}. \end{aligned} \tag{7.2.4}$$

$$\alpha_n = \frac{(-1)^n q^{-2n}(1-q^{4n+2})}{(1-q^2)}, \qquad (a=q) \tag{7.2.5}$$
$$\beta_n = \frac{(-1)^n q^{-2n}}{(-q,q;q)_n}.$$

$$\alpha_n = (-1)^n (q^{\frac{n}{2}} + q^{-\frac{n}{2}}), \qquad (a=1) \tag{7.2.6}$$
$$\beta_n = \frac{(-1)^n q^{-\frac{n}{2}}}{(-q^{\frac{1}{2}},q;q)_n}.$$

We next give corrected versions of the Bailey pairs ("BP" in the tables below) in the Slater papers [40, 41], which have typographic errors.

Corrected Bailey pairs

BP	α_n	β_n
E.6*	$\frac{(-1)^n q^{n^2-n}(1-q^{4n+2})}{(1-q^2)}$	$\frac{q^n}{(-q,q;q)_n}$
E.7*	$(-1)^n q^{-n}\frac{\left(1-q^{2n+1}\right)}{(1-q)}$	$\frac{(-1)^n q^{-n}}{(-q,q;q)_n}$
H.1*	$q^{n^2}(q^{\frac{n}{2}}+q^{-\frac{n}{2}})$	$\frac{1}{(q^{\frac{1}{2}},q;q)_n}$
I.15*	$q^{\frac{n^2}{2}}\frac{(1+q^{n+\frac{1}{2}})}{(1+q^{\frac{1}{2}})}$	$\frac{(-q^{\frac{1}{2}};q)_n}{(q^{\frac{3}{2}};q)_n(q^2;q^2)_n}$
L.5*	$q^{\frac{n(n-1)}{2}}(1+q^n)$	$\frac{(-1;q)_n}{(q;q)_n(q;q^2)_n}$
M.2*	$q^{\frac{n(2n+1)}{4}}\frac{\left(1+q^{\frac{(2n+1)}{4}}\right)}{(1+q^{\frac{1}{4}})}$	$\frac{(-q^{\frac{3}{4}};q^{\frac{1}{2}})_{2n}}{(q^2;q)_{2n}}$
M.3*	$(-1)^n q^{\frac{n(2n+1)}{4}}\frac{\left(1-q^{\frac{(2n+1)}{4}}\right)}{(1-q^{\frac{1}{4}})}$	$\frac{(q^{\frac{3}{4}};q^{\frac{1}{2}})_{2n}}{(q^2;q)_{2n}}$

Corrected Bailey pairs

BP	α_{2n}	α_{2n+1}	β_n
G.2*	$\frac{q^{\frac{6n^2+n}{2}}\left(1-q^{2n+\frac{1}{2}}\right)}{(1-\sqrt{q})}$	$\frac{q^{\frac{6n^2+11n+5}{2}}\left(1-q^{-2n-\frac{3}{2}}\right)}{(1-\sqrt{q})}$	$\frac{1}{(-q^{\frac{3}{2}};q)_n(q^2;q^2)_n}$
G.5*	$\frac{q^{\frac{2n^2-n}{2}}\left(1-q^{2n+\frac{1}{2}}\right)}{(1-\sqrt{q})}$	$\frac{q^{\frac{2n^2+5n+3}{2}}\left(1-q^{-2n-\frac{3}{2}}\right)}{(1-\sqrt{q})}$	$\frac{(-1)^n q^{\frac{n^2}{2}}}{(-q^{\frac{3}{2}};q)_n(q^2;q^2)_n}$
I.2*	$(-1)^n q^{n^2}\left(q^{\frac{n}{2}}+q^{-\frac{n}{2}}\right)$	$(-1)^{n+1}q^{n^2}\left(q^{\frac{n}{2}}-q^{\frac{3n+1}{2}}\right)$	$\frac{q^{\frac{n^2}{2}}}{(\sqrt{q};q)_n(q^2;q^2)_n}$

I.8*	$(-1)^n q^{2n^2}\left(q^n + q^{-n}\right)$	$(-1)^n q^{2n^2}\left(q^{3n+1} - q^n\right)$	$\frac{q^n(-1;q^2)_n}{(q,q^2;q^2)_n}$
I.16*	$(-1)^n q^{2n^2}\frac{(1+q^{2n+\frac{1}{2}})}{(1+q^{\frac{1}{2}})}$	$(-1)^{n+1} q^{2n^2+2n+\frac{1}{2}}\frac{(1+q^{2n+\frac{3}{2}})}{(1+q^{\frac{1}{2}})}$	$\frac{(-q;q^2)_n}{(q^2;q^2)_n(-q^{\frac{1}{2}},q^{\frac{3}{2}};q)_n}$
L.4*	$(-1)^n q^{n(n-1)}\left(1+q^{2n}\right)$	0	$\frac{q^{\frac{n(n-1)}{2}}}{(-q^{\frac{1}{2}};q)_n(q^{\frac{1}{2}};q^{\frac{1}{2}})_{2n}}$

Corrected Bailey pair

BP	α_{3n-1}	α_{3n}	α_{3n+1}	β_n
J.1*	0	$(-1)^n q^{\frac{3n(3n-1)}{2}}\left(1+q^{3n}\right)$	0	$\frac{(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}}$

8 Conclusion

We have only touched on a small part of the Rogers-Ramanujan story in this survey. The main goal has been to present an expanded version of Slater's list with the earliest known reference to each identity in the literature. Slater's list contained only a few references to the earlier literature, and of course, Ramanujan's lost notebook was unknown to the mathematical community in 1952. Accordingly, the authors believe it was a useful endeavor to bring together Slater's list with Ramanujan's lost notebook, and the dozens of additional identities of similar type which have been scattered throughout the literature over the years. Since Slater's main tool was Bailey's lemma and Bailey pairs, we included an exposition of this material in the introduction.

We have not attempted to address the numerous advances in the theory of q-series which have occurred in the past few decades (e.g. Andrews's discovery of the "Bailey Chain," and its application to multisum–product identities; the extension of the Bailey chain to the WP Bailey chain by Andrews, Berkovich; the extension of the Bailey chain to elliptic hypergeometric series by Spiridonov and Warnaar; Lepowsky, Milne, and Wilson's connection of q-series to Lie algebras and vertex operator algebras; the work of Baxter, Berkovich, Forrester, McCoy, Melzer, Warnaar connecting q-series to models in statistical mechanics; the work on finite analogs by Andrews, Berkovich, Forrester, McCoy, Melzer, Paule, Riese, the second author, Warnaar, Wilf, and Zeilberger. Further, we have not considered any of the combinatorial consequences of these identities, of which there are too many to even begin to list the most important contributors.

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As with any long list of identities, keyed into LaTeX by hand, there are bound to be typographical errors, and errors of omission. Accordingly, the authors would like to thank in advance the kind readers who will bring such errors, omissions, and updates to our attention, so that we can make this survey as useful to our readers as possible.

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