Hamiltonian cycles in torical lattices

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We establish sufficient conditions for a toric lattice $T_{m,n}$ to be Hamiltonian. Also, we give some asymptotics for the number of Hamiltonian cycles in $T_{m,n}$.

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Let $T_{m,n} = J_m \times J_n$ be a toric lattice, i.e., the Cartesian product of two directed cycles lengths m and n respectively.

Erdös problem [1]. When $T_{m,n}$ contains Hamiltonian cycles?

The next theorem was proved by A.A.Evdokimov [2].

Theorem 1 *$T_{m,n}$ is Hamiltonian iff there are solutions of the following Diophantine system*

$$\begin{array}{l} x + y = \gcd(m, n), \\ \gcd(x, m) = 1,\ \gcd(y, n) = 1 \end{array} \tag{1}$$

($\gcd$ means the greatest common divisor).

Let $J_{m,n}$ be the number of solutions of the system (1). We obtain estimates for $J_{m,n}$ in two special cases. Let

$$m = \prod_{i=1}^{r} p_i^{\alpha_i}, \quad n = \prod_{j=1}^{s} q_j^{\beta_j}$$

are prime decompositions for m, n. We use the following notations

$$P = \prod_{i=1}^{r} p_i, \quad Q = \prod_{j=1}^{s} q_j, \quad \lambda(P, Q) = \prod_{r|\mathrm{lcm}(P,Q)} \Big(1 - \frac{1}{r}\Big)$$

(lcm means the least common multiple).

Theorem 2 $J_{m,n} \geq 1$ *if* $\gcd(m, n) > \Big[\prod_{i=1}^{r}(1 + p_i) + \prod_{j=1}^{s}(1 + q_j)\Big]\big(4\lambda(P, Q)\big)^{-1}$.

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The proofs of the theorems 1, 2 are based on the following analytic and combinatorial results.

Let

$$J_N(u) = \sum_{(a,N)=1} u^a, \quad N = p_1^{\alpha_1} \dots p_k^{\alpha_k}.$$

Lemma 1 $J_N(u) = \dfrac{1}{1-u} - \displaystyle\sum_{i=1}^{k} \dfrac{1}{1-u^{p_i}} + \sum_{1 \le i < j \le k}^{k} \dfrac{1}{1-u^{p_i p_j}} - \dots$

This formula can be easily proved by inclusion - exclusion principle.

Let $S_r(m,n)$ be the number of solutions of the system

$$\begin{aligned} & x + y = r, \\ & \gcd(x,m) = 1, \ \gcd(y,n) = 1. \end{aligned} \tag{2}$$

The generating function for $S_r(m,n)$ is related with $J_n(u)$ by the following formula.

Lemma 2

$$\sum_{r=1}^{\infty} S_r(m,n) u^r = J_m(u) J_n(u). \tag{3}$$

Formula (3) implies an expression for the number of solutions of the system (1).

Lemma 3 *Let $N = \gcd(m,n) + 1$. Then the following equation holds*

$$\begin{aligned} J_{m,n} = \gcd(m,n) \sum_{u|P,\ v|Q} \frac{\mu(u)\mu(v)}{\operatorname{lcm}(u,v)} + \sum_{u|P,\ v|Q} \frac{\mu(u)\mu(v)(u+v)}{2\operatorname{lcm}(u,v)} + \\ \sum_{u|P,\ v|Q} \frac{\mu(u)}{u} \sum_{\alpha^u=1} \frac{1}{\alpha^{N-1}(\alpha^v - 1)} + \sum_{u|P,\ v|Q} \frac{\mu(v)}{v} \sum_{\alpha^v=1} \frac{1}{\alpha^{N-1}(\alpha^u - 1)}. \end{aligned} \tag{4}$$

In sums of type

$$\sum_{\alpha^u=1} \frac{1}{\alpha^{N-1}(\alpha^v - 1)} \tag{5}$$

the summation is over those roots of equation $\alpha^u = 1$ that are not the roots of equation $\alpha^v = 1$.

Sums (5) are called Dedekind sums. They are well-known in combinatorial analysis (e.g., see [3]).

To simplify (4) we use identities about Möbius function. They are 2-dimensional analogues of the classical formula

$$\sum_{d|n} \frac{\mu(d)}{d} = \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

An example of these identities is given by the following Lemma.

Lemma 4 ([4]) $\displaystyle\sum_{u|m,\ v|n} \frac{\mu(u)\mu(v)}{\operatorname{lcm}(u,v)} = \prod_{r|\operatorname{lcm}(P,Q)} \left(1 - \frac{1}{r}\right).$

Dealing with Dedekind sums (5) we use the following useful statement. Let

$$S_n(a) = \sum_{\alpha^b=1} \frac{1}{\alpha^n(\alpha^a - 1)}, \tag{6}$$

where summation is over those roots of equation $x^b = 1$ that are not the roots of equation $x^a = 1$. By m_0 we denote the smallest positive solution of equation

$$ax \equiv -(n+a) \pmod{b}.$$

Let $w(a,b) = m_0 - 1$.

Lemma 5

$$S_n(a) = \frac{b}{2} - \frac{\gcd(a,b)}{2\,\mathrm{lcm}(a,b)} - w(a,b). \tag{7}$$

References

[1] Trotter W.T., Erdös P. When the cartesian product of directed cycles is hamiltonian. J. Graph Theory. V.2, 1978. P. 137–142.

[2] Evdokimov A.A. Numeration of subsets of a finite set. (In Russian) Metody Diskret. Analiz. V. 34, 1980. P. 8–26.

[3] Ira M. Gessel. Generating functions and generalized Dedekind sums. The electronic Journal of Combinatorics. V. 4, no. 2, 1997.

[4] Leontiev V.K. Hamiltonian cycles in toric lattices. (In Russian) DAN, V. 395, no 5, 2004. P. 590–591.