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Acyclic Coloring of Graphs of Maximum Degree Δ

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An acyclic coloring of a graph G is a coloring of its vertices such that: (i) no two neighbors in G are assigned the same color and (ii) no bicolored cycle can exist in G. The acyclic chromatic number of G is the least number of colors necessary to acyclically color G, and is denoted by $a(G)$. We show that any graph of maximum degree Δ has acyclic chromatic number at most $\frac{\Delta(\Delta-1)}{2}$ for any $\Delta \geq 5$, and we give an $O(n\Delta^2)$ algorithm to acyclically color any graph of maximum degree Δ with the above mentioned number of colors. This result is roughly two times better than the best general upper bound known so far, yielding $a(G) \leq \Delta(\Delta-1)+2$ [ACK$^+$04]. By a deeper study of the case $\Delta = 5$, we also show that any graph of maximum degree 5 can be acyclically colored with at most 9 colors, and give a linear time algorithm to achieve this bound.

Keywords: Acyclic chromatic number, acyclic coloring algorithm, maximum degree

1 Introduction

In this paper, we address the acyclic coloring problem. An acyclic coloring of a graph G is a coloring of its vertices such that: (i) no two neighbors in G are assigned the same color and (ii) no bicolored cycle can exist in G. In other words, an acyclic coloring of G is a proper coloring of G such that any two classes of colors induce a graph G' which is a forest (that is, an acyclic graph). The minimum number of colors necessary to acyclically color G is called the *acyclic chromatic number* of G, and is denoted by $a(G)$.

For a family $\mathcal{F}$ of graphs, the acyclic chromatic number of $\mathcal{F}$, denoted by $a(\mathcal{F})$, is defined as the maximum $a(G)$ over all graphs $G \in \mathcal{F}$. Acyclic coloring has been largely studied in the past 25 years ; in particular, $a(\mathcal{F})$ has been determined for several families $\mathcal{F}$ of graphs such as planar graphs [Bor79], planar graphs with "large" girth [BKW99], 1-planar graphs [BKRS01], outerplanar graphs (see for instance [Sop97]), d-dimensional grids [FGR03], graphs of maximum degree 3 [Grü73] and of maximum degree 4 [Bur79]. In the last two cases, in particular, it was shown that:

- 4 colors are sufficient to color any graph of maximum degree 3, and there exists a graph G of maximum degree 3 for which any acyclic coloring requires 4 colors. It was proved recently in [Sku04] that there exists a linear time algorithm that acyclically colors any graph of maximum degree 3 in 4 colors.

- 5 colors are sufficient to color any graph of maximum degree 4, and there exists a graph G of maximum degree 4 for which any acyclic coloring requires 5 colors.

However, little is known concerning the acyclic chromatic number of graphs of maximum degree Δ, for general Δ. Most of the results on this topic come from Alon et al. [AMR90], where the following results were proved: (1) asymptotically, there exist graphs of maximum degree Δ with acyclic chromatic number in $\Omega(\frac{\Delta^{\frac{4}{3}}}{(\log\Delta)^{\frac{1}{3}}})$; (2) asymptotically, it is possible to acyclically color any graph of maximum degree Δ with $O(\Delta^{\frac{4}{3}})$ colors ; (3) a trivial greedy polynomial time algorithm exists that acyclically colors any graph of maximum degree Δ in Δ^2+1 colors. More recently, the latter result was improved by Albertson et al. [ACK$^+$04]. In that paper, the authors showed a result concerning another type of coloring, called *star coloring*. However, their result directly implies that in any graph G of maximum degree Δ, $a(G) \leq \Delta(\Delta-1)+2$.

The main drawback concerning results (1) and (2) is that they come from probabilistic arguments ; thus their proofs are existential but not constructive. Moreover, as mentioned in [AMR90], at this time there is no example of graphs of maximum degree Δ needing $\Omega(\frac{\Delta^{\frac{4}{3}}}{(\log\Delta)^{\frac{1}{3}}})$ colors, and we have no algorithm that performs an acyclic coloring of graphs of maximum degree Δ with $O(\Delta^{\frac{4}{3}})$ colors (or even $o(\Delta^2)$ colors).

Acyclic coloring algorithms for graphs of maximum degree Δ, to our knowledge, has barely been studied, except for the two above mentioned results and the result from Skulrattanakulchai [Sku04] that is specialized to the case $\Delta = 3$. In this paper, the algorithms we provide can be considered as a follow-up and a generalization of Skulrattanakulchai's work; they also consist in an improvement of the result from [ACK$^+$04], since we roughly divide by 2 the upper bound on $a(G)$ for graphs G of maximum degree Δ. However, our method still uses $O(\Delta^2)$ colors. Nevertheless, when applied to small values of Δ, our method gives upper bounds that prolongate the knowledge we already have for the cases $\Delta = 3$ [Grü73] and $\Delta = 4$ [Bur79]. In particular, for $\Delta = 5$, we obtain that any graph of degree 5 can be acyclically colored in 9 colors (as well as a linear time algorithm to do so).

The paper is organized as follows: in Section 2, we first give an $O(n\Delta^2)$ time algorithm to acyclically color any graph of order n and degree $\Delta \geq 4$ in $\frac{\Delta(\Delta-1)}{2} - odd(\Delta) + 1$ colors, where $odd(\Delta) = 1$ if Δ is odd, and 0 otherwise. Then, we improve this bound by 1 (while preserving the algorithm complexity) in Section 3, for any $\Delta \geq 6$; this gives us the main theorem of this paper, namely Theorem 2. Finally, in Section 4, we focus on the case $\Delta = 5$, where we prove that there exists a linear time algorithm to color any graph of maximum degree 5 with 9 colors. Due to space considerations, most of the proofs will be omitted in this extended abstract.

2 An Algorithm using at most $\frac{\Delta(\Delta-1)}{2} - \mathrm{odd}(\Delta) + 1$ Colors ($\Delta \geq 4$)

In this section, we give a polynomial time algorithm to acyclically color any graph of maximum degree $\Delta \geq 4$ in $\frac{\Delta(\Delta-1)}{2} - odd(\Delta) + 1$ colors, where $odd(\Delta) = 1$ if Δ is odd, and 0 otherwise. So far, the best constructive algorithm to acyclically color any graph of degree Δ uses $\Delta(\Delta-1)-2$ colors [ACK$^+$04]. Our algorithm improves this upper bound by, roughly, dividing it by 2 (cf. Theorem 1). Before proving the theorem, we will describe the main ideas of the algorithm, introduce some useful notations, and give intermediate lemmas that will lead us to the result.

First, let us explain the main ideas of the proposed algorithm. The algorithm, somewhat intuitive, works in a greedy way: at any step, we suppose that we have colored a certain number of vertices in an acyclic way. The next step then consists in taking arbitrarily an uncolored vertex u, and depending on the colors of its neighbors and of the vertices at distance 2 from u, of assigning a "valid" color to u (that is, a color that still satisfies the acyclicity of the current coloring). The algorithm processes as such for every uncolored vertex, and thus stops after n loops, where n is the order (ie, number of vertices) of the graph. We will show that each loop can be achieved in $O(\Delta^2)$ time ; hence, since $\Delta \leq n-1$, this clearly shows that our algorithm is polynomial. We also note that if $\Delta = O(1)$, then our algorithm is linear.
Before going into further details, we introduce some notations: for any integer $\Delta \geq 4$, let $C(\Delta) = \frac{\Delta(\Delta-1)}{2}$ when Δ is odd, and $C(\Delta) = \frac{\Delta(\Delta-1)}{2} + 1$ when Δ is even. For any vertex u of graph G, let n_u be the number of neighbors of u that are already colored. Clearly, $n_u \leq \Delta$. Among those n_u vertices, p_u different colors are used, with $p_u \leq n_u$. Among those p_u classes of colors, we also distinguish the p'_u colors having strictly more than one representative among the neighbors of u. Any colored neighbor of u whose color is represented at least twice among the neighbors of u will be called a *non single vertex*, or *NS* vertex. Similarly, any colored neighbor of u whose color is represented only once among the neighbors of u will be called a *single* vertex. Let I be a partially (properly and acyclically) colored instance of our problem. If I is such that any NS neighbor v of u has itself an NS neighbor, we will say that instance I **satisfies property** $P_I(u)$.

Lemma 1 *Let G be a graph of maximum degree Δ. For any partially colored instance I using at most $C(\Delta)$ colors, and any uncolored vertex u, if property $P_I(u)$ is not satisfied, then it is possible to modify I into I', such that $P_{I'}(u)$ is satisfied, while preserving the use of at most $C(\Delta)$ colors.*

Sketch of Proof: Consider a graph G of maximum degree Δ, and a partially colored instance I, using at most $C(\Delta)$ colors. Consider also an uncolored vertex u such that property $P_I(u)$ is not satisfied. We can show that it is possible to change the color of every NS neighbor v of u such that all its colored neighbors are single vertices, in such a way that (i) v becomes a single vertex, and (ii) the new instance I' remains proper and acyclic. We do this simply by changing the color of vertex v, that is by assigning to v a color that is not used among the neighbors of u (an illustration of such a modification is given in Figure 1). We know that no bicolored cycle can be induced by this modification ; moreover, we can prove that we have enough colors available to do so.
This implies that if we carry out this operation for every such neighbor of u, we end up in the situation where the modified instance I' remains proper and acyclic, and satisfies property $P_{I'}(u)$. □

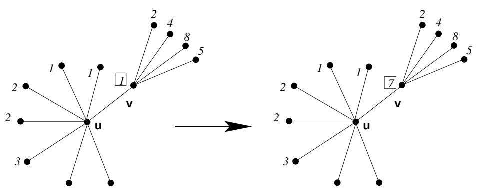

Fig. 1: A way to modify an instance I not satisfying $P_I(u)$

Lemma 2 *For any graph G of maximum degree Δ, and any uncolored vertex u such that the partially colored instance I satisfies property $P_I(u)$, it is always possible to find a color for u in $[1; C(\Delta)]$ such that the coloring remains acyclic.*

These two results now allow us to detail the acyclic coloring algorithm $\mathcal{A}$ we suggest.

Acyclic Coloring of G with at most $C(\Delta)$ Colors

```
1: while there remains at least one uncolored vertex in G do
2:    Let u be an uncolored vertex of G
3:    if property P_I(u) is not satisfied then
4:       Modify the color of some neighbor(s) of u, so that the new instance I' satisfies P_I'(u)
5:    end if
6:    Color u in such a way that the new instance remains proper and acyclic
7: end while
```

We now, thanks to Lemma 1, that line 4 of $\mathcal{A}$ can be carried out without exceeding the maximum number of colors allowed. Moreover, Lemma 2 shows that the same thing holds for line 6 of $\mathcal{A}$. Thus, we can state the following theorem.

Theorem 1 *There exists an $O(n\Delta^2)$ time algorithm to acyclically color any graph of order n and maximum degree Δ. This algorithm uses at most:*

(a) $\frac{\Delta(\Delta-1)}{2}+1$ *colors for any even* $\Delta \geq 4$

(b) $\frac{\Delta(\Delta-1)}{2}$ *colors for any odd* $\Delta \geq 5$

Sketch of Proof: The algorithm is $\mathcal{A}$, given previously. As mentioned earlier, its correctness is given by Lemmas 1 and 2. What remains to be proved here is the time complexity of $\mathcal{A}$. In each loop, the most time-consuming parts of $\mathcal{A}$ are lines 4 and 6. However, both can be achieved in $O(\Delta^2)$ time (in each case, we just need to look at the neighbors and vertices at distance 2 from u, and there are $O(\Delta^2)$ such vertices). Since there are n loops altogether, the $O(n\Delta^2)$ time complexity follows. □

3 How to spare one Color when $\Delta \geq 6$

As mentioned in the introduction, more detailed computations can lead to an improvement in the number of colors used by our algorithm. More precisely, it is possible to spare one color when $\Delta \geq 6$. This is the purpose of Theorem 2 of this section. Before proving this theorem, we give two intermediate results, in Lemmas 3 and 4 below.

Lemma 3 *Let $\Delta \geq 6$ be even, and let I be a partially and acyclically colored instance using at most $C'(\Delta) = \frac{\Delta(\Delta-1)}{2}$ colors. Let u be an uncolored vertex of I. If $n_u = \Delta$ and $p_u = p'_u = \frac{\Delta}{2}$, then there exists a way to recolor one of the neighbors of u, still using at most $C'(\Delta)$ colors, in such a way that p'_u strictly decreases.*

Lemma 4 *Let $\Delta \geq 7$ be odd, and let I be a partially and acyclically colored instance using at most $C'(\Delta) = \frac{\Delta(\Delta-1)}{2} - 1$ colors. Let u be an uncolored vertex of I. If $n_u = \Delta$ and $p_u = p'_u = \frac{\Delta-1}{2}$, then there exists a way to recolor one of the neighbors of u, still using at most $C'(\Delta)$ colors, in such a way that p'_u strictly decreases.*

The acyclic coloring algorithm $\mathcal{A}$ presented in proof of Theorem 1 can then be adapted to use one color less, provided that $\Delta \geq 6$. This is the purpose of the following theorem.

Theorem 2 *There exists an $O(n\Delta^2)$ time algorithm to acyclically color any graph of order n and maximum degree $\Delta \geq 6$. This algorithm uses at most:*

(a) $\frac{\Delta(\Delta-1)}{2}$ *colors when Δ is even*

(b) $\frac{\Delta(\Delta-1)}{2} - 1$ *colors when Δ is odd*

Sketch of Proof: Here, we use a slight variant, say $\mathcal{A}'$, of algorithm $\mathcal{A}$ of proof of Theorem 1, as shown below. The idea is the following: we can show that for any uncolored vertex u, there are enough colors to choose from, even if we decrease $C(\Delta)$ by one. That is, we can show that there is a way (1) to modify any instance I not satisfying property $P_I(u)$ into an instance I' (in which the coloring remains proper and acyclic) satisfying property $P_{I'}(u)$ and (2) to color u in such a way that the coloring remains proper and acyclic, thereby proving the correctness of our modified algorithm $\mathcal{A}'$.

Acyclic Coloring of G with at most $C'(\Delta)$ Colors - $\Delta \geq 6$

while there remains at least one uncolored vertex in G **do**
  Let u be an uncolored vertex
  if property $P_I(u)$ is not satisfied **then**
    Modify the color of a neighbor of u, such that the new instance I' satisfies $P_{I'}(u)$
  end if
  if Δ is even and $p'_u = p_u = \frac{\Delta}{2}$ and $n_u = \Delta$ **then**
    Modify the color of any neighbor of u, in such a way that p'_u strictly decreases (Lemma 3)
  end if
  if Δ is odd and $p'_u = p_u = \frac{\Delta-1}{2}$ and $n_u = \Delta$ **then**
    Modify the color of a neighbor of u whose color appears twice, in such a way that p'_u strictly decreases (Lemma 4)
  end if
  Color u in such a way that the new instance remains proper and acyclic
end while

Let us now comment the complexity of algorithm $\mathcal{A}'$: for each "operation" (modification of the instance, or coloring of u), $\mathcal{A}'$ takes $O(\Delta^2)$ time (roughly, in each case, we only need to look at distance 2 from u). Since we have n loops (where n is the order of the graph G), we end up with a complexity of $O(n\Delta^2)$. □

4 A Linear Time Algorithm using at most 9 Colors when $\Delta = 5$

By an even deeper analysis, we are also able to extend Theorem 2 to the case where $\Delta = 5$.

Proposition 1 *For any graph G of maximum degree 5, $a(G) \leq 9$ and there exists a linear time algorithm to acyclically color G in at most 9 colors.*

Sketch of Proof: The coloring algorithm for $\Delta = 5$ is not much different from algorithm $\mathcal{A}'$, but there are more specific cases to treat. For sake of readability, it is not included here, but can be seen in Appendix. During each loop, the coloring decision is made in constant time, since we need to look at distance at most 3 from u, and thus search for the colors of at most $\Delta^3 = 125$ vertices. Hence, altogether, we end up with a linear time algorithm that colors acyclically any graph of degree 5 with at most 9 colors. $\square$

5 Conclusion

In this paper, we have provided a non trivial polynomial time algorithm to acyclically color any graph of maximum degree $\Delta \geq 5$ in $\frac{\Delta(\Delta-1)}{2} - odd(\Delta)$ colors. This algorithm takes $O(n\Delta^2)$ time, hence it is linear when Δ is a constant. Roughly, this algorithm uses half of the colors used by the best deterministic algorithm known so far [ACK$^+$04]. The above results can also be applied for small values of Δ in order to obtain upper bounds for $a(G)$; for instance, if we take $\Delta = 5$, our results show that any graph of maximum degree Δ can be acyclically colored in at most 9 colors. This result can be seen as a follow up to the results of [Grü73] and [Bur79], yielding respectively that any graph G of maximum degree 3 satisfies $a(G) \leq 4$ and any graph of maximum degree 4 satisfies $a(G) \leq 9$. Though we believe the upper bound of 9 colors in the case $\Delta = 5$ is not tight, it still is a great improvement compared to the bound of $\Delta(\Delta - 1) + 2 = 22$ given previously by the result in [ACK$^+$04]. We also note that there exist graphs of maximum degree 5 with acyclic chromatic number 6, such as the complete graph K_6, the complete bipartite graph $K_{5,5}$ or the circulant graph $C_8(1, 2, 4)$. However, we do not know any graph of maximum degree 5 needing strictly more than 6 colors. Finally, we would like to end this paper by recalling some open problems already mentioned in [AMR90], that is: is it possible to display examples of graphs G of maximum degree Δ for which (1) $a(G) = O(\Delta^{\frac{4}{3}})$ (or even $a(G) = o(\Delta^2)$) ? (2) $a(G) = \Omega(\frac{\Delta^{\frac{4}{3}}}{(\log \Delta)^{\frac{1}{3}}})$?

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6 Appendix : An $O(n)$ Algorithm using at most 9 Colors - $\Delta = 5$

We present here the detailed algorithm that we use in the case $\Delta = 5$.

while there remains at least one uncolored vertex in G **do**
  Let u be an uncolored vertex
  if property $P_I(u)$ is not satisfied **then**
    Modify the color of a neighbor of u, such that the new instance I' satisfies $P_{I'}(u)$
  end if
  if $p'_u = 2$ and $n_u = 5$ **then**
    if $p'_u = p_u = 2$ **then**
      Let v be a neighbor of u whose color is represented twice (v has at most 4 colored neighbors)
      if $n_v \leq 3$ **then**
        Color u in such a way that the new instance remains proper and acyclic
      else
        (thus $n_v = 4$)
        if $p'_v = p_v = 2$ **then**
          if all the neighbors of v are such that $p' \geq 2$ **then**
            Color u in such a way that the new instance remains proper and acyclic
          else
            Let w a neighbor of v such that $p_w \leq 1$.
            Recolor w in such a way that p'_v decreases ($\Rightarrow p'_v \leq 1$)
          end if
        else
          Color u in such a way that the new instance remains proper and acyclic
        end if
      end if
    else if $p'_u = p_u + 1 = 2$ **then**
      Let v be a neighbor of u whose color is represented exactly twice (v has at most 4 colored neighbors)
      if $n_v \leq 3$ **then**
        Color u in such a way that the new instance remains proper and acyclic
      else
        (thus $n_v = 4$)
        if $p'_v = p_v = 1$ **then**

```
            if all the neighbors of $v$ are such that $p' \geq 2$ then
                Color $u$ in such a way that the new instance remains proper and acyclic
            else
                Let $w$ a neighbor of $v$ such that $p_w \leq 1$.
                Recolor $w$ in such a way that $p_v$ increases ($\Rightarrow p_v = 2$)
            end if
        else if $p'_v = p_v = 2$ then
            if All the neighbors of $v$ are such that $p' \geq 2$ then
                Color $u$ in such a way that the new instance remains proper and acyclic
            else
                Let $w$ a neighbor of $v$ such that $p_w \leq 1$.
                Recolor $w$ in such a way that $p'_v$ decreases ($\Rightarrow p'_v = 1$)
            end if
        else
            Color $u$ in such a way that the new instance remains proper and acyclic
        end if
      end if
    else
      Color $u$ in such a way that the new instance remains proper and acyclic
    end if
  else
    Color $u$ in such a way that the new instance remains proper and acyclic
  end if
end while
```