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Maximal sets of integers not containing $k+1$ pairwise coprimes and having divisors from a specified set of primes

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We find the formula for the cardinality of maximal set of integers from $[1,\dots,n]$ which does not contain $k+1$ pairwise coprimes and has divisors from a specified set of primes. This formula is defined by the set of multiples of the generating set, which does not depend on n.

Keywords: greatest common divisor, coprimes, squarefree numbers

1 Formulation of the result

Let $\mathbb{P} = \{p_1 < p_2, \dots\}$ be the set of primes and $\mathbb{N}$ be the set of natural numbers. Write $\mathbb{N}(n) = \{1,\dots,n\}$, $\mathbb{P}(n) = \mathbb{P} \bigcap \mathbb{N}(n)$. For $a, b \in \mathbb{N}$ denote the greatest common divisor of a and b by (a,b). Let $S(n,k)$ be the family of sets $A \subset \mathbb{N}(n)$ of positive integers which does not contain $k+1$ coprimes. Define

$$f(n,k) = \max_{A \in S(n,k)} |A|.$$

In the paper [1] the following was proved.

Theorem 1 *For all sufficiently large*

$$f(n,k) = |\mathbb{E}(n,k)|,$$

where

$$\mathbb{E}(n,k) = \{a \in \mathbb{N}(n) : \ a = up_i, \ for \ some \ i = 1,\dots,k\}. \tag{1}$$

Let now $\mathbb{Q} = \{q_1 < q_2 < \dots < q_r\} \subset \mathbb{P}$ be finite set of primes and $R(n,\mathbb{Q}) \subset S(n,1)$ is such family of sets of positive integers that for the arbitrary $a \in A \in R(n,\mathbb{Q})$, $\left(a, \prod_{j=1}^r q_j\right) > 1$. In [2] was proved the following

Theorem 2 *Let $n \geq \prod_{j=1}^r q_j$, then*

$$f(n,\mathbb{Q}) \stackrel{\Delta}{=} \max_{A \in R(n,\mathbb{Q})} |A| = \max_{1 \leq t \leq r} |M(2q_1,\dots,2q_t, q_1 \dots q_t) \bigcap \mathbb{N}(n)|, \tag{2}$$

where $M(B)$ is the set of multiples of the set of integers B.

In [2] the problem was stated of finding the maximal set of positive integers from $\mathbb{N}(n)$ which satisfies the conditions of Theorems 1 and 2 simultaneously i.e. which is a set A without $k+1$ coprimes and such that each element of this set has a divisor from $\mathbb{Q}$. This paper is devoted to the solution of this problem. In our work we use the methods from the paper [1].

Denote $R(n,k,\mathbb{Q}) \subset S(n,k)$ the family of sets of positive integers with the property that an arbitrary $a \in A \in R(n,k,\mathbb{Q})$ has divisor from $\mathbb{Q}$. For given s and $\mathbb{T} = \{r_1 < r_2 < \ldots\} = \mathbb{P} - \mathbb{Q}$ let $F(n,k,s,\mathbb{Q}) \subset R(n,k,\mathbb{Q})$ is the family of sets of squarefree positive numbers such that for the arbitrary $a \in A \in F(n,k,s,\mathbb{Q})$ we have $(r_i,a) = 1,\ i > s$. For given s, r cardinality of the family $F(n,k,s,\mathbb{Q})$ and cardinalities of each $A \in F(n,k,s,\mathbb{Q})$ are bounded from above as $n \to \infty$.

Next we formulate our main result which extent the result of the Theorems 1, 2 and in some sense include both of them.

Theorem 3 *If $\mathbb{Q} \neq \emptyset$, then for sufficiently large n the following relation is valid*

$$\varphi(n,k,\mathbb{Q}) \stackrel{\Delta}{=} \max_{A \in R(n,k,\mathbb{Q})} |A| = \max_{F \in F(n,k,s-1,\mathbb{Q})} |M(F) \bigcap \mathbb{N}(n)|, \tag{3}$$

where s is the minimal integer which satisfies the inequality $r_s > r$.

2 Proof of the Theorem 3

Let's remind the definition of the left pushing which the reader can find in [2]. For the arbitrary

$$a = up_j^{\alpha},\ p_i < p_j,\ (p_i p_j, u) = 1,\ \alpha > 0 \ and\ p_j \notin \mathbb{Q} \ or\ p_i, p_j \in \mathbb{Q} \tag{4}$$

define

$$L_{i,j}(a,\mathbb{Q}) = p_i^{\alpha} u.$$

For a not of the form (4) we set $L_{i,j}(a,\mathbb{Q}) = a$. For $A \subset \mathbb{N}$ denote

$$L_{i,j}(a,A,\mathbb{Q}) = \begin{cases} L_{i,j}(a,\mathbb{Q}), & L_{i,j}(a,\mathbb{Q}) \notin A, \\ a, & L_{i,j}(a,\mathbb{Q}) \in A. \end{cases}$$

At last set

$$L_{i,j}(A,\mathbb{Q}) = \{L_{i,j}(a,A,\mathbb{Q});\ a \in A\}.$$

We say that A is left compressed if for the arbitrary $i < j$

$$L_{i,j}(A,\mathbb{Q}) = A.$$

It can be easily seen that every finite $A \subset \mathbb{N}$ after finite number of left pushing operations can be made left compressed,

$$|L_{i,j}(A,\mathbb{Q})| > |A|$$

and if $A \in R(n,k,\mathbb{Q})$, then $L_{i,j}(A,\mathbb{Q}) \in R(n,k,\mathbb{Q})$.

If we denote $O(n,k,\mathbb{Q}) \subset R(n,k,\mathbb{Q})$ the families of sets on which achieved $\max$ in (3) and $C(n,k,\mathbb{Q}) \subset R(n,k,\mathbb{Q})$ is the family of left compressed sets from $R(n,k,\mathbb{Q})$, then it follows that $O(n,k,\mathbb{Q}) \bigcap C(n,k,\mathbb{Q}) \neq \emptyset$. Next we assume that $A \in C(n,k,\mathbb{Q}) \bigcap O(n,k,\mathbb{Q})$.

For the arbitrary $a \in A$ we have the decomposition $a = a^1a^2$, where $a^1 = r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f},\ r_i < r_j,\ i < j,\ a^2 = q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell};\ q_{j_m} < q_{j_s},\ m < s;\ \alpha_j, \beta_j > 0$. If $a = r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f} q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell} \in A,\ \alpha_j, \beta_j > 0$,then $\bar{a} = r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell} \in A$ as well and also $\hat{a} = ua \in A$ for all $u \in \mathbb{N}:\ ua \le n$. Consider all squarefree numbers $A^* \subset A$ and for given a^2 the set of all a^1 such that $a^1a^2 \in A^*$. This set is the ideal generated by the division. The set of minimal elements from this ideal denote by $P(a^2, A^*)$. It follows that $(A \in O(n,k,\mathbb{N}))$,

$$A = M(\{a^1a^2;\ a^1 \in P(a^2, A^*)\}) \bigcap \mathbb{N}(n),$$

For each a^2 we order $\{a_1^1 < a_2^1 < \dots\} = P(a^2, A^*)$ lexicographically according to their decomposition $a_i^1 = r_{i_1} \dots r_{i_f}$. Let ρ is the maximal over the choice of a^2 positive integer such that r_ρ divide some a_i^1 for which $a_i^1 a^2 \in A^*$. From the left compressedness of the set A it follows that $a' = a_j^1 a^2,\ j < i$ also belongs to A. Then the set B of elements $b = b^1b^2 \le n,\ \left(b^1, \prod_{j=1}^r q_j\right) = 1$ such that $b^2 = q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell},\ \beta_j > 0$ and $a_i^1 | b^1,\ a_j^1 \nmid b^1,\ j < i$ is exactly the set

$$B(a) = \left\{ u \le n:\ u = r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f} r_\rho^{\alpha_\rho} q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell} F;\ \alpha_i, \beta_i > 0,\ \left(F, \prod_{j=1}^{\rho} r_j \prod_{j=1}^{r} q_j\right) = 1 \right\}.$$

Denote

$$P^\rho(a^2, A^*) = \left\{a \in P(a^2, A^*) : (a, r_\rho) = r_\rho\right\},$$

$$P_s^\rho(A^*) = \left\{ a \in P^\rho(a^2, A^*)\ \mathit{for\ some}\ a^2,\ \mathit{such\ that}\ \left(a^2, \prod_{j=1}^{s} q_j\right) = q_s \right\}$$

and

$$L^\rho(a^2) = \bigcup_{a \in P^\rho(a^2, A^*)} B(a).$$

Then the set $\bigcup_{s=1}^r P_s^\rho(A^*)$ is exactly the set $\bigcup_{a^2} P^\rho(a^2, A^*)$ of numbers which are divisible by r_ρ. Because each $a \in P(a^2, A^*)$ for all a^2 has divisor from $\mathbb{Q}$ it follows that for some $1 \le s \le r$

$$\left| \bigcup_{a \in P_s^\rho(A^*)} B(a) \right| \ge \frac{1}{r} \left| \bigcup_{a^2} L^\rho(a^2) \right|. \tag{5}$$

Next for this s we define the transformation

$$\bar{P}(a^2, A^*) = \left(P(a^2, A^*) - P^\rho(a^2, A^*)\right) \bigcup R_s^\rho(a^2, A^*),$$

where

$$\begin{aligned} R_s^\rho(a^2, A^*) &= \left\{v \in \mathbb{N};\ vr_\rho \in P_s^\rho(a^2, A)\right\}, \\ P_s^\rho(a^2, A^*) &= \left\{a = a^1a^2 \in P_s^\rho(A^*)\right\}. \end{aligned}$$

It is easy to see that

$$\bigcup_{a^2} \bar{P}(a^2, A^*) \subset S(n, k, \mathbb{Q}).$$

Next we prove that if $r_\rho > r$, then

$$\left| M\left(\bigcup_{a^2} \bar{P}(a^2, A^*)\right) \bigcap \mathbb{N}(n) \right| > |A| \tag{6}$$

which gives the contradiction to the maximality of A.

For $a \in R_s^\rho(a^2, A^*),\ a = r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell},\ r_{i_1} < \dots < r_{i_f} < r_\rho,\ q_{j_1} \dots q_{j_\ell} = a^2$ denote

$$D(a) = \left\{ v \in \mathbb{N}(n) :\ v = r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f} q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell} T,\ \alpha_j, \beta_j \geq 1,\ \left(T, \prod_{j=1}^{\rho-1} r_j \prod_{j=1}^{r} q_j \right) = 1 \right\}.$$

It can be easily seen that

$$D(a) \bigcap D(a') = \emptyset, a \neq a'$$

and

$$M\left(\bigcup_{a^2} \left(P(a^2, A^*) - P^\rho(a^2, A^*) \right) \right) \bigcap D(a) = \emptyset.$$

Thus to prove (6) it is sufficient to show, that for large $n > n_0$

$$|D(a) > r|B(ar_\rho)|. \tag{7}$$

To prove (7) we consider three cases.

First case when $n/(ar_\rho) \geq 2$ and $\rho > \rho_0$. It follows that

$$\begin{aligned} |B(ar_\rho)| &\leq c_2 \sum_{\alpha_i, \alpha, \beta_i \geq 1} \frac{n}{r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f} r_\rho^{\alpha} q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell}} \prod_{j=1}^{\rho} \left(1 - \frac{1}{r_j}\right) \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right) \qquad (8) \\ &= c_2 \frac{n}{(r_{i_1} - 1) \dots (r_{i_f} - 1)(r_\rho - 1)(q_{j_1} - 1) \dots (q_{j_\ell} - 1)} \prod_{j=1}^{\rho} \left(1 - \frac{1}{r_j}\right) \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right). \end{aligned}$$

At the same time

$$\bar{D}(a) \triangleq \left\{ v \in \mathbb{N}(n);\ v = r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell} F_1,\ \left(F_1, \prod_{j=1}^{\rho-1} r_j \prod_{j=1}^{r} q_j \right) = 1 \right\} \subset D(a)$$

and we obtain the inequalities

$$|D(a)| \geq |\bar{D}(a)| \geq c_1 \frac{n}{r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell}} \prod_{j=1}^{\rho-1} \left(1 - \frac{1}{r_j}\right) \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right). \tag{9}$$

Thus from (8), (9) it follows that

$$\begin{aligned} \frac{|D(a)|}{B(ar_\rho)} &\geq \frac{c_1}{c_2} r_\rho \frac{(r_{i_1} - 1) \dots (r_{i_f} - 1)}{r_{i_1} \dots r_{i_f}} \prod_{j \in [r] - \{j_1, \dots, j_\ell\}} \left(1 - \frac{1}{q_j}\right) \\ &\geq \frac{c_1}{c_2} \prod_{j=1}^{f} \left(1 - \frac{1}{r_j}\right) r_\rho \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right) > r. \end{aligned}$$

Now let's $n/(ar_\rho) \geq 2,\ \rho < \rho_0$. Then we obtain the inequalities

$$|B(ar_\rho)| \quad < \quad (1+\epsilon)\frac{n}{(r_{i_1}-1)\ldots(r_{i_f}-1)(r_\rho-1)(q_{j_1}-1)\ldots(q_{j_\ell}-1)}\prod_{j=1}^{\rho}\left(1-\frac{1}{r_j}\right)\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right),$$

$$|D(a)| \quad > \quad (1-\epsilon)\frac{n}{(r_{i_1}-1)\ldots(r_{i_f}-1)(q_{j_1}-1)\ldots(q_{j_\ell}-1)}\prod_{j=1}^{\rho-1}\left(1-\frac{1}{r_j}\right)\prod_{j=1}^{r}\left(1-\frac{1}{q_j}\right).$$

From these inequalities it follows that

$$\frac{|D(a)|}{|B(ar_\rho)|} > \frac{1-\epsilon}{1+\epsilon}r_\rho > r.$$

Here the last inequality is valid for sufficiently small ϵ because $r_\rho > r$.

The last case is when $1 \leq n/(ar_\rho) < 2$. In this case $|B(ar_\rho)| = 1$. Let's $r_{i_1}\ldots r_{i_f}r_\rho q_{j_1}\ldots q_{j_\ell} = B(ar_\rho)$. Then we choose $r_g > (q_{j_1})^r$ and $n > \prod_{j=1}^{q} r_j \prod_{j=1}^{r} q_j$. We have $r_\rho > r_g$. Indeed, otherwise

$$n > \prod_{j=1}^{g} r_j \prod_{j=1}^{r} q_j > 2\prod_{j=1}^{\rho}\prod_{j=1}^{r} q_j > 2ar_\rho$$

which is the contradiction to our case.

Hence

$$\{r_{i_1}\ldots r_{i_f}q_{j_1}\ldots q_{j_\ell}, r_{i_1}\ldots r_{i_f}q_{j_1}^2\ldots q_{j_\ell},$$
$$\ldots, r_{i_1}\ldots r_{i_f}q_{j_1}^r\ldots q_{j_\ell}, r_{i_1}\ldots r_{i_f}q_{j_1}\ldots q_{j_\ell}r_\rho\} \subset D(a).$$

Thus in this case also $|D(a)| > r = r|B(ar_\rho)|$.

From the above follows that for sufficiently large $n > n_0(\mathbb{Q})$ for all $a \in R_s^\rho(a^2, A^*)$ inequality (7) is valid and taking into account (5) we obtain (6). This gives the contradiction to the maximality of A. Hence the maximal $r_\rho \in \mathbb{P} - \mathbb{Q}$ which appear as the divisor of some $a \in \bigcup_{a^2} P(a^2, A^*)$ such that $M(A^*)\bigcap \mathbb{N}(n) \in O(n,k,\mathbb{Q})$ satisfies the condition $r_\rho \leq r$. This inequality gives the statement of Theorem.

This is joint work with R.Ahlswede.

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