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A Probabilistic Counting Lemma for Complete Graphs

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We prove the existence of many complete graphs in almost all sufficiently dense partitions obtained by an application of Szemerédi's Regularity Lemma. More precisely, we consider the number of complete graphs K_ℓ on ℓ vertices in ℓ-partite graphs where each partition class consists of n vertices and there is an ε-regular graph on m edges between any two partition classes. We show that for all $\beta > 0$, at most a β^m-fraction of graphs in this family contain less than the expected number of copies of K_ℓ provided ε is sufficiently small and $m \geq Cn^{2-1/(\ell-1)}$ for a constant $C > 0$ and n sufficiently large. This result is a counting version of a restricted version of a conjecture by Kohayakawa, Łuczak and Rödl [8] and has several implications for random graphs.

1 Introduction

The celebrated Regularity Lemma by Szemerédi [15] states, roughly speaking, that the vertex set of every sufficiently large graph can be partitioned into a constant number of classes in such a way that the edges between most partition classes are distributed regularly. It is particularly useful for finding subgraphs in large graphs since one can employ the structure of the partition obtained from the lemma and show the existence of the subgraph in there. Lemmas that assert the existence of subgraphs in such a partition are called embedding or counting lemmas. For numerous applications of Szemerédi's Regularity Lemma and various powerful embedding lemmas for dense graphs, we refer the reader to the excellent survey articles [11, 12].

We are interested in a counting lemma for sparse graphs. It is well known that Szemerédi's original lemma is only helpful for dense graphs (that is graphs with n vertices and $\Theta(n^2)$ edges), but Kohayakawa [7] and Rödl (unpublished) independently introduced a variant of Szemerédi's Regularity Lemma for sparse graphs. Unfortunately, there do not (yet) exist corresponding counting or embedding lemmas, and moreover one can show that straight-forward generalizations of these are false as was noted by Łuczak (see [5, 9]). However, Kohayakawa, Łuczak, and Rödl [8] formulated a probabilistic embedding lemma (see Conjecture 3) that, if true, would solve some long-standing open problems in random graph theory [5, 6, 8, 13].

Conjecture 3, an embedding lemma, states the existence of a fixed graph H in a certain structure. We generalize this conjecture to a counting version and ask for several copies of H in this structure. We then prove this generalized version for some special cases. In order to state the conjecture, we need the following definitions.

Definition 1 *For a graph $G = (V, E)$ and two disjoint sets $V_1, V_2 \subseteq V$, we denote the set of edges with one endpoint in V_1 and one endpoint in V_2 by $E(V_1, V_2)$. The* density *$d(V_1, V_2)$ is defined as $d(V_1, V_2) = |E(V_1, V_2)|/(|V_1||V_2|)$. We say that the graph induced by V_1, V_2 is* (ε)-regular *if for all $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ with $|V_1'| \geq \varepsilon|V_1|$ and $|V_2'| \geq \varepsilon|V_2|$, $|d(V_1', V_2') - d(V_1, V_2)| \leq \varepsilon d(V_1, V_2)$.*

Note that a bipartite graph $B = (V_1 \cup V_2, E)$ is (ε)-regular if for all sufficiently large sets V_1' and V_2' the density is only $\varepsilon d(V_1, V_2)$ away from the density of the graph. In particular, the smaller the density of the underlying graph and ε are, the closer are the densities of large sets of an (ε)-regular graph to their expected value.

We have defined (ε)-regularity for pairs of vertex sets, but we are mainly interested in ℓ-partite graphs such that certain pairs of partition classes form an (ε)-regular graph.

Definition 2 *For a graph H, let $\mathcal{G}(H, n, m)$ be the family of graphs on the vertex set $V = \bigcup_{x \in V(H)} V_x$ with pairwise disjoint sets of vertices V_x of size n and the edge set $E = \bigcup_{\{x,y\} \in E(H)} E_{xy}$, where $E_{xy} \subseteq V_x \times V_y$ and $|E_{xy}| = m$. Let $\mathcal{G}(H, n, m, \varepsilon) \subseteq \mathcal{G}(H, n, m)$ denote the set of graphs in $\mathcal{G}(H, n, m)$ satisfying that each $(V_x \cup V_y, E_{xy})$ is an (ε)-regular graph.*

Note that a graph $G \in \mathcal{G}(H, n, m, \varepsilon)$ looks like H with every vertex blown up to a class of n vertices and every edge to a set of m edges, which form an (ε)-regular graph between the corresponding vertex classes. One may obtain graphs of $\mathcal{G}(H, n, m, \varepsilon)$ by applying Szemerédi's Regularity Lemma to a large graph G (and removing some edges). Thus, if one wants to find a copy of H in G, it suffices to find H in all graphs of $\mathcal{G}(H, n, m, \varepsilon)$. As we have already mentioned, this is not always possible if $m = o(n^2)$. Moreover, if one considers graphs in $G(H, n, m)$, then one can show that there exists a constant c such that at least a c^m fraction does not contain H. Kohayakawa, Łuczak and Rödl conjectured that the fraction of graphs in $G(H, n, m, \varepsilon)$ that does not contain a graph H is substantially smaller.

Conjecture 3 (KŁR-Conjecture [8]) *Let H be a fixed graph and let*

$$\mathcal{F}(H, n, m) = \{G \in \mathcal{G}(H, n, m) : H \text{ is not a subgraph of } G\}.$$

For any $\beta > 0$, there exist constants $\varepsilon_0 > 0$, $C > 0$, $n_0 > 0$ such that for all $m \geq Cn^{2-1/d_2(H)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$, $|\mathcal{F}(H, n, m) \cap \mathcal{G}(H, n, m, \varepsilon)| \leq \beta^m \binom{n^2}{m}^{|E(H)|}$, where

$$d_2(H) = \max\left\{\frac{|E(F)| - 1}{|V(F)| - 2} : F \subseteq H, |V(F)| \geq 3\right\}.$$

Conjecture 3 can easily be verified for trees and it is also known to be true when H is a cycle [1] or when H is the complete graph on 3 [13], 4 [3] and 5 [4] vertices.

Our main result is a counting version of Conjecture 3 for H being the complete graph K_ℓ of size ℓ and $m \geq Cn^{2-1/(\ell-1)}$ (instead of $m \geq Cn^{2-2/(\ell+1)}$ as stated in Conjecture 3). In order to state it, we need the following definition.

Definition 4 *We denote the family of graphs in $\mathcal{G}(H, n, m)$ that contain less than $(1-\delta)n^{|V(H)|}\left(m/n^2\right)^{|E(H)|}$ copies of H by $\mathcal{F}(H, n, m, \delta)$.*

Observe that apart from the factor $(1 - \delta)$ this quantity is equal to the expected number of subgraphs isomorphic to H in $G \in \mathcal{G}(H, n, m)$ if the edges are randomly distributed between the vertex classes. We conjecture that one can replace $\mathcal{F}(H, n, m)$ by $\mathcal{F}(H, n, m, \delta)$ in Conjecture 3.

Conjecture 5 (Counting Lemma) *Let H be a fixed graph. For any $\beta > 0$ and $\delta > 0$, there exist constants $\varepsilon_0 > 0$, $C > 0$, $n_0 > 0$ such that for all $m \geq Cn^{2-1/d_2(H)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$,*

$$|\mathcal{F}(H, n, m, \delta) \cap \mathcal{G}(H, n, m, \varepsilon)| \leq \beta^m \binom{n^2}{m}^{|E(H)|}.$$

Our main result states that Conjecture 5 (and thus Conjecture 3) holds true if H is a complete graph and m is slightly greater than conjectured.

Theorem 6 *For all $\ell \geq 3$, $\delta > 0$, and $\beta > 0$, there exist constants $n_0 \in \mathbb{N}$, $C > 0$, and $\varepsilon > 0$ such that*

$$|\mathcal{F}(K_\ell, n, m, \delta) \cap \mathcal{G}(K_\ell, n, m, \varepsilon)| \leq \beta^m \cdot \binom{n^2}{m}^{\binom{\ell}{2}}. \tag{1}$$

provided that $m \geq Cn^{2-1/(\ell-1)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$.

It is well known [6, 8, 13] that Conjecture 3 has several implications. In particular Theorem 6 implies the following.

Corollary 7 *[13] For all $\ell \geq 3$ and every $\delta > 0$, there exists $c = c(\delta, \ell)$ such that the probability that a graph chosen uniformly at random from the family of all K_ℓ-free labeled graphs on n vertices and $m \geq cn^{2-1/(\ell-1)}$ edges can be made $(\ell-1)$-partite by removing δm edges tends to one as n tends to infinity.*

Corollary 8 *[13] For all $\ell \geq 3$ and for every $\varepsilon > 0$, there exist $c = c(\varepsilon, \ell)$ and $n_0 = n_0(\varepsilon, \ell)$ such that for $n \geq n_0$ and $cn^{2-1/(\ell-1)} \leq m \leq n^2/c$, a graph $G_{n,m}$ drawn uniformly at random from all labeled graphs on n vertices and m edges satisfies*

$$\left(\frac{\ell-2}{\ell-1} - \varepsilon\right)^m \leq \mathbb{P}\left(G(n,m) \text{ does not contain } K_\ell\right) \leq \left(\frac{\ell-2}{\ell-1} + \varepsilon\right)^m.$$

Another implication of Theorem 6 concerns the maximal number of edges $\mathrm{ex}(G, H)$ that a subgraph of G may have without containing a copy of H if $G = G_{n,p}$. As usual, $G_{n,p}$ is the probability space of all graphs on n labeled vertices, where each edge is present with probability p independently of all other edges. Let $\ell \geq 3$ and $0 < p = p(n) \leq 1$ be such that $pn^{1/(\ell-1)} \to \infty$ as $n \to \infty$. Then asymptotically almost surely (a.a.s.), that is, with probability tending to one as n tends to infinity

$$\mathrm{ex}(G_{n,p}, K_\ell) \leq \left(1 - \frac{1}{\ell-1} + o(1)\right)\binom{n^2}{m}^{\binom{\ell}{2}}.$$

This result was already shown in [10, 14]. Let us remark that using Theorem 6 together with the sparse version of the Regularity Lemma in fact implies a slightly stronger result. Namely, we obtain that a.a.s. every subgraph of $G_{n,p}$ with $(1+\delta)\mathrm{ex}(G_{n,p}, K_\ell)$ edges contains not just one copy of a K_ℓ (which is the essential statement of the above result) but $c_\delta n^\ell p^{\ell(\ell-1)/2}$ many copies.

We shall prove Theorem 6 by induction on ℓ. Let us remark that the lower bound on m in the theorem depends on the base case of the induction which is $\ell = 3$ in our proof. If we want to prove Conjecture 3, we can chose $\ell = 5$ as the base case for the induction since Conjecture 3 has been verified in this setting in [4]. The induction would work analogously and yield the threshold $m \geq n^{2-1/(\ell-2)}$. This includes the main result of [14].

The remainder of this extended abstract is organized as follows. In Sect. 2 we explain our notation and state several results about regular graphs. Sect. 3 is devoted to the exposition of our main lemma, Lemma 12, the proof of which will appear elsewhere [2]. Roughly speaking, this lemma asserts that for $\varepsilon' \gg \varepsilon$, certain subgraphs of $G \in \mathcal{G}(K_\ell, n, m, \varepsilon)$ are (ε')-regular *and* have further "typical" properties with very high probability. Suppose the members of a family of bad graphs $\mathcal{B}(K_\ell, x, y) \subseteq \mathcal{G}(K_\ell, x, y)$ are very rare, that is, if we choose a graph $H \in \mathcal{G}(K_\ell, x, y, \varepsilon)$ uniformly at random, then the probability of $H \in \mathcal{B}(K_\ell, x, y)$ is exponentially small for x sufficiently large, $y \geq y_0(x)$, and ε sufficiently small. Now consider a graph $G \in \mathcal{G}(K_\ell, n, m, \varepsilon)$ with $n \gg x$ vertices, and choose subsets $X_i \subseteq V_i$, $1 \leq i \leq \ell$, of size x randomly. The main result of [1] essentially states that for all $\varepsilon' > 0$, there exists $\varepsilon > 0$ such that with high probability the induced graph $G[X_1, \dots, X_\ell]$ contains a large (ε')-regular subgraph G', i.e., $G' \in \mathcal{G}(K_\ell, x', y, \varepsilon')$ with $x' \sim x$ and $y \sim x^2 m/n^2$. The lemma asserts that G' satisfies even more, namely that $G' \in \mathcal{G}(K_\ell, x', y, \varepsilon') \setminus \mathcal{B}(K_\ell, x', y)$ provided $x^2 m/n^2 \geq y_0(x)$. Since the size of the neighborhood into one vertex class of a typical vertex is about m/n, we can apply the lemma for $x \approx m/n$ and $\ell - 1$ and deduce that most vertices in one class have nicely structured neighborhoods within the other $\ell - 1$ classes. This allows the application of the induction hypothesis in the proof of our main result Theorem 6, which is given in Sect. 4.

2 Notation and Preliminaries

All graphs are labeled, undirected, and simple, i.e., contain no multiple edges. We denote the vertex set of a graph G by $V(G)$ and its edge set by $E(G)$. For any vertex $v \in V(G)$, we denote its neighborhood by $\Gamma(v)$. For any vertex set $X \subseteq V(G)$, $G[X]$ denotes the graph induced by X.

The logarithm is always the natural logarithm. All variables denoted by small greek letters have a tiny value, say, less than 10^{-10}. We will omit floors and ceilings in order to round expressions like n/k to an integral value. As we assume that k is fixed and n tends to infinity, this does not make any essential difference, but greatly simplifies notation. We write $a \sim_\varepsilon b$ if $(1-\varepsilon)b \leq a \leq (1+\varepsilon)b$.

As in Definitions 2 and 4, we use capital, curly letters to denote families of graphs. To simplify notation slightly we shall write $\mathcal{G}(\ell, n, m)$ for $\mathcal{G}(K_\ell, n, m)$ and $\mathcal{G}(\ell, n, m, \varepsilon)$ for $\mathcal{G}(K_\ell, n, m, \varepsilon)$ respectively. We define the family $\widetilde{\mathcal{G}}(\ell, n, m, \varepsilon)$, which is a superset of $\mathcal{G}(\ell, n, m, \varepsilon)$, as follows. $\widetilde{\mathcal{G}}(\ell, n, m, \varepsilon)$ is the family of ℓ-partite graphs on ℓ pairwise disjoint vertex sets $V_1, \dots, V_\ell$ of size n such that for all $1 \leq i < j \leq \ell$ the bipartite graph between V_i and V_j

(i) has $m_{i,j} \sim_\varepsilon m$ edges and

(ii) is (ε)-regular.

In contrast to $\mathcal{G}(\ell, n, m, \varepsilon)$ the graphs in $\widetilde{\mathcal{G}}(\ell, n, m, \varepsilon)$ do not have a fixed number of edges, but the number of edges may vary by some ε-fraction in every regular pair of vertex classes.

The following theorem from [1] states that most small sets of an (ε)-regular graph also contain a large regular subgraph.

Theorem 9 ([1]) *For $0 < \beta, \varepsilon' < 1$, there exist $\varepsilon_0 = \varepsilon_0(\beta, \varepsilon') > 0$ and $C = C(\varepsilon')$ such that for any $0 < \varepsilon \leq \varepsilon_0$, every (ε)-regular graph $G = (V_1 \cup V_2, E)$ with density d satisfies that the number of sets*

$Q \subset V_1$ of size $q = |Q| \geq Cd^{-1}$ that contain a set $\widetilde{Q}$ of size at least $(1-\varepsilon')|Q|$ inducing an (ε')-regular graph of density $d' \sim_\varepsilon d$ with V_2, is at least $(1-\beta^q)\binom{|V_1|}{q}$.

Theorem 9 can be applied to obtain the following consequence [5].

Theorem 10 ([5]) *Let $\ell \geq 3$ be any integer. For all $\alpha > 0$, $\beta > 0$ and $\varepsilon' > 0$, there exist constants $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \varepsilon) > 0$ and $C = C(\varepsilon') > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, n sufficiently large, and $m \geq Cn^{3/2}$ all but at most $\beta^m \binom{n^2}{m}^{\binom{\ell}{2}}$ graphs in $\mathcal{G}(\ell, n, m, \varepsilon)$ satisfy that all but αn vertices v in V_1 have neighborhoods that contain a graph in $\mathcal{G}(\ell-1, x, y, \varepsilon')$ with $x = (1-\varepsilon')m/n$ and $y = (1-\varepsilon')x^2 m/n^2$.*

3 Typical Tuples of Sublinear Subsets

Before we state the main result of this section, let us discuss some consequences of the inheritance property of ε-regularity stated in Theorem 9. It was proved in [5] that given $\varepsilon' > 0$ and $\beta > 0$ we can find $\varepsilon > 0$ and $C > 0$ such that all graphs in $\mathcal{G}(\ell, n, m, \varepsilon)$ satisfy for all $x \geq Cn^2/m$ the following property: if we choose sets $X_i \subseteq V_i$ of size $(1+\varepsilon')x$ randomly, then with probability $1-\beta^x$ these sets contain a graph from $\widetilde{\mathcal{G}}(\ell, x, x^2m/n^2, \varepsilon')$.

In this section we want to generalize this result as follows. Instead of just requiring that the graph induced by the tuple $(X_1, \ldots, X_\ell)$ contains a regular subgraph that has approximately the expected number of edges, we want to deduce that it also has further "typical" properties. Here we call a property "typical" if for suitable n's and m's, all but a β^m-fraction of the graphs in $\mathcal{G}(\ell, n, m, \varepsilon)$ satisfy it. We formalize this as follows.

Definition 11 *We say that a family $\mathcal{B}(\ell, n, m) \subseteq \mathcal{G}(\ell, n, m)$ is* small with respect to a function $m_0(n)$ *if for all $\beta > 0$, there exist constants $n_\beta \in \mathbb{N}$, $C_\beta > 0$, and $\varepsilon_\beta > 0$ such that*

$$|\mathcal{B}(\ell, n, m) \cap \mathcal{G}(\ell, n, m, \varepsilon_\beta)| \leq \beta^m \cdot \binom{n^2}{m}^{\binom{\ell}{2}} \qquad \forall\, n \geq n_\beta,\ m \geq C_\beta m_0(n). \tag{2}$$

Thus our main result Theorem 6 states that for $\ell > 3$ and $\delta > 0$, the family $\mathcal{F}(\ell, n, m, \delta)$ is small with respect to $n^{2-1/(\ell-1)}$. The key for proving this theorem will be our next lemma.

Lemma 12 ([2]) *Let $y_0(x) \geq x$ be a monotone increasing function and let $\mathcal{B}(\ell-1, x, y)$ be small with respect to y_0. For all $\beta, \varepsilon' > 0$, there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all ε, x, and y satisfying*

$$0 < \varepsilon \leq \varepsilon_0, \quad x = (1-\varepsilon')\frac{m}{n}, \quad m \gg n^{3/2}\sqrt{\log n}, \quad Cy_0(x) \leq \frac{m^3}{n^4}, \quad \varepsilon' \frac{m^3}{n^4} \leq y \leq (1-\varepsilon')^3 \frac{m^3}{n^4},$$

and n sufficiently large, all but at most $\beta^m \binom{n^2}{m}^{\binom{\ell}{2}}$ graphs $G \in \mathcal{G}(\ell, n, m, \varepsilon)$ satisfy the following property: there exist at least $(1-\varepsilon')n$ vertices in V_1 that contain a member of the family $\mathcal{G}(\ell-1, x, y, \varepsilon') \setminus \mathcal{B}(\ell-1, x, y)$ in their neighborhood. □

4 Proof of Theorem 6

Before we come to the proof of Theorem 6, we shall proof the base case $\ell = 3$ of the induction.

Lemma 13 *For all $\delta > 0$, the family $\mathcal{F}(3, n, m, \delta)$ is small with respect to $m_3 = n^{3/2}$.*

Proof: Let $\delta' > 0$ satisfy $(1-\delta) \leq (1-\delta')^4$. Apply Theorem 10 with parameters $\ell \leftarrow 3, \alpha \leftarrow \delta', \delta \leftarrow \delta'$, and β. Hence, we obtain constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $m \geq Cn^{3/2}$, and n sufficiently large, all but at most $\beta^m \binom{n^2}{m}^3$ graphs $G \in \mathcal{G}(3, n, m, \varepsilon)$ satisfy that at least $(1-\delta')n$ vertices v in V_1 have neighborhoods containing a bipartite graph with at least $(1-\delta')^3 m^3/n^4$ edges. Since each edge forms a triangle with v, the number of triangles in G is at least $(1-\delta')^4 nm^3/n^4 \geq (1-\delta)m^3/n^3$. $\square$

Proof of Theorem 6: We prove the theorem by induction on ℓ. The base case is settled by Lemma 13. So assume that the Theorem holds for some $\ell - 1 \geq 3$, i.e., for all $\delta' > 0$, the family $\mathcal{F}(\ell-1, x, y, \delta')$ is small with respect to $y_{\ell-1}(x) = x^{2-1/(\ell-2)}$. Choose $\delta' > 0$ such that $(1-\delta')^{\binom{\ell}{2}+2} \geq 1-\delta$. Apply Lemma 12 with the family $\mathcal{F}(\ell-1, x, y, \delta')$ and the parameters β and $\varepsilon' \leftarrow \delta'$ in order to obtain constants $\varepsilon_0 > 0$ and $C_{\ell-1} > 0$. Let $C = C_{\ell-1}$. Observe that $m \geq Cn^{2-1/(\ell-1)} \geq Cn^{5/3} \gg n^{3/2}\log n$ since $\ell \geq 4$, and that

$$C_{\ell-1} y_{\ell-1}(x) = C_{\ell-1}\left[(1-\delta')\frac{m}{n}\right]^{\frac{2\ell-5}{\ell-2}} \leq C_{\ell-1} m^{-\frac{\ell-1}{l-2}} n^{\frac{2\ell-3}{\ell-2}} \frac{m^3}{n^4} \leq C\frac{m^3}{n^4}$$

since $m \geq Cn^{2-1/(\ell-1)}$. Thus, for $y = (1-\delta')^3 m^3/n^4$ and n sufficiently large, all but at most $\beta^m \binom{n^2}{m}^{\binom{\ell}{2}}$ graphs $G \in \mathcal{G}(\ell, n, m, \varepsilon)$ satisfy the following property: there are at least $(1-\delta')n$ vertices $v \in V_1$ that contain a member of the family $\mathcal{G}(\ell-1, x, y, \delta') \setminus \mathcal{F}(\ell-1, x, y, \delta')$ in their neighborhood. But this means that each such vertex v has at least $(1-\delta')x^{\ell-1}\left(y/x^2\right)^{\binom{\ell-1}{2}}$ cliques of size $\ell - 1$ in its neighborhood. Hence, the number of complete graphs on ℓ vertices in G is at least

$$\begin{aligned}(1-\delta')n(1-\delta')x^{\ell-1}\left(\frac{y}{x^2}\right)^{\binom{\ell-1}{2}} &= (1-\delta')^2 n\left[(1-\delta')\frac{m}{n}\right]^{\ell-1}\left[(1-\delta')\frac{m}{n^2}\right]^{\binom{\ell-1}{2}} \\ &= (1-\delta')^{\binom{\ell-1}{2}+\ell+1} n^\ell \left(\frac{m}{n^2}\right)^{\binom{\ell-1}{2}+\ell-1} \geq (1-\delta)n^\ell\left(\frac{m}{n^2}\right)^{\binom{\ell}{2}}.\end{aligned}$$

$\square$

As mentioned in the introduction, the threshold $m \geq n^{2-1/(\ell-1)}$ can be improved by lifting the base case of the induction. If we used $\ell = 4$, the induction would work for $m \geq n^{2-1/(\ell-3/2)}$. The case $\ell = 5$ yields $m \geq n^{2-1/(\ell-2)}$ and so forth. Rough calculations reveal that in order to prove the conjectured threshold $m \geq n^{2-2/(\ell+1)}$ inductively, one has to use the induction assumption $\ell - 2$ instead of $\ell - 1$.

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