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Equivalent Subgraphs of Order 3

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It is proved that any graph of order $14n/3 + O(1)$ contains a family of n induced subgraphs of order 3 such that they are vertex-disjoint and equivalent to each other.

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1 Introduction

A graph is finite and non-directed with no multiple edge or loop. For a graph G, we denote the vertex set G by $V(G)$. Let G and H be a pair of graphs and let n be a positive integer. A partition $V(G)$ into $V_0, V_1, \ldots, V_n$ is called an (n, H)-*decomposition* of G, if $\langle V_i \rangle_G \cong H$ for $1 \leq i \leq n$, where $\langle V_i \rangle_G$ is a subgraph of G induced by V_i. Let $N(G, H)$ be the maximum integer n such that G admits an (n, H)-decomposition. For a family of graphs $\mathcal{H}$, we denote $\max\{N(G, H) : H \in \mathcal{H}\}$ by $N(G, \mathcal{H})$. Moreover, for a positive integer n, we define $f(n, \mathcal{H})$ as the minimum integer s such that $N(G, \mathcal{H}) \geq n$ for any graph G of order s.

The function $f(n, \mathcal{H})$ has a close connection to Ramsey numbers. The classical Ramsey number $R(k, l)$ is defined as the minimum integer s such that any graph G of order s contains K_k or $\overline{K_l}$ as a subgraph. In our definition, $R(k, l) = f(1, \{K_k, \overline{K_l}\})$.

It is not difficult to show that $f(n, \{K_2, \overline{K_2}\}) = 3n - 1$. Burr, Erdös, and Spencer showed that $f(n, \{K_3, \overline{K_3}\}) = 5n$ for $n \geq 2$ [3]. Let $k, l \geq 2$. Burr proved that $f(n, \{K_k, \overline{K_l}\}) = (k + l - 1)n + f(1, \{K_{k-1}, \overline{K_{l-1}}\}) - 2$ for sufficiently large n [1, 2].

Let $\mathcal{G}_k$ be the family of all graphs of order k. For $k = 3$, $\mathcal{G}_3$ consists of four graphs K_3, $\overline{K_3}$, $K_{1,2}$ and $\overline{K_{1,2}}$. Let $\mathcal{D}_k = \{K_k, \overline{K_k}, K_{1,k-1}, \overline{K_{1,k-1}}\}$ for $k \geq 3$. Our main result is as follows.

Theorem 1. *Let $k \geq 3$. Then $f(n, \mathcal{D}_k) = (2k - 1 - \frac{1}{k})n + O(1)$.*

Since $\mathcal{G}_3 = \mathcal{D}_3$, we have an immediate consequence of Theorem 1.

Corollary 2. $f(n, \mathcal{G}_3) = \frac{14}{3}n + O(1)$.

In Section 2 and Section 3, we outline the proof of Theorem 1.

2 Proof of Theorem 1—Lower Bound

For a pair of graphs G_1 and G_2, we denote the union(the join) of G_1 and G_2 by $G_1 \cup G_2 (G_1 + G_2)$. Let $k-2 < n$. Let $\alpha = \lfloor \{(k-1)n + (k-2)\}/k \rfloor$ and $\beta = (k-1)n - 1$. Let us define $G = K_\alpha + (K_\beta \cup \overline{K_\beta})$. It turns out that $N(G, \mathcal{D}_k) < n$. Hence, we have $f(n, \mathcal{D}_k) \geq |V(G)| + 1 > (2k - 1 - \frac{1}{k})n - 2$ for $k - 2 < n$.

3 Proof of Theorem 1—Upper Bound

For a given graph G, we consider the following inequalities.
(I1) $N(G, K_k) \geq n$,
(I2) $N(G, \overline{K_k}) \geq n$,
(I3) $k \cdot N(G, K_k) + k \cdot N(G, \overline{K_k}) + N(G, K_{1,k-1}) \geq (2k+1)n$,
(I4) $k \cdot N(G, K_k) + k \cdot N(G, \overline{K_k}) + N(G, \overline{K_{1,k-1}}) \geq (2k+1)n$.

We say that a graph G is (n,k)-*good* if G satisfies at least one of the inequalities from (I1) to (I4).

Let $G_0 = K_{k(k^2-1)} + (K_{k(k^2-1)} \cup \overline{K_{2k^2(k-1)}})$. Set $n_0 = 2k^2$. Note that $|V(G_0)| = (2k - 1 - \frac{1}{k})n_0$.

Lemma 3. *Both G_0 and $\overline{G_0}$ satisfy all of the inequalities from (I1) to (I4) with $n = n_0$.*

Proposition 4. *There exists a positive integer c depending on k such that any graph G with $|V(G)| \geq (2k - 1 - \dfrac{1}{k}) + c$ is (n,k)-good.*

Note that Proposition 4 implies that $f(n, \mathcal{D}_k) \leq (2k - 1 - \frac{1}{k})n + c$.
Proof of Proposition 4. Let us take a constant c sufficiently large. We proceed by induction on n. There are two cases.
Case 1. G contains G_0 or $\overline{G_0}$ as an induced subgraph.
We may assume G contains G_0. We decompose $V(G)$ into $V_1 = V(G_0)$ and $V_2 = V(G) - V_1$. Let $G' = \langle V_2 \rangle_G$. We have $|V(G')| \geq (2k - 1 - \frac{1}{k})(n - n_0) + c$. Hence, by the inductive hypothesis, G' is $(n - n_0, k)$-good. By Lemma 3, G becomes (n,k)-good.
Case 2. G does not contain either G_0 or $\overline{G_0}$.
In this case, possible structures of G are considerably restricted. Hence, by a relatively short argument, we can show that G is (n,k)-good.

4 Further Discussions

1. For $k \geq 4$, $f(n, \mathcal{G}_k)$ is not known well. For $k = 4$, let $G = K_{2n-1} \cup (K_{n-1} + \overline{K_{3n-1}})$. Then we have $N(G, \mathcal{G}_4) < n$. It follows that $f(n, \mathcal{G}_4) \geq 6n - 2$. We conjecture $f(n, \mathcal{G}_4) = 6n + O(1)$.
2. There are some related results. Let $\mathcal{C}_k$ be the family of graphs G such that G is a disjoint union of complete graphs with $|V(G)| = k$. Let $g(n,k)$ be the minimum integer s such that $N(G, \mathcal{C}_k) \geq n$ for any graph $G \in \mathcal{C}_s$. First we consider the case $n = 2$ [4, 5].

Theorem 5. $g(2,k) = 2k + \min\{r : k \leq c_r\}$, *where* $c_0 = 1$, $c_1 = 4$, *and* $c_r = c_{r-1} + c_{r-2} + 2r + 1$ *for* $r \geq 2$.

For $k \geq 3$, $g(n,k)$ is not determined in general. However, if n is large enough with respect to k, we have the following result [5].

Theorem 6. *Let $k, n \geq 2$ with $k - 2 \leq n$. Then $g(n, k) = (k + 1)n - 1$.*

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