Improving the Gilbert-Varshamov bound for q-ary codes

Van H. Vu[1†] and Lei Wu[1‡]

[1] *Department of Mathematics, University of California, San Diego, 9500 Gilman Dr., La Jolla, CA 92093-0112, USA.*

Given positive integers q, n and d, denote by $A_q(n,d)$ the maximum size of a q-ary code of length n and minimum distance d. The famous Gilbert-Varshamov bound asserts that

$$A_q(n,d+1) \geq q^n/V_q(n,d),$$

where $V_q(n,d)=\sum_{i=0}^{d}\binom{n}{i}(q-1)^i$ is the volume of a q-ary sphere of radius d.

Extending a recent work of Jiang and Vardy on binary codes, we show that for any positive constant α less than $(q-1)/q$ there is a positive constant c such that for $d \leq \alpha n$, $A_q(n,d+1) \geq c\frac{q^n}{V_q(n,d)}n$. This confirms a conjecture by Jiang and Vardy.

1 Introduction

Given a set $\mathbf{\Omega}$ of q symbols, without loss of generality, let $\mathbf{\Omega}=\{0,1,\ldots,q-1\}$. A q-ary word of length n is a sequence $x=(x_1,\ldots,x_n)$, where $x_i \in \mathbf{\Omega}$. The number of non-zero symbols in a word x is the weight of x. Given two words x and y, the (Hamming) distance between x and y is the number of coordinates i in which x_i and y_i are different. A set $\mathcal{C}$ of words is called a code with minimum distance d if any two codewords in $\mathcal{C}$ have distance at least d. For a word x, the Hamming sphere of radius d centered at x has volume

$$V_q(n,d)=\sum_{i=0}^{d}\binom{n}{i}(q-1)^i.$$

Thanks to symmetry, the volume of the sphere does not depend on x.

For integers q, n and d, let $A_q(n,d)$ denote the maximum size of a q-ary code of length n and minimum distance d. Estimating $A_q(n,d)$ is one of the most important problems in coding theory. The famous Gilbert-Varshamov bound [4, 11] asserts that

$$A_q(n,d+1) \geq \frac{q^n}{V_q(n,d)}.$$

†Email address: vanvu@ucsd.edu; webpage:, http://www.math.ucsd.edu/˜vanvu/ Research supported in part by by an NSF CAREER award and by an Alfred P. Sloan fellowship.

‡Email: lwu@math.ucsd.edu; http://math.ucsd.edu/˜lwu

This bound is used extensively in numerous contexts and has been generalized in many different settings [7, 8, 6]. Improving upon the Gilbert-Varshamov bound asymptotically is a notoriously difficult task [8]. Tsfasman, Vlădut, and Zink [10] made a breakthrough for the case when $q \geq 49$. More recently, Jiang and Vardy [6] improved the Gilbert-Varshamov bound, for the case $q = 2$, for certain range of d:

Theorem 1.1 *Let α be a constant satisfying $0 < \alpha \leq .4994$. Then there is a positive constant c depending on α such that the following holds. For $d \leq \alpha n$,*

$$A_2(n, d+1) \geq c \frac{2^n}{V_2(n,d)} \log_2 V_2(n,d) \tag{1}$$

If $d \geq \alpha' n$ for some constant $\alpha' > 0$, then $V_2(n,d)$ is exponential in n. Thus, Theorem 1.1 improved Gilbert-Varshamov bound by a factor linear in n. We can rewrite (1) in the following more pleasant form (the constant c here, of course, would be different):

$$A_2(n, d+1) \geq c \frac{2^n}{V_2(n,d)} n. \tag{2}$$

Jiang and Vardy asked if one can get to $\alpha < 0.5$ using a different method than computer simulations as they did (the strange constant .4994 resulted from these simulations). They also conjectured that an improvement similar to (2) can be achieved for q-ary codes, for any $q \geq 3$.

The main result of this paper resolves both of these issues. For the binary case, our main theorem (Theorem 1.2) extends the assumption $\alpha < 0.4994$ in [6] to its natural limit $\alpha < 0.5$. The proof of Theorem 1.2 does not rely on computers, and reflects, in a clean way, the necessity of the assumption $\alpha < (q-1)/q$.

Throughout the paper, asymptotic notations are used under the assumption that n goes to infinity. We also emphasize the case when d is proportional to n, namely, $d = \alpha n$ for some positive constant α. This case is of special interest in coding theory.

Theorem 1.2 *Let q be a fixed positive integer and α be a constant satisfying $0 < \alpha < \frac{q-1}{q}$. There is a positive constant c depending on q and α such that for $d = \alpha n$,*

$$A_q(n, d+1) \geq c \frac{q^n}{V_q(n,d)} n \tag{3}$$

In general, the constant α can take any value less than or equal to one. However, it is well known and easy to show that for $\alpha \geq (q-1)/q$, the volume $V_q(n,d)$ is close to q^n, namely, $q^n \leq 2V_q(n,d)$. In this case, the Gilbert-Varshamov bound gives no useful information. Thus, the value $(q-1)/q$ serves as a natural threshold and we will assume $\alpha < (q-1)/q$.

2 Graph theoretic frame work

We recall a folklore in graph theory.

Proposition 2.1 *Let G be a D-regular graph on n vertices. Then G contains an independent set of size $n/(D+1)$.*

Given q, n and d, we follow [6] and define a graph $\mathcal{G}$ whose vertices are the q-ary words of length n and two words are adjacent if their Hamming distance is at most d. It's easy to see that $\mathcal{G}$ has q^n vertices, the degree of every vertex is $D = V_q(n,d) - 1$, and $A_q(n, d+1)$ is the independence number of $\mathcal{G}$, denoted by $I(\mathcal{G})$. The Gilbert-Varshamov bound is simply the realization of Proposition 2.1 on this graph.

For a D-regular graph, each neighborhood has at most $\binom{D}{2}$ edges. We say that such a graph is *locally sparse* if in every neighborhood the number of edges is much less than $\binom{D}{2}$. In the extreme case when the graph is triangle-free, i.e., when the number of edges in each neighborhood is zero, Proposition 2.1 was improved by a logarithmic factor by Ajtai, Komlós and Szemerédi in [1]. Namely, they obtained $I(G) \geq cn \log D/D$. This result has been extended to locally sparse graphs (i.e. with few triangles) by Shearer [9].

Lemma 2.2 (Shearer) *For any positive constant $\epsilon \leq 2$ there is a positive constant c such that the following holds. Let G be a D-regular graph on N vertices. Assume that each neighborhood in G contains at most $D^{2-\epsilon}$ edges. Then the independence number of G, denoted by $I(G)$, satisfies:*

$$I(G) \geq c\frac{N}{D}\ln D.$$

In order to prove Theorems 1.1 and 1.2, one needs to verify the hypothesis of Lemma 2.2 for $\mathcal{G}$. Due to symmetry, every neighborhood in $\mathcal{G}$ has the same number of edges. Thus, for convenience, we can consider the neighborhood of the word consisting of only zeros. Let T be the number of edges in this neighborhood and $\mathcal{G}_0$ be the graph spanned by these edges. Our goal is to show that there is a positive constant ε such that

$$T \leq D^{2-\varepsilon}. \tag{4}$$

It is not hard to give explicit formulae for T and D. Fixed $q \geq 2$, we have

$$D = V_q(n,d) - 1 = \sum_{i=1}^{d} \binom{n}{i}(q-1)^i,$$

$$T = \Theta\left(\sum_{w=1}^{d} \binom{n}{w}(q-1)^w \sum_{\{i,j,k\}\in N} \binom{w}{i}\binom{w-i}{k}\binom{n-w}{j}(q-2)^k(q-1)^j\right),$$

where N is the set of all triples $\{i, j, k\}$ that satisfies:

$$i + k \leq w, \quad j \leq n - w, \quad w - i + j \leq d, \textit{ and } d(x,y) = i + j + k \leq d$$

One can easily see the difficulty of dealing with these two variables directly, especially T. In fact, this was the main hurdle for further improvement of [6].

Our approach is to translate (4) into simpler inequalities which we are able to prove using the following notion. Let X and Y be two functions in n. We call X and Y polynomially equivalent and write $X \sim Y$ if there are positive constants c_1, c_2 such that

$$n^{-c_1}X \leq Y \leq n^{c_2}X.$$

We find new parameters $T' \sim T$, $D' \sim D$ where both T' and D' are relatively simple. Since both T and D are exponential functions in n, if we can show

$$T' \leq D'^{2-\delta}, \tag{5}$$

for a positive constant δ, then it follows that for all sufficiently large n, $T \leq D^{2-\epsilon}$, where, say, $\epsilon = .999\delta$.

Finding D' is easy. For T', we will apply a technique which can be viewed as a discrete analogue of Lagrange's multiplier. Once D' and T' are determined, (5) becomes equivalent to a reasonable inequality concerning entropy functions, which serve as good estimates of binomial coefficients. This inequality is not obvious, but can be proved using the assumption $\alpha < (q-1)/q$ and an analytic argument. The readers are invited to check the full version of the paper for the (rather technical) details.

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