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Density of universal classes of series-parallel graphs

Jaroslav Nešetřil[1†] and Yared Nigussie[1‡]

[1] *Department of Applied Mathematics*
Institute for Theoretical Computer Science(ITI)
Charles University
Malostranské nám.25
11800 Praha 1 Czech Republic

A class of graphs $\mathcal{C}$ ordered by the homomorphism relation is *universal* if every countable partial order can be embedded in $\mathcal{C}$. It was shown in [1] that the class $\mathcal{C}_k$ of k-colorable graphs, for any fixed $k \geq 3$, induces a universal partial order. In [4], a surprisingly small subclass of $\mathcal{C}_3$ which is a proper subclass of K_4-minor-free graphs ($\mathcal{G}/K_4$) is shown to be universal. In another direction, a density result was given in [9], that for each rational number $a/b \in [2, 8/3] \cup \{3\}$, there is a K_4-minor-free graph with circular chromatic number equal to a/b. In this note we show for each rational number a/b within this interval the class $\mathcal{K}_{a/b}$ of K_4-minor-free graphs with circular chromatic number a/b is universal if and only if $a/b \neq 2$, 5/2 or 3. This shows yet another surprising richness of the K_4-minor-free class that it contains universal classes as dense as the rational numbers.

Keywords: circular chromatic number, homomorphism, series-parallel graphs, universality

1 Introduction

We assume graphs are finite and simple (with no loops and parallel edges). Let G, G' be graphs. A *homomorphism* from G to G' is a mapping $f{:}V(G) \to V(G')$ which preserves adjacency. That is, $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(G')$. We write $G \leq G'$ if there is a homomorphism from G to G'. The notation $G < G'$ means $G \leq G' \not\leq G$, whereas $G \sim G'$ means $G \leq G' \leq G$. If $G \sim G'$, we say G and G' are *hom-equivalent*. The smallest graph H for which $G \sim H$ is called the *core* of G. For finite graphs, the core is uniquely determined up to an isomorphism. It can also be seen that H is an induced subgraph of G. This will be denoted by $H \subseteq G$. Let $\mathcal{C}$ and $\mathcal{C}'$ be two classes of graphs. We also write $\mathcal{C} \sim \mathcal{C}'$ if for each graph $G \in \mathcal{C}$ there exists a $G' \in \mathcal{C}'$ such that $G \sim G'$ and vice versa. See [2] for introduction to graphs and their homomorphisms.

Let $k \geq d \geq 1$ be integers. The *circular chromatic number* of G, written $\chi_c(G)$, is the smallest rational k/d such that $G \leq K_{k/d}$, where $K_{k/d}$ is the *circular graph* with $V(K_{k/d}) = \{0, 1, 2, \ldots, k-1\}$ and

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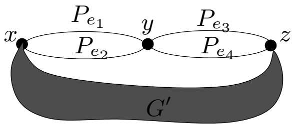

Fig. 1: Unavoidable configuration of G (a minimal counterexample to Lemma 6) with odd girth $2k+1$ and $l_{e_1}+l_{e_2} = l_{e_3}+l_{e_4} = 2k+1$.

$E(K_{k/d}) = \{\{i,j\} : d \le |i-j| \le k-d\}$. Note that when $d=1$ we have the usual vertex coloring of G. Let $\mathcal{K}_{a/b}\{G \in \mathcal{G}/K_4 : \chi_c(G) = a/b\}$. See [10] for some other equivalent definitions. It is trivial to see the following:

Theorem 1 $\mathcal{K}_2 \sim \{K_2\}$.

It is well known that graphs in $\mathcal{G}/K_4$ are 3-colorable. Hell and Zhu [3] have shown that triangle-free graphs in $\mathcal{G}/K_4$ have circular chromatic number at most 8/3. Hence no graph in $\mathcal{G}/K_4$ has circular chromatic number in the interval (8/3,3). Hence, we have:

Theorem 2 $\mathcal{K}_3 \sim \{C_3\}$.

The main results of this note are the following two theorems establishing nice dichotomy between universality and homomorphism finiteness of the class $\mathcal{K}_{a/b}$:

Theorem 3 $\mathcal{K}_{5/2} \sim \{C_5\}$.

Somewhat surprisingly we show that Theorem 1, 2, and 3 cover all cases when $\mathcal{K}_{a/b}$ is a finite set.

Theorem 4 *$\mathcal{K}_{a/b}$ is universal if $a/b \in (2, 5/2) \cup (5/2, 8/3]$.*

In section 2 we prove Theorem 3 using a folding lemma. In section 3 we prove Theorem 4.

2 $\mathcal{K}_{5/2}$ is equivalent to $\{C_5\}$

Let G be a graph. A *thread* in G is a path $P \subseteq G$ such that the two endpoints of P have degree at least 3 and all internal vertices of P are degree 2 in G. We shall often use the fact that if P and P' are two edge-disjoint paths and if the lengths of P and P' have same parity such that P is a thread and has at least equal length as P', then there is a homomorphism that maps P to P' sending the two ends of P to the two ends of P'. Such a homomorphism is said to *fold* P to P'. Let G be a graph and let G^s denote the multi-graph we obtain from G by "smoothing" all degree 2 vertices of G. For each edge e of G^s, let P_e denote the thread of G represented by e in G^s, and let l_e denote the length of P_e.

The following Folding Lemma for K_4-minor-free graphs is an analogy of the Folding Lemma of Klostermeyer and Zhang [6] for planar graphs. Its proof is easy (see [7]).

Lemma 5 *(**Edge folding lemma**) Let $G \in \mathcal{G}/\{K_4\}$ be of odd girth $2k+1$ and let e, e' be parallel edges in G^s, with common end vertices x, y. If G is not homomorphic to a strictly smaller graph of the same odd girth, then $l_e + l_{e'} = 2k+1$. Moreover, $P_e \cup P_{e'}$ is the unique cycle of length $2k+1$ containing both x and y.*

For short let K^m denote $K_{(7+5m)/(3+2m)}$. Recall that $V(K^m) = \{0, 1, \ldots, 6+5m\}$.

Lemma 6 *Let $G \in \mathcal{G}/\{K_4\}$ be of odd girth at least 7. Then $\chi_c(G) \le (7+5m)/(3+2m) < 5/2$, for some $m < |V(G)|/2$.*

Proof: Let $G \in \mathcal{G}/\{K_4\}$ be a core of odd girth $g \ge 7$. It suffices to show $G \le K^m$ for some $m \ge 0$. Let $\bar{G^s}$ be the graph we get by identifying parallel edges of G^s. Then, $\bar{G^s} \in \mathcal{G}/K_4$ and so there exists a $y \in V(\bar{G^s})$ such that the degree $deg_{\bar{G^s}}(y) = 2$. Then $3 \le deg_{G^s}(y) \le 4$ (here we use a parity argument that, the multiplicity of edges of G^s is at most two, as G is a core). By Lemma 5, and assuming G is a minimal counterexample we can get $deg_{G^s}(y) = 4$. Hence, a configuration depicted in Figure 1 is unavoidable. Let $G' = G - (\bigcup_{i=1}^{4} P_{e_i} - \{x,z\})$. By induction $G' \le K^m$, for some $m \ge 0$. We can assume $f(x) = 0$. By investigating a few cases for values of $f(y)$, it is not hard to see $G \le K^{m+1}$, contrary to assumption (see [7] for detailed proof). □ **Proof of Theorem 3:** Let $G \in \mathcal{K}_{5/2}$ be of odd girth g. Then $G \le C_5$. By Lemma 6, we have $g \le 5$. By Theorem 2, $g > 3$. Hence $g = 5$ and so $C_5 \le G \le C_5$. Hence $G \sim C_5$. The converse is obvious since $\chi_c(C_5) = 5/2$.

3 Universal sets of $\mathcal{G}/K_4$ are dense in $(2, 5/2) \cup (5/2, 8/3]$

In this section we shall show that we obtain a universal class $\mathcal{K}_{p/q} \subset \mathcal{G}/K_4$ for arbitrary $p/q \in (2, 5/2) \cup (5/2, 8/3]$. We use a graph $G_{p/q}$ with $\chi_c(G) = p/q$ as a generator of $\mathcal{K}_{p/q}$. We assume $G_{p/q}$ has the following two properties:

(P1) $G_{p/q}$ is hom-equivalent neither to a cycle nor to a vertex.
(P2) if $G' \in \mathcal{G}/K_4$ satisfies (P1) and $\chi_c(G') = p/q$, then $|V(G')| \ge |V(G)|$.

Lemma 7 *Let $G \in \mathcal{G}/K_4$ have properties (P1) and (P2). Then, G is 2-connected. Moreover, G is a core and it is not vertex-transitive.*

Proof: Since the circular graph $K_{k/d}$ is a vertex-transitive graph, for all k, d, we have $\chi_c(G) = \max_i(\chi_c(H_i)), 1 \le i \le p$, where each H_i is a 2-connected component of G. Here, (P2) implies that $p = 1$ and so G is 2-connected. Next, note that any graph $G \in \mathcal{G}/K_4$ is vertex-transitive if and only if G is an odd cycle or K_1 or K_2. This is because all other 2-connected graphs in $\mathcal{G}/K_4$ have at least one degree-2 vertex and one non-degree-2 vertex. Hence by (P1), G is not vertex-transitive. Moreover, by (P1) the core of G also is not vertex-transitive. By (P2), we deduce G is a core. □ For any rational number $p/q \in (2, 8/3]$, Pan and Zhu have shown in [9] a recursive method of constructing a 2-connected graph $G_{p/q}$ with $\chi_c(G) = p/q$. If $p/q \ne (2k+1)/k$ then $G_{p/q}$ satisfies (P1). If $p/q = (2k+1)/k$, the graph given in [9] is the cycle C_{2k+1} which is the natural candidate. Cycles do not satisfy (P1), hence we introduce a graph denoted by G^k of odd girth $2k+3$ as follows: Take a triangle and double each edge to obtain a multi-graph H. For $i = 0, 1, 2$, let $\{e_1^i, e_2^i\}$, be the three parallel pairs of edges of H. To obtain a thread of length $k+2$, subdivide e_1^0 and e_1^1 each $k+1$ times. Next subdivide e_2^0 and e_2^1 each k times. Finally, subdivide e_1^3 three times and e_2^3, $2k-2$ times to obtain the graph G^k. We have:

Lemma 8 *$\chi_c(G^k) = (2k+1)/k$ for all $k \ge 3$.*

Proof: It is easy to see that $G^k \le C_{2k+1}$, and $G^k \not\le C_{2k+3}$. Hence, we have $(2k+3)/(k+1) < \chi_c(G^k) \le (2k+1)/k$. Note that $\gcd(4k+4, 2k+1) = 1$. From basic number theory [8], using what is known as the *Farey sequence*, we can see that any rational strictly between $(2k+3)/(k+1)$ and $(2k+1)/k$

has numerator $a \geq 4k+4$. But then, if $k \geq 3$ the circumference of G^k is $4k+3$. It is well known [10] that the numerator a of a circular chromatic number a/b of a graph G is at most its circumference. We deduce $\chi_c(G^k) = (2k+1)/k$. $\square$

Corollary 9 *For every rational number $p/q \in (2, 5/2) \cup (5/2, 8/3]$ there is a graph $G_{p/q}$ satisfying (P1) and (P2).*

Next we prove that $\mathcal{K}_{p/q}$ inherits universality from the class $\mathcal{P}$ of directed finite paths [5]. We take several isomorphic copies $H_1, \ldots, H_n$ of a fixed graph H, such that $\chi_c(H) = p/q$ and construct a 'path-like' structured graph H' by identifying a vertex of H_i with a vertex of H_{i+1}. Then $\chi_c(H') = \chi_c(H)$ because the circular graphs are vertex-transitive. We call such a construction K_1-*concatenation*. A more precise definition of 'K_1-concatenation' of a graph H:

Let $P \in \mathcal{P}$ be an oriented path of length $n \geq 1$, $V(P) = \{v_1, v_2, \ldots, v_{n+1}\}$. Then either $v_i v_{i+1}$ or $v_{i+1} v_i \in E(P)$ (but not both). Let H be a graph and $a, b \in V(H)$. Let $H_1, H_2, \ldots, H_n$ be isomorphic copies of H and let a_i, b_i be the vertices of H_i corresponding to a and b. The K_1-*concatenation* of H by P is a graph $P * (H, a, b)$ constructed as follows: For $i = 1, \ldots, n$, if $v_i v_{i+1} \in E(P)$, choose b_i (otherwise choose a_i). Then identify every chosen vertex of H_i with the unchosen vertex of H_{i+1}. To make the construction non-trivial, we choose a and b so that there is no automorphism sending a to b or b to a. If H satisfies (P1), then we know there exists such a pair.

Lemma 10 *Let $G_{p/q} \in \mathcal{G}/K_4$ satisfy $(P1), (P2)$. Then, $\mathcal{K}_{p/q}$ is universal.*

Proof: Since the class of oriented paths $\mathcal{P}$ is universal we show for every $P, P' \in \mathcal{P}$, we have $P \leq P'$ if and only if $P * (G_{p/q}, a, b) \leq P' * (G_{p/q}, a, b)$. This proves the lemma.

The forward implication is straightforward. To prove the reverse implication, let $P, P' \in \mathcal{P}$ of length n and n' such that P is a core and suppose there exists a homomorphism $f : P * (G_{p/q}, a, b) \to P' * (G_{p/q}, a, b)$. Assume further that P is not an edge, since this case is trivial, and without loss of generality, assume that the first edge of P is directed forward. Let $H_1, \ldots, H_n$ and $H'_1, \ldots, H'_{n'}$ be isomorphic copies of $G_{p/q}$. First we show that $f|_{H_i}$ is induced by an automorphism of $G_{p/q}$ for each i. Suppose not. Then the image $f(H_i)$ is connected. If $f(H_i)$ is 2-connected then, it is isomorphic to $G_{p/q}$, since $G_{p/q}$ is a core. Suppose $f(H_i)$ is not 2-connected. Then each 2-connected component F of $f(H_i)$ is a proper subgraph of $G_{p/q}$. By (P2), we have $\chi_c(F) < p/q$. Then, $H \not\leq f(H_i)$, a contradiction. Hence $f|_{H_i}$ is induced by an automorphism of $G_{p/q}$.

Now we claim a stronger assertion that for any i, $f(a_i) = a'_j$ and $f(b_i) = b'_j$, for some $j, 1 \leq j \leq m$. let $H'_j = f(H_1)$. Since $f|_{H_1}$ is an automorphism, $f(b_1)$ must be in the same automorphism class of b'_j. Suppose that $f(b_1) \neq b'_j$, then $f(b_1)$ is not a cut-vertex of $P' * (G_{p/q}, a, b)$. As b_1 is a cut-vertex of $P * (G_{p/q}, a, b)$, it is identified with either a_2 or b_2. If it were identified with a_2, then $f|_{H_2}$ would be an automorphism of $G_{p/q}$ such that $f(a_2) = b'_j$, contrary to the choice of a, b in $V(G_{p/q})$. Hence b_1 is identified with b_2. Similarly, we get a_2 is identified with a_3, and b_3 with b_4 and so on. This implies P is a 'zig-zag' which is hom-equivalent to an edge, contrary to P being a core. So $f(b_1) = b'_j$. The claim follows by induction on the length of P.

We define a homomorphism $g : V(P) \to V(P')$ so that if $f(a_i) = a'_j$ then $g(v_i) = v'_j$ and similarly for b_i. By our construction g preserves the adjacency condition and so $P \leq P'$. $\square$ **Proof of Theorem 4:** Let $a/b \in (2, 5/2) \cup (5/2, 8/3]$. By Lemma 9, there is a graph $G_{a/b}$ with properties (P1),(P2). By Lemma 10, $\mathcal{K}_{a/b}$ is universal. This concludes our result.

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