Multigraph decomposition into multigraphs with two underlying edges

Miri Priesler[1] and Michael Tarsi[2]

[1] *Engineering Department, Ruppin Academic Centre, Emek Hefer 40250, Israel*
[2] *School of Computer Science, Tel-Aviv University, Tel-Aviv 69978, Israel*

Due to some intractability considerations, reasonable formulation of necessary and sufficient conditions for decomposability of a general multigraph G into a fixed connected multigraph H, is probably not feasible if the underlying simple graph of H has three or more edges. We study the case where H consists of two underlying edges. We present necessary and sufficient conditions for H-decomposability of G, which hold when certain size parameters of G lies within some bounds which depends on the multiplicities of the two edges of H. We also show this result to be "tight" in the sense that even a slight deviation of these size parameters from the given bounds results intractability of the corresponding decision problem.

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1 Extended Abstract

Given two graphs H and G, an H-decomposition of G is a partition of the edge set of G into disjoint isomorphic copies of H. The study of Graph decomposition started back at the mid 19^{th} century, with the seminal concept of *Steiner triple systems* Steiner (1853), and has since become the subject of some hundreds of research papers, with active research still carried out today. R. Wilson's fundamental theorem Wilson (1976) states that for any fixed graph H there exists an H-decomposition of the complete graph K_n if the obvious necessary divisibility conditions hold and n is *large enough*. A considerable amount of research was indeed devoted to thoroughly studying the existence of H decompositions of complete graphs for specific graphs H, such as: some small graphs, complete graphs, complete multipartite graphs, paths and cycles (a finite problem for every fixed graph H, in light of Wilson's theorem). For a review of methods and results see e.g. Bermond and Sotteau (1975) and Bosak (1990).

Hopes for similar accurate results where H decomposition of a general graph G is considered are slim, due to the following negative result:

Theorem 1.1 *Deciding whether there exists an H-decomposition of an input graph G is* NP-*complete for any fixed simple graph H which contains a connected component with at least 3 edges*

The above was conjectured by I. Holyer Holyer (1981) on 1981 and proved sixteen years later in Dor and Tarsi (1997). On the other hand, the existence of a polynomial time algorithm to decide H-decomposability of an input G, where every component of H consists of at most two edges was proved (though not in terms of an explicit necessary and sufficient condition) in Bryś and Lonc (1995).

In this research we study *Multigraph decomposition*, that is the case where multiple edges are allowed in both graphs H and G. Although Theorem 1.1 was not (yet?) generalized to multigraphs, a graph decomposition decision problem most probably remains at least as hard when extended to multigraphs. Furthermore, we have managed to prove the following intractability results Priesler and Tarsi (2002b):

Theorem 1.2 *Deciding the decomposability of an input multigraph G with a constant multiplicity λ on all its edges, into the star $K_{1,t}$ is NP-Complete for every fixed λ and $t \geq 3$.*

Theorem 1.3 *Deciding H-decomposability of an input multigraph G into any fixed* multistar *(a multigraph whose underlying simple graph is $K_{1,t}$, with any sequence of positive multiplicities on its t edges) with at least three underlying edges, is NP-Complete.*

In an attempt to find the conditions for decomposability of a general "input" multigraph G into a "fixed" connected multigraph H, serious hopes for results are limited, in light of the theorems above, to the case were H consists of two underlying edges.

Quite surprisingly we found out this limited setting to be rather involved, producing somewhat unexpected results: In a previous result Priesler and Tarsi (2004), we considered the simplest case, where $H = S^{1,2}$ is a multigraph on an underlying $K_{1,2}$ with multiplicity 1 on one edge and 2 on the other, and G is a multigraph on any underlying simple graph with a constant multiplicity λ on all its edges. We gave necessary and sufficient condition for such a decomposition to exist if $\lambda \neq 2$ and $\lambda \neq 5$. We also showed that similar conditions for $\lambda = 2$ and for $\lambda = 5$ do most probably not exist, by proving the corresponding decision problems to be NP-complete.

In our article we investigate the decomposition of a general multigraph G into $S^{\alpha,\beta}$ - an underlying $K_{1,2}$ with multiplicities α and β. We show some necessary divisibility conditions to be also sufficient if certain size parameters of G lie between certain bounds which depend on α, β and $\frac{\alpha}{\beta}$. The following theorem summarizes this main result:

Theorem 1.4 *Let α be a larger integer than an integer β and let ε be any constant $0 < \varepsilon < 1 - \frac{\beta}{\alpha}$, then there exist $\lambda_0(\alpha, \beta, \varepsilon)$ and $M_0(\alpha, \beta, \varepsilon)$ such that for every $\lambda > \lambda_0$*
If $G = (V, E)$ is

- *Any connected graph, other than an odd regular tree, or*
- *An odd regular tree where $|E| \geq M_0$,*
  and
- $w : E \rightarrow [\lambda, \frac{\alpha}{\beta}(1 - \varepsilon)\lambda]$*, represents the multiplicity function of the edges (clearly integer values)*

then (G, w) admits an $S^{\alpha,\beta}$-decomposition if and only if

1. *$\sum_{e \in E} w(e)$ is divisible by $\alpha + \beta$*
2. *if G is either a path, or an even circuit, then*
   $\alpha \sum_{e \text{ is odd}} w(e) \equiv \beta \sum_{e \text{ is even}} w(e)$ *(mod $\alpha^2 - \beta^2$).*

Finally we show this result to be "best possible" in the sense that the corresponding decision problem becomes NP-complete when the relevant size requirements are not met.

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