Largest cliques in connected supermagic graphs

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A graph $G = (V, E)$ is said to be *magic* if there exists an integer labeling $f : V \cup E \longrightarrow [1, |V \cup E|]$ such that $f(x) + f(y) + f(xy)$ is constant for all edges $xy \in E$. Enomoto, Masuda and Nakamigawa proved that there are magic graphs of order at most $3n^2 + o(n^2)$ which contain a complete graph of order n. Bounds on Sidon sets show that the order of such a graph is at least $n^2 + o(n^2)$. We close the gap between those two bounds by showing that, for any given graph H of order n, there are connected magic graphs of order $n^2 + o(n^2)$ containing H as an induced subgraph. Moreover it can be required that the graph admits a supermagic labelling f, which satisfies the additional condition $f(V) = [1, |V|]$.

Keywords: Labelings of graphs, magic graphs, Sidon sets.

1 Introduction

A simple finite graph $G = (V, E)$ is said to be *magic* if there is a bijection $f : V \cup E \to [1, |V \cup E|]$ and a constant k such that $f(x) + f(y) + f(xy) = k$ for each edge $xy \in E$. This notion was introduced by Kotzig and Rosa [8] in 1966 under the name of *magic valuations*. When $f(V) = [1, |V|]$ then the graph is *supermagic*; see for instance [2, 10]. There are several related notions under the name of magic labellings; see the dynamic survey of Gallian [5].

An upper bound for the size of a magic graph containing a clique had been given already by Kotzig and Rosa [9], where they proved that, if $G = (V, E)$ is a magic graph containing a complete graph of order $n > 8$, then

$$|V| + |E| \geq n^2 - 5n + 14.$$

This result was improved by Enomoto, Masuda and Nakamigawa [3] to

$$|V| + |E| \geq 2n^2 - O(n^{3/2}), \tag{1}$$

by using the known bound for the size of a Sidon set given in ([4]).

[†]Supported by the Ministry of Science and Technology of Spain, and the European Regional Development Fund (ERDF) under project-BFM-2002-00412 and by the Catalan Research Council under grant 2001SGR-00258.

Recall that a set A of integers is said to be a Sidon set if all sums of pairs of elements (non necessarily different) of A, are pairwise distinct.

In 1941 Erdös and Turan [4] proved that a Sidon set $A \subset [1, N]$ always satisfies,

$$|A| \leq N^{1/2} + N^{1/4} + 1. \tag{2}$$

Kotzig [7] calls a set $A \subset \mathbb{Z}$ a *well spread sequence* if all sums of distinct elements in A are pairwise different. He showed that, if $A \subset [1, N]$ with $N \geq 8$, then $N \geq 4 + \binom{|A|-1}{2}$. Ruzsa [11] calls such a set a *weak Sidon* set. He gives a very nice short proof that a weak Sidon set in $[1, N]$ satisfies

$$|A| \leq N^{1/2} + 4N^{1/4} + 11. \tag{3}$$

If $A \subset V$ induces a clique in a magic graph $G = (V, E)$ with magic labelling f then $f(A)$ is a weak Sidon set. That is, for each pair of vertices $x, y \in A$, we have $f(x) + f(y) = k - f(xy)$, so that the sums of labels of pairs of vertices in A are pairwise distinct. Therefore $|A|$ is bounded by (3) with $N = |V \cup E|$, or $N = |V|$ if f is super magic.

We want to point out that in 1972 Kotzig using well spread sequences proved in a long paper that K_n is not magic for $n \geq 7$, [8]. The same result was reproved in 1999 by Craft and Tesar [1]. Note that inequality (3) shows directly this result for n large enough.

There are explicit constructions of (weak) Sidon sets whose cardinality is close to the upper bound in (3). For instance, for any prime p, Singer gives a construction of a Sidon set of cardinality $p + 1$ in $[1, N]$ with $N = p^2 + p + 1$ and Bose gives one of cardinality p with $N = p^2 - 1$; see for instance [6]. Ruzsa [11] gives also such a construction of a Sidon set with $p-1$ elements in $[1, p^2 - p]$. Since for each positive integer n there is a prime p such that $p \leq n + o(n)$, these constructions provide a Sidon set of order n in $[1, N]$ with $N \leq n^2 + o(n^2)$; see for instance [3, Lemma 3].

The existence of dense Sidon sets provide the means to obtain lower bounds for the largest possible clique in a connected magic graph. By using the construction of Singer [12] for dense difference sets Enomoto, Masuda and Nakamigawa [3] show that, for any graph H with n vertices and m edges, there is a connected supermagic graph G which contains H as an induced subgraph such that

$$|V(G)| \leq 2m + 2n^2 + o(n^2). \tag{4}$$

In particular, there are supermagic graphs G containing the complete graph K_n, such that

$$|V(G)| \leq 3n^2 + o(n^2). \tag{5}$$

On the other hand, if G is a supermagic graph which contains a clique of order n, then (1) becomes

$$|V(G)| \geq n^2 - O(n^{3/2}), \tag{6}$$

so that there is a gap between these upper and lower bounds. Our main result, which closes the gap between the bounds (5) and (6), is the following.

Theorem 1 *Let $s(n)$ denote the minimum order of a connected supermagic graph containing a clique of order n. Then*

$$s(n) = n^2 + o(n^2).$$

The proof of Theorem 1 can be adapted to show the following improvement of inequality (4).

Theorem 2 *For any graph H with n vertices there is a connected supermagic graph G of order $N = n^2 + o(n^2)$ which contains H as an induced subgraph.*

2 Magic graphs from Sidon sets

We have already mentioned that the existence of a connected supermagic graph of order N containing a clique of order n implies the existence of a weak Sidon set of order n in $[1, N]$. We will show that these two facts are actually equivalent.

Recall that a set of positive integers A is a weak Sidon set if for any diferent elements $x, y, u, v \in A$, $x + y \neq u + v$. This is equivalent to say that $x - u \neq v - y$.

We first give a bound for a weak Sidon set A which is good for any $|A| \geq 3$. The proof is similar to Ruzsa [11, Theorem 4.7].

Lemma 1 *Let $A \subset [1, N]$ be a weak Sidon set with $|A| \geq 3$. Then*

$$N \geq \frac{|A|(|A| - 3)}{2} + 3 + \epsilon(|A|),$$

where $\epsilon(n) = 1$ for $n \geq 6$ and $\epsilon(n) = 0$ otherwise.

The following easy Lemma gives a simple criteria for a vertex labeling to extend to a super magic labeling.

Lemma 2 *Let $G = (V, E)$ be a graph of order n and $f : V \to [1, n]$ a bijection. Suppose that the edge sums of f, $\{f(x)) + f(y),\ xy \in E\}$, form a consecutive set of integers. Then f can be extended to a supermagic labeling of G.*

Lemma 3 *Let $G = (V, E)$ be a supermagic connected graph of order N. For each $N' > N$ there is a supermagic connected graph G' which contains G as an induced subgraph.*

Lemma 4 *Let $G = (V, E)$ be a graph of order N and $f : V \to [1, N]$ a bijection such that the edge sums $f(x) + f(y),\ xy \in E$ are pairwise different. Then there is a super magic graph G' of order N which contains G as a spanning subgraph.*

We are now ready for the proof of our main result.

Theorem 3 *There is a connected super magic graph of order N containing a clique of order n if and only if there is a weak Sidon set $A \subset [1, N]$ of cardinality n.*

Theorem 1 follows from Theorem 3 and the bounds described in the Introduction. Let $s(n)$ denote the minimum order of a supermagic graph containing a clique of order n. By (3) we have $s(n) \geq n^2 + o(n^2)$. On the other hand, the known constructions of dense Sidon sets together with results on the distribution of primes provide in particular weak Sidon sets of cardinality n in $[1, N]$ with $N = n^2 + o(n^2)$, see for instance [3, Lemma 3]. By Theorem 3 we then have $s(n) \leq n^2 + o(n^2)$.

Theorem 1 can be extended to construct connected supermagic graphs which contain any given graph H as induced subgraph.

Corollary 1 *Let H be a connected graph of order n. If there is a weak Sidon set $A \subset [1, N]$ of cardinality n then there exists a connected super magic graph G of order N which contains H as an induced subgraph.*

As a result of Corollary 1, there are connected supermagic graphs of order $N \leq n^2 + o(n^2)$ which contain a given graph H of order n as an induced graph. This proves Theorem 2.

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