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Finding a Strong Stable Set or a Meyniel Obstruction in any Graph

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A strong stable set in a graph G is a stable set that contains a vertex of every maximal clique of G. A Meyniel obstruction is an odd circuit with at least five vertices and at most one chord. Given a graph G and a vertex v of G, we give a polytime algorithm to find either a strong stable set containing v or a Meyniel obstruction in G. This can then be used to find in any graph, a clique and colouring of the same size or a Meyniel obstruction.

Keywords: stable set, independent set, graph colouring, Meyniel graph, perfect graph

A *Meyniel graph* is a graph which does not contain an odd circuit with at least five vertices and at most one chord. Such a circuit is called a *Meyniel obstruction.* Meyniel [6] proved that Meyniel graphs are perfect. Meyniel's theorem can be stated in the following way.

Theorem 1 (Meyniel's Theorem) *For any graph G, either G contains a Meyniel obstruction, or G contains a clique and colouring of the same size, or both.*

We give a polytime algorithm to find, in any graph G, some instance of what Meyniel's Theorem says exists.

Burlet and Fonlupt [1] and Roussel and Rusu [7] gave polytime algorithms for recognizing whether or not a graph is a Meyniel graph. In the case that the graph is Meyniel, they do not find a clique and colouring of the same size. Our algorithm is incomparable with Meyniel graph recognition. It may give a clique and colouring the same size in a non-Meyniel graph without recognizing that the graph is non-Meyniel.

Algorithms for finding a minimum colouring of a Meyniel graph have been given by Hoàng [4], Hertz [3], Roussel and Rusu [8], and Lévêque and Maffray [5]. Any polytime algorithm for finding a minimum colouring in a perfect graph, in particular a Meyniel graph, can be used to find in polytime a clique in the graph which is the same size as the colouring [2, 4]. However, none of these algorithms provide a way to find in any graph an instance of what Meyniel's Theorem asserts to exist. All of them, as well as ours, can be used to find a clique and colouring the same size in any graph which does not have a Meyniel obstruction. However our algorithm can also be described as finding a Meyniel obstruction in any graph which does not have a clique and colouring the same size.

A *stable set* in a graph G is a set of vertices, no two of which are joined by an edge of G. A *strong stable set* is a stable set that contains a vertex of every maximal clique. (By maximal, we mean maximal with respect to inclusion, not largest.) It is easy to see that a polytime algorithm for finding a strong stable set in a graph can be applied repeatedly to find a colouring of a graph, and it is also then easy to construct a clique of the same size as the colouring.

Theorem 2 (Hoàng [4]) *For any graph G and vertex w of G, either G contains a strong stable set containing w, or G contains a Meyniel obstruction, or both.*

We give a polytime algorithm to find an instance of what Theorem 2 says exists. We now describe the ideas we use for developing this algorithm. As usual, we use P_4 to denote the path with four vertices.

A *nice set* S is a maximal stable set linearly ordered so there is no induced P_4 between any vertex u of S and the pseudonode obtained by identifying all vertices of S which precede u.

Theorem 3 *Every nice set is a strong stable set.*

Note that the definition of nice set is an NP description, but the definition of strong stable set is not.

Algorithm.
Input: Graph G and vertex w of G.
Output: Nice stable set of G containing w or a Meyniel obstruction.
Let $w = u_1$.
Suppose $u_1, u_2, ..., u_k$ have been chosen. If every vertex of $V(G) - \{u_1, u_2, ..., u_k\}$ is adjacent to one of $u_1, u_2, ..., u_k$, then the chosen vertices form a nice set. Otherwise, choose u_{k+1} to be a vertex of $V(G) - \{u_1, u_2, ..., u_k\}$ not adjacent to any of $u_1, u_2, ..., u_k$ and such that it has the largest number of common neighbours with the pseudonode $v(u_1, u_2, ..., u_k)$ obtained by identifying $u_1, u_2, ..., u_k$. If there is a P_4 from $v(u_1, u_2, ..., u_k)$ to u_{k+1}, then G contains a Meyniel obstruction. To find this circuit, we use a "pseudonode expansion algorithm", which we cannot describe here. The simple lemmas below help us to find the circuit.

A chord of a circuit C is called *short* if it joins two vertices at distance 2 in C (i.e., if it creates a triangle with C). Two short chords of C are *overlapping* if one is ac and the other is bd, where a, b, c, d are consecutive vertices on C.

Lemma 1 *In an odd circuit of size at least 7 with two chords, either there is an odd circuit of size at least 5 with at most one chord, or the two chords are overlapping short chords..*

Lemma 2 *In an odd circuit of size at least 5 with all chords hitting the same vertex h and at least one of these possible chords missing, there is an odd circuit of size at least 5 with at most one chord, and the chord is short and hits h.*

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