An upper bound for the chromatic number of line graphs

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It was conjectured by Reed [12] that for any graph G, the graph's chromatic number $\chi(G)$ is bounded above by $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$, where $\Delta(G)$ and $\omega(G)$ are the maximum degree and clique number of G, respectively. In this paper we prove that this bound holds if G is the line graph of a multigraph. The proof yields a polynomial time algorithm that takes a line graph G and produces a colouring that achieves our bound.

1 Introduction

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colours required to colour the vertex set of G so that no two adjacent vertices are assigned the same colour. That is, the vertices of a given colour form a *stable set*. Determining the exact chromatic number of a graph efficiently is very difficult, and for this reason it has proven fruitful to explore the relationships between $\chi(G)$ and other invariants of G. The *clique number* of G, denoted by $\omega(G)$, is the largest set of mutually adjacent vertices in G and the *degree* of a vertex v, written $\deg(v)$, is the number of vertices to which v is adjacent; the maximum degree over all vertices in G is denoted by $\Delta(G)$. It is easy to see that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. Brooks' Theorem (see [1]) tightens this:

Brooks' Theorem $\chi(G) \leq \Delta(G)$ *unless G contains a clique of size* $\Delta(G) + 1$ *or* $\Delta(G) = 2$ *and G contains an odd cycle.*

So for $\chi(G)$ we have a trivial upper bound in terms of $\Delta(G)$ and a trivial lower bound in terms of $\omega(G)$. We are interested in exploring upper bounds on $\chi(G)$ in terms of a convex combination of $\Delta(G) + 1$ and $\omega(G)$. In [12], Reed conjectured a bound on the chromatic number of any graph G:

Conjecture 1 *For any graph G,* $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

Several related results exist. In the same paper, Reed proved that the conjecture holds if $\Delta(G)$ is sufficiently large and $\omega(G)$ is sufficiently close to $\Delta(G)$. Using this, he proved that there exists a positive constant α such that $\chi(G) \leq \alpha(\omega(G)) + (1-\alpha)(\Delta(G)+1)$ for all graphs. Some results are also known for generalizations of the chromatic number.

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A *fractional vertex c-colouring* of a graph G can be described as a set of stable sets $\{S_1, S_2, \dots S_l\}$ with weights $\{w_1, w_2, \dots w_l\}$ such that for every vertex v, $\sum_{S_i : v \in S_i} w_i = 1$ and $\sum_{i=1}^{l} w_i = c$. The *fractional chromatic number* of G, written $\chi^*(G)$, is the smallest c for which G has a *fractional vertex c-colouring*. Note that it is always bounded above by the chromatic number. The *list chromatic number* of a graph G, written $\chi_l(G)$, is the smallest r such that if each vertex is assigned any list of r colours, the graph has a colouring in which every vertex is coloured with a colour on its list. For any graph we clearly have $\chi^*(G) \leq \chi(G) \leq \chi_l(G)$.

In [10], Molloy and Reed proved the fractional analogue of Conjecture 1 for all graphs, i.e. that

$$\chi^*(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil \text{ for any graph } G. \tag{1}$$

In fact, the round-up is not needed in the fractional case. In this paper we prove that Conjecture 1 holds for line graphs, which are defined in the next section.

2 Fractional and Integer Colourings in Line Graphs of Multigraphs

A *multigraph* is a graph in which multiple edges are permitted between any pair of vertices – all multigraphs in this paper are loopless. Given a multigraph $H = (V, E)$, the *line graph* of H, denoted by $L(H)$, is a graph with vertex set E; two vertices of $L(H)$ are adjacent if and only if their corresponding edges in H share at least one endpoint. We say that G is a line graph if there is a multigraph H for which $G = L(H)$.

The *chromatic index* of H, written $\chi'(H)$, is the chromatic number of $L(H)$. Similarly, the *fractional chromatic index* $\chi'^*(H)$ is equal to the fractional chromatic number of $L(H)$. In [6], Holyer proved that determining the chromatic index of an arbitrary multigraph is NP-complete, so practically speaking we are bound to the task of approximating the chromatic index of multigraphs and hence the chromatic number of line graphs.

Vizing's Theorem (see [14]) bounds the chromatic index of a multigraph in terms of its maximum degree, stating that $\Delta(H) \leq \chi'(H) \leq \Delta(H) + d$, where d is the maximum number of edges between any two vertices in H. Both bounds are achievable, but a more meaningful bound should consider other invariants of H. Of course, $\chi'(H)$ is always bounded below by $\chi'^*(H)$, and Edmond's theorem for matching polytopes (presented in [3], also mentioned in [8]) tells us that given

$$\Gamma(H) = \max \left\{ \frac{2|E(W)|}{|V(W)| - 1} : W \subseteq H, |V(W)| \text{ is odd} \right\},$$

$$\chi'^*(H) = \max\{\Delta(H), \Gamma(H)\}. \tag{2}$$

Does this necessarily translate into a good upper bound on the chromatic index of a multigraph? The following long-standing conjecture, proposed by Goldberg [4] and Seymour [13], implies that $\chi'^*(H) \leq \chi'(H) \leq \chi'^*(H) + 1$:

Goldberg-Seymour Conjecture *For a multigraph H for which $\chi'(H) > \Delta(H) + 1$, $\chi'(H) = \lceil \Gamma(H) \rceil$.*

Asymptotic results are known: Kahn [7] proved that the fractional chromatic index asymptotically agrees with the integral chromatic index, i.e. that $\chi'(H) \leq (1 + o(1))\chi'^*(H)$. This implies the Goldberg-Seymour Conjecture asymptotically. He later proved that in fact, the fractional chromatic index asymptotically agrees with the list chromatic index [8].

Another result that supports the Goldberg-Seymour Conjecture is the following theorem:

Theorem 2 (Caprara and Rizzi [2]) *For any multigraph H, $\chi'(H) \leq \max\{\lfloor 1.1\Delta(H)+0.7\rfloor, \lceil\Gamma(H)\rceil\}$.*

This theorem is a slight improvement of an earlier result of Nishizeki and Kashiwagi [11], lowering the additive factor from 0.8 to 0.7. Note that this implies the Goldberg-Seymour Conjecture for any multigraph H with $\Delta(H) \leq 12$, since in this case we have $\lfloor 1.1\Delta(H) + 0.7\rfloor \leq \Delta(H) + 1$.

3 The Main Result

We will now prove our main result:

Theorem 3 *For any line graph G, $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.*

Consider a multigraph H for which $G = L(H)$. The proof consists of two cases: the case where $\Delta(G)$ is large compared to $\Delta(H)$, and the case where $\Delta(G)$ is close to $\Delta(H)$. In both cases we use the fact that $\omega(G) \geq \Delta(H)$. The first case is given by the following lemma, which follows easily from Theorem 2.

Lemma 4 *If G is the line graph of a multigraph H, and $\Delta(G) \geq \frac{3}{2}\Delta(H)-1$, then $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.*

Proof of Theorem 3:

Consider a counterexample $G = L(H)$ such that the theorem holds for every line graph on fewer vertices. We know that $\Delta(G) < \frac{3}{2}\Delta(H) - 1$. Our approach is as follows: We find a maximal stable set $S \subset V(G)$ that has a vertex in every maximum clique in G, and let G' be the subgraph of G induced on $V(G) \setminus S$. We can see that $\Delta(G') \leq \Delta(G) - 1$ (since S is maximal) and $\omega(G') = \omega(G) - 1$, and that the theorem holds for G', as any induced subgraph of a line graph is clearly a line graph. So we know that $\chi(G') \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil - 1$. We can now construct a proper $\chi(G') + 1$-colouring of $V(G)$ by taking a proper $\chi(G')$-colouring of G' and letting S be the final colour class, hence $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$, a contradiction.

It suffices, then, to show the existence of such a stable set S in G. We actually need only find a stable set that hits all the maximum cliques of G, as we can extend any such stable set until it is maximal. We will do this in terms of a *matching* in H, i.e. a set of edges in $E(H)$, no two of which share an endpoint – a matching in H exactly represents a stable set in G. We need some notation first. For a pair of vertices $u, v \in V(H)$, the *multiplicity* of uv is the number of edges in $E(H)$ between u and v; we denote it by $\mu(u, v)$. A *triangle* in H is a set of three mutually adjacent vertices, and we denote the maximum number of edges of any triangle in H by tri(H); the edges of a triangle are those edges in $E(H)$ joining the triangle's vertices. Note the following facts that relate invariants of H and G:

Fact 1 $\Delta(G) = \max_{uv\in E(H)}\{\deg(u) + \deg(v) - \mu(u, v) - 1\}$.

Fact 2 $\omega(G) = \max\{\Delta(H), \text{tri}(H)\}$.

We say that a matching *hits* a vertex v if v is an endpoint of an edge in the matching. We will find a maximal matching M in H which corresponds to a desired stable set because it hits every vertex of maximum degree in H and contains an edge of every triangle with $\max\{\Delta(H), \text{tri}(H)\}$ edges in H.

To this end, let S_Δ be the set of vertices of degree $\Delta(H)$ in H and let T be the set of triangles in H that contain $\max\{\Delta(H), \text{tri}(H)\}$ edges. It is instructive to consider how the elements of T interact; we omit the straightforward proofs of these observations from this extended abstract.

Observation 1 *If two triangles of T intersect in exactly the vertices a and b then ab has multiplicity greater than $\Delta(H)/2$.*

Observation 2 *If abc is a triangle of T intersecting another triangle ade of T in exactly the vertex a then $\mu(b,c)$ is greater than $\Delta(H)/2$.*

Observation 3 *If there is an edge of H joining two vertices a and b of S_Δ then $\mu(a,b) > \Delta(H)/2$.*

Guided by these observations, we let T' be those triangles in T that contain no pair of vertices of multiplicity $> \Delta(H)/2$ and S'_Δ be those elements of S_Δ which are in no pair of vertices of multiplicity greater than $\Delta(H)/2$. We treat $T' \cup S'_\Delta$ and $(T \setminus T') \cup (S_\Delta \setminus S'_\Delta)$ separately. A few more observations regarding S'_Δ and T' will serve us well. Again, we omit the proofs.

Observation 4 *For any $S \subseteq S'_\Delta$, $|N(S)| \geq |S|$.*

Observation 5 *If an edge ab in H has exactly one endpoint in a triangle bcd of T', then the degree of a is less than $\Delta(H)$.*

Observation 6 *If an edge ab in H has exactly one endpoint in a triangle bcd of T', then $\mu(a,b) \leq \Delta(H)/2$.*

Observation 7 *For any vertex v with two neighbours u and w, $\deg(u) + \mu(vw) \leq \frac{3}{2}\Delta(H)$.*

It is now straightforward to show that the desired matching exists. We begin with a matching M consisting of one edge between each vertex pair with multiplicity greater than $\Delta(H)/2$ – this hits $S_\Delta \setminus S'_\Delta$ and contains an edge of each triangle in $T \setminus T'$. Observation 4 tells us that we can apply Hall's Theorem (see [5]) to get a matching in H that hits S'_Δ; Observation 7 dictates that this matching cannot hit M, so the union M' of these two matchings is a matching in H that hits S_Δ and contains an edge of each triangle in $T \setminus T'$. Every edge in this matching either hits a maximum-degree vertex in H or has endpoints with multiplicity greater than $\Delta(H)/2$.

What, then, can prevent us from extending this M' to contain an edge of every triangle in T'? Observations 1 and 2 tell us that any two triangles in T' are vertex-disjoint, so our only worry is that M' hits two vertices of some triangle in T'. Observations 3, 5 and 6 guarantee that at most one such vertex in a given triangle is hit, and if there is such a vertex, it has degree $\Delta(H)$. We can therefore extend M' to contain an edge of every triangle in T'. The result is a matching that satisfies all of our requirements, so the proof of the theorem is complete. □

4 Algorithmic Considerations

We have presented a new upper bound for the chromatic number of line graphs, i.e. $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$. Our proof of the bound yields an algorithm for constructing a colour class in G but we have an initial condition in the proof (i.e. $\Delta(G) < \frac{3}{2}\Delta(H) - 1$) that does not necessarily remain if we remove these vertices. However, the bound given by Caprara and Rizzi in Theorem 2 can be achieved in $O(|E(H)|(|V(H)| + \Delta(H)))$ time [2]. It is easy to see that in the proof of Theorem 3 we can find our matching in polynomial time, so we can formulate a polytime algorithm for $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$-colouring a line graph G with root graph H as follows.

1. While $\Delta(L(H)) < \frac{3}{2}\Delta(H) - 1$, remove a matching M from H as in the proof of Theorem 3 (and let it be a colour class).

2. Employ Caprara and Rizzi's algorithm to complete the edge colouring of H.

This, of course, assumes that we have the root graph H such that $G = L(H)$. Lehot provides an $O(|E(G)|)$ algorithm that detects whether or not G is the line graph of a simple graph H and outputs H if possible [9]. Two vertices u and v in G are *twins* if they are adjacent and their neighbourhoods are otherwise identical. We can extend Lehot's algorithm to line graphs of multigraphs by contracting each set of k mutually twin vertices in G into a single vertex, which we say has multiplicity k. This can be done trivially in $O(|E(G)|\Delta(G))$ time. The resulting graph G' is the line graph of a simple graph H' if and only if G is the line graph of a multigraph H; we can generate H from H' by considering the multiplicities of the vertices in G' and duplicating edges in H' accordingly.

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