Packing non-returning A-paths algorithmically

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In this paper we present an algorithmic approach to packing A-paths. It is regarded as a generalization of Edmonds' matching algorithm, however there is the significant difference that here we do not build up any kind of alternating tree. Instead we use the so-called 3-way lemma, which either provides augmentation, or a dual, or a subgraph which can be used for contraction. The method works in the general setting of packing non-returning A-paths. It also implies an ear-decomposition of criticals, as a generalization of the odd ear-decomposition of factor-critical graph.

Keywords: A-paths, matching

1 Introduction

The paper is devoted to the problem of packing fully node-disjoint non-returning A-paths in a graph $G = (V, E)$. Given a graph and a subset $A \subseteq V$, a path is said to be an A-path if its ends are two distinct nodes in A. Packing fully node-disjoint A-paths reduces to maximum matching in an auxiliary graph, see T. Gallai (3). The special case $A = V$ is in fact equivalent to maximum matching. W. Mader considered a more difficult problem. We are given a subset $A \subseteq V$ with a partition $\mathcal{A}$. An A-path is called an $\mathcal{A}$-path if its ends are in two distinct members of $\mathcal{A}$. Mader (5) gave a min-max formula for the maximum number of fully node-disjoint $\mathcal{A}$-paths. A polynomial time algorithm to find these paths was given by L. Lovász using his matroid parity apparatus. Matroid parity is still a challanging topic in combinatorial optimization. If a problem turns out to be an instance for matroid parity, this does not necessarily imply a polynomial time algorithm or a good characterization. Lovász disentangled some techincal details to construct an algorithm, see (4). Later, A. Schrijver gave a funny reduction to linear matroid parity – which by itself also implies an algorithm. It was a challange to construct directly an algorithm for packing $\mathcal{A}$-paths. Such an algorithm was given by Chudnovsky et al. (2). They in fact work with the concept of non-zero A-paths, which is a generalization of $\mathcal{A}$-paths, see also (1). The main goal of this paper is to construct an algorithm which presents the "dual" in a more structured form. Our method implies an ear-decomposition of "criticals" – this generalizes the ear-decomposition of factor-critical graphs.

Maximum matching is a special case of the problem discussed in this paper, let us briefly sketch how the method works for maximum matching. For a given matching $M \subseteq E$ in G, we call an odd cycle $C \subseteq E$ an M-alternating odd cycle if $|C \cap M| = (|C| - 1)/2$ and C is incident to an M-exposed node. The following lemma can be proved directly, a proof "on the level of bipartite matching" can be given. In

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fact, Edmonds' alternating forests provide an alternative proof of this lemma. Our crucial observation is that a matching algorithm can be constructed by only using the below lemma as a black box. This black box is regarded as a compact formulation of some consequences of alternating forests. However, one can also give a short, inductive proof without alternating forests.

Lemma 1.1 (3-Way Lemma for Matching) *Given an undirected graph G with a matching M, then at least one of the following alternatives holds:*

1. *There is a matching N with $|N| = |M| + 1$.*
2. *There is a matching N with $|N| = |M|$ and an N-alternating odd cycle in G.*
3. *There is a vector $c \in \{0, 1, 2\}^V$ such that the weight of any edge is at least 2, and the sum of its entries is exactly $2|M|$.*

This lemma allows us to interpret of Edmonds' algorithm as follows. Consider a matching M in graph G, try Lemma 1.1. Alternative 1 gives an augmentation, alternative 3 verifies optimality. Alternative 2 provides an odd cycle for contraction. Contraction of an alternating odd cycle has the property that augmentation, or a Berge-Tutte-dual in G/C can be expanded to G.

2 Packings in p-graphs — Definitions

The most important notion in this paper is a **permutation labeled graph** or **p-graph**, for short. A p-graph comes in the form of G, A, ω, π, where G is a graph, A is a set of nodes, π are edge-labels. This notion provides a generalization of some well-known packing problems – matching, node-disjoint $\mathcal{A}$-paths, non-zero A-paths. The motivation for this version is that important reduction principles used by our algorithm stay within the concept of a p-graph, but does not stay within well-known previous concepts. The precise definition of a p-graph is formulated as follows.

Let $G = (V, E)$ be an undirected graph with node-set V, edge-set E with a reference orientation. Let $A \subseteq V$ be a fixed set of **terminals**. Let Ω be an arbitrary **set of "potentials"** and let $\mathbf{jj}, \mathbf{JJ}$ be called **Jolly Joker** (some imaginary labels). Let $\omega : A \to \Omega$ define the **potential of origin** for the terminals. Let $\pi : E \to S(\Omega) \cup \{\mathbf{JJ}\}$ where $S(\Omega)$ is the set of all permutations of Ω. For an edge $ab = e \in E$, let $\pi(e, a) := \pi(e)$ and $\pi(e, b) := \pi^{-1}(e)$ be the **mapping of potential** on edge ab. (We use $\circ$ for the composition of permutations. We define $\mathbf{JJ}^{-1} := \mathbf{JJ} \circ \pi := \pi \circ \mathbf{JJ} := \mathbf{JJ}$ and $\mathbf{JJ}(\omega) := \pi(\mathbf{jj}) := \mathbf{jj}$ for any $\pi \in S(\Omega) \cup \{\mathbf{JJ}\}$ and for any $\omega \in \Omega \cup \{\mathbf{jj}\}$.) A **walk** in G is a sequence of nodes and edges, say $W = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ where $e_i = v_i v_{i+1}$ or $e_i = v_{i+1} v_i$ for all $0 \le i \le k-1$. W is called an A**-walk** in G if $v_0, v_k \in A$ and $v_j \notin A$ (for $j \neq 0, k$). $\chi_W \in \mathbb{N}^V$ denotes the **traversing multiplicity vector** of walk W, defined by $\chi_W(v) := |\{j : v_j = v\}|$. A walk W is called a **path** if $\chi_W \le \mathbf{1}$. We will usually use letters P, R for paths. For an A-walk let $\pi(W) := \pi(e_0, v_0) \circ \pi(e_1, v_1) \circ \ldots \circ \pi(e_{k-1}, v_{k-1})$ define the **mapping of potentials on** W. W is called **non-returning** if $\pi(W)(\omega(v_0)) \neq \omega(v_k)$. (Hence, an empty A-walk (having a single node and no edge) is not considered to be non-returning. Notice, if W traverses any edge with label **JJ**, then W is non-returning.) A family $\mathcal{P}$ of fully node-disjoint non-returning A-paths is called a **packing**. $\nu = \nu(G) = \nu(G, A, \omega, \pi)$ denotes the **maximum cardinality of a packing**. Also, a "node-capacited packing problem" can be defined. Consider a function $b \in \mathbb{N}^V$ of **node capacities**. A family $\mathcal{W}$ of A-walks (we allow walks to be taken multiply) is called a b**-packing** if $\sum_{W \in \mathcal{W}} \chi_W \le b$. Let $\nu_b = \nu_b(G) = \nu_b(G, A, \omega, \pi)$ denotes the maximum cardinality of a b-packing. $b = \mathbf{1}$ defines packings, $b = \mathbf{2}$ defines 2-packings.

3 Min-max Theorems for packings

For a set $F \subseteq E$ of edges, let $A^F := A \cup V(F)$. F is called A-**balanced** if ω can be extended to a function $\omega^F : A^F \to \Omega$ s.t. each edge $ab \in F$ is ω^F**-balanced** – i.e. $\pi(ab, a)(\omega^F(a)) = \omega^F(b)$. Let $c_{odd}(G, A)$ be the number of components in G having an odd number of nodes in A – these will be called **odd components of** G, A. Let $c_1(G, A)$ be the number of nodes in A which are isolated nodes of G.

Theorem 3.1 *In a p-graph the maximum cardinality of a packing is determined by*

$$\nu(G, A, \omega, \pi) = \min_{F,X} \; |X| + \frac{1}{2}\left(|A^F - X| - c_{odd}(G - F - X, A^F - X)\right) \, , \tag{1}$$

where the minimum is taken over an A-balanced edge-set F and a set $X \subseteq V$.

Theorem 3.2 *In a p-graph the maximum cardinality of a 2-packing is determined by*

$$\nu_2(G, A, \omega, \pi) = \min_{F,X} \; 2|X| + |A^F - X| - c_1(G - F - X, A^F - X) \, , \tag{2}$$

where the minimum is taken over an A-balanced edge-set F and a set $X \subseteq V$.

In Theorem 3.2 we do not count odd components to determine a maximum 2-packing, this indicates that 2-packings are simpler than packings. A similar relation there is between matchings and 2-matchings, the latter admitting a reduction to bipartite matching, Kőnig's Theorem. The following theorem is in fact a reformulation of Theorem 3.2, here we formulate a Kőnig-type condition for 2-packings.

Theorem 3.3 *In a p-graph the maximum cardinality of a 2-packing is determined by*

$$\nu_2(G, A, \omega, \pi) = \; \min \; ||c|| \, , \tag{3}$$

where $||c|| := \sum_{v \in V} c(v)$ and the minimum is taken over ***2-covers*** *c, i.e. vectors $c \in \{0, 1, 2\}^V$ such that $c \cdot \chi_W \geq 2$ for any non-returning A-walk.*

4 Contraction of dragons

A path P is called a **half-A-path** if it starts in a terminal $s \in A$, ends in a node $t \in V$ and $V(P) \cap A = \{s\}$. We say P **ends in** t **with potential** $\pi(P)(\omega(s))$. Consider a node $v \in V$ and a potential $\omega_0 \in \Omega \cup \{\mathbf{jj}\}$. We say a node v **is** ω_0**-reachable** (or ω_0 is reachable at v), if there is a pair $\mathcal{P}, P_v$ such that P_v is a half-A-path ending in v with ω_0, and $\mathcal{P}$ is a packing of ν non-returning A-paths each of which is fully node-disjoint from P_v. We say a node is **reachable** if it is ω_0-reachable for some $\omega_0 \in \Omega \cup \{\mathbf{jj}\}$. v is called **uniquely reachable** if it is ω_0-reachable only with a single element $\omega_0 \neq \mathbf{jj}$. Otherwise – if v is **jj**-reachable or there are at least two different elements of Ω which are reachable at v, then v is called **multiply reachable**. The definition implies that a reachable terminal is uniquely reachable. We call a p-graph G a **dragon** if $|A| = 2\nu + 1$ and every node is reachable. A p-graph is called **critical** if it is a dragon such that every non-terminal is multiply reachable. (The notion of criticals is analogue to the notion used in (1). The notion of dragons should be considered as a weak version of criticality.) Let us use the expression **odd cycle** for p-graphs s.t. $G = (V, E)$ is an odd cycle, $A = V$, and all the edges in E give one-edge non-returning A-walks (which are in fact non-returning A-paths except for 1-edge odd cycles). A p-graph with $V = \{a, b\}$, $E = \{ab\}$, $A = \{a\}$ is called a **rod**.

Claim 4.1 *Odd cycles and rods are dragons.* □

A crucial lemma is the following, saying that the min-max formula holds for dragons.

Lemma 4.2 (A dragon has a special dual) *Suppose a G is a dragon with exactly its nodes in V_1 being uniquely reachable, say $v \in V_1$ is $\omega'(v)$-reachable. Let $F := \{e \in E[V_1] : e$ is ω'-balanced$\}$. Then $2\nu = |V_1| - c(G - F, V_1)$.*

The notion "reachability" is in fact motivated by the goal to define the contraction of dragon subgraphs.

Definition 4.3 (Contraction of a dragon) *Consider a set $Z \subseteq V$ such that $G[Z]$ is dragon. We define the contracted p-graph on G/Z as follows. Let Z_1 be the uniquely reachable nodes in $G[Z]$, say $a \in Z_1$ is ω_a-reachable. Let $A/Z := A - Z + \{Z\}$. Let $\Omega' := \Omega + \bullet$ for some new element $\bullet \notin \Omega$. Let $\omega_Z(s) := \omega(s)$ for all $s \in A/Z - \{Z\}$, and let $\omega_Z(\{Z\}) := \bullet$. We define $\pi_Z(e)$ by the following case splitting. If e is disjoint from Z, then we define $\pi_Z(e)$ by extending $\pi(e)$ to Ω' by mapping $\bullet$ to $\bullet$. For an edge ab with $a \in Z_1$, $b \notin Z$ we label its image $\{Z\}b$ s.t. $\pi_Z(\{Z\}b)(\{Z\}) = \pi(ab)(\omega_a)$. For an edge ab with $a \in Z - Z_1$, $b \notin Z$ we define let $\pi_Z(\{Z\}b) :=$ **JJ**.*

We define the **contraction of a node-disjoint family $\mathcal{Z}$ of dragons** $G/\mathcal{Z}, A/\mathcal{Z}, \omega_{\mathcal{Z}}, \pi_{\mathcal{Z}}$ by contracting the dragons in $\mathcal{Z}$ one-by-one. By definition, a contraction has the following properties.

Claim 4.4 (Expansion of a packing) *From any packing in $G/\mathcal{Z}$ one can construct a packing in G which exposes the same number of terminals.*

Claim 4.5 (Pre-image of a dragon) *The pre-image of a dragon Z_1 in $G/\mathcal{Z}$ is dragon. (Thus, $\mathcal{Z}/Z_1 := \{Z : Z \in \mathcal{Z}, \{Z\} \notin Z_1\} \cup \{$the pre-image of $Z_1\}$ is a finer node-disjoint family of dragons.)*

5 The 3-Way Lemma and the algorithm

Our main tool in the algorithm is the 3-Way Lemma for packings. Consider a packing $\mathcal{P}$ in G and a dragon Z in G. We say $\mathcal{P}$ is **equipped with** Z if $\mathcal{P}$ consists of some paths disjoint from $V(Z)$ and exactly $\nu(G[Z]) = (|A \cap V(Z)| - 1)/2$ paths inside Z.

Lemma 5.1 (The 3-way Lemma) *Consider a p-graph with a packing $\mathcal{P}$. Then at least one of the following alternatives holds:*

1. *There is a packing $\mathcal{R}$ with $|\mathcal{R}| = |\mathcal{P}| + 1$.*
2. *There is a packing $\mathcal{R}$ s.t. $|\mathcal{R}| = |\mathcal{P}|$, and is equipped with a rod or an odd cycle.*
3. *There is a 2-cover c such that $2|\mathcal{P}| = ||c||$. (I.e. a verifying 2-cover for $2 \times \mathcal{P}$)*

The 3-Way Lemma is applied sequentially in the algorithm to construct sequences of contractions. A **sequence of contractions** is a sequence $(\mathcal{Z}_1, G_1, \mathcal{P}_1, \mathcal{R}_1, S_1)$, $\cdots$, $(\mathcal{Z}_m, G_m, \mathcal{P}_m, \mathcal{R}_m, S_m)$, $(\mathcal{Z}_{m+1}, G_{m+1}, \mathcal{P}_{m+1})$ with $m \geq 0$, and the following properties. $\mathcal{Z}_0 = \emptyset$, and $\mathcal{Z}_i$ is a node-disjoint family of dragons in G. $G_i = (V_i, E_i) := G/\mathcal{Z}_i$. $G_i[S_i]$ is an odd cycle or a rod, where $S_i \subseteq V_i$. $\mathcal{R}_i$ is a packing in G_i which is equipped with S_i. $\mathcal{P}_{i+1} := \mathcal{R}_i/S_i$, $\mathcal{Z}_{i+1} := \mathcal{Z}_i/S_i$ for $i = 1, \cdots, m$. Each $\mathcal{P}_i, \mathcal{R}_i$ leaves the same number of terminals uncovered.

The proof of Theorem 3.1 and the algorithm relies on the following key observation, which provides a tool to construct a verifying pair. It says that from a 2-packing verification in a contraction we can construct a packing verification in the original p-graph.

Lemma 5.2 (Constructing a verifying pair) *Suppose we have a sequence of contractions, and a 2-cover c in G_{m+1} with $2|\mathcal{P}_{m+1}| = ||c||$. Then for all i, $\mathcal{P}_i$ is a maximum packing in G_i and one can construct a verifying pair for $\mathcal{P}_i$.*

Now we are in position to sketch the algorithm. Our algorithm has an input of a p-graph G and a packing $\mathcal{P}$. The output is either a larger packing, or a verifying pair for $\mathcal{P}$. The algorithm starts off with initiating the trivial sequence of contractions, $m = 0$. In a general step, apply Lemma 5.1 to $G_{m+1}, \mathcal{P}_{m+1}$! If alternative 1 holds, then by Claim 4.4 one can construct a packing in G larger than $\mathcal{P}$. If alternative 2 holds, then by Claim 4.5 one can construct a longer sequence of contractions. If alternative 3 holds, then by Claim 5.2 $\mathcal{P}$ is maximum, and a verifying pair can be constructed. Full proofs are given in (7). Detailed analysis of the algorithm implies that dragons have a so-called dragon-decomposition.

Definition 5.3 *A **dragon-decomposition** is given by a forest $F \subseteq E$ which has the following properties.*

1. *The components of forest $(V(F) \cup A, F)$ are exactly $\{F_a :$ for each $a \in A\}$ s.t. for each $a \in A$ we have $A \cap V(F_a) = \{a\}$.*
2. *Let $\omega^F : V(F) \cup A \to \Omega$ be the (uniquely defined) function s.t. each edge in F is ω^F-balanced. Let F' be the set of ω^F-balanced edges. Let $\mathcal{K}$ is the family of components of $G - F'$. $F/\mathcal{K}$ is a tree.*
3. *$K, V(F) \cap V(K), \omega^F, \pi$ is critical.*

Lemma 5.4 *Dragons are exactly those p-graphs which have a dragon-decomposition. $V(F) \cup A$ is exactly the set of uniquely reachable nodes.*

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