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A characterization of extremal graphs with no matching-cut

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A graph is called (matching-)immune if it has no edge cut that is also a matching. Farley and Proskurowski proved that for all immune graphs $G = (V, E)$, $|E| \geq \lceil 3(|V| - 1)/2 \rceil$, and constructed a large class of immune graphs that attain this lower bound for every value of $|V(G)|$, called ABC graphs. They conjectured that every immune graph that attains this lower bound is an ABC graph. We present a proof of this conjecture.

Keywords: matching-cut, matching immune, extremal graphs

1 Introduction

All graphs in this paper are allowed to be multi-graphs. For a graph $G = (V, E)$, an edge cut $[S, \overline{S}]$ with $S \subset V$ is called a *matching-cut* if $[S, \overline{S}]$ is a matching. If a graph has no matching-cut, it is called *(matching-)immune*. Farley and Proskurowski [Farley and Proskurowski(1984)] proved the following extremal result on immune graphs:

Theorem 1 (Farley and Proskurowski) *If $G = (V, E)$ is immune, then*

$$|E| \geq \lceil 3(|V| - 1)/2 \rceil.$$

In addition, they constructed a class of multi-graphs called *ABC graphs* that have the following properties:

- ABC graphs are immune.
- If $G = (V, E)$ is an ABC graph, then $|E| = \lceil 3(|V| - 1)/2 \rceil$.

The definition of ABC graphs is based on the following three operations:

- An *A operation* on vertex u introduces vertices v and w and edges uv, uw and vw.
- A *B operation* on edge uv introduces vertices w and x and edges uw, vw, ux and vx, and removes edge uv.
- A *C operation* on vertices u and v ($u = v$ is allowed) introduces vertex w and edges uw and vw.

Note that the C operation is the only operation that can introduce parallel edges.

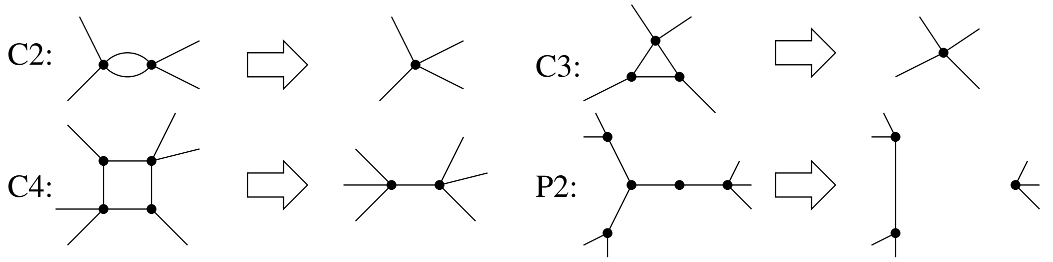

Fig. 1: The four reduction operations

Definition 1 *An* ABC graph *is a graph that can be obtained from K_1 with a sequence of A and B operations and* at most one *C operation.*

A sequence of these operations that constructs a graph G from a graph H is called a *decomposition* of G from H, or simply a decomposition of G if $H = K_1$. For every integer $n \geq 1$, an ABC graph exists. For every integer $n \geq 1$, $n \neq 2$, a simple ABC graph exists. This shows that the lower bound from Theorem 1 is tight. This inspires the following definition:

Definition 2 *A graph $G = (V, E)$ is called* extremal immune *if $|E| = \lceil 3(|V| - 1)/2 \rceil$ and G is immune.*

Farley and Proskurowski stated the following conjecture:

Conjecture 2 (Farley and Proskurowski) *If a graph G is extremal immune, then G is ABC.*

In this paper, we sketch our proof of this conjecture, and show some of the techniques that are used. First we will go into more detail on Farley and Proskurowski's proof of Theorem 1.

2 Contraction operations

To prove Theorem 1, Farley and Proskurowski [Farley and Proskurowski(1984)] introduce four graph operations, named after the structure they reduce. These operations are explained in Figure 1. Below are formal definitions, which show that all of these operations can be expressed with edge deletions and contractions. C_2 stands for the cycle of length two, which is a multi-graph.

C2 Let vertices u and v induce a C_2 in graph G. The C2 operation consists of deleting one of the edges of this C_2 and contracting the other.

C3 Let vertices u, v and w induce a C_3 in G. The C3 operation consists of deleting uv and contracting vw and wu.

C4 Let vertices u, v, w and x induce a C_4 in G such that $uw \notin E(G)$. The C4 operation consists of deleting uv and contracting ux and vw. Note that for one induced C_4 the C4 operation can be applied in two different ways.

P2 Let vertices u and v be neighbors in G with $d(u) = 3$ and $d(v) = 2$. Let v have another neighbor $z \neq u$, and let u have another neighbor $x \neq v$. The P2 operation consists of deleting uv and contracting ux and vz.

It can be checked that these four operations have the following properties:

Lemma 3 (Farley and Proskurowski) *Suppose G' can be obtained from G by a C2, C3, C4 or P2 operation. Then:*

- *If G is immune, then G' is immune.*
- *If $|E(G)| = \lceil 3(|V(G)| - 1)/2 \rceil$, then $|E(G')| \leq \lceil 3(|V(G')| - 1)/2 \rceil$. This inequality is always an equality for the C3, C4 and P2 operation.*

To prove Theorem 1, the following lemma was used:

Lemma 4 (Farley and Proskurowski) *If G is an extremal immune graph not equal to K_1, then one of the operations C2, C3, C4 or P2 can be applied to G.*

On this lemma our proof of Conjecture 2 is based.

3 Overview of the proof

The proof of Conjecture 2 is by contradiction, so first we assume an extremal immune graph exists that is not an ABC graph. Then we consider a graph with minimum number of vertices among all such graphs. This explains the following definition:

Definition 3 *A graph G is a* minimum counterexample *if it is extremal immune, it is not an ABC graph, and has minimum number of vertices among all such graphs.*

We first consider the case that a minimum counterexample G contains a C_2, so a C2 operation can be applied, resulting in vertex v. After applying this C2 operation, another extremal indecomposable graph G' is obtained (Lemma 3, Theorem 1). By our choice of G, G' must be an ABC graph. We consider a number of cases for G', for the choice of $v \in V(G')$ and for the possible graphs G that can correspond to this, and in every case we obtain one of the following contradictions: G has a matching-cut, G is also an ABC graph, or a smaller counterexample exists.

If a minimum counterexample G contains a triangle, we can apply operation C3 and find a contradiction in a similar way, and if G contains a C_4, applying a C4 operation leads to a contradiction. Finally, we can show that if a P2 operation can be applied resulting in the ABC graph G', then there is always a triangle or C_4 in G' that corresponds to a triangle or C_4 in G, so the previous cases can be applied. Since every operation leads to a contradiction, Lemma 4 shows that no counterexamples for the conjecture can exist.

4 The structure of ABC graphs

Definition 4 *A graph G that can be obtained from a graph H using only B operations is called an H-*component*.*

These graphs will often be used for H in this case:

K_2 A K_2-component G is also called an *edge component*. If G can be constructed from a K_2 on vertices u and v, then G is called an edge component *between u and v*.

C_3 A C_3-component is also called a *triangle component*.

We can partition the edges of ABC graphs into edge induced components that are all H-components for $H = C_3$, C_2 or P_3, where C_3-components correspond to A operations in the decomposition, and the C operation corresponds to a C_2 or P_3-component:

Observation 5 *For any decomposition of ABC graph G using k A operations, we can partition the edges of G into sets $A_1, \ldots, A_k$ and at most one set C such that for every i, $G[A_i]$ is a triangle component, and $G[C]$ is a C_2 or P_3-component.*

Observation 6 *For any decomposition of an H-component G, we can partition $E(G)$ into sets $\{E_{uv} : uv \in E(H)\}$ such that $G[E_{uv}]$ is an edge component between u and v. u and v are the only vertices of $G[E_{uv}]$ that can be incident with other edges in G.*

Lemma 7 *For every edge component G between u and v, and every edge $e \in E(G)$, a matching-cut M exists that separates u from v with $e \in M$.*

Lemma 8 *If u is a vertex in a triangle component T, then a decomposition of T exists that starts on u.*

5 An example case

Consider the case that minimum counterexample G contains a triangle on vertices u_1, u_2 and u_3, and a C3 operation on this triangle yields ABC graph G' that consists of a single triangle component. Vertex $u \in V(G')$ corresponds to the contracted triangle.

Edges of G' correspond to edges of G, since the C3 operation consists of contraction and edge deletion operations. $G[M]$ denotes the graph induced by the edges of G that correspond to $M \subset E(G')$ this way. If $G[E(G')]$ is isomorphic to G', then all edges incident with $u \in V(G')$ must w.l.o.g. be incident with $u_1 \in V(G)$. In this case, G can be obtained from G' with a single A operation on u introducing u_2 and u_3, and renaming u as u_1. So G is an ABC graph in this case, a contradiction.

Now suppose $G[E(G')]$ is not isomorphic to G'. By Lemma 8, we know that a decomposition of G' exists that starts with vertex u. Suppose the first A operation introduces vertices v and w. Now the edges of G' can be partitioned into edge components F_{uv}, F_{uw} and F_{vw} that correspond to the edges introduced in this first A operation (Observation 6). Since $G[E(G')]$ is not isomorphic to G', it follows that there are edges $e \in E(F_{uv})$ and $f \in E(F_{uw})$ that are both incident with u in G', but that correspond w.l.o.g. to edges incident with u_1 resp. u_2 in G. Since G' is a simple graph, e and f therefore correspond to non-adjacent edges in G. Now consider a matching-cut M_1 in F_{uv} that separates u from v, such that $e \in M_1$, and a matching-cut M_2 in F_{uw} that separates u from w, such that $f \in M_2$ (Lemma 7). Using Observation 6, it can be checked that $M_1 \cup M_2$ corresponds to a matching-cut in G, which is a contradiction.

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References

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