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On the enumeration of uniquely reducible double designs

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A double 2-(v,k,2λ) design is a design which is reducible into two 2-(v,k,λ) designs. It is called uniquely reducible if it has, up to equivalence, only one reduction. We present properties of uniquely reducible double designs which show that their total number can be determined if only the designs with non-trivial automorphisms are classified with respect to their automorphism group. As an application, after proving that a reducible 2-(21,5,2) design is uniquely reducible, we establish that the number of all reducible 2-(21,5,2) designs is 1 746 461 307.

Keywords: double design, projective plane, enumeration

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1 Preliminaries

Let V be a finite set of v points and $\mathcal{B}$ a finite collection of k-element subsets of V, called blocks. $D = (V, \mathcal{B})$ is a t-(v,k,λ) *design* if any t-subset of V is contained in exactly λ blocks of $\mathcal{B}$. Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence. An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that transforms the blocks into blocks. The set of all automorphisms of a design form a group called its *full group of automorphisms*. Each subgroup of this group is a group of automorphisms of the design.

Each 2-(v,k,λ) design determines the existence of 2-(v,k,$m\lambda$) designs for any integer $m > 1$. A 2-(v,k,$m\lambda$) design is called a *quasimultiple* of a 2-(v,k,λ) design. A quasimultiple 2-(v,k,$m\lambda$) is *reducible* into m 2-(v,k,λ) designs if there is a partition of its blocks into m subcollections each of which forms a 2-(v,k,λ) design. For $m = 2$, quasimultiple designs are called *quasidoubles* and the reducible quasidouble designs are called *doubles*. We shall denote by $[D_1|D_2]$ a double design which can be reduced to the two designs D_1 and D_2. Here we will consider doubles of designs for which, up to isomorphism, only one design of its parameter set exists. So instead of $[D_1|D_2]$ we will often use the notation $[D|\varphi D]$, where

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the constituent design φD is obtained from D by a permutation φ of its points. By G we denote the automorphism group of D; by G_φ we denote the common subgroup of the automorphism groups of D and φD; by $\widehat{G}_\varphi$ we denote the automorphism group of the double design $[D|\varphi D]$.

In this paper we study uniquely reducible designs and show that the reducible 2-(21,5,2) designs are uniquely reducible. These are the doubles of the projective plane $PG(2,4)$ of order 4, which is a unique 2-(21,5,1) design. Lower bounds on the number of doubles of projective planes can be found in [1] and [2]. We enumerate the reducible 2-(21,5,2) designs by constructing those which have non-trivial automorphisms. For other examples of enumerating designs which contain incidence structures see for instance [3], [4], [5].

2 Uniquely reducible double designs

Doubles can be reducible in more than one way. Two reductions $[D_1|D_2]$ and $[D_3|D_4]$ of one and the same double design are *equivalent* if and only if there exists some point permutation μ such that $D_3 = \mu D_1$ and $D_4 = \mu D_2$. A double which has, up to equivalence, only one reduction is *uniquely reducible*.

¿From now on we consider only uniquely reducible double designs. The following propositions give some properties of uniquely reducible designs, which will be useful for their enumeration.

Proposition 1 *The set $\mathcal{C}_G(\varphi)$ of all doubles of the form $[D|\psi D]$ which are isomorphic to $[D|\varphi D]$ is given by $\mathcal{C}_G(\varphi) = G\varphi G \cup G\varphi^{-1}G$. The number of such doubles is*

$$|\mathcal{C}_G(\varphi)| = \begin{cases} |G|^2/|G_\varphi| & \text{if } G\varphi G = G\varphi^{-1}G, \\ 2|G|^2/|G_\varphi| & \text{otherwise, i.e. } G\varphi G \cap G\varphi^{-1}G = \emptyset\,. \end{cases} \tag{1}$$

Proposition 2 *If $G\varphi G = G\varphi^{-1}G$, then there exists $\omega \in \widehat{G}_\varphi$ such that $[D|\varphi D] = [D|\omega D]$. This ω transforms D into φD and vice versa. If $|G_\varphi| = 1$, then ω is of order 2.*

Corollary 3 *If $|\widehat{G}_\varphi| = 1$, then $|\mathcal{C}_G(\varphi)| = 2|G|^2$.*

Let N_i (resp. N'_i) denote the number of classes $\mathcal{C}_G(\varphi)$ for which $|G_\varphi| = i$ and $G\varphi G \cap G\varphi^{-1}G = \emptyset$ (resp. $G\varphi G = G\varphi^{-1}G$), then

$$v! = \sum_{\mathcal{C}_G(\varphi)} |\mathcal{C}_G(\varphi)| = 2|G|^2 N_1 + |G|^2 N'_1 + \sum_{i>1} \frac{2|G|^2}{i} N_i + \sum_{i>1} \frac{|G|^2}{i} N'_i\,. \tag{2}$$

Let N be the total number of non-isomorphic doubles of D, then

$$N = N_1 + N'_1 + \sum_{i>1} N_i + \sum_{i>1} N'_i\,. \tag{3}$$

Constructing the doubles of D with non-trivial automorphisms gives the numbers N'_1 as well as N_i and N'_i for all $i > 1$. Equation (2) can be used to obtain the number N_1 of doubles of D with trivial automorphisms. Equation (3) can be used to calculate the total number N of doubles of D.

Corollary 4 *A uniquely reducible 2-(v,k,λ) design D with automorphism group G has at least $v!/2|G|^2$ non-isomorphic doubles.*

3 Enumeration of the reducible 2-(21,5,2) designs

Proposition 5 *A reducible 2-(21,5,2) design is uniquely reducible.*

Proof: We give a sketch of the proof, which is based on the following observation.

Let $[D_1|D_2]$ be a double 2-(v,k,2λ) design where D_1 and D_2 are isomorphic 2-(v,k,λ) designs. Let this double design be reducible in two different ways. The blocks of D_1 and D_2 can then be partitioned into two collections D_1^a and D_1^b, and D_2^a and D_2^b, respectively, such that the blocks of D_1^a and D_2^b form a 2-(v,k,λ) design, and the blocks of D_1^b and D_2^a also form a 2-(v,k,λ) design.

$$\begin{tikzcd} & & \text{2-}(v,k,\lambda) & \\ D_1^a \arrow[dr] & D_1^b \arrow[ur] \arrow[d] & D_2^a \arrow[u] & D_2^b \\ & \text{2-}(v,k,\lambda) & & \end{tikzcd} \qquad \begin{tikzcd} & \text{2-}(v,k,\lambda) & & \\ D_1^a \arrow[ur] & D_1^b \arrow[dr] & D_2^a \arrow[d] & D_2^b \arrow[ull] \\ & & \text{2-}(v,k,\lambda) & \end{tikzcd}$$

Without loss of generality we can demand the a parts to be always larger than the b parts, and D_1^b and D_2^b to have no blocks in common. Let n be the number of blocks in D_1^b, and a_i the number of blocks in D_1^b containing point i ($i = 1, 2, \ldots, v$). The following considerations can be made:

1. Any point is in the same number of blocks of D_1^a and D_2^a (D_1^b and D_2^b).
2. Any pair of points is in the same number of blocks of D_1^a and D_2^a (D_1^b and D_2^b).
3. If D_1 is a projective plane of order q (i.e. a 2-(q^2+q+1,q+1,1) design), the following holds:

$$a_i \neq 1 \quad ; \; i = 1, 2, \ldots, v \tag{4}$$

$$\sum_{i=1}^{v} a_i = n(q+1) \tag{5}$$

$$\sum_{i=1}^{v} a_i^2 = n(n+q) \tag{6}$$

$$n \leq \frac{q^2+q}{2} \tag{7}$$

A careful study of all possible solutions of the system (4)–(7) for $q = 4$, shows that a 2-(21,5,2) design cannot have two inequivalent reductions, i.e. it is uniquely reducible. □

It follows from Corollary 4 that the number of the non-isomorphic doubles of the projective plane of order 4 is at least 1 745 944 200. To determine their exact number all designs with non-trivial automorphisms should be constructed.

In [6] all 2-(21,5,2) designs with automorphisms of odd prime orders were constructed, their number was determined to be 22 998 and 4 170 of them were found to be reducible. In [7] the authors constructed the designs with automorphisms of order 2 which transform each of the constituent 2-(21,5,1) designs into itself. 40 485 such doubles were found, 305 of which have also an automorphism of odd prime order, so they were already counted among the 4 170 doubles found above. This leaves only the reducible 2-(21,5,2) designs with automorphisms of order 2 which transform one of the constituent 2-(21,5,1) designs into the other and vice versa ($|G_\varphi| = 1$) to be constructed.

We generate all designs $[D|\varphi D]$, where φ is a permutation of order 2. We first find all automorphisms of D and then generate all possible permutations of order 2 in lexicographic order. As the automorphism group of the projective plane of order 4 is doubly transitive, we can fix one non-trivial orbit. Moreover a simple pruning condition allows to filter out many isomorphic copies, such that the final full isomorphism check does not filter much more. We checked the results by two different implementations. We construct 991 957 non-isomorphic designs which have an automorphism of order 2 transforming the constituent designs into one another. We establish that for 984 549 of them the order of the full group of automorphisms is 2, and $|G_\varphi| = 1$.

Having constructed all 1 028 899 doubles which possess non-trivial automorphisms, we use equations (2) and (3) to calculate the number of non-isomorphic designs without automorphisms, which turns out to be 1 745 432 408. The total number of 2-(21,5,2) doubles is 1 746 461 307.

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