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Chromatic Turán problems and a new upper bound for the Turán density of $\mathcal{K}_4^-$

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We consider a new type of extremal hypergraph problem: given an r-graph $\mathcal{F}$ and an integer $k \geq 2$ determine the maximum number of edges in an $\mathcal{F}$-free, k-colourable r-graph on n vertices.

Our motivation for studying such problems is that it allows us to give a new upper bound for an old problem due to Turán. We show that a 3-graph in which any four vertices span at most two edges has density less than $\frac{33}{100}$, improving previous bounds of $\frac{1}{3}$ due to de Caen [1], and $\frac{1}{3} - 4.5305 \times 10^{-6}$ due to Mubayi [9].

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Given an r-graph $\mathcal{F}$ the Turán number $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an n-vertex r-graph not containing a copy of $\mathcal{F}$. The Turán density of $\mathcal{F}$ is

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

For 2-graphs the Turán density is determined completely by the chromatic number of $\mathcal{F}$, but for $r \geq 3$ there are very few r-graphs for which $\pi(\mathcal{F})$ is known. (Examples of 3-graphs for which $\pi(\mathcal{F})$ is now known include the Fano plane [2], $\mathcal{F} = \{abc, abd, abe, cde\}$ [8] and $\mathcal{F} = \{abc, abd, cde\}$ [7].)

The two most well-known problems in this area are to determine $\pi(\mathcal{K}_4)$ and $\pi(\mathcal{K}_4^-)$, where $\mathcal{K}_4 = \{abc, abd, acd, bcd\}$ is the complete 3-graph on 4 vertices and $\mathcal{K}_4^- = \{abc, abd, acd\}$ is the complete 3-graph on 4 vertices with an edge removed. For $\pi(\mathcal{K}_4)$ we have the following bounds due to Turán and Chung and Lu [3] respectively

$$\frac{5}{9} \leq \pi(\mathcal{K}_4) \leq \frac{3+\sqrt{17}}{12} = 0.59359\ldots.$$

Although the problem of determining $\pi(\mathcal{K}_4)$ is an extremely natural question in some respects the problem of determining $\pi(\mathcal{K}_4^-)$ is even more fundamental since $\mathcal{K}_4^-$ is the smallest 3-graph satisfying $\pi(\mathcal{F}) \neq 0$. Note also that this problem may be restated as follows: determine the maximum density of a 3-graph in which any four vertices span at most two edges.

The problem of determining $\pi(\mathcal{K}_4^-)$ has been considered by many people, including Turán [12], Erdős and Sós [5], Frankl and Füredi [6], de Caen [1] and Mubayi [9]. Previously the best known bounds were

$$\frac{2}{7} \leq \pi(\mathcal{K}_4^-) \leq \frac{1}{3} - (4.5305 \times 10^{-6}).$$

The upper bound was proved by Mubayi [9], improving on the upper bound $\pi(\mathcal{K}_4^-) \leq 1/3$, due to de Caen [1]. The lower bound is from a construction due to Frankl and Füredi [6].

Our main result is the following theorem, improving the upper bound for $\pi(\mathcal{K}_4^-)$.

Theorem 1 *The Turán density of $\mathcal{K}_4^-$ satisfies*

$$\frac{2}{7} \leq \pi(\mathcal{K}_4^-) < \frac{33}{100}.$$

The proof of this result leads naturally to a new general class of extremal problems which we call *chromatic Turán problems*. These are questions of the form: given an r-graph $\mathcal{F}$ and an integer $k \geq 2$ determine the maximum number of edges in an $\mathcal{F}$-free, k-colourable r-graph on n vertices. (Recall that an r-graph is *k-colourable* iff its vertices can be partitioned into k classes, none of which contain an edge.) We denote this quantity by $\mathrm{ex}_k(n, \mathcal{F})$. The corresponding k-chromatic Turán density is then defined to be

$$\pi_k(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}_k(n, \mathcal{F})}{\binom{n}{r}}.$$

The following lemma shows that any upper bound for the chromatic Turán density $\pi_3(\mathcal{K}_4^-)$ will yield an upper bound for the ordinary Turán density $\pi(\mathcal{K}_4^-)$. (The proof of Theorem 1 actually requires a slightly stronger result but the principle is the same.)

Lemma 1 *If $\pi_3 = \pi_3(\mathcal{K}_4^-)$ and $\pi = \pi(\mathcal{K}_4^-)$ then*

$$\pi \leq \frac{2 + \pi_3/\pi}{9}.$$

To prove Lemma 1 we first note that if $\mathcal{F}$ is a $\mathcal{K}_4^-$-free 3-graph with n vertices and m edges then it satisfies

$$mn = \sum_{xy \in V^{(2)}} d_{xy}^2 + q_1, \tag{1}$$

where q_1 is the number of 4-sets in $\mathcal{F}$ spanning exactly one edge. Lemma 1 is proved by giving a lower bound for q_1.

This idea of giving a lower bound for q_1 was previously used by Mubayi [9] (he used supersaturation to achieve this). Our approach is quite different in that it involves a chromatic Turán problem.

To be more precise the key idea used in the proof of Lemma 1 is as follows. Let $\mathcal{F}$ be a $\mathcal{K}_4^-$-free 3-graph of order n and let $uvw \in \mathcal{F}$. If $E_{ij} = \{k : ijk \in \mathcal{F}\}$ then E_{uv}, E_{uw} and E_{vw} are pairwise disjoint. We say that an edge $xyz \in \mathcal{F}$ is *bad* relative to the edge uvw if $xyz \cap uvw = \emptyset$ and none of the following

4-sets span exactly one edge: $uvwx, uvwy, uvwz, xyzu, xyzv, xyzw$. It is straightforward to check that if xyz is bad relative to uvw then $xyz \subset E_{uv} \cup E_{uw} \cup E_{vw}$ and xyz is not contained entirely in either E_{uv}, E_{uw} or E_{vw}. Hence the collection of bad edges relative to uvw is 3-colourable. This fact can then be used to give a lower bound for q_1 in terms of the 3-chromatic Turán density $\pi_3(\mathcal{K}_4^-)$.

One obvious reason why chromatic Turán problems do not seem to have been considered before is that for 2-graphs they are rather uninteresting. If K_t is the complete 2-graph of order t then Turán's theorem determines not only their Turán numbers but also their chromatic Turán numbers.

Theorem 2 (Turán[11]) *If $t \geq 3$ then the unique K_t-free graph with n vertices and $ex(n, K_t)$ edges is the complete $(t-1)$-partite graph whose vertex classes are as equal as possible in size.*

Corollary 1 *If $t \geq 3$ and $k \geq 2$ then*

$$ex_k(n, K_t) = \begin{cases} ex(n, K_t), & k \geq t-1, \\ ex(n, K_{k+1}), & k \leq t-2. \end{cases}$$

For general r-graphs with $r \geq 3$ the problems of determining chromatic and ordinary Turán numbers seem to be genuinely different. While the extremal K_t-free 2-graph is not only K_t-free but also $(t-1)$-colourable this does not seem to be the case in general. For example the conjectured extremal $\mathcal{K}_4^-$-free 3-graph has rather large chromatic number.

In order to make use of Lemma 1 we need an upper bound on $\pi_3(\mathcal{K}_4^-)$. To achieve this we first consider $\pi_2(\mathcal{K}_4^-)$.

Theorem 3 *The 2-chromatic Turán density $\pi_2(\mathcal{K}_4^-)$ satisfies*

$$\frac{4}{9\sqrt{3}} \leq \pi_2(\mathcal{K}_4^-) \leq \frac{3}{10}.$$

Proof: The lower bound follows from a simple construction. For the upper bound let $\mathcal{F} = ((A, B), E)$ be a bipartite $\mathcal{K}_4^-$-free 3-graph with n vertices and m edges. Let $|A| = \alpha n$, so $|B| = (1-\alpha)n$. Then

$$\sum_{xy \in A \times B} d_{xy} = 2m, \qquad \sum_{xy \in A^{(2)} \cup B^{(2)}} d_{xy} = m. \tag{2}$$

From (1) and the fact that $q_1 \geq 0$ we obtain

$$mn \geq \sum_{xy \in A \times B} d_{xy}^2 + \sum_{xy \in A^{(2)} \cup B^{(2)}} d_{xy}^2$$

Convexity and (2) then imply that

$$m \leq \frac{n^3}{\frac{4}{\alpha(1-\alpha)} + \frac{2}{\alpha^2 + (1-\alpha)^2}} \leq \frac{n^3}{20}.$$

The result then follows directly. □

With some more work we can obtain an upper bound for $\pi_3(\mathcal{K}_4^-)$. (Again the lower bound follows from a simple construction.)

Theorem 4 *The 3-chromatic Turán density $\pi_3(\mathcal{K}_4^-)$ satisfies*

$$\frac{5}{18} \leq \pi_3(\mathcal{K}_4^-) < \frac{58}{177}.$$

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