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Human and constructive proof of combinatorial identities: an example from Romik

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It has become customary to prove binomial identities by means of the method for automated proofs as developed by Petkovšek, Wilf and Zeilberger [12]. In this paper, we wish to emphasize the role of "human" and constructive proofs in contrast with the somewhat lazy attitude of relaying on "automated" proofs. As a meaningful example, we consider the four formulas by Romik [14], related to Motzkin and central trinomial numbers. We show that a proof of these identities can be obtained by using the method of coefficients, a human method only requiring hand computations.

Keywords: Combinatorial identities, method of coefficients, Lagrange inversion, Riordan Arrays, WZ-method

1 Introduction

Computers have pervasively invaded our life, and Mathematics is not an exception. Computer Algebra Systems have become very popular, and tedious computations can be performed safely by a computer, thus saving a lot of time and energy to the mathematician, who can concentrate on the conceptual aspects of a problem instead of being distracted by technical details. This is also true for more delicate situations, as for example some kind of theorem proving. In that case, several psychological problems have been expressed since automated proofs (or, at least, proofs obtained by the aid of a computer) have been introduced. The classical example is the four colors problem, but the number of cases is growing. One of the most striking situations is the automated proof of combinatorial identities, and related problems, as is obtained by the method of Petkovšek, Wilf and Zeilberger [12] (see also Krattenthaler [10] and Paule and Schorn [11]). The Zeilberger point of view on the automated proof of combinatorial identities can be found in [13] as a letter posted by H. Prodinger on the Analysis of Algorithms bulletin board and goes as follows:

> Since Everything that we can prove today will soon be provable, faster and better, by computers, it is a waste time to keep proving, in the same old-way, either by only pencil and paper, and even doing 'computer-assisted' proofs, regarding the computer, as George Andrews put it, as a 'pencil with power-stirring'. Very soon all our awkwardly phrased proofs, in semi-natural language, with their endless redundancy, will seem just as ludicrous as Levi's half-page statement of (a+b)c=ac+bc, and his subsequent half-page proof. We could be much more useful than we are now, if, instead of proving yet another theorem, we would start teaching the computer everything we know, so that it would have a headstart. Of course, eventually computers will be able to prove everything humans did (and much more!) ab initio, but if we want to reap the fruits of the computer revolution as soon as possible, and see the proofs of the Riemann Hypothesis and the Goldbach conjecture in OUR lifetime, we better get to work, and transcribe our human mathematical heritage into Maple, Mathematica, or whatever. Hopefully we will soon have super-symbolic programming languages, of higher and higher levels . . ..

Combinatorial identities have received attention since very old times, and a quantity of techniques have been proposed to solve single cases or special classes thereof (see [4] and the related literature for a recent example of a combinatorial identity solved with different methodologies). The vast literature goes from Laplace to McMahon and Netto, from Riordan and Carlitz to Egorichev, Knuth, Wilf and Comtet. Gould [6] collected about 500 identities and more recent authors have added many other examples. Part of this literature has a strong connection with the analysis of algorithms: the work of Knuth on combinatorial sums in [9] was purposely done in this direction. Also a very old result, apparently far away from analysis of algorithms, the Abel binomial sum, has applications in this field: see the papers of Hwang and Szpankowski on coding theory [8, 19] and Sedgewick and Flajolet [15, p. 450].

At the beginning of '90's, Zeilberger and then Wilf proposed several techniques to solve, in an algorithmic way, combinatorial identities concerning sums of hypergeometric terms. In particular, computer programs were written which 'certify' that a given combinatorial identity (of the kind indicated) is true or not, and the certificate can be checked manually.

Obviously, these proofs are legitimate and many complicated identities have been proved in this way (see, e.g., [1, 3, 5]). However, in many cases one wonders whether the computer promotes a lazy attitude. In other words, the doubt is: is it so often necessary to resort on mechanical devices or, on the contrary, is it possible to produce a *human and constructive proof*? We think that this kind of proofs is much more satisfactory; actually, automated proofs give the same unpleasant feeling that proofs by verification give: just think of proofs by induction. You have a way to verify your assert, and you can consider this check a legitimate proof; however, you continue to consider yourself rather lucky (or, more often, very clever) to have guessed the right formula. There is a gap between finding out a formula, an identity, and simply verifying it after having somehow imagined or guessed the result.

In the present paper we wish to discuss a specific case, as a meaningful example of a general situation. Actually, the method we use has previously been applied to the collection of Gould, and a large part of these combinatorial sums has been proved in [17] by formulas (A) and (B) below and the Lagrange Inversion Formula, most times by a couple of lines proof. The recent paper of Romik [14] deals with well-known quantities, Motzkin and central binomial numbers, often used in the analysis of algorithms. His formulas are not obvious and therefore are well suited to show how our method can be applied in problems requiring some attention.

In [14] Romik gives two formulas for the central trinomial and the Motzkin numbers. He states that such formulas "can be proved automatically by computer, using the methods and software of Petkovšek, Wilf and Zeilberger" [12]. However, he offers a non-automatic proof based on two "auxiliary identities ... [which] are also automatically verifiable and shall not be proved". Romik's basic aim is to give "proofs that involve a certain symmetry idea which might lead to the discovery of other such identities". This can justify Romik's approach, but what we wish to show here is that the two identities of Romik, and his auxiliary formulas, can be proved in a direct way, without involving any special software. Actually, the proofs we produce are *human*, in the specific sense that they can be developed entirely by hand: some passages are better and more safely performed by some computer algebra system, but this is only a matter of speed and does not influence the "human" aspect of our proofs.

Our proofs are based on the "method of coefficients" (see Greene and Knuth [7], who cite I. Gessel and G. P. Egorychev [2]) and in particular on the concept of a Riordan Array, as introduced in [16] and used in [18]. In particular, in [18] we find the two following formulas, which will be used in developing Romik's identities:

$$\sum_k \binom{n+ak}{m+bk} f_k = [t^n]\frac{t^m}{(1-t)^{m+1}} f\left(\frac{t^{b-a}}{(1-t)^b}\right) \qquad b>a \qquad (A)$$

$$\sum_k \binom{n+ak}{m+bk} f_k = [t^m](1+t)^n f(t^{-b}(1+t)^a) \qquad b<0 \qquad (B).$$

In (A) and (B) $f(t)$ is the generating function of the sequence $\{f_k\}_{k\in N}$, $f(t)=\sum_k f_k t^k$, and $[t^n]$ is the "coefficient of" operator, that is $[t^n]f(t)=f_n$. The only other tool we need is the Lagrange Inversion Formula (LIF), in the form of the *diagonalization rule*:

$$c_n = [t^n]F(t)\phi(t)^n \rightsquigarrow C(t) = \sum_{k=0}^{\infty} c_n t^n = \left[\frac{F(w)}{1-t\phi'(w)}\;\middle|\; w=t\phi(w)\right]$$

where $\phi(0)\neq 0$ and $C(t)$ is the generating function of the sequence $\{c_n\}_{n\in N}$.

As already observed, most of the identities of Gould [6] have been proved with this method by Sprugnoli [17].

2 Romik's identities

In this section we consider the two main formulas found by Romik in his paper [14] and concerning the central trinomial and Motzkin numbers. Central trinomial coefficients are the coefficients $[t^n](1+t+t^2)^n$

and their generating function is:

$$c(t) = \sum_{k=0}^{\infty} c_k t^k = \frac{1}{\sqrt{1-2t-3t^2}} = 1 + t + 3t^2 + 7t^3 + 19t^4 + 51t^5 + 141t^6 + \cdots.$$

Motzkin numbers count several kinds of paths on the integral lattice $\mathbb{Z}^2$ and their generating function is:

$$m(t) = \sum_{k=0}^{\infty} m_k t^k = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2} = 1 + t + 2t^2 + 4t^3 + 9t^4 + 21t^5 + 127t^6 + \cdots.$$

The formulas found by Romik are:

$$m_n = \sum_{k=\lceil (n+2)/3 \rceil}^{\lfloor (n+2)/2 \rfloor} \frac{(3k-2)!}{(2k-1)!(n+2-2k)!(3k-n-2)!} \tag{1}$$

$$c_n = (-1)^{n+1} + 2n \sum_{k=\lceil n/3 \rceil}^{\lfloor n/2 \rfloor} \frac{(3k-1)!}{(2k)!(n-2k)!(3k-n)!} \tag{2}$$

and are valid for $k \geq 1$ (see however below).

We begin by developing the following identity:

$$\sum_k \frac{(3k-2)!}{(2k-1)!(n+2-2k)!(3k-n-2)!} = \sum_k \frac{(3k-2)!(3k-(3k-1))}{(2k-1)!(n+2-2k)!(3k-n-2)!}$$

We have added and subtracted the same quantity, so that nothing is changed, but much can be simplified in the passage to binomial coefficients:

$$\sum_k \frac{3(3k-2)!}{(2k-1)!(k-1)!} \cdot \frac{k(k-1)!}{(n+2-2k)!(3k-n-2)!} - \sum_k \frac{(3k-1)!}{(2k-1)!k!} \cdot \frac{k!}{(n+2-2k)!(3k-n-2)!} =$$

$$= \sum_k \left(3\binom{3k-2}{k-1} - \binom{3k-1}{k} \right) \binom{k}{n+2-2k}.$$

At this point, we have to find the generating function for the part only depending on k; we find:

$$3\binom{3k-2}{k-1} - \binom{3k-1}{k} = 3[t^k]t(1+t)^{3k-2} - [t^k](1+t)^{3k-1} = [t^k]\frac{2t-1}{(1+t)^2}(1+t)^{3k}.$$

The rule of diagonalization can be applied and we find the following implicit generating function:

$$G(t) = \left[\frac{2w-1}{(1+w)^2} \frac{1}{1-t\cdot 3(1+w)^2} \;\middle|\; w = t(1+w)^3 \right] =$$

$$= \left[\frac{2w-1}{(1+w)^2} \frac{1}{1-3w/(1+w)} \;\middle|\; w = t(1+w)^3 \right] = \left[\frac{-1}{1+w} \;\middle|\; w = t(1+w)^3 \right].$$

We do not even try to obtain an explicit formula, but apply the Riordan Array rule (B) and find:

$$\sum_k \binom{k}{n+2-2k} \left(3\binom{3k-2}{k-1} - \binom{3k-1}{k} \right) = [t^{n+2}]G(t^2(1+t)) =$$

$$= [t^{n+2}] \left[\frac{-1}{1+w} \;\middle|\; w = t^2(1+t)(1+w)^3 \right].$$

Here we have to solve the equation in terms of $w = w(t)$, looking for the solution such that $w(0) = 0$. A simple observation is that $w = -(1+t)/t$ is in fact a solution, as can be checked directly:

$$-\frac{1+t}{t} = t(1+t)^2 \left(1 - \frac{1+t}{t}\right)^3.$$

This is not the solution we are looking for, because $w(0) \neq 0$; however, we can now reduce the degree of our equation $w = t^2(1+t)(1+w)^3$ or:

$$t^2(1+t)w^3 + 3t^2(1+t)w^2 + (3t^2(1+t)-1)w + t^2(1+t) = 0,$$

for example by means of the method of indeterminate coefficients. If $\alpha w^2 + \beta w + \gamma$ is the second degree equation we are looking for, we should have:

$$\left(w + \frac{1+t}{t}\right)(\alpha w^2 + \beta w + \gamma) = \alpha w^3 + \left(\beta + \alpha\frac{1+t}{t}\right)w^2 + \left(\gamma + \beta\frac{1+t}{t}\right)w + \gamma\frac{1+t}{t},$$

and this is to be equal to the previous equation. By equating like coefficients, we start with $\alpha = t^2(1+t)$ and obtain β:

$$\beta + t^2(1+t)\frac{1+t}{t} = 3t^2(1+t)$$
$$\beta = 3t^2 + 3t^3 - t - 2t^2 - t^3 = -t(1-t-2t^2) =$$
$$= -t(1+t-2t-2t^2) = -t(1+t)(1-2t).$$

Let us pass to γ:

$$\gamma - t(1+t)(1-2t)\frac{1+t}{t} = 3t^2(1+t) - 1;$$
$$\gamma = (1+t)^2(1-2t) + 3t^2(1+t) - 1 = (1+t)(1-t-2t^2+3t^2) - 1 =$$
$$= (1+t)(1-t+t^2) - 1 = 1 - t + t^2 + t - t^2 + t^3 - 1 = t^3$$

and this clearly checks with the last equation: $\gamma(1+t)/t = t^2(1+t)$.

In this way, we have reduced our equation to:

$$t(1+t)w^2 - (1+t)(1-2t)w + t^2 = 0.$$

Let us compute the discriminant:

$$\Delta = (1+t)^2(1-2t)^2 - 4t^3(1+t) = 1 - 2t - 3t^2$$

and this is the traditional trinomial of the Motzkin and central trinomial numbers. We find:

$$w = \frac{(1+t)(1-2t) \pm \sqrt{1-2t-3t^2}}{2t(1+t)} = \frac{1-t-2t^2 \pm \sqrt{1-2t-3t^2}}{2t(1+t)}.$$

Since we should have $w(0) = 0$, we cannot consider the "$+$" sign.

Returning to our sum, we find:

$$f(t) = \left[\frac{-1}{1+w} \;\middle|\; w = \frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{2t(1+t)}\right] =$$
$$= \frac{-2t(1+t)}{2t+2t^2+1-t-2t^2-\sqrt{1-2t-3t^2}} =$$
$$= \frac{-2t(1+t)}{1+t-\sqrt{1-2t-3t^2}} = \frac{-2t(1+t)(1+t+\sqrt{1-2t-3t^2})}{1+2t+t^2-1+2t+3t^2} =$$
$$= \frac{-2t(1+t)(1+t+\sqrt{1-2t-3t^2})}{4t(1+t)} = -\frac{1+t+\sqrt{1-2t-3t^2}}{2} = -t^{-2} + m(t).$$

The reader can be puzzled by the appearance of $-t^{-2}$ in this generating function. However, if we consider formula (1) for negative values of n, we observe that it reduces to an empty sum for $n \neq -2$ (and therefore its value is 0), but for $n = -2$ the index value $k = 0$ is legal. In that case, we have $m_{-2} = (-2)!/(-1)!$. The factorial of negative integers is not defined, but if we compute:

$$m_{-2} = \lim_{\epsilon \to 0} \frac{(-2+\epsilon)!}{(-1+\epsilon)!} = -1$$

and this justifies the term $-t^{-2}$, a fact that apparently was not noticed by Romik.

Let us now come to identity (2). We can take the n inside the summation and write it as $3k-(3k-n)$. This passage, as the analogous one in the proof of the previous identity, is made in order to obtain "pure" binomial coefficients in the sum, so, in a sense, they are forced by the formula and are not arbitrary.

$$\sum_k \frac{(3k-1)!(3k-(3k-n))}{(2k)!(n-2k)!(3k-n)!} =$$

$$= \sum_k \frac{(3k)!}{(2k)!k!}\,\frac{k!}{(n-2k)!(3k-n)!} - \sum_k \frac{(3k-1)!}{(2k)!(k-1)!}\,\frac{(k-1)!}{(n-2k)!(3k-n-1)!} =$$

$$= \sum_k \binom{3k}{k}\binom{k}{n-2k} - \sum_k \binom{3k-1}{k-1}\binom{k-1}{n-2k} = a_n - b_n.$$

This time we cannot merge the two sums in a single one, but we have:

$$\binom{3k}{k} = [t^k](1+t)^{3k} = [t^k]\left[\frac{1+w}{1-2w}\;\middle|\; w = t(1+w)^3\right];$$

$$\binom{3k-1}{k-1} = [t^k]t(1+t)^{3k-1} = [t^k]\left[\frac{w}{1-2w}\;\middle|\; w = t(1+w)^3\right].$$

By using the rule (B) derived by the Riordan Array theory, we have:

$$\sum_k \binom{k}{n-2k} f_k = [t^n]f(t^2(1+t));$$

$$\sum_k \binom{k-1}{n-2k} f_k = [t^n]\frac{1}{1+t}f(t^2(1+t)).$$

Therefore, we perform separately the two sums:

$$a_n = \sum_k \binom{k}{n-2k}\binom{3k}{k} = [t^n]\left[\frac{1+w}{1-2w}\;\middle|\; w = t^2(1+t)(1+w)^3\right].$$

¿From the previous case, we have already the solution of $w = t^2(1+t)(1+w)^3$ and we can develop the generating function:

$$\frac{1+w}{1-2w} = \left(1 + \frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{2t(1+t)}\right) \Big/ \left(1 - \frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{t(1+t)}\right) =$$

$$= \frac{1+t-\sqrt{1-2t-3t^2}}{2t(1+t)} \cdot \frac{t(1+t)}{-1+2t+3t^2+\sqrt{1-2t-3t^2}} =$$

$$= \frac{1+t-\sqrt{1-2t-3t^2}}{2\sqrt{1-2t-3t^2}(1-\sqrt{1-2t-3t^2})}.$$

Now, for the second sum we have:

$$b_n = \sum_k \binom{k-1}{n-2k}\binom{3k-1}{k-1} = [t^n]\frac{1}{1+t}\left[\frac{w}{1-2w}\;\middle|\; w = t^2(1+t)(1+w)^3\right]$$

and we develop the generating function:

$$\frac{1}{1+t}\cdot\frac{w}{1-2w} = \frac{1}{1+t}\cdot\frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{2t(1+t)}\cdot\frac{t(1+t)}{\sqrt{1-2t-3t^2}(1-\sqrt{1-2t-3t^2})} =$$

$$= \frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{2(1+t)\sqrt{1-2t-3t^2}(1-\sqrt{1-2t-3t^2})}.$$

Finally, by observing that the generating function for $(-1)^{n+1}$ is $-1/(1+t)$, we perform the final computation:

$$c_n = (-1)^{n+1} + a_n - b_n = -[t^n]\frac{1}{1+t} + [t^n]\frac{1+t-\sqrt{1-2t-3t^2}}{\sqrt{1-2t-3t^2}(1-\sqrt{1-2t-3t^2})} -$$

$$-[t^n]\frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{(1+t)\sqrt{1-2t-3t^2}(1-\sqrt{1-2t-3t^2})} =$$

$$= [t^n]\frac{1+t-(1+t)\sqrt{1-2t-3t^2}}{(1+t)\sqrt{1-2t-3t^2}(1-\sqrt{1-2t-3t^2})} = [t^n]\frac{1}{\sqrt{1-2t-3t^2}}$$

and this actually is the generating function for the central trinomial coefficients.

3 The other two Romik's identities

Romik considers other two identities, only quoting an automated proof by the method of Petkovšek, Wilf and Zeilberger. The first one is:

$$\sum_k \frac{(-1)^k(3k-2)!}{(k-1)!(n-k)!(3k-n)!} = 0 \qquad \forall\, n > 1. \tag{3}$$

If we try to transform the l.h.s. into a sum of binomial coefficients, we find that something has to be added to change $(3k-2)!$ into $(3k-1)!$, and $(3k-n)!$ into $(3k-n-1)!$. To this purpose, we consider the identity $n = (3k-1) - (3k-n) + 1$; we have:

$$n\sum_k \frac{(-1)^k(3k-2)!}{(k-1)!(n-k)!(3k-n)!} = \sum_k \frac{(-1)^k(3k-2)!((3k-1)-(3k-n)+1)}{(k-1)!(n-k)!(3k-n)!} =$$

$$= \sum_k \frac{(-1)^k(3k-1)!(2k)!}{(k-1)!(2k)!(n-k)!(3k-n)!} - \sum_k \frac{(-1)^k(3k-2)!(2k-1)!}{(k-1)!(2k-1)!(n-k)!(3k-n-1)!} +$$

$$+\sum_k \frac{(-1)^k(3k-2)!}{(k-1)!(n-k)!(3k-n)!}$$

which is equivalent to:

$$(n-1)\sum_k \frac{(-1)^k(3k-2)!}{(k-1)!(n-k)!(3k-n)!} =$$

$$= \sum_k (-1)^k\binom{3k-1}{k-1}\binom{2k}{n-k} - \sum_k (-1)^k\binom{3k-2}{k-1}\binom{2k-1}{n-k} = S_n - T_n.$$

We now proceed by using the LIF to find the generating functions for S_n and T_n. We have:

$$(-1)^k\binom{3k-1}{k-1} = -[t^k]\frac{t}{1-t}(1-t)^{3k} = -[t^k]\left[\frac{w}{1-w}\cdot\frac{1}{1+3\cdot t(1-w)^2}\;\middle|\; w = t(1-w)^3\right] =$$

$$= -[t^k]\left[\frac{w}{1-w}\cdot\frac{1-w}{1-w+3w}\;\middle|\; w = t(1-w)^3\right] = -[t^k]\left[\frac{w}{1+2w}\;\middle|\; w = t(1-w)^3\right].$$

Analogously:

$$(-1)^k\binom{3k-2}{k-1} = -[t^k]\left[\frac{w}{(1-w)(1+2w)}\;\middle|\; w = t(1-w)^3\right].$$

On the other hand, by the rule (B) of the Riordan Array method, we have:

$$\sum_k \binom{2k}{n-k} f_k = [t^n] f(t(1+t)^2)$$

$$\sum_k \binom{2k-1}{n-k} f_k = [t^n]\frac{1}{1+t} f(t(1+t)^2).$$

For the first sum we therefore have:

$$S_n = \sum_k \binom{2k}{n-k}(-1)^k\binom{3k-1}{k-1} = -[t^n]\left[\frac{w}{1+2w} \;\Big|\; w = t(1+t)^2(1-w)^3\right].$$

We now look for the solution $w = w(t)$ of the equation $w = t(1+t)^2(1-w)^3$ having $w(0) = 0$. It is immediate to observe that a possible solution is $w = t/(1+t)$; in fact, by substituting this value in the equation, we find:

$$\frac{t}{1+t} = t(1+t)^2\left(1-\frac{t}{1+t}\right) \qquad \text{or} \qquad \frac{1}{1+t} = \frac{1}{1+t}.$$

Since $w(0) = 0$, this is the solution we are looking for; the other two solutions can be found as in the previous section, but are not relevant to our purposes. Now we have:

$$\sum_k \binom{2k}{n-k}(-1)^k\binom{3k-1}{k-1} = -[t^k]\frac{t}{1+t}\Big/\left(1+\frac{2t}{1+t}\right) = -[t^n]\frac{t}{1+t}\cdot\frac{1+t}{1+3t} =$$

$$= -[t^{n-1}]\frac{1}{1+3t} = -(-1)^{n-1}3^{n-1} = (-1)^n 3^{n-1} \qquad \text{for } n \geq 1.$$

For the second sum we find:

$$T_n = \sum_k \binom{2k-1}{n-k}(-1)^k\binom{3k-2}{k-1} = -[t^n]\frac{1}{1+t}\left[\frac{w}{(1-w)(1+2w)} \;\Big|\; w = t(1+t)^(1-w)^3\right] =$$

$$= -[t^n]\frac{1}{1+t}\cdot\frac{t}{1+t}\Big/\left(\left(1-\frac{t}{1+t}\right)\left(1+\frac{2t}{1+t}\right)\right) = -[t^n]\frac{t}{(1+t)^2}\cdot\frac{(1+t)^2}{1+3t} =$$

$$= -[t^n]\frac{t}{1+3t} = (-1)^n 3^{n-1} \qquad \text{for } n \geq 1.$$

This proves that:

$$\frac{S_n - T_n}{n-1} = \sum_k \frac{(-1)^k(3k-2)!}{(k-1)!(n-k)!(3k-n)!} = \frac{(-1)^n 3^{n-1} - (-1)^n 3^{n-1}}{n-1} = 0 \qquad \text{for } n > 1.$$

Although similar, the proof of the last identity:

$$(-1)^n \sum_{k=\lceil n/3\rceil}^{n} \frac{(-1)^k(3k-1)!}{k!(n-k)!(3k-n)!} = \frac{1}{n}, \qquad n \geq 1 \tag{4}$$

is a bit more interesting. In fact, it emphasizes a peculiar aspect of the Riordan array transformations. First of all, by multiplying both members by $n = 3k - (3k-n)$ we have:

$$(-1)^n n \sum_{k=\lfloor n/3\rfloor}^{n} \frac{(-1)^k(3k-1)!}{k!(n-k)!(3k-n)!} =$$

$$= (-1)^n \sum_{k=\lfloor n/3\rfloor}^{n} \frac{(-1)^k(3k)!(2k)!}{k!(n-k)!(3k-n)!(2k)!} - (-1)^n \sum_{k=\lfloor n/3\rfloor}^{n} \frac{(-1)^k(3k-1)!(2k-1)!}{k!(n-k)!(3k-n-1)!(2k)!} =$$

$$= (-1)^n\left(\sum_{k=\lfloor n/3\rfloor}^{n}(-1)^k\binom{3k}{k}\binom{2k}{n-k} - \sum_{k=\lfloor n/3\rfloor}^{n}(-1)^k\binom{3k-1}{k}\binom{2k-1}{n-k}\right) =$$

$$= (-1)^n(U_n - W_n).$$

Let us now compute the two sums, ignoring the $(-1)^n$. Proceeding as in the previous case, we find:

$$(-1)^k\binom{3k}{k} = [t^k]\left[\frac{1-w}{1+2w} \;\Big|\; w = t(1-w)^3\right]$$

$$(-1)^k\binom{3k-1}{k} = [t^k]\left[\frac{1}{1+2w}\;\middle|\; w = t(1-w)^3\right]$$

The first one of these two sums can be extended from 0 to ∞ and by applying rule (B) we find:

$$U_n = \sum_k \binom{2k}{n-k}\binom{3k}{k}(-1)^k = [t^n]\left[\left[\frac{1-w}{1+2w}\;\middle|\; w = u(1-w)^3\right]\;\middle|\; u = t(1+t)^2\right].$$

As we know, the solution of the equation $w = t(1+t)^2(1-w)^3$ is $w = t/(1+t)$ and therefore we get:

$$U_n = [t^n]\left(1 - \frac{t}{1+t}\right)\Big/\left(1 + \frac{2t}{1+t}\right) = [t^n]\frac{1}{1+3t} = (-3)^n$$

(this result is worth of being considered also on its own). The second sum represents the interesting part. If we try to find the range of values for which the sum is defined, we find that, besides the requested range $\lfloor n/3\rfloor \le k \le n$, also the value $k = 0$ makes sense. In that case the term is:

$$\binom{-1}{n}\binom{-1}{0}(-1)^0 = (-1)^n.$$

If we call W_n the sum we are looking for, we actually have:

$$W_n = \sum_k \binom{2k-1}{n-k}\binom{3k-1}{k}(-1)^k - (-1)^n.$$

The extended sum can now be evaluated by rule (B):

$$\sum_k \binom{2k-1}{n-k}\binom{3k-1}{k}(-1)^k = [t^k]\frac{1}{1+t}\left[\left[\frac{1}{1+2w}\;\middle|\; w = u(1-w)^3\right]\;\middle|\; u = t(1+t)^2\right] =$$

$$= [t^n]\frac{1}{1+t}\cdot\frac{1+t}{1+3t} = (-3)^n.$$

In conclusion we have:

$$U_n - W_n = (-3)^n - ((-3)^n - (-1)^n) = (-1)^n,$$

and the required sum is $(-1)^n(U_n - W_n) = (-1)^n \cdot (-1)^n = 1$, as desired.

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